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THE SQUARE MODEL FOR RANDOM GROUPS

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Abstract. We introduce a new random group model called the *square model*: we quotient a free group on n generators by a random set of relations, each of which is a reduced word of length 4. We prove that, just as in the Gromov model, for densities $> 1/2$ a random group in the square model is trivial with overwhelming probability and for densities $< 1/2$ a random group is hyperbolic with overwhelming probability. Moreover, we show that for densities $d < 1/3$ a random group in the square model does not have Property (T). Inspired by the results for the triangular model, we prove that for densities $< 1/4$ in the square model, a random group is free with overwhelming probability. We also introduce abstract diagrams with fixed edges and prove a generalization of the isoperimetric inequality.

1. Introduction. Each group can be obtained by quotienting a free group by a normal subgroup generated by a set of relators. In [Gro93] Gromov introduced the notion of a random finitely presented group on $m \geq 2$ generators at density $d \in (0, 1)$. The idea was to fix a set of m generators and consider presentations with $(2m-1)^{dl}$ relators, each of which is a random reduced word of length l. Gromov investigated the properties of random groups when l goes to infinity. We say that a property occurs in the Gromov density model *with overwhelming probability* if the probability that a random group has this property converges to 1 when $l \to \infty$. Significant results of this theory are the following: for densities $> 1/2$ a random group is trivial with overwhelming probability [Oll05, Theorem 11]; for densities $< 1/2$ a random group is, with overwhelming probability, infinite, hyperbolic and torsion-free [Oll05, Theorem 11]; for densities $< 1/5$ a random group does not have Property (T) with overwhelming probability [OW11, Corollary 7.5].

One modification of Gromov's idea is the triangular model: length of relators in the presentation is always 3, but we let the number of generators go to infinity. More precisely, for a fixed density d, we consider a presentation on n generators with n^{3d} relations, each of which is a random cyclically reduced word of length 3. We say that some property occurs in the triangular model *with overwhelming probability* if the probability that a random group

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has this property converges to 1 when $n \to \infty$. This model was introduced in [Żuk96] and further studied in [KK13]. The triangular model was a way to prove that random groups in the Gromov density model for densities $> 1/3$ have Property (T) with overwhelming probability [KK13, Theorem B], [Żuk03].

We introduce a new random group model by considering a random set of relations, each of which is a random cyclically reduced word of length 4. The following notation will be used in the whole paper.

Consider the set $A_n = \{a_1, \ldots, a_n\}$, which we will treat as an alphabet. Let W_n be the set of positive words of length 4 over A_n and W'_n be the set of all cyclically reduced words of length 4 over A_n. Note that $|W_n| = n^4$ and $|W'_n| = (2n-1)^4$ up to a multiplicative constant. We will denote by F_n the free group generated by the elements of A_n. By *relators* we will understand words over generators and by *relations* equalities holding in the group.

Definition 1.1. For $d \in (0,1)$ let us choose randomly, with uniform distribution, a subset $R_n \subset W_n$ such that $|R_n| = \lfloor n^{4d} \rfloor$. Quotienting F_n by the normal closure of the set R_n, we obtain a *random group in the positive square model at density d*.

Definition 1.2. We say that property P occurs in the positive square model *with overwhelming probability* if the probability that a random group has property P converges to 1 when $n \to \infty$.

The most important group properties which we consider are: being trivial, being free, being hyperbolic and having Property (T). We prove, as in the Gromov model, that for densities $> 1/2$ a random group in the positive square model is trivial with overwhelming probability (Theorem 2.1), and that for densities $< 1/2$ a random group is hyperbolic with overwhelming probability (Corollary 3.8). Moreover, we show that for densities $< 1/3$ a random group in the positive square model does not have Property (T) (Theorem 5.1). Inspired by the results in the triangular model, we prove that for densities $< 1/4$ in the positive square model a random group is free with overwhelming probability (Theorem 4.1). We also introduce abstract diagrams with fixed edges (Definition 3.9) and prove a generalization of the isoperimetric inequality (Theorem 3.10).

In the Gromov density model the optimal density threshold for not having Property (T) is not known. This model seems to be much harder to analyze than the triangular model, where we know that for densities $< 1/3$ a random group is free with overwhelming probability [Oll05, Proposition 30], and for densities $> 1/3$ a random group has Property (T) with overwhelming probability [KK13, Theorem A], [Żuk03].

In our model we expect to find the critical density threshold for having Property (T) in further research. We also expect that for densities $< 1/3$ a random group in the square model can be cubulated (see [OW11] for discussion of cubulating random groups in the Gromov density model). The advantage of the positive square model is that since the length of relators is even, we can consider the hypergraphs in the presentation complex and Cayley complex of the random group (Definition 4.3), which is not possible in the triangular model. As we will see, hypergraphs are a useful tool to investigate many group-theoretic and topological properties.

We have decided to consider as relators only positive words to avoid technical annoyances, but we will show that all of our results remain true in a model where we allow all cyclically reduced words. Firstly, we will define the model:

Definition 1.3. For $d \in (0,1)$ let us choose randomly, with uniform distribution, a subset $R_n \subset W_n'$ such that $|R_n| = \lfloor (2n-1)^{4d} \rfloor$. Quotienting F_n by the normal closure of the set R_n, we obtain a *random group in the square model at density d.*

2. Triviality.

Firstly we are going to investigate a case where there are many relations. Inspired by the results in Gromov's theory, we prove that when density is greater than 1/2, random groups in positive square models are as trivial as possible (with overwhelming probability). Let us determine "how trivial" such a group can be.

Note that there is always an epimorphism of our group onto $\mathbb{Z}_4$ sending all generators to a fixed generator of $\mathbb{Z}_4$. Therefore, a random group in the positive square model cannot have order less than 4. Hence, by the *trivial* group in the positive square model we will mean $\mathbb{Z}_4$.

Our goal in this section is to prove the following theorem:

Theorem 2.1. *In the positive square model at density $d > 1/2$ a random group is trivial* (*in the sense described above*) *with overwhelming probability.*

Before providing the proof we need to introduce random graphs and recall several facts about them.

Definition 2.2 (Erdős–Rényi random graph). $G(n,m)$ is the graph obtained by sampling uniformly from all graphs with n vertices and m edges.

Definition 2.3 (Gilbert random graph). $G(n,p)$ is the random graph obtained by starting with vertex set $V = \{1,\dots,n\}$, letting $0 \le p \le 1$, and connecting each pair of vertices by an edge with probability p.

In general, m and p are functions of n. We will say that a random graph in the $G(n,p)$ model has some property *asymptotically almost surely* if the probability that this property occurs converges to 1 when $n \to \infty$.

REMARK 2.4 ([ER60]). There is a strong link between the two models. First note that the expected number of edges in the $G(n,p)$ model equals $\binom{n}{2}p$. If P is any graph property which is monotone with respect to the subgraph ordering (meaning that if A is a subgraph of B and A satisfies P, then so does B), then the statements "P holds in the Gilbert $G(n,p)$ model with overwhelming probability" and "P holds in the Erdős and Rényi $G(n, \lfloor\binom{n}{2}p\rfloor)$ model with overwhelming probability" are equivalent if $pn^2 \to \infty$.

Graph properties which are monotone in the above sense and are relevant for us are: being connected and having a cycle of odd length.

Gilbert's model is much easier for calculations than Erdős–Rényi's, so we will prove two lemmas in Gilbert's model.

LEMMA 2.5. *Let G be a random graph in the $G(n,p)$ model, where $p \geq n^{\delta-1}$ for some $\delta > 0$. Then G is connected asymptotically almost surely.*

More general statements about connectedness of random graphs can be found for example in [Bol01, Theorem 7.3], but since we repeat the argument in the proof of Lemma 2.6 we present here a simple proof of Lemma 2.5.

Proof. Denote by V the set of vertices of G. Note that disconnectedness means that there exist two nonempty sets $S, T \subset V$ such that $S \cup T = V$, $S \cap T = \emptyset$ and there is no edge between S and T. For a fixed S and T the probability that there is no edge between S and T equals $(1-p)^{|S|\cdot|T|}$. Hence the probability P_d of disconnectedness can be estimated (as of now we assume $n > 2$):

$$P_d \leq \sum_{l=1}^{n} \binom{n}{l}(1-p)^{l(n-l)}.$$

The right hand side can be estimated as follows:

$$\sum_{l=1}^{n} \binom{n}{l}(1-p)^{l(n-l)} \leq 2\sum_{l=1}^{\lceil n/2\rceil} \binom{n}{l}(1-p)^{l(n-l)} \leq 2\sum_{l=1}^{\lceil n/2\rceil} n^l(1-p)^{l\lfloor n/2\rfloor}. \tag{1}$$

From our assumption on p we know that

$$n(1-p)^{\lfloor n/2\rfloor} \leq n(1-n^{\delta-1})^{n/2-1}.$$

Let us denote $z_n = n(1-n^{\delta-1})^{n/2-1}$. Because $z_n > 0$, instead of proving that $\lim_{n\to\infty} z_n = 0$ we can prove $\lim_{n\to\infty} \ln z_n = -\infty$. It is well known that $|\ln(1-x)| > x$ for $x \in (0,1)$. Hence, we estimate

$$\ln z_n = \ln n + \left(\frac{n}{2} - 1\right)\ln\left(1 - \frac{n^\delta}{n}\right) < \ln n - \left(\frac{n}{2} - 1\right)\frac{n^\delta}{n}.$$

Therefore, $\lim_{n\to\infty} \ln z_n = -\infty$. Thus the geometric series on the right hand side of (1) converges to 0 as $n \to \infty$. ∎

LEMMA 2.6. *Let G be a random graph in the $G(n,p)$ model, where $p \geq n^{\delta-1}$ for some $\delta > 0$. Then asymptotically almost surely there is a cycle of odd length in G.*

Proof. We will estimate the probability that our graph is connected and has no cycle of odd length. First we will prove that if the graph is connected and has no cycle of odd length then it is bipartite.

Let V be the set of vertices of G. Denote by T the spanning tree of G. Such a tree exists by connectedness. Fix some vertex $v \in V$. Then we can define

$$A = \{u \in V : \text{there is a path of odd length in } T \text{ between } u \text{ and } v\},$$
$$B = \{u \in V : \text{there is a path of even length in } T \text{ between } u \text{ and } v\}.$$

The sets A and B form a partition of V: $A \cap B = \emptyset$, $A \cup B = V$. Hence, it is sufficient to show that the graph is connected and not bipartite.

The probability that there is no edge with both ends in A equals $(1-p)^{\binom{|A|}{2}}$. Thus the probability P_c that there is no cycle of odd length can be estimated by

$$P_c \leq \sum_{l=1}^{n-1} \binom{n}{l} (1-p)^{\binom{n}{l}\binom{n}{n-l}} \leq \sum_{l=1}^{n-1} n^l (1-p)^{nl}. \tag{2}$$

The summation is over all sets A and B. In the last inequality we have used the fact that $\binom{n}{l}\binom{n}{n-l} > nl$. In the proof of Lemma 2.5 we have already shown that the right hand side of (2) converges to 0 as $n \to \infty$. This ends the proof. ∎

REMARK 2.7. Let G be a connected graph that has a cycle of odd length. Let x, y be vertices of G. Then there exists an edge path in G of even length joining x and y.

Proof. Denote by γ_c the closed edge path in G of odd length. Let v be the beginning vertex of γ_c. From connectedness of G there exist edge paths γ_{xv} from x to v, and γ_{vy} from v to y. If $|\gamma_{xv} \cup \gamma_{vy}|$ is even, then $\gamma_{xv} \cup \gamma_{vy}$ is the desired edge path joining x and y. If $|\gamma_{xv} \cup \gamma_{vy}|$ is odd, then $\gamma_{xv} \cup \gamma_c \cup \gamma_{vy}$ is as desired. ∎

Now we are able to prove our main statement.

Proof of Theorem 2.1. Let D_n be the set of positive words of length 2 over A_n, i.e. $D_n = \{a_i a_j \mid i, j = 1, \dots, n\}$. The set of positive words of length 4 over A_n coincides with the set of positive words of length 2 over D_n.

Let R be the set of relators in the presentation of the random group, and denote by R_0 the set of elements of R of the form $a_i a_j a_i a_j$ for $1 \leq i, j \leq n$. Let $P_{k,n}$ be the conditional probability that the group $\langle A_n \mid R - R_0 \rangle$ is trivial under the condition that $|R_0| = k$. We denote by $\widetilde{P}_{k,n}$ the probability that

$|R_0| = k$. Then from the Bayes formula the probability that a random group is trivial is greater than

$$\sum_{k=0}^{n^2} \widetilde{P}_{k,n} P_{k,n}. \tag{3}$$

It can be easily seen that $P_{0,n} > P_{1,n} > \cdots > P_{n^2,n}$ and $\sum_{k=0}^{n^2} \widetilde{P}_{k,n} = 1$. We will prove that $P_{n^2,n} \to 1$ when $n \to \infty$, which will imply that (3) converges to 1 when $n \to \infty$.

Let us assume that $|R_0| = n^2$. Consider the following graph G with the set of vertices D_n: when the relator $a_i a_j a_k a_l$ belonging to $R - R_0$ is drawn, we add the edge in G connecting the vertices $a_i a_j$ and $a_k a_l$. Thus G is a random graph in the $G(n^2, \lfloor n^{4d} \rfloor - n^2)$ Erdős and Rényi model.

Let us consider a random graph G' in the $G\big(n^2, 1/\binom{n}{2}(\lfloor n^{4d} \rfloor - n^2)\big)$ Gilbert model. From Lemmas 2.5 and 2.6 we know that asymptotically almost surely the graph G' is connected and has a cycle of odd length. Hence, from Remark 2.4 asymptotically almost surely the graph G is connected and has a cycle of odd length. Therefore, from Remark 2.7 asymptotically almost surely for any two vertices of G there is a path of even length joining them.

An edge between the vertices $a_i a_j$ and $a_k a_l$ of G corresponds to the relation $a_i a_j = (a_k a_l)^{-1}$ in our random group. An adjacent edge connecting $a_k a_l$ and $a_t a_s$ implies that $a_i a_j = a_t a_s$. Therefore, by induction, if there is a path of even length joining $a_i a_j$ and $a_k a_l$ then $a_i a_j = a_k a_l$. According to the previous observations about G this means that with overwhelming probability all words $a_i a_j$ are equal. In particular, for any i, j, k we have $a_i a_k = a_j a_k$, which implies that $a_i = a_j$. Therefore, all generators are equal. This ends the proof. ■

2.1. Triviality in the square model. Note that in the square model there is always an epimorphism of a random group onto $\mathbb{Z}_2$ sending all generators to the nontrivial element of $\mathbb{Z}_2$. Hence, by the *trivial* group in the square model we will mean $\mathbb{Z}_2$.

Our goal is to prove the following

THEOREM 2.8. *In the square model at density $d > 1/2$ a random group is trivial with overwhelming probability.*

First we will prove

LEMMA 2.9. *Let $W_n \subset W'_n$ be the set of positive words of length 4 over A_n. Let $G = \langle A_n \mid R_n \rangle$ be the random group in the square model at density d. Then, for any $d' < d$,*

$$\mathbb{P}(|R_n \cap W_n| > n^{4d'}) \to 1 \quad \text{as } n \to \infty.$$

Proof. First we will prove that

$$\mathbb{P}\big(|R_n \cap (W'_n - W_n)| > \tfrac{16}{17}\lfloor(2n-1)^{4d}\rfloor\big) \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Drawing at random a set of relators can be treated as sampling without replacement: we draw the first relator r_1, then from the set $W'_n - \{r_1\}$ we draw the next relator r_2 and inductively in the kth step we draw r_k from $W'_n - \{r_1, \dots, r_{k-1}\}$. Thus, we obtain a sequence of $\lfloor(2n-1)^{4d}\rfloor$ randomly selected relators $r_1, \dots, r_{\lfloor(2n-1)^{4d}\rfloor}$ and we define $R_n := \{r_1, \dots, r_{\lfloor(2n-1)^{4d}\rfloor}\}$.

For $1 \le i \le |R_n|$ we define a random variable X_i by setting $X_i = 1$ when $r_i \in W'_n - W_n$ and 0 otherwise. Note that $|W'_n - W_n|/|W'_n| < 15/16$. Hence for each $1 \le i \le |R_n|$ we have $\mathbb{E}X_i < 15/16$. Let $S := \sum_{i=1}^{|R_n|} X_i$. By [Ser74, Corollary 1.1] for $\mu = |W_n|/|W'_n|$, $a = 1$, $b = 1$ and $f_* = 0$ we obtain

$$\mathbb{P}(S \ge \lfloor(2n-1)^{4d}\rfloor(\mu + t)) < \exp(-2\lfloor(2n-1)^{4d}\rfloor t^2).$$

Let $t = 16/17 - \mu$. Then

$$\mathbb{P}\big(S \ge \lfloor(2n-1)^{4d}\rfloor \tfrac{16}{17}\big) < \exp(-2\lfloor(2n-1)^{4d}\rfloor t^2). \tag{5}$$

Since $\mu < 15/16$ we have $t > 16/17 - 15/16$ and the right hand side of (5) converges to 0 as $n \to \infty$. Observe that $S = |R_n \cap (W'_n - W_n)|$, so the proof of (4) is complete. Consequently,

$$\mathbb{P}\big(|R_n \cap W_n|) > \tfrac{1}{17}\lfloor(2n-1)^{4d}\rfloor\big) \xrightarrow[n\to\infty]{} 1. \tag{6}$$

Since for sufficiently large n we have $\frac{1}{17}\lfloor(2n-1)^{4d}\rfloor > n^{4d'}$, the proof of Lemma 2.9 is complete. ∎

Proof of Theorem 2.8. Let G be a random group in the square model at density $d > 1/2$. Choose any $1/2 < d' < d$. Lemma 2.9 guarantees that there are at least $n^{4d'}$ positive relators in the presentation of G. From the proof of Theorem 2.1 we know that it is sufficient to show that all generators of G are equal. Moreover, with overwhelming probability, the random set of relators at a positive density contains a word of the form $a_i a_j a_k a_l^{-1}$, where $1 \le i, j, k, l \le n$. Combining this with the fact that all generators are equal gives $a_1^2 = e$. Hence, G is generated by one element of order 2, thus it is $\mathbb{Z}_2$. ∎

3. Isoperimetric inequality. In this section we introduce van Kampen diagrams, then prove the "isoperimetric inequality" and discuss its consequences. As we will see, it implies (with overwhelming probability) freeness of random groups for densities $< 1/4$ and hyperbolicity for densities $< 1/2$.

The van Kampen diagram represents in a geometric way how all equalities holding in a group are derived from combinations of relators. The definitions and notation below largely follow [LS77, Ch. V].

Definition 3.1 (Van Kampen diagrams). Let $G = \langle A \mid R \rangle$ be a group presentation where all $r \in R$ are cyclically reduced words in the free group $F(A)$. We will denote this presentation by (†). The alphabet A and the set R of defining relations are often assumed to be finite, which corresponds to a finite group presentation, but this assumption is not necessary for the general definition of a van Kampen diagram. Let R_* be the symmetrized closure of R, that is, R_* is obtained from R by adding all cyclic permutations of elements of R and of their inverses.

A *van Kampen diagram* over the presentation (†) is a planar finite cell complex $\mathcal{D}$ together with a specific embedding $\mathcal{D} \subseteq \mathbb{R}^2$ and the following data, satisfying additional properties:

1. The complex $\mathcal{D}$ is connected and simply connected.
2. Each edge (1-cell) of $\mathcal{D}$ is given some orientation and is labelled by a letter $a \in A$.
3. Some vertex (0-cell) which belongs to the topological boundary of $\mathcal{D} \subseteq \mathbb{R}^2$ is specified as a base-vertex.
4. For each face (2-cell) of $\mathcal{D}$ and every vertex of the boundary cycle of that face and for each of the two choices of direction (clockwise or counter-clockwise) the label of the boundary cycle of the face read from that vertex and in that direction is a freely reduced word in $F(A)$ that belongs to R_*.

A van Kampen diagram $\mathcal{D}$ is called *nonreduced* if there exists a *reduction pair* in $\mathcal{D}$, that is, a pair of distinct faces of $\mathcal{D}$ such that their boundary cycles share a common edge and these boundary cycles, read starting from that edge, clockwise for one of the faces and counter-clockwise for the other, are equal as words in $A \cup A^{-1}$. If no such pair of faces exists, $\mathcal{D}$ is called *reduced*.

An *internal edge* is an edge e such that $\mathrm{Int}(e) \subset \mathrm{Int}(\mathcal{D})$. An *internal vertex* is a vertex contained in $\mathrm{Int}(\mathcal{D})$.

By $|\partial\mathcal{D}|$ we will denote the length of the boundary word of the diagram $\mathcal{D}$, and by $|\mathcal{D}|$ the number of faces of $\mathcal{D}$.

The following two variations of the concept of the van Kampen diagram will be needed in the proof of Theorem 4.5 in the next section.

Definition 3.2. If in Definition 3.1 we replace simple connectedness and planarity conditions with the assumption that the diagram $\mathcal{D}$ is homeomorphic to an annulus or a Möbius strip, we get the definitions of an *annular diagram* and a *twisted diagram* respectively.

An important theorem of van Kampen states that the boundary words of van Kampen diagrams are exactly those words which are equal to the identity element in the presentation.

LEMMA 3.3. *If in the presentation $\langle A_n \mid R\rangle$ the set of relators R consists of only positive words then for every van Kampen diagram $\mathcal{D}$ with respect to this presentation each internal vertex of $\mathcal{D}$ has even valence.*

Proof. Let v be a vertex of $\mathcal{D}$ and denote by $e_1, \ldots, e_k$ the consecutive edges of $\mathcal{D}$ ending in v. For $1 \leq i \leq k$ let F_i be the face containing the edges e_i and e_{i+1} (where $e_{k+1} = e_1$). Since the presentation consists of only positive words, for $1 \leq i \leq k-1$ the faces F_i and F_{i+1} have opposite orientations and similarly F_k and F_1 have opposite orientations. Therefore, the number of faces must be even, which implies that k is an even number. ■

This lemma will be useful in the proof of Theorem 5.10. Now we can prove the following theorem (inspired by [Oll05, Theorem 13]), which we will call the "isoperimetric inequality in the positive square model".

THEOREM 3.4. *For any $\varepsilon > 0$, in the positive square model at density $d < 1/2$, with overwhelming probability, all reduced van Kampen diagrams associated to the group presentation satisfy*

$$|\partial D| \geq 4(1-2d-\varepsilon)|D|.$$

Proving Theorem 3.4 we will be mimicking the proof of the analogous theorem in Gromov's model. There are only a few details that have to be changed. Let us recall some definitions and propositions from "Proof of the density one half theorem" in [Oll05].

An *abstract diagram* $\mathcal{A}$ is a van Kampen diagram in which we forget the actual relators associated with the faces, but only remember: the geometry of the diagram, which faces bear the same relator, the orientation, and the starting points of the relators. More precisely, there is a number N (called the *number of distinct abstract relators*), an epimorphism from the set of faces of $\mathcal{A}$ onto $\{1, \ldots, N\}$ (called the *set of abstract relators*) and for every given face of $\mathcal{A}$: one of its vertices (called the *starting point*) and the orientation of that face.

An N-tuple $(w_1, \ldots, w_N)$ of cyclically reduced words is said to *fulfill the abstract diagram* $\mathcal{A}$ if there exists a van Kampen diagram $\mathcal{D}$ formed by polygons bearing these words, such that after prescribing for each $1 \leq i \leq N$ an abstract relator i to all faces bearing the word w_i and also prescribing to every face the starting point and the orientation of its labeling word and then forgetting the actual relators, we obtain $\mathcal{A}$. An abstract diagram is *reduced* if no edge is adjacent to two faces bearing the same relator with opposite orientations such that the edge is the kth edge of both faces for some k. See [Oll05, p. 83] for details. The following proposition is inspired by [Oll05, Proposition 58].

PROPOSITION 3.5. *Let R be a set of $\lfloor n^{dl} \rfloor$ relators chosen randomly, with uniform distribution, from the set of positive words of length l on n*

generators. Let D be a reduced abstract diagram and let $\varepsilon > 0$. Then either $|\partial D| \geq |D|l(1-2d-2\varepsilon)$ or the probability that there exists a tuple of relators in R fulfilling D is less than $n^{-\varepsilon l}$.

We will present the proof of this proposition in a more general form in Subsection 3.2. Now we will recall the theorem known as the "local-global principle" or the "Gromov–Cartan–Hadamard theorem". This principle has many different formulations. The variant best suited to our context is [Oll05, Theorem 60], which is a slight modification of [Oll07, Theorem 8].

DEFINITION 3.6. For a van Kampen diagram D we define its *area* by

$$A(D) := \sum_{f \text{ face of } D} |\partial f|,$$

where $|\partial f|$ is the length of the boundary path of the face f.

THEOREM 3.7 (The "local-global principle"). *Let $G = \langle a_1, \ldots, a_n \mid R \rangle$ be a finite group presentation and let l_1 and l_2 be the minimal and maximal lengths of relators in R. Let $C > 0$. Choose $\varepsilon > 0$. Suppose that for some $K > 10^{50}(l_2/l_1)^3\varepsilon^{-2}C^{-3}$ any reduced van Kampen diagram D with $A(D) \leq Kl_2$ satisfies*

$$|\partial D| \geq CA(D).$$

Then any reduced van Kampen diagram D satisfies

$$|\partial D| \geq (C-\varepsilon)A(D)$$

and in particular the group is hyperbolic.

3.1. Isoperimetric inequality in the positive square model. Now we are able to prove Theorem 3.4. Our argument is very close to Ollivier's.

Proof of Theorem 3.4. In our case all relators in the presentation have the same length $l = 4$, so $A(D) = 4|D|$. In particular, the assumption $A(D) \leq Kl_2$ in the Theorem 3.7 becomes $|D| \leq K$, i.e. we have to check diagrams with at most K faces.

Choose any $\varepsilon > 0$. Set $C = 1-2d-2\varepsilon$ and $K = 10^{50}\varepsilon^{-2}(1-2d-2\varepsilon)^{-3}$. Let n be the number of letters in the generating alphabet. Let $N(K,n)$ be the number of abstract reduced diagrams with at most K square faces. We know from Proposition 3.5 that for any fixed reduced abstract diagram D violating the inequality $|\partial D| \geq 4(1-2d-2\varepsilon)|D|$ the probability that it appears as a van Kampen diagram of the presentation is $\leq n^{-4\varepsilon}$. So the probability that there exists a reduced van Kampen diagram with at most K faces and violating the inequality is $\leq N(K,n)n^{-4\varepsilon}$.

Observe that there are finitely many planar diagrams with at most K square faces. There are also finitely many ways to decide which faces would bear the same relator, and also finitely many ways to choose the starting point of each relator. Therefore, the values $\{N(K,n)\}_{\{n\in\mathbb{N}\}}$ have a uniform bound M (independent of n).

Hence, for fixed ε, $\lim_{n\to\infty} N(K,n)n^{-4\varepsilon} = 0$. Applying Theorem 3.7 (with our choice of C and ε) shows that all reduced van Kampen diagrams D satisfy $|\partial D| \geq 4(1-2d-3\varepsilon)|D|$ as needed. ∎

Corollary 3.8. *In the positive square model at density $d < 1/2$ a random group is hyperbolic with overwhelming probability.*

3.2. A generalization of the isoperimetric inequality. In this section we consider a more general case where there are some fixed letters in the diagrams. These considerations, and in particular Theorem 3.10, will be used later in Section 5, in the proof of Theorem 5.10.

Definition 3.9. Let $\mathcal{A}$ be an abstract diagram. Let $e_1,\dots,e_K$ be a set of K distinct edges in $\mathcal{A}$. We call them *fixed edges* (*of* $\mathcal{A}$). Let $a_{i_1},\dots,a_{i_K}$ be a sequence of generators labeling the edges $e_1,\dots,e_K$ successively and according to the orientation of the face or faces containing e_j. We call these generators *fixed letters* (*of* $\mathcal{A}$). We call such a diagram with some labellings on edges an *abstract diagram with K fixed edges* (*in the positive square model*).

We say that a tuple of relators *fulfills* $\mathcal{A}$ if this tuple fulfills $\mathcal{A}$ as an abstract diagram and this fulfilling is consistent with the labels on the fixed edges. For k not larger than the number of distinct abstract relators in $\mathcal{A}$ we say that a k-tuple $(w_1,\dots,w_k)$ of words *partially fulfills* $\mathcal{A}$ if the edges of $\mathcal{A}$ can be labelled with the generators in a way that all faces labelled by an abstract relator $1\leq i\leq k$ bear the word w_i consistently with the prescribed starting points, orientations and fixed letters.

Our goal is to prove the following statement.

Theorem 3.10. *Let R be a set of $\lfloor n^{4d}\rfloor$ relators chosen randomly, with uniform distribution, from the set of positive words of length 4 on n generators. Let $\mathcal{A}$ be an abstract diagram with K fixed edges and let $\varepsilon > 0$. Then either $|\partial\mathcal{A}| - 2K \geq |\mathcal{A}|l(1-2d-\varepsilon)$ or the probability that there exists a tuple of relators in R fulfilling $\mathcal{A}$ is $< n^{-4\varepsilon}$.*

Our proof is based on the proof of [Oll05, Proposition 58]. To prove our theorem we need some more definitions. Let N be the number of distinct abstract relators in $\mathcal{A}$. For $1\leq i\leq N$ let m_i be the number of faces bearing relator i. Up to reordering the relators we can suppose that $m_1\geq\cdots\geq m_N$.

For $1\leq i_1,i_2\leq N$ and $1\leq k_1,k_2\leq 4$ we say that $(i_1,k_1) > (i_2,k_2)$ if $i_1 > i_2$ or $i_1 = i_2$ but $k_1 > k_2$. Let e be an edge of $\mathcal{A}$ adjacent to faces f_1 and f_2 bearing relators i_1 and i_2, which is the k_1th edge of f_1 and the k_2th edge of f_2. If e is not a fixed edge we say that e *belongs* to f_1 if $(i_1,k_1) > (i_2,k_2)$, and e *belongs* to f_2 if $(i_2,k_2) > (i_1,k_1)$. If e is a fixed edge then we say that *e belongs* to both faces: f_1 and f_2. If e' is a fixed edge which is adjacent to a face f and e' is a boundary edge, then we say that e' *belongs* to f.

Note that since $\mathcal{A}$ is reduced, each internal edge which is not a fixed edge belongs to some face: indeed, if $(i_1, k_1) = (i_2, k_2)$ then either the corresponding two faces have opposite orientation and then $\mathcal{A}$ is not reduced, or they have the same orientation and the diagram is never fulfillable since a letter would be its own inverse.

Let $\delta(f)$ be the number of edges belonging to the face f. Since each internal edge which is not a fixed edge belongs to some face, we have

$$|\partial\mathcal{A}| - 2K = 4|\mathcal{A}| - 2 \sum_{f \text{ face of } \mathcal{A}} \delta(f).$$

For $1 \le i \le N$ let

$$\kappa_i = \max_{f \text{ face bearing relator } i} \delta(f).$$

Then

$$|\partial\mathcal{A}| - 2K \ge 4|\mathcal{A}| - 2 \sum_{1 \le i \le N} m_i \kappa_i. \tag{7}$$

LEMMA 3.11. *For $1 \le i \le N$ let p_i be the probability that i randomly chosen positive words $w_1, \ldots, w_i$ partially fulfill $\mathcal{A}$ and let $p_0 = 1$. Then*

$$\frac{p_i}{p_{i-1}} \le n^{-\kappa_i}. \tag{8}$$

Proof. Suppose that $i - 1$ words $w_1, \ldots, w_{i-1}$ partially fulfilling $\mathcal{A}$ are given. Then successively choose the letters of the word w_i so as to fulfill the diagram. Let $k \le 4$ and suppose that the first $k - 1$ letters of w_i are chosen. Let f be the face realizing the maximum of κ_i and let e be the kth edge of f.

If e belongs to f, this means that there is another face f' meeting e which bears relator $i' < i$ or bears i too, but u appears in f' as the k'th edge for some $k' < k$ or e is a fixed edge. In all these cases the letter on the edge e is imposed by some letter already chosen, so drawing it at random has probability $\le 1/n$.

Combining all these observations we see that the probability that the correct word w_i is chosen at random is at most $p_{i-1} n^{-\kappa_i}$. ■

Proof of Theorem 3.10. Recall that N denotes the number of distinct abstract relators in $\mathcal{A}$. For $1 \le i \le N$ let P_i be the probability that there exists an i-tuple of words partially fulfilling $\mathcal{A}$ in the random set of relators R. We trivially have

$$P_i \le |R|^i p_i = n^{4id} p_i, \tag{9}$$

where p_i are as in Lemma 3.11. Combining (7) and (8) we get

$$\begin{aligned}|\partial\mathcal{A}| - 2K &\geq 4|\mathcal{A}| + 2\sum_{i=1}^{N} m_i(\log_n p_i - \log_n p_{i-1}) \\ &= 4|\mathcal{A}| + 2\sum_{i=1}^{N-1}(m_i - m_{i+1})\log_n p_i + 2m_N \log_n p_N - 2m_1 \log_n p_0.\end{aligned}$$

Now $p_0 = 1$, so $\log_n p_0 = 0$ and we have

$$|\partial\mathcal{A}| - 2K \geq 4|\mathcal{A}| + 2\sum_{i=1}^{N-1}(m_i - m_{i+1})\log_n p_i + 2m_N \log_n p_N.$$

Using (9) and again the fact that $m_{i+1} \leq m_i$ we obtain

$$|\partial\mathcal{A}| - 2K \geq 4|\mathcal{A}| + 2\sum_{i=1}^{N-1}(m_i - m_{i+1})(\log_n P_i - 4id) + 2m_N \log_n(P_N - 4Nd).$$

Observe that $\sum_{i=1}^{N-1}(m_i - m_{i+1})id + m_N Nd = d\sum_{i=1}^{N} m_i = d|\mathcal{A}|$. Hence

$$|\partial\mathcal{A}| - 2K \geq 4|\mathcal{A}|(1-2d) + 2\sum_{i=1}^{N-1}(m_i - m_{i+1})\log_n P_i + 2m_N \log_n P_N.$$

Setting $P = \min_i P_i$ and using $m_i \geq m_{i+1}$ we get

$$\begin{aligned}|\partial\mathcal{A}| - 2K &\geq 4|\mathcal{A}|(1-2d) + 2(\log_n P)\sum_{i=1}^{N-1}(m_i - m_{i+1}) + 2m_N \log_n P \\ &= 4|\mathcal{A}|(1-2d) + 2m_1 \log_n P \geq |\mathcal{A}|(4(1-2d) + 2\log_n P),\end{aligned}$$

since $m_1 \leq |\mathcal{A}|$. Of course a diagram is fulfillable if it is partially fulfillable for any $i \leq N$, and so

$$\text{Prob}(\mathcal{A} \text{ is fullfillable by relators of } R) \leq P \leq n^{\frac{1}{2}\left(\frac{|\partial\mathcal{A}|-2K}{|\mathcal{A}|} - 4(1-2d)\right)},$$

which was to be proven. ■

Proposition 3.5 follows as a special case of Theorem 3.10 when there are no fixed letters.

3.3. Isoperimetric inequality in the square model. In this section we are going to prove that all previous results remain true in the square model. We start with

Definition 3.12. If in Definition 3.9 we replace the assumption that fixed edges are labelled by generators with the assumption that fixed edges are labelled by generators and their inverses, we obtain the definition of an *abstract diagram with K fixed edges* (*in the square model*).

THEOREM 3.13. *Let R be a set of $\lfloor (2n-1)^{4d} \rfloor$ relators chosen randomly, with uniform distribution, from the set of cyclically reduced words of length 4 on n generators. Let $\mathcal{A}$ be an abstract diagram with K fixed edges and let $\varepsilon > 0$. Then either $|\partial\mathcal{A}| - 2K \geq |\mathcal{A}|4(1-2d-\varepsilon)$ or the probability that there exists a tuple of relators in R fulfilling $\mathcal{A}$ is $< (2n-1)^{-4\varepsilon}$.*

Actually the proof of this theorem goes in complete analogy to the proof of Theorem 3.10. Again, let N be the number of distinct abstract relators in $\mathcal{A}$, and for $1 \leq i \leq N$ let m_i be the number of faces bearing relator i. Up to reordering the abstract relators we can assume that $m_1 \geq \cdots \geq m_N$. Again, let $\delta(f)$ be the number of edges belonging to the face f, and for $1 \leq i \leq N$ we define

$$\kappa_i = \max_{f \text{ face bearing relator } i} \delta(f).$$

Then we have (as in the previous section)

$$|\partial\mathcal{A}| - 2K \geq 4|\mathcal{A}| - 2 \sum_{1 \leq i \leq N} m_i \kappa_i. \tag{10}$$

LEMMA 3.14. *For $1 \leq i \leq N$ let p_i be the probability that i randomly chosen cyclically reduced words $w_1, \ldots, w_i$ partially fulfill $\mathcal{A}$ and let $p_0 = 1$. Then*

$$\frac{p_i}{p_{i-1}} \leq (2n-1)^{-\kappa_i}. \tag{11}$$

Proof. To prove this lemma one has to change only one thing in the proof of Lemma 3.11: replace n with $2n-1$. ∎

Proof of Theorem 3.13. For $1 \leq i \leq N$ let P_i be the probability that there exists an i-tuple of words partially fulfilling $\mathcal{A}$ in the random set of relators. We have (analogously to (9)):

$$P_i \leq |R|^i p_i \leq (2n-1)^{dl} p_i. \tag{12}$$

Repeating the reasoning in the proof of Theorem 3.10 but replacing n with $2n-1$ we obtain

$$\mathrm{Prob}(\mathcal{A} \text{ is fullfillable by relators of } R) \leq (2n-1)^{\frac{1}{2}\left(\frac{|\partial\mathcal{A}|-2K}{|\mathcal{A}|} - 4(1-2d)\right)},$$

which was to be proven. ∎

Now we can prove the isoperimetric inequality for the square model:

THEOREM 3.15. *For any $\varepsilon > 0$, in the square model at density $d < 1/2$ with overwhelming probability all reduced van Kampen diagrams associated to the group presentation satisfy*

$$|\partial D| \geq 4(1-2d-\varepsilon)|D|.$$

Proof. The proof is completely analogous to the proof of Theorem 3.4: the only change is that we have to use Theorem 3.13 (for $K = 0$) instead of Proposition 3.5, to find that for any fixed reduced abstract diagram D violating the inequality $|\partial D| \geq 4(1-2d-2\varepsilon)|D|$ the probability that it appears as a van Kampen diagram of the presentation is $\leq (2n-1)^{-4\varepsilon}$, which is $< n^{-4\varepsilon}$. ∎

COROLLARY 3.16. *In the square model at density $d < 1/2$ a random group is hyperbolic with overwhelming probability.*

4. Freeness. In this section we are going to consider the case where density of relations is small. Our goal is to prove the following statement:

THEOREM 4.1 (Freeness theorem). *In the positive square model at density $d < 1/4$ a random group is free with overwhelming probability.*

To prove this theorem we will introduce several geometric objects:

DEFINITION 4.2. A *square complex* is a metric polyhedral complex in which each cell is isometric to the Euclidean square $[-1/2, 1/2]^2$ and the gluing maps are isometries.

Observe that we allow gluing a cell to itself and gluing two cells several times along distinct pairs of faces. Notice that van Kampen diagrams in the positive square model are square complexes.

Now we will introduce a useful tool in geometric group theory: hypergraphs. The following definitions are taken from [OW11, Definition 2.1].

DEFINITION 4.3. Let X be a connected square complex. We define a graph Γ as follows: The set of vertices of Γ is the set of 1-cells of X. There is an edge in Γ between two vertices if there is some 2-cell R of X such that the vertices correspond to opposite 1-cells in the boundary of R (if there are several such 2-cells, we put as many edges in Γ). The 2-cell R is the 2-cell of X *containing* the edge.

There is a natural map φ from Γ to X, which sends each vertex of Γ to the midpoint of the corresponding 1-cell of X and each edge of Γ to a segment joining two opposite points in the 2-cell R. Note that the images of two edges contained in the same 2-cell R always intersect, so that in general φ is not an embedding.

A *hypergraph* in X is a connected component of Γ. The 1-cells of X through which a hypergraph passes are *dual* to it. The hypergraph Λ *embeds* if φ is an embedding from Λ to X, that is, if no two distinct edges of Λ are mapped to the same 2-cell of X.

The *carrier* of Λ is a subspace of X equal to the union of all open 1-cells and open 2-cells of X intersected by Λ. Note that $X -$ (carrier of Λ) is a subcomplex of X, homotopically equivalent to $X - \Lambda$.

The *hypergraph segment* in X is a finite path in a hypergraph immersed into X.

In the next definition we give one of the typical ways to construct a topological space X such that $\pi_1(X) = G$ for a given finitely generated group G.

DEFINITION 4.4 (Presentation complex). Let $G = \langle A_n \mid R\rangle$ be a group generated by n elements. Consider a bouquet of n circles labelled with elements of A_n. For every relator $r \in R$ there is a polygon with as many edges as letters in r, which is glued to the bouquet in the following way: the edge labelled by $a \in A_n$ is glued to the circle with label a respecting the orientation. For a random group in the square or positive square model this construction results in a square complex, which we call the *presentation complex*.

There is a natural map from any van Kampen diagram to the presentation complex of G. One of the main steps in our proof of Theorem 4.1 is the following statement:

THEOREM 4.5. *In the positive square model at density $d < 1/4$, with overwhelming probability, all hypergraphs in the presentation complex are embedded trees.*

Proof. Denote the presentation complex of a random group by X. We will estimate the probability P of drawing the set R_n of relators for which the statement does not hold, i.e. there exists a hypergraph in X which is not an embedded tree.

Hence, assume that such a hypergraph Λ exists. The image of Λ under the natural map is not a tree, so Λ contains a *circuit* (an edge path $(e_1, \dots, e_k)$ in Λ such that images of e_1 and e_k intersect in X). Without loss of generality we can assume that k is the minimal possible length of a circuit. For i in $\{1, \dots, k\}$, let F_i be the 2-cell of X containing the edge e_i. We have chosen the circuit of minimal length, so $F_1 = F_k$ and $F_i \neq F_j$ for $i < j$, except where $i = 1, j = k$.

Let D be a diagram consisting of the faces $F_1, \dots, F_k$ glued in the following way: for $1 \le i \le k-1$ the faces F_i and F_{i+1} are glued along 1-cells which contain the common vertex of e_i and e_{i+1}. It can be easily seen that D is either an annular diagram, a twisted diagram (as introduced in Definition 3.2) or a van Kampen diagram, where the latter corresponds to the case $k = 1$. We will estimate the probability P_k of drawing the set of relators which allows one to construct a diagram with k faces and exactly $3k$ edges, and consisting of distinct relators (D has these properties).

Let E be the abstract diagram obtained from D. There are n^{3k} k-tuples of relators fulfilling E. Denote by L the set of these k-tuples. To fulfill E one of the elements $\alpha \in L$ must be a subset of the set R_n of words drawn.

For $\alpha \in L$ let P_α be the probability of drawing the set R_n which contains α. Then

$$P_\alpha = \frac{\binom{n^4-k}{\lfloor n^{4d}\rfloor - k}}{\binom{n^4}{\lfloor n^{4d}\rfloor}}.$$

Hence

$$P_\alpha = \frac{\lfloor n^{4d}\rfloor(\lfloor n^{4d}\rfloor - 1)\dots(\lfloor n^{4d}\rfloor - k+1)}{n^4(n^4-1)\dots(n^4-k+1)} < \frac{n^{k4d}}{n^{4k}}. \tag{13}$$

We have $P_k \le n^{3k} P_\alpha$. To estimate the probability of the existence of a hypergraph that is not an embedded tree, we sum P_k over all possible k:

$$P = \sum_{k=1}^{n^4} P_k \le \sum_{k=1}^{n^4} n^{3k}\frac{n^{k4d}}{n^{4k}} < \sum_{k=1}^{\infty} n^{(4d-1)k} < \frac{1}{1-n^{4d-1}} - 1. \tag{14}$$

We have assumed that $d < 1/4$, so the right hand side of (14) converges to 0 when $n \to \infty$. ∎

Let us recall one of the applications of the HNN extension construction:

THEOREM 4.6 ([SW79, Proposition 1.2]). *Let V be a Hausdorff topological space and let $Y_1, Y_2 \subset V$ be two disjoint, simply-connected and path-connected subsets such that there is a homeomorphism $f : Y_1 \to Y_2$. By $V*_f$ we denote the topological space $V/{\sim}$, where $y \sim f(y)$ for all $y \in Y_1$. Then $\pi_1(V*_f) = \pi_1(V) * \mathbb{Z}$.*

Now we are ready to prove the freeness theorem.

Proof of Theorem 4.1. From Theorem 4.5 we know that with overwhelming probability all hypergraphs in the presentation complex X are embedded trees. Let us take an arbitrary hypergraph Λ. Let H be the carrier of Λ.

Let us consider the complex $\overline{X - \Lambda}$ (by $\overline{A}$ we denote the completion of the complex A in the path metric). Note that $\overline{X - \Lambda} - (X - \Lambda)$ consists of two isometric copies of Λ, denoted Λ_1 and Λ_2. Let $\phi : \Lambda_1 \to \Lambda_2$ be the homeomorphism between Λ_1 and Λ_2 induced by the identity of Λ. The space $X - \Lambda$ is homotopically equivalent to $\overline{X - \Lambda}$ and $X - H$. Moreover, $(\overline{X - \Lambda})*_\phi$ is equal to the complex X. Hence $\pi_1(X) = \pi_1((\overline{X - \Lambda})*_\phi)$, and since $\overline{X - \Lambda}$ is connected and Λ_1, Λ_2 are disjoint subspaces of it, by Theorem 4.6 we obtain $\pi_1(X) = \pi_1(X - H) * \mathbb{Z}$.

We now perform the same procedure for the subcomplex $X_1 := X - H$. Note that X_1 is connected, and its hypergraphs are subgraphs of the hypergraphs of X, so they are also embedded trees. We choose an arbitrary hypergraph in X_1 and remove its carrier from X_1, obtaining a smaller complex X_2. By Theorem 4.6, we have $\pi_1(X) = \pi_1(X_1) * \mathbb{Z} = \pi_1(X_2) * \mathbb{Z} * \mathbb{Z}$.

We now inductively repeat this procedure. Note that the presentation complex is finite and each time we remove at least one cell, so this process must stop after a finite number m of steps. Let X_m be the subcomplex obtained after m steps. Then there are no hypergraphs in X_m. But the only square complex with no hypergraphs is the square complex consisting of one vertex, which has the trivial fundamental group. Therefore

$$\pi_1(X) = \pi_1(X_m) * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m} = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m}.$$

Hence $\pi_1(X)$ is a free group with overwhelming probability. ■

4.1. Freeness in the square model. First, we will prove the following

THEOREM 4.7. *In the square model at density $d < 1/4$, with overwhelming probability, all hypergraphs in the presentation complex are embedded trees.*

Proof. As in the proof of Theorem 4.5, we only need to prove that, with overwhelming probability, there are no diagrams with k faces and exactly $3k$ edges in the presentation complex for any k. Let E be such an abstract diagram with the minimal number of faces. We will estimate the probability P_k of drawing a set of relators such that E can be fulfilled. There are at most $(2n)^{k-1}(2n-1)^{2k-1}(2n-2)^2$ k-tuples of relators fulfilling E. Note that for any δ_1 this number is smaller than $(2n-1)^{(3+\delta_1)k}$ for sufficiently large n. Denote by L the set of these k-tuples. To fulfill E, one of the k-tuples $\alpha \in L$ must be a subset of the set R_n of words drawn.

For $\alpha \in L$ let P_α be the probability of drawing a set R_n which contains α. Then

$$P_\alpha = \frac{\binom{|W'_n|-k}{\lfloor (2n-1)^{4d}\rfloor - k}}{\binom{|W'_n|}{\lfloor (2n-1)^{4d}\rfloor}}.$$

Hence

$$P_\alpha = \frac{(\lfloor (2n-1)^{4d}\rfloor)(\lfloor (2n-1)^{4d}\rfloor - 1)\dots(\lfloor (2n-1)^{4d}\rfloor - k)}{|W'_n|(|W'_n|-1)\dots(|W'_n|-k)} \tag{15}$$
$$< \frac{(2n-1)^{k4d}}{|W'_n|^k}.$$

Note that $|W'_n| \geq 2n(2n-1)^2(2n-2)$, which, for any $\delta_2 > 0$, is greater than $(2n-1)^{4-\delta_2}$ for a sufficiently large n. Therefore, we can estimate the right hand side of (15) by $(2n-1)^{(4d-4+\delta_2)k}$. As in the proof of Theorem 4.5, we estimate the sum of P_k over all possible k:

$$P = \sum_{k=1}^{|W'_n|} P_k < \sum_{k=1}^{\infty} (2n-1)^{(4d-1+\delta_1+\delta_2)k} < \frac{1}{1-(2n-1)^{4d-1+\delta_1+\delta_2}}. \tag{16}$$

We assumed $d < 1/4$, so we can choose $\delta_1, \delta_2 > 0$ such that $4d-1+\delta_1+\delta_2 < 0$. Then the right hand side of (16) converges to 0 as $n \to \infty$. ■

Theorem 4.8 (Freeness theorem in the square model). *In the square model at density $d < 1/4$, a random group is free with overwhelming probability.*

Proof. The proof is identical to the proof of Theorem 4.1 with the only change that we use Theorem 4.7 instead of Theorem 4.5. ∎

5. Groups without Property (T). In this section our goal is to prove the following statement.

Theorem 5.1. *In the positive square model at density $d < 1/3$, with overwhelming probability a random group does not have Property* (T).

We refer the reader to [BdlHV08] for the definition and discussion of Property (T). For our purpose we only need the following criterion:

Theorem 5.2 ([NR98]). *If a group G has a subgroup H with the number of relative ends at least* 2 *then G does not have Property* (T).

We will be mimicking the proof of the analogous theorem for Gromov's model which states that for densities $< 1/5$ a random group in the Gromov density model does not have Property (T) with overwhelming probability [OW11].

Until the end of Subsection 5.3 let G be a random group in the positive square model and $\widetilde{X}$ its Cayley complex, that is, the universal cover of the presentation complex.

5.1. Hypergraphs in the Cayley complex are embedded trees. First, let us recall

Definition 5.3. The universal cover of the presentation complex of a group (introduced in Definition 4.4) we call the *Cayley complex* of that group. Note that the Cayley complex of a random group in the square or positive square model is a square complex. We will denote by $\widetilde{X}$ the Cayley complex of a random group (in both models).

Lemma 5.4. *In the positive square model for densities $< 1/3$, hypergraphs in the Cayley complex of a random group are embedded trees.*

To prove the lemma we need a notion of a collared diagram which was introduced by Ollivier and Wise to investigate hypergraphs in the Gromov model.

Definition 5.5. We say that a reduced van Kampen diagram D is a *collared diagram* if there is a vertex v in the boundary such that for any other boundary vertex there is exactly one internal edge which ends in this vertex. Moreover, we assume that v is the end of at most two internal edges.

Let us denote this set of internal edges by L. Let $\lambda \subset D$ be the hypergraph segment consisting of all edges dual to the elements of L.

If there is exactly one internal edge ending in v we say that a diagram is *cornerless*. In this case it can be easily seen that λ is a circuit.

If the diagram is collared and not cornerless then λ is not a loop, but there is a 2-cell called a *corner* which contains two edges of λ.

Moreover, there is a natural combinatorial map $\varphi : D \to \widetilde{X}$ such that the image $\varphi(\lambda)$ is a hypergraph segment in $\widetilde{X}$ (as in Definition 4.3). For such λ we say that D is *collared* by the segment $\varphi(\lambda)$. The definition is illustrated in Figure 1.

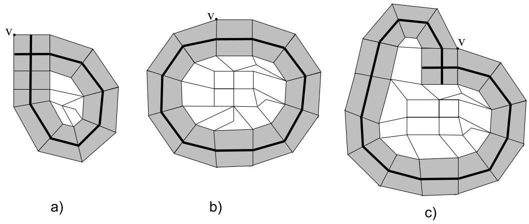

Fig. 1. Collared diagrams. The segment λ is drawn with a thick line.

In [OW11, Definition 3.11] Ollivier and Wise defined diagrams collared by hypergraphs and paths for an arbitrary length l of relators. Their definition coincides with ours for $l = 4$, the number of collaring hypergraphs equal to 1 and the number of collaring paths equal to 0. The following theorem shows the relationship between collared diagrams and hypergraphs:

THEOREM 5.6 ([OW11, Theorem 3.5]). *Let Λ be some hypergraph in $\widetilde{X}$. The following conditions are equivalent:*

(i) *Λ is an embedded tree.*
(ii) *There is no collared diagram collared by a segment of Λ.*

Proof of Lemma 5.4. Assume on the contrary that there is a hypergraph which is not an embedded tree. From Theorem 5.6 we know that there is a diagram D collared by some segment λ. Let $k = |\partial D|$. For $\varepsilon < 2(1/3 - d)$ by Theorem 3.4 (isoperimetric inequality) we have

$$k = |\partial D| \geq 4|D|(1 - 2d - \varepsilon) > \tfrac{4}{3}|D|. \tag{17}$$

We have two possibilities: either D is cornerless or not. In the first case, $|D| \geq k$. By (17) we know that with overwhelming probability all collared cornerless diagrams satisfy

$$k > \tfrac{4}{3}k, \tag{18}$$

which is a contradiction. Therefore, with overwhelming probability there are no such diagrams.

Now suppose D is not cornerless. Then $|D| \geq k-1$. We have two possibilities: $|D| \geq k$ or $|D| = k-1$. If $|D| \geq k$ we again obtain (18), which is a contradiction. Therefore, with overwhelming probability there are no such diagrams. The only remaining case is where $|\partial D| = k$ and $|D| = k-1$. Again we use (17) to obtain

$$k > \tfrac{4}{3}(k-1),$$

impossible for $k > 3$. So we only have to exclude the diagram $|D| = 2$, $|\partial D| = 3$. But there are no diagrams with odd boundary length. ∎

LEMMA 5.7 ([OW11, Lemma 2.3]). *Suppose a hypergraph Λ is an embedded tree in $\widetilde{X}$. Then $\widetilde{X} - \Lambda$ consists of two connected components.*

5.2. Pair of hypergraphs which intersect in a single point. We now introduce a new type of diagram:

DEFINITION 5.8 (Diagram collared by two segments). Let D be a reduced van Kampen diagram and let $x_1, \ldots, x_n$ be all the vertices on its boundary in that order. Suppose that for some $2 \leq i \leq n-2$ the following holds: for every $k \in \{2, \ldots, i-1\} \cup \{i+1, \ldots, n\}$ there is exactly one internal edge e_k ending in x_k. Moreover, we assume that for $v \in \{x_1, x_i\}$ there are exactly 0 or 2 internal edges ending in v.

It can be easily seen that there are two hypergraph segments λ_1, λ_2 in D such that for $k \in \{2, \ldots, i-1\}$ the edges e_k are dual to λ_1, and for $k \in \{i+1, \ldots, n\}$ the edges e_k are dual to λ_2. There are exactly two cells containing edges of both segments λ_1, λ_2, called *corners*. There is a natural combinatorial map $\varphi : D \to \widetilde{X}$ such that $\varphi(\lambda_1)$ and $\varphi(\lambda_2)$ are hypergraph segments in $\widetilde{X}$. In that case we say that D is *collared by the segments $\varphi(\lambda_1)$ and $\varphi(\lambda_2)$*.

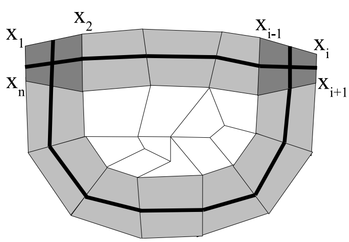

Fig. 2. Diagram collared by two segments. The segments are drawn with a solid thick line and the corners are highlighted in dark gray.

Our definition coincides with the one given by Ollivier and Wise [OW11, Definition 3.11] for the length of relators equal 4, the number of collaring segments equals 2 and the number of collaring paths is 0.

Lemma 5.9 ([OW11, Lemma 3.12]). *Let Λ_1 and Λ_2 be two distinct hypergraphs in $\widetilde{X}$ that are embedded trees. There is more than one point in $\Lambda_1 \cap \Lambda_2$ if and only if there exists a reduced diagram E collared by segments of Λ_1 and Λ_2. Moreover, if Λ_1 and Λ_2 cross at a 2-cell C, we can choose E so that C is one of its corners.*

Theorem 5.10. *With overwhelming probability, in the positive square model, there exists a pair of hypergraphs Λ_1, Λ_2 in $\widetilde{X}$ such that Λ_1 and Λ_2 intersect only in one point.*

Proof. Suppose Λ, Λ' are hypergraphs intersecting at least in two points. Assume that Λ and Λ' cross at a 2-cell C. From Theorem 5.9 we know that there exists a diagram E collared by segments $\lambda \subset \Lambda$ and $\lambda' \subset \Lambda'$ such that C is its corner. Note that $|\partial E| = |\lambda| + |\lambda'|$ and $|E| \geq |\lambda| + |\lambda'| - 2$. By Theorem 3.4 (isoperimetric inequality), with overwhelming probability,

$$|\lambda| + |\lambda'| = |\partial E| \geq \tfrac{4}{3}|E| \geq \tfrac{4}{3}(|\lambda| + |\lambda'| - 2), \tag{19}$$

which is equivalent to $|\lambda| + |\lambda'| < 8$, which implies $|\partial E| \leq 6$. Moreover, $|E| \geq |\partial E| - 2$. Therefore, all possibilities which do not violate (19) are: $|E| = 4$, $|\partial E| = 6$ and $|E| = 2, |\partial E| = 4$. There are only three 2-collared diagrams satisfying (19) (see Figure 3).

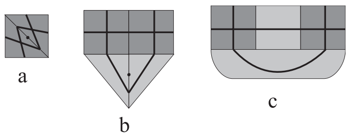

Fig. 3. Diagrams collared by two segments. The segments are drawn with a solid thick line and the corners are highlighted in dark gray.

According to Lemma 3.3 the diagram c cannot be fulfilled by positive relators. Therefore, we are left only with potential cases: a and b.

For each 2-cell $O \in \widetilde{X}$ consider two hypergraphs passing through O. Assume, contrary to our conclusion, that any two such hypergraphs have at least two intersection points. According to our previous discussion this means that for each relator r in the presentation there is a van Kampen diagram E of the form a or b (see Figure 3) such that its corner bears r.

The probability that no two hypergraphs in $\widetilde{X}$ intersect in one point is not larger than the probability that the two hypergraphs passing through any 2-cell O in $\widetilde{X}$ corresponding to the first relator r appearing in R_n do not intersect in a singe point.

We can draw relators in two steps: in the first step we draw one relator r and in the second step we draw $\lfloor n^{4d} \rfloor - 1$ remaining relators from the set $W_n - \{r\}$. This way of drawing the presentation gives us a specific relator r.

We will show that with overwhelming probability the relator r is not borne by a corner in any van Kampen diagram of the shape a or b.

The probability that there exists a van Kampen diagram of type a or b such that one of its corners bears r is the same as the probability that one of the abstract diagrams in Figure 4 (where x, y are two consecutive edges of r) can be fulfilled by a tuple from a random set of relators.

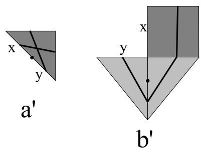

Fig. 4. Diagrams with fixed letters. x and y are two consecutive letters of r.

Note that with overwhelming probability the relator r consists of four distinct letters (in fact this probability equals $n(n-1)(n-2)(n-3)/n^4$).

To fulfill E we can use r and other relators. Observe that since with overwhelming probability, r consists of different letters and it is a positive word, the two faces bearing r cannot be adjacent (such a pair of faces would give a reduction pair).

Hence, the diagram a' can be fulfilled only by a relator different from r. Let $P_{a'}$ be the probability of fulfilling a'. Observe that a' is an abstract diagram with two fixed letters and one face. Moreover, note that $|\partial a'| - 2\cdot 2 < \frac{4}{3}|a'|$, so from Theorem 3.10 (used for $l = 4$) the probability of fulfilling a' is less than $n^{-4\varepsilon}$ for any $\varepsilon < 2(1/3 - d)$. Let us fix some $\varepsilon < 2(1/3 - d)$.

According to the previous observation about faces bearing r, there can be at most two faces bearing r in b in order to fulfill the diagram. Hence, there can be at most one face bearing r in b'.

Let $P_{b'}$ be the probability of fulfilling b' without using the relator r. The diagram b' is an abstract diagram with two fixed letters and satisfies $|\partial b'| - 2 \cdot 2 < \frac{4}{3}|b'|$, so again using Theorem 3.10 we find that $P_{b'} \leq n^{-4\varepsilon}$.

Now we will estimate the probability $P_{b''}$ of fulfilling b using the relator r twice. The only face in b' which can bear r is the right bottom face. We can therefore consider a diagram b'' where we remove this face and label the new boundary edges with three consecutive letters of r (see Figure 5).

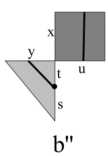

Fig. 5. Diagram with five fixed edges: x, y and s, t, u are two tuples of consecutive letters of r.

Observe that $|\partial b''| - 2 \cdot 5 < \frac{4}{3}|b''|$, so from Theorem 3.10 (used for $l = 4$) the probability of fulfilling b'' is less than $n^{-4\varepsilon}$.

For a fixed relator r there are eight possible pairs x, y and also eight possible triples s, t, u. Hence, we can estimate the probability P_r that r is a corner of a 2-collared diagram:

$$P_r < 8(P_{a'} + P_{b'} + 8P_{b''}) < 80n^{-4\varepsilon}.$$

Therefore, with overwhelming probability, the two hypergraphs passing through any 2-cell O in $\widetilde{X}$ corresponding to the first relator r appearing in R_n intersect in a single point, which ends the proof. ■

LEMMA 5.11. *In the positive square model, at density $d > 1/4$, all hypergraphs are leafless trees.*

Proof. Note that a hypergraph can have a leaf only if there exists a generator which appears in exactly one relator. Consider a fixed generator a. The number of positive words of length 4 containing a equals $(n^4 - (n-1)^4)$, so $(n^4 - (n-1)^4)\binom{(n-1)^4}{\lfloor n^{4d}\rfloor - 1}$ is the number of presentations where exactly one relator contains a. Hence the probability that a appears in exactly one relator equals

$$p = \frac{(n^4 - (n-1)^4)\binom{(n-1)^4}{\lfloor n^{4d}\rfloor - 1}}{\binom{n^4}{\lfloor n^{4d}\rfloor}}.$$

Since $d < 1$, for n large enough we have $\binom{n^4}{\lfloor n^{4d}\rfloor} \geq \binom{n^4}{\lfloor n^{4d}\rfloor - 1}$, so we estimate:

$$p \leq (n^4 - (n-1)^4)\frac{\binom{(n-1)^4}{\lfloor n^{4d}\rfloor - 1}}{\binom{n^4}{\lfloor n^{4d}\rfloor - 1}}.$$

We continue the estimation:

$$\begin{aligned}\frac{\binom{(n-1)^4}{\lfloor n^{4d}\rfloor - 1}}{\binom{n^4}{\lfloor n^{4d}\rfloor - 1}} &= \frac{(n-1)^4((n-1)^4 - 1)\dots((n-1)^4 - \lfloor n^{4d}\rfloor)}{n^4(n^4-1)\dots(n^4 - \lfloor n^{4d}\rfloor)} \\ &< \left(\frac{n-1}{n}\right)^{4(\lfloor n^{4d}\rfloor - 1)}.\end{aligned}$$

The probability that there exists a generator that is contained in exactly one relator is bounded by np. Set $z_n = n(n^4 - (n-1)^4 + 1)\left(\frac{n-1}{n}\right)^{4(\lfloor n^{4d}\rfloor - 1)}$. Note that $pn < z_n$, so it suffices to show that $\lim_{n\to\infty} \ln z_n = -\infty$. Note that $z_n < 2n^5\left(\frac{n-1}{n}\right)^{4(\lfloor n^{4d}\rfloor - 1)}$. From the fact that $|\ln(1-x)| > x$ for $x \in (0,1)$, we can estimate

$$\begin{aligned} \ln z_n &< \ln 2 + 5\ln n + 4(\lfloor n^{4d} \rfloor - 1)\ln\left(\frac{n-1}{n}\right) \\ &< \ln 2 + 5\ln n - 4(\lfloor n^{4d} \rfloor - 1)\frac{1}{n}. \end{aligned} \tag{20}$$

Since $d > 1/4$ the right hand side of (20) converges to $-\infty$ as $n \to \infty$. ∎

5.3. For densities $d < 1/3$ a random group in the positive square model does not have Property (T).

Definition 5.12. For a hypergraph Λ in $\widetilde{X}$ the *orientation preserving stabilizer* $\mathrm{Stab}^+(\Lambda)$ is the index ≤ 2 subgroup of $\mathrm{Stab}(\Lambda)$ that also stabilizes each of the two components of $\widetilde{X} - \Lambda$.

We now recall the following

Lemma 5.13 ([OW11, Lemma 7.2]). *Suppose that a group G acts cocompactly and freely on $\widetilde{X}$, and the system of hypergraphs in $\widetilde{X}$ is locally finite and cocompact (meaning that the hypergraphs in $\widetilde{X}/G$ are compact and finitely many). Suppose that two distinct leafless hypergraphs Λ_1 and Λ_2, which are embedded trees, intersect in a single point.*

Then for $i = 1, 2$ the group $H_i = \mathrm{Stab}^+(\Lambda_i)$ is a subgroup of G with relative number of ends $e(G, H_i) = 2$.

Theorem 5.14. *In the positive square model at density $1/4 < d < 1/3$, with overwhelming probability a random group G has a subgroup H which is free, quasiconvex and such that the relative number of ends $e(G, H)$ is at least 2. In particular with overwhelming probability G does not have Property (T).*

Proof. By Lemma 5.4 we know that with overwhelming probability hypergraphs in $\widetilde{X}$ are embedded trees.

The presentation complex X of G is finite since our group is finitely presented, so G acts cocompactly on $\widetilde{X}$ and the system of hypergraphs is locally finite and cocompact. By Theorem 5.10 we know that with overwhelming probability there is a pair of hypergraphs which intersect in exactly one point, and by Lemma 5.11 with overwhelming probability all hypergraphs are leafless trees.

Hence, by Lemma 5.13 with overwhelming probability there is a subgroup H in G such that the relative number of ends $e(G, H)$ is at least 2.

Finally using Theorem 5.2 we conclude that G does not have Property (T). ∎

To prove Theorem 5.1 we need the following

Lemma 5.15. *Suppose that for some density $d \in (0,1)$ a random group in the positive square model does not have Property* (T) *with overwhelming probability. Then for any $0 < d' < d$ a random group in the positive square model at density d' does not have Property* (T) *with overwhelming probability.*

Proof. Denote by $R_{n,d}$ a random set of n^{4d} positive relators of length 4. Let $\varphi(R_{n,d})$ be the restriction of the set $R_{n,d}$ to the first $n^{4d'}$ relators. It can be easily seen that $\varphi(R_{n,d})$ is a random set of $n^{4d'}$ relators with the uniform distribution. Therefore a random group in the positive square model at density d' is a group with presentation $\langle A_n \mid \varphi(R_{n,d})\rangle$.

Now, note that Property (T) is preserved by epimorphisms, meaning that the image of a group with Property (T) has Property (T) as well. Hence, removing relations from the presentation of a group without Property (T) results in a group without Property (T). Therefore, the probability that a group with presentation $\langle A_n \mid \varphi(R_{n,d})\rangle$ does not have Property (T) is not smaller than the probability that a group with presentation $\langle A_n \mid R_{n,d}\rangle$ does not have Property (T). From our assumption we know that the latter tends to 1 as $n \to \infty$. ∎

Theorem 5.1 follows from Theorem 5.14 combined with Lemma 5.15.

5.4. Groups without Property (T) in the square model. We proved in Section 3.3 that the isoperimetric inequality holds in the square model. Some of the lemmas and theorems proved earlier in this section can be generalized to the square model.

Lemma 5.16. *In the square model for densities $< 1/3$, hypergraphs in the Cayley complex of a random group are embedded trees.*

Proof. The proof is identical to the proof of Lemma 5.4. ∎

Lemma 5.17. *In the square model at density $d > 1/4$ all hypergraphs in the Cayley complex of a random group are leafless trees.*

Proof. The proof is analogous to the proof of Lemma 5.11. ∎

Theorem 5.18. *With overwhelming probability, in the square model at density $1/4 < d < 1/3$ there exists a pair of hypergraphs Λ_1, Λ_2 in the Cayley complex of a random group such that Λ_1 and Λ_2 intersect in one point.*

Proof. The proof is analogous to the proof of Theorem 5.10. ∎

Lemma 5.19. *Suppose that for some density $d \in (0,1)$ a random group in the square model does not have Property* (T) *with overwhelming probability.*

Then for any $0 < d' < d$ *a random group in the square model at density* d' *does not have Property* (T) *with overwhelming probability.*

Proof. The proof is analogous to the proof of Lemma 5.15. ■

Combining Theorem 5.18 with Lemma 5.17, by Lemma 5.13, Theorem 5.2 and Lemma 5.19 we obtain the following:

THEOREM 5.20. *In the square model at density* $d < 1/3$ *a random group does not have Property* (T) *with overwhelming probability.*

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