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SOME NEW INFINITE FAMILIES OF CONGRUENCES MODULO 3 FOR OVERPARTITIONS INTO ODD PARTS

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Abstract. Let $\bar{p}_o(n)$ denote the number of overpartitions of n in which only odd parts are used. Some congruences modulo 3 and powers of 2 for the function $\bar{p}_o(n)$ have been derived by Hirschhorn and Sellers, and Lovejoy and Osburn. In this paper, employing 2-dissections of certain quotients of theta functions due to Ramanujan, we prove some new infinite families of Ramanujan-type congruences for $\bar{p}_o(n)$ modulo 3. For example, we prove that for $n, \alpha \geq 0$,

$$\bar{p}_o(4^\alpha(24n+17)) \equiv \bar{p}_o(4^\alpha(24n+23)) \equiv 0 \pmod 3.$$

1. Introduction. The aim of this paper is to establish some infinite families of congruences modulo 3 for the number of overpartitions in which only odd parts are used.

We start with an overview of the terminology and notation on partitions and q-series. An *overpartition* of n is a partition of n where we may overline the first occurrence of a part (see Corteel and Lovejoy [CL]). Arithmetic properties of overpartitions have been established in, for example, Chen and Xia [CX], Fortin, Jacob and Mathieu [FJM], Hirschhorn and Sellers [HS-1], Kim [K], Lovejoy and Osburn [LO], Mahlburg [M], and Xia and Yao [XY]. For any positive integer n, let $\bar{p}_o(n)$ denote the number of overpartitions of n using only odd parts. Set $\bar{p}_o(0) = 1$. As noted in [HS-2], the generating function for $\bar{p}_o(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}_o(n) q^n = \frac{f_2^3}{f_1^2 f_4}; \tag{1.1}$$

here and throughout, for any positive integer k, f_k is defined by

$$f_k = \prod_{n=1}^{\infty} (1 - q^{kn}), \qquad |q| < 1. \tag{1.2}$$

The generating function for $\bar{p}_o(n)$ has appeared in the works of Ardonne, Kedem and Stone [AKS], Bessenrodt [Bes], Santos and Sills [SS]. Hirschhorn

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and Sellers [HS-2] proved some functional equations involving the generating functions for overpartitions and overpartitions into odd parts. They also used these functional equations to develop characterizations of $\bar{p}_o(n)$ modulo small powers of 2. They discovered two infinite families of congruences modulo 3 for $\bar{p}_o(n)$, namely, for all $n, \alpha \geq 0$,

$$\bar{p}_o(9^\alpha(9n+6)) \equiv 0 \pmod 3, \tag{1.3}$$
$$\bar{p}_o(9^\alpha(27n+9)) \equiv 0 \pmod 3. \tag{1.4}$$

Later, Lovejoy and Osburn [LO] proved that for $n, \alpha \geq 0$,

$$\bar{p}_o(4^\alpha(24n+9)) \equiv 0 \pmod 3, \tag{1.5}$$
$$\bar{p}_o(4^\alpha(24n+15)) \equiv 0 \pmod 3. \tag{1.6}$$

Recently, Chen [C] proved an identity involving $\bar{p}_o(n)$, and established many explicit Ramanujan-like congruences for $\bar{p}_o(n)$ modulo 32 and 64.

In this paper, employing 2-dissections of certain quotients of theta functions due to Ramanujan, we establish further Ramanujan-type congruences modulo 3 for $\bar{p}_o(n)$. We list our main results in the following theorem.

From this point on, all congruences are to modulus 3.

THEOREM 1.1. *For all nonnegative integers n and α, we have*

$$\bar{p}_o(4^\alpha(12n+8)) \equiv \bar{p}_o(12n+8), \tag{1.7}$$
$$\bar{p}_o(4^\alpha(12n+5)) \equiv \bar{p}_o(12n+5), \tag{1.8}$$
$$\bar{p}_o(4^\alpha(48n+44)) \equiv -\bar{p}_o(12n+11), \tag{1.9}$$
$$\bar{p}_o(4^\alpha(48n+8)) \equiv -\bar{p}_o(12n+2), \tag{1.10}$$
$$\bar{p}_o(4^\alpha(24n+17)) \equiv 0, \tag{1.11}$$
$$\bar{p}_o(4^\alpha(24n+23)) \equiv 0, \tag{1.12}$$
$$\bar{p}_o(4^\alpha(48n+26)) \equiv 0, \tag{1.13}$$
$$\bar{p}_o(4^\alpha(48n+38)) \equiv 0. \tag{1.14}$$

2. Proof of Theorem 1.1.

The following lemma gives generating functions of $\bar{p}_o(6n+2)$ and $\bar{p}_o(6n+5)$.

LEMMA 2.1. *We have*

$$\sum_{n=0}^{\infty} \bar{p}_o(6n+2)q^n \equiv \frac{f_1^2 f_4^{15}}{f_2^3 f_8^6} + \frac{f_4^{25}}{f_1^2 f_2^5 f_8^{10}} + q\frac{f_4^{13}}{f_1^2 f_2 f_8^2} + 2q^2\frac{f_2^3 f_4 f_8^6}{f_1^2}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \bar{p}_o(6n+5)q^n \equiv 2\frac{f_4^{19}}{f_1^2 f_2^3 f_8^6} + q\frac{f_2 f_4^7 f_8^2}{f_1^2} + 2q\frac{f_1^2 f_2^3 f_8^6}{f_4^3} + q^2\frac{f_2^5 f_8^{10}}{f_1^2 f_4^5}. \tag{2.2}$$

Proof. From [HS-2, proof of Theorem 3.5], we have

$$\sum_{n=0}^{\infty}\bar{p}_o(n)q^n=\frac{D(q^9)}{D^4(q^3)}(D(q^{18})-2q^2Y(q^6))\times\left(D^2(q^9)+2qD(q^9)Y(q^3)+4q^2Y^2(q^3)\right), \tag{2.3}$$

where $D(q)$ and $Y(q)$ are defined by

$$D(q)=\frac{f_1^2}{f_2}, \tag{2.4}$$

$$Y(q)=\frac{f_1f_6^2}{f_2f_3}. \tag{2.5}$$

It follows from (2.3)–(2.5) that

$$\sum_{n=0}^{\infty}\bar{p}_o(3n+2)q^n=4\frac{f_2^2f_6^5}{f_1^6f_{12}}-2\frac{f_2^5f_3^6f_{12}^2}{f_1^8f_4f_6^4}. \tag{2.6}$$

By the binomial theorem, it is easy to see that for any positive integer k,

$$f_{3k}\equiv f_k^3. \tag{2.7}$$

Employing (2.7), we can rewrite (2.6) as

$$\sum_{n=0}^{\infty}\bar{p}_o(3n+2)q^n\equiv\frac{f_2^{17}}{f_1^6f_4^3}+\frac{f_1^{10}f_4^5}{f_2^7}. \tag{2.8}$$

The following 2-dissections are consequences of Ramanujan's dissection formulas collected in Entry 25 in Berndt's book [Ber, p. 40]:

$$f_1^2=\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}, \tag{2.9}$$

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we deduce that

(2.11)

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(3n+2)q^n&\equiv\frac{f_2^{17}}{f_4^3}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)^3+\frac{f_4^5}{f_2^7}\left(\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}\right)^5\\&\equiv\frac{f_2^2f_8^{15}}{f_4^3f_{16}^6}+\frac{f_8^{25}}{f_2^2f_4^5f_{16}^{10}}+2q\frac{f_8^{19}}{f_2^2f_4^3f_{16}^6}+q^2\frac{f_8^{13}}{f_2^2f_4f_{16}^2}\\&\quad+q^3\frac{f_4f_8^7f_{16}^2}{f_2^2}+2q^3\frac{f_2^2f_4^3f_{16}^6}{f_8^3}+2q^4\frac{f_4^3f_8f_{16}^6}{f_2^2}+q^5\frac{f_4^5f_{16}^{10}}{f_2^2f_8^5},\end{aligned}$$

which yields (2.1) and (2.2). ∎

LEMMA 2.2. *We have*

$$\sum_{n=0}^{\infty}\bar{p}_o(12n+2)q^n \equiv \frac{f_2^{13}}{f_1^2f_4f_8^2}+\frac{f_2^{25}}{f_1^{10}f_4^5f_8^2}+2q\frac{f_2^{15}f_8^2}{f_1^6f_4^3}+2q\frac{f_2f_4^{11}}{f_1^2f_8^2}, \tag{2.12}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(12n+5)q^n \equiv 2\frac{f_2^{19}}{f_1^8f_4f_8^2}+2q\frac{f_1^4f_4^5f_8^2}{f_2^3}+2q\frac{f_2^9f_4f_8^2}{f_1^4}+q\frac{f_4^{15}}{f_2^5f_8^2}, \tag{2.13}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(12n+8)q^n \equiv \frac{f_2^{15}f_8^2}{f_1^2f_4^7}+2\frac{f_2^{27}f_8^2}{f_1^{10}f_4^{11}}+\frac{f_2^{13}f_4^3}{f_1^6f_8^2}+q\frac{f_2^3f_4^5f_8^2}{f_1^2}, \tag{2.14}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(12n+11)q^n \equiv 2\frac{f_1^4f_4^{11}}{f_2^5f_8^2}+\frac{f_2^{21}f_8^2}{f_1^8f_4^7}+\frac{f_2^7f_4^7}{f_1^4f_8^2}+2q\frac{f_4^9f_8^2}{f_2^3}. \tag{2.15}$$

Proof. Substituting (2.9) and (2.10) into (2.1) and (2.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(6n+2)q^n &\equiv \frac{f_4^{15}}{f_2^3f_8^6}\left(\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}\right)\\
&\quad+\frac{f_4^{25}}{f_2^5f_8^{10}}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\quad+q\frac{f_4^{13}}{f_2f_8^2}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\quad+2q^2f_2^3f_4f_8^6\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\equiv \frac{f_4^{13}}{f_2^2f_8f_{16}^2}+\frac{f_4^{25}}{f_2^{10}f_8^5f_{16}^2}+q\frac{f_4^{15}f_{16}^2}{f_2^2f_8^7}\\
&\quad+q\frac{f_4^{13}f_8^3}{f_2^6f_{16}^2}+2q\frac{f_4^{27}f_{16}^2}{f_2^{10}f_8^{11}}+2q^2\frac{f_4^{15}f_{16}^2}{f_2^6f_8^3}\\
&\quad+2q^2\frac{f_4f_8^{11}}{f_2^2f_{16}^2}+q^3\frac{f_4^3f_8^5f_{16}^2}{f_2^2},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(6n+5)q^n &\equiv 2\frac{f_4^{19}}{f_2^3f_8^6}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\quad+qf_2f_4^7f_8^2\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\quad+2q\frac{f_2^3f_8^6}{f_4^3}\left(\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}\right)\\
&\quad+q^2\frac{f_2^5f_8^{10}}{f_4^5}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\
&\equiv 2\frac{f_4^{19}}{f_2^8f_8f_{16}^2}+q\frac{f_4^{21}f_{16}^2}{f_2^8f_8^7}+q\frac{f_4^7f_8^7}{f_2^4f_{16}^2}\end{aligned} \tag{2.17}$$

$$+ 2q\frac{f_2^4 f_8^{11}}{f_4^5 f_{16}^2} + 2q^2\frac{f_4^9 f_8 f_{16}^2}{f_2^4} + 2q^2\frac{f_2^4 f_8^5 f_{16}^2}{f_4^3} + q^2\frac{f_8^{15}}{f_4^5 f_{16}^2} + 2q^3\frac{f_8^9 f_{16}^2}{f_4^3}.$$

Lemma 2.2 follows from (2.16) and (2.17). ∎

Lemma 2.3. *We have*

$$\sum_{n=0}^{\infty} \bar{p}_o(24n+2)q^n \equiv \frac{f_1^8 f_4^3}{f_2 f_8^2} + \frac{f_4^{23}}{f_2^5 f_8^{10}} + q\frac{f_4^{11}}{f_2 f_8^2} + q\frac{f_2^{13} f_8^2}{f_1^4 f_4^3}, \tag{2.18}$$

$$\sum_{n=0}^{\infty} \bar{p}_o(24n+8)q^n \equiv \frac{f_1^{10} f_4^7}{f_2^7 f_8^2} + 2q\frac{f_1^2 f_4^{15}}{f_2^7 f_8^2}, \tag{2.19}$$

$$\sum_{n=0}^{\infty} \bar{p}_o(24n+14)q^n \equiv 2\frac{f_1^8 f_2 f_8^2}{f_4^3} + 2q f_2 f_4^5 f_8^2 + 2q^2\frac{f_2^5 f_8^{10}}{f_4^7} + 2\frac{f_2^{11} f_4^3}{f_1^4 f_8^2}, \tag{2.20}$$

$$\sum_{n=0}^{\infty} \bar{p}_o(24n+20)q^n \equiv 2\frac{f_1^{10} f_4 f_8^2}{f_2^5} + q\frac{f_1^2 f_4^9 f_8^2}{f_2^5}, \tag{2.21}$$

and for $n \geq 0$,

$$\bar{p}_o(24n+17) \equiv 0, \tag{2.22}$$

$$\bar{p}_o(24n+23) \equiv 0. \tag{2.23}$$

Proof. Substituting (2.10) into (2.12), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{p}_o(12n+2)q^n \equiv\ & \frac{f_2^{13}}{f_4 f_8^2}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \\ & + \frac{f_2^{25}}{f_4^5 f_8^2}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)^5 \\ & + 2q\frac{f_2^{15} f_8^2}{f_4^3}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)^3 \\ & + 2q\frac{f_2 f_4^{11}}{f_8^2}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \\ \equiv\ & \frac{f_2^8 f_8^3}{f_4 f_{16}^2} + \frac{f_8^{23}}{f_4^5 f_{16}^{10}} + 2q\frac{f_4^{11} f_8^3}{f_2^4 f_{16}^2} + 2q\frac{f_2^8 f_4 f_{16}^2}{f_8^3} \\ & + q^2\frac{f_8^{11}}{f_4 f_{16}^2} + q^2\frac{f_4^{13} f_{16}^2}{f_2^4 f_8^3} + 2q^3 f_4 f_8^5 f_{16}^2 + 2q^5\frac{f_4^5 f_{16}^{10}}{f_8^7},\end{aligned} \tag{2.24}$$

which implies (2.18) and (2.20).

The following identities are consequences of Ramanujan's dissection formulas collected in Entry 25 in Berndt's book [Ber, p. 40]:

$$f_1^4=\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}, \tag{2.25}$$

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.26}$$

Substituting (2.10) and (2.26) into (2.14), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(12n+8)q^n&\equiv\frac{f_2^{15}f_8^2}{f_4^7}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\&\quad+2\frac{f_2^{27}f_8^2}{f_4^{11}}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\\&\quad+qf_2^3f_4^5f_8^2\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\\&\quad+\frac{f_2^{13}f_4^3}{f_8^2}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\\&\equiv\frac{f_2^{10}f_8^7}{f_4^7f_{16}^2}+2q\frac{f_2^{10}f_8f_{16}^2}{f_4^5}+2q^2\frac{f_2^2f_8^{15}}{f_4^7f_{16}^2}+q^3\frac{f_2^2f_8^9f_{16}^2}{f_4^5}.\end{aligned} \tag{2.27}$$

Congruences (2.19) and (2.21) follow from (2.27).

Substituting (2.25) and (2.26) into (2.13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(12n+5)q^n&\equiv2\frac{f_2^{19}}{f_4f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\\&\quad+2q\frac{f_4^5f_8^2}{f_2^3}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)\\&\quad+2qf_2^9f_4f_8^2\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)+q\frac{f_4^{15}}{f_2^5f_8^2}\\&\equiv2\frac{f_4^{27}}{f_2^9f_8^{10}}+2q^2\frac{f_4^3f_8^6}{f_2},\end{aligned} \tag{2.28}$$

which implies (2.22).

Substituting (2.25) and (2.26) into (2.15), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(12n+11)q^n&\equiv2\frac{f_4^{11}}{f_2^5f_8^2}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)\\&\quad+\frac{f_2^{21}f_8^2}{f_4^7}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\end{aligned} \tag{2.29}$$

$$+\frac{f_2^7 f_4^7}{f_8^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4}+4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)+2q\frac{f_4^9 f_8^2}{f_2^3}$$
$$\equiv \frac{f_4^{21}}{f_2^7 f_8^6}+q^2\frac{f_2 f_8^{10}}{f_4^3},$$

which yields (2.23). ∎

LEMMA 2.4. *We have*

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+8)q^n \equiv \frac{f_2^{25}}{f_1^{10} f_4^5 f_8^2}+q\frac{f_2 f_4^{11}}{f_1^2 f_8^2}, \tag{2.30}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+20)q^n \equiv 2\frac{f_2^{19}}{f_1^8 f_4 f_8^2}+2q\frac{f_4^{15}}{f_2^5 f_8^2}, \tag{2.31}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+32)q^n \equiv \frac{f_2^{27} f_8^2}{f_1^{10} f_4^{11}}+q\frac{f_2^3 f_4^5 f_8^2}{f_1^2}, \tag{2.32}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+44)q^n \equiv 2\frac{f_2^{21} f_8^2}{f_1^8 f_4^7}+2q\frac{f_4^9 f_8^2}{f_2^3}, \tag{2.33}$$

and for $n \geq 0$,

$$\bar{p}_o(48n+26) \equiv 0, \tag{2.34}$$
$$\bar{p}_o(48n+38) \equiv 0. \tag{2.35}$$

Proof. Substituting (2.9) and (2.25) into (2.19), we deduce that

$$\sum_{n=0}^{\infty}\bar{p}_o(24n+8)q^n \equiv \frac{f_4^7}{f_2^7 f_8^2}\left(\frac{f_4^{10}}{f_2^2 f_8^4}-4q\frac{f_2^2 f_8^4}{f_4^2}\right)^2\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2}-2q\frac{f_2 f_{16}^2}{f_8}\right) \tag{2.36}$$
$$+2q\frac{f_4^{15}}{f_2^7 f_8^2}\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2}-2q\frac{f_2 f_{16}^2}{f_8}\right)$$
$$\equiv \frac{f_4^{25}}{f_2^{10} f_8^5 f_{16}^2}+q^2\frac{f_4 f_8^{11}}{f_2^2 f_{16}^2}+q\frac{f_4^{27} f_{16}^2}{f_2^{10} f_8^{11}}+q^3\frac{f_4^3 f_8^5 f_{16}^2}{f_2^2},$$

which yields (2.30) and (2.32).

Substituting (2.9) and (2.25) into (2.21), we find that

$$\sum_{n=0}^{\infty}\bar{p}_o(24n+20)q^n \equiv 2\frac{f_4 f_8^2}{f_2^5}\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2}-2q\frac{f_2 f_{16}^2}{f_8}\right)\left(\frac{f_4^{10}}{f_2^2 f_8^4}-4q\frac{f_2^2 f_8^4}{f_4^2}\right)^2 \tag{2.37}$$
$$+q\frac{f_4^9 f_8^2}{f_2^5}\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2}-2q\frac{f_2 f_{16}^2}{f_8}\right)$$
$$\equiv 2\frac{f_4^{19}}{f_2^8 f_8 f_{16}^2}+2q\frac{f_4^{21} f_{16}^2}{f_2^8 f_8^7}+2q^2\frac{f_8^{15}}{f_4^5 f_{16}^2}+2q^3\frac{f_8^9 f_{16}^2}{f_4^3},$$

which implies (2.31) and (2.33).

Substituting (2.25) and (2.26) into (2.18), we obtain

$$\sum_{n=0}^{\infty}\bar{p}_o(24n+2)q^n \equiv \frac{f_4^3}{f_2f_8^2}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)^2+\frac{f_4^{23}}{f_2^5f_8^{10}} + q\frac{f_4^{11}}{f_2f_8^2}+q\frac{f_2^{13}f_8^2}{f_4^3}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right) \equiv 2\frac{f_4^{23}}{f_2^5f_8^{10}}+2q^2\frac{f_2^3f_8^6}{f_4}. \tag{2.38}$$

Congruence (2.34) follows from (2.38).

Substituting (2.25) and (2.26) into (2.20), we get

$$\sum_{n=0}^{\infty}\bar{p}_o(24n+14)q^n \equiv 2\frac{f_2f_8^2}{f_4^3}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)^2+2qf_2f_4^5f_8^2 + 2q^2\frac{f_2^5f_8^{10}}{f_4^7}+2\frac{f_2^{11}f_4^3}{f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right) \equiv \frac{f_4^{17}}{f_2^3f_8^6}+q^2\frac{f_2^5f_8^{10}}{f_4^7}, \tag{2.39}$$

which yields (2.35). ∎

Lemma 2.5. *We have*

$$\sum_{n=0}^{\infty}\bar{p}_o(96n+8)q^n \equiv \frac{f_2^{23}}{f_1^8f_4^5f_8^2}+q\frac{f_4^{11}}{f_2f_8^2}, \tag{2.40}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(96n+32)q^n \equiv \frac{f_2^{17}}{f_1^6f_4f_8^2}+q\frac{f_1^2f_4^{15}}{f_2^7f_8^2}, \tag{2.41}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(96n+56)q^n \equiv 2\frac{f_2^{25}f_8^2}{f_1^8f_4^{11}}+2qf_2f_4^5f_8^2, \tag{2.42}$$

$$\sum_{n=0}^{\infty}\bar{p}_o(96n+80)q^n \equiv 2\frac{f_2^{19}f_8^2}{f_1^6f_4^7}+2q\frac{f_1^2f_4^9f_8^2}{f_2^5}, \tag{2.43}$$

and for $n \geq 0$,

$$\bar{p}_o(96n+68) \equiv 0, \tag{2.44}$$

$$\bar{p}_o(48n+20) \equiv \bar{p}_o(12n+5), \tag{2.45}$$

$$\bar{p}_o(96n+92) \equiv 0, \tag{2.46}$$

$$\bar{p}_o(48n+44) \equiv -\bar{p}_o(12n+11). \tag{2.47}$$

Proof. Substituting (2.10) and (2.26) into (2.30), we get

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+8)q^n \equiv \frac{f_2^{25}}{f_4^5f_8^2}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2 \tag{2.48}$$

$$+ q\frac{f_2 f_4^{11}}{f_8^2}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)$$
$$\equiv \frac{f_4^{23}}{f_2^8 f_8^5 f_{16}^2} + q^2\frac{f_8^{11}}{f_4 f_{16}^2} + 2q\frac{f_4^{25} f_{16}^2}{f_2^8 f_8^{11}} + 2q^3 f_4 f_8^5 f_{16}^2,$$

which implies (2.40) and (2.42).

Substituting (2.10) and (2.26) into (2.32), we obtain

(2.49)
$$\sum_{n=0}^{\infty}\bar{p}_o(48n+32)q^n \equiv \frac{f_2^{27} f_8^2}{f_4^{11}}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2$$
$$+ q f_2^3 f_4^5 f_8^2\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)$$
$$\equiv \frac{f_4^{17}}{f_2^6 f_8 f_{16}^2} + 2q\frac{f_4^{19} f_{16}^2}{f_2^6 f_8^7} + q^2\frac{f_2^2 f_8^{15}}{f_4^7 f_{16}^2} + 2q^3\frac{f_2^2 f_8^9 f_{16}^2}{f_4^5}.$$

Congruences (2.41) and (2.43) follow from (2.49).

Substituting (2.26) into (2.31), we get

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+20)q^n \equiv 2\frac{f_2^{19}}{f_4 f_8^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2 + 2q\frac{f_4^{15}}{f_2^5 f_8^2} \tag{2.50}$$
$$\equiv 2\frac{f_4^{27}}{f_2^9 f_8^{10}} + 2q^2\frac{f_4^3 f_8^6}{f_2},$$

which yields (2.44). Moreover, from (2.28) and (2.50), we derive (2.45).

Substituting (2.26) into (2.33), we see that

$$\sum_{n=0}^{\infty}\bar{p}_o(48n+44)q^n \equiv 2\frac{f_2^{21} f_8^2}{f_4^7}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2 + 2q\frac{f_4^9 f_8^2}{f_2^3} \tag{2.51}$$
$$\equiv 2\frac{f_4^{21}}{f_2^7 f_8^6} + 2q^2\frac{f_2 f_8^{10}}{f_4^3}.$$

Congruence (2.46) follows from (2.51). Furthermore, by (2.29) and (2.51), we obtain (2.47). ∎

Lemma 2.6. *For $n \geq 0$,*

$$\bar{p}_o(48n+8) \equiv -\bar{p}_o(12n+2), \tag{2.52}$$
$$\bar{p}_o(48n+32) \equiv \bar{p}_o(12n+8), \tag{2.53}$$
$$\bar{p}_o(192n+104) \equiv 0, \tag{2.54}$$
$$\bar{p}_o(192n+152) \equiv 0. \tag{2.55}$$

Proof. Jacobi [J] discovered the identity

$$\prod_{n=1}^{\infty}(1+q^{2n-1})^8-\prod_{n=1}^{\infty}(1-q^{2n-1})^8=16q\prod_{n=1}^{\infty}(1+q^{2n})^8, \tag{2.56}$$

which is known as "aequatio identica satis abstrusa". We can rewrite (2.56) as follows:

$$\frac{f_2^{16}}{f_1^8}-\frac{f_1^8f_4^8}{f_2^8}=16q\frac{f_4^{16}}{f_2^8}. \tag{2.57}$$

Then we deduce from (2.19), (2.41) and (2.57) that

$$\begin{aligned}\sum_{n=0}^{\infty}\big(\bar{p}_o(24n+8)-\bar{p}_o(96n+32)\big)q^n &\equiv \frac{f_1^{10}f_4^7}{f_2^7f_8^2}+2q\frac{f_1^2f_4^{15}}{f_2^7f_8^2}-\left(\frac{f_2^{17}}{f_1^6f_4f_8^2}+q\frac{f_1^2f_4^{15}}{f_2^7f_8^2}\right)\\ &\equiv \frac{f_1^2f_2}{f_4f_8^2}\left(\frac{f_1^8f_4^8}{f_2^8}-\frac{f_2^{16}}{f_1^8}+q\frac{f_4^{16}}{f_2^8}\right)\equiv 0,\end{aligned} \tag{2.58}$$

which implies that for $n\geq 0$,

$$\bar{p}_o(96n+32)\equiv \bar{p}_o(24n+8). \tag{2.59}$$

By means of (2.21), (2.43) and (2.57), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}(\bar{p}_o(24n+20)-\bar{p}_o(96n+80))q^n &\equiv 2\frac{f_1^{10}f_4f_8^2}{f_2^5}+q\frac{f_1^2f_4^9f_8^2}{f_2^5}-\left(2\frac{f_2^{19}f_8^2}{f_1^6f_4^7}+2q\frac{f_1^2f_4^9f_8^2}{f_2^5}\right)\\ &\equiv \frac{f_1^2f_2^3f_8^2}{f_4^7}\left(2\frac{f_1^8f_4^8}{f_2^8}-2\frac{f_2^{16}}{f_1^8}-q\frac{f_4^{16}}{f_2^8}\right)\equiv 0,\end{aligned} \tag{2.60}$$

which implies that for $n\geq 0$,

$$\bar{p}_o(96n+80)\equiv \bar{p}_o(24n+20). \tag{2.61}$$

Combining (2.59) and (2.61), we obtain (2.53).

Substituting (2.26) into (2.40), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_o(96n+8)q^n &\equiv \frac{f_2^{23}}{f_4^5f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2+q\frac{f_4^{11}}{f_2f_8^2}\\ &\equiv \frac{f_4^{23}}{f_2^5f_8^{10}}+q^2\frac{f_2^3f_8^6}{f_4},\end{aligned} \tag{2.62}$$

which yields (2.54).

Moreover, in view of (2.38) and (2.62), we see that for $n\geq 0$,

$$\bar{p}_o(96n+8)\equiv -\bar{p}_o(24n+2). \tag{2.63}$$

Substituting (2.26) into (2.42), we find that

$$\sum_{n=0}^{\infty}\bar{p}_o(96n+56)q^n \equiv 2\frac{f_2^{25}f_8^2}{f_4^{11}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2+2qf_2f_4^5f_8^2 \equiv 2\frac{f_4^{17}}{f_2^3f_8^6}+2q^2\frac{f_2^5f_8^{10}}{f_4^7}, \tag{2.64}$$

which yields (2.55).

It follows from (2.39) and (2.64) that for $n \geq 0$,

$$\bar{p}_o(96n+56) \equiv -\bar{p}_o(24n+14). \tag{2.65}$$

Congruence (2.52) follows from (2.63) and (2.65). ■

Proof of Theorem 1.1. Congruence (1.7) follows from (2.53) and mathematical induction. Replacing n by $4n+1$ in (1.7), we deduce that for $n, \alpha \geq 0$,

$$\bar{p}_o(4^\alpha(48n+20)) \equiv \bar{p}_o(48n+20). \tag{2.66}$$

Congruence (1.8) follows from (2.45) and (2.66).

Replacing n by $4n+3$ in (1.7), we find that for $n, \alpha \geq 0$,

$$\bar{p}_o(4^\alpha(48n+44)) \equiv \bar{p}_o(48n+44). \tag{2.67}$$

Congruence (1.9) follows from (2.47) and (2.67).

Replacing n by $4n$ in (1.7), we find that for $n, \alpha \geq 0$,

$$\bar{p}_o(4^\alpha(48n+8)) \equiv \bar{p}_o(48n+8). \tag{2.68}$$

Congruence (1.10) follows from (2.52) and (2.68).

Replacing n by $8n+5$ in (1.7), and employing (2.22) and (2.44), we arrive at (1.11). Replacing n by $8n+7$ in (1.7), and using (2.23) and (2.46), we deduce (1.12). Replacing n by $16n+8$ in (1.7), and employing (2.34) and (2.54), we obtain (1.13). Replacing n by $16n+12$ in (1.7), and utilizing (2.35) and (2.55), we derive (1.14). ■

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