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AXIAL PERMUTATIONS OF ω^2

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Abstract. We prove that every permutation of ω^2 is a composition of a finite number of axial permutations, where each axial permutation moves only a finite number of elements on each axis.

1. Introduction. We consider axial permutations of the infinite matrix ω^2, where ω denotes $\{0, 1, 2, \ldots\}$, the set of natural numbers. We say that a permutation $f\colon \omega^2 \to \omega^2$ is *horizontal* if there exists $g\colon \omega^2 \to \omega$ such that $f(x, y) = (x, g(x, y))$. Analogously, f is *vertical* if there exists $g\colon \omega^2 \to \omega$ such that $f(x, y) = (g(x, y), y)$. A permutation is *axial* if it is either horizontal or vertical.

In 1935 Stefan Banach posed a question in *The Scottish Book* whether every permutation of ω^2 is a composition of a finite number of axial permutations. A positive answer was given by Nosarzewska [N], stating that a sufficient number of axial permutations is 5. This number was reduced to 4 by Ehrenfeucht and Grzegorek [EG], [G]. The subject of axial permutations has also been treated in [Sz1] and [Sz2].

In this paper, we are interested in permutations for which the support is finite, i.e.

$$|\mathrm{supp}(\sigma)| = |\{n \in \omega\colon \sigma(n) \neq n\}| < \aleph_0.$$

We will prove that every permutation of ω^2 is a composition of a finite number of axial permutations, where each axial permutation has a finite support on each axis. In the final part of the paper we generalize one of our results to the case of ideals of subsets of natural numbers.

2. Permutations with finite supports. By $[n, m]$ and $[n, m)$ we will denote the sets $\{n, n+1, \ldots, m\}$ and $\{n, n+1, \ldots, m-1\}$, respectively. We will say that a subset of ω^2 is an *L-area* if it is of the form $[0, m]^2 \setminus [0, n)^2$. Additionally, we require all L-areas to be sufficiently *thick*, i.e. $2n \leq m$.

In the proof of Lemma 2.2, we are going to use the following result from [EG].

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THEOREM 2.1 ([EG]). *Let A be a set of an arbitrary cardinality and $|B| < \aleph_0$. Every permutation of $A \times B$ is a composition of three axial permutations, where the first one is horizontal.*

For each L-area we consider permutations that we will call *good*. These are horizontal or vertical permutations $f\colon [0,m]^2 \setminus [0,n)^2 \to [0,m]^2 \setminus [0,n)^2$ of the following form:

$$f(x,y) = \begin{cases} (x,y), & x \in [0,n), \\ (x, g(x,y)), & x \in [n,m], \end{cases}$$

or

$$f(x,y) = \begin{cases} (x,y), & y \in [0,n), \\ (h(x,y), y)), & y \in [n,m]. \end{cases}$$

LEMMA 2.2. *Each permutation of an L-area can be represented as a composition of at most* 24 *good permutations.*

Proof. Fix some L-area and its permutation σ. Divide the L-area into three subsets: $L_1 = [n,m] \times [0,n)$, $L_2 = [n,m]^2$, $L_3 = [0,n) \times [n,m]$. We will be using the following notation:

$$(x,y) \oplus (n,m) = (x+n, y+n),$$
$$A(i,j) = \{(x,y) \in L_j \colon \sigma(x,y) \in L_i\}.$$

The final composition will be constructed within the following twelve steps.

STEP 1. Define a permutation f_1 of the L-area by

$$f_1(x,y) = \begin{cases} \sigma(x,y) \oplus (0,n) & \text{if } (x,y) \in A(1,2), \\ (x,y) & \text{otherwise.} \end{cases}$$

STEP 2. Define a horizontal permutation f_2 by

$$f_2(x,y) = \begin{cases} (x, y+n) & \text{if } (x,y) \in [n,m) \times [0,n), \\ (x, y-n) & \text{if } (x,y) \in [n,m] \times [n,2n), \\ (x,y) & \text{otherwise.} \end{cases}$$

Then $(f_2 \circ f_1)(x,y) = \sigma(x,y)$ for every $(x,y) \in A(1,2)$.

STEP 3. Since L_2 now contains some elements which originally belonged to L_1, we repeat the method from the first two steps, i.e. let f_3 be a permutation such that for each $(x,y) \in A(1,2)$ we have $f_3(x,y) = \sigma(x,y) \oplus (0,n)$, and f_3 is the identity elsewhere. While f_3 acts similarly to f_1, it is not the same permutation, as $A(1,2)$ now has a different form.

STEP 4. We shift the aforementioned elements into L_1 using a horizontal permutation f_4 where

$$f_4(x,y) = \begin{cases} (x, y+n) & \text{if } (x, y+n) \in A(1,2), \\ (x, y-n) & \text{if } (x,y) \in A(1,2), \\ (x,y) & \text{otherwise.} \end{cases}$$

STEP 5. We now use a vertical permutation f_5 constructed in an analogous way to the horizontal permutation f_2:

$$f_5(x,y) = \begin{cases} (x+n, y) & \text{if } (x,y) \in [0,n) \times [n,m], \\ (x-n, y) & \text{if } (x,y) \in [n,2n) \times [n,m], \\ (x,y) & \text{otherwise.} \end{cases}$$

STEP 6. Since L_2 now contains some elements which originally belonged to L_3, we repeat the method of rearranging those $(x,y) \in L_2$ for which $\sigma(x,y) \in L_1$. Let f_6 be a permutation such that for each $(x,y) \in A(1,2)$ we have $f_6(x,y) = \sigma(x,y) \oplus (0,n)$.

STEP 7. We proceed similarly to Step 4. Define a horizontal permutation f_7 by

$$f_7(x,y) = \begin{cases} (x, y+n) & \text{if } (x, y+n) \in A(1,2), \\ (x, y-n) & \text{if } (x,y) \in A(1,2), \\ (x,y) & \text{otherwise.} \end{cases}$$

The rearranging of L_1 is now complete.

STEP 8. Let f_8 be a permutation such that for each $(x,y) \in A(3,2)$ we have $f_8(x,y) = \sigma(x,y) \oplus (n,0)$.

STEP 9. We set $f_9(x,y) = f_5(x,y)$.

STEP 10. Let f_{10} be a permutation such that for each $(x,y) \in A(3,2)$ we have $f_{10}(x,y) = \sigma(x,y) \oplus (n,0)$.

STEP 11. Similarly to Step 7, we shift the aforementioned elements into L_3 using a vertical permutation f_{11} where

$$f_{11}(x,y) = \begin{cases} (x+n, y) & \text{if } (x+n, y) \in A(3,2), \\ (x-n, y) & \text{if } (x,y) \in A(3,2), \\ (x,y) & \text{otherwise.} \end{cases}$$

The rearranging of L_3 is now complete.

STEP 12. Let f_{12} be a permutation such that for each $(x,y) \in L_2$ we have $f_{12}(x,y) = \sigma(x,y)$. This completes the rearranging of L_2 and therefore of the entire L-area.

Each of the twelve permutations is either good or by Theorem 2.1 can be represented as a composition of three axial permutations, which are also good for the L-area. One can check that the total sufficient number of good permutations is 24. ∎

We will say that a partition of ω^2 is an *L-partition* if it consists only of disjoint L-areas. For every permutation σ of ω^2 we may consider its decomposition into disjoint cycles. Such a decomposition defines a partition of ω^2, which we will denote $\pi(\sigma)$. Recall that a partition $\mathcal{F}$ is a *refinement* of a partition $\mathcal{G}$, if for every $A \in \mathcal{F}$ there exists $B \in \mathcal{G}$ such that $A \subseteq B$. In this case we write $\mathcal{F} \prec \mathcal{G}$. We will say that a permutation σ of ω^2 is an *L-permutation* if $\pi(\sigma)$ is a refinement of some L-partition.

Corollary 2.3. *Every L-permutation can be represented as a composition of at most* 24 *axial permutations which have finite supports on each axis.*

Proof. Fix an L-permutation σ. The partition $\pi(\sigma)$ is a refinement of some L-partition $\mathcal{F}$. According to Lemma 2.2, for each member $A \in \mathcal{F}$ it is sufficient to use 24 good permutations to obtain $\sigma_{\restriction A}$. To prove the assertion, simply take such compositions simultaneously on all members of $\mathcal{F}$. ∎

Lemma 2.4. *Let A be such that $|A| = \aleph_0$. Every permutation of A can be represented as a composition of two permutations σ_1, σ_2, where each σ_i has only finite cycles.*

Proof. Since every permutation of a countable set A can consist of finite and infinite cycles, it suffices to consider a permutation that consists only of one cycle. Also, it suffices to assume that this permutation is a shift on the set of integers, i.e. $\sigma\colon \mathbb{Z} \to \mathbb{Z}$, $\sigma(k) = k + 1$. Set $\sigma_1(k) = -(k + 1)$ and $\sigma_2(k) = -k$. Then $\sigma = \sigma_2 \circ \sigma_1$ and the length of all cycles of both σ_1 and σ_2 is 2. ∎

Lemma 2.5. *Let π be a partition of ω^2 such that every element of π is finite. Then there exist partitions τ_1, τ_2 such that $\pi \prec \tau_1 \cup \tau_2$ and every element of both τ_1 and τ_2 is an L-area.*

Proof. Let us denote $\pi = \{P_n \colon n \in \omega\}$. Inductively define the following sets: $B_0 = \emptyset$, $A_0 = P_0$, $A_n = \bigcup\{P \in \pi \colon P \cap B_n \neq \emptyset\} \cup P_n$. Pick B_n so that it is of the form $[0, b_n)^2$, $B_n \setminus B_{n-1}$ is an L-area and $B_n \supseteq A_{n-1}$. Notice that $B_n \subseteq A_n \subseteq B_{n+1}$. Also, since $P_n \subseteq A_n$, we have $\bigcup_{n\in\omega} B_n = \omega^2$.

Define

$$\tau_1 = \{B_2, B_4 \setminus B_2, B_6 \setminus B_4, \ldots\}, \qquad \tau_2 = \{B_1, B_3 \setminus B_1, B_5 \setminus B_3, \ldots\}.$$

Obviously τ_1 and τ_2 are partitions consisting of L-areas.

We will show that $\pi \prec \tau_1 \cup \tau_2$. Fix $P \in \pi$. Define $l_0 = \min\{l \in \omega \colon B_l \cap P \neq \emptyset\}$. Of course $l_0 > 0$. We have $B_{l_0-1} \cap P = \emptyset$. Also $P \subseteq A_{l_0} \subseteq B_{l_0+1}$. Therefore we obtain $P \subseteq B_{l_0+1} \setminus B_{l_0-1}$. If l_0 is odd, then $B_{l_0+1} \setminus B_{l_0-1} \in \tau_1$, and if l_0 is even, then $B_{l_0+1} \setminus B_{l_0-1} \in \tau_2$. ∎

LEMMA 2.6. *Each permutation $\sigma\colon \omega^2 \to \omega^2$ which has only finite cycles can be represented as a composition of two L-permutations.*

Proof. Since σ has only finite cycles, each member of $\pi(\sigma)$ is finite. By Lemma 2.5, there exist two partitions τ_1, τ_2 such that $\pi(\sigma) \prec \tau_1 \cup \tau_2$ and all members of both partitions are L-areas.

We will now define two permutations σ_1 and σ_2. Fix $p \in \omega^2$. Let $C \in \pi(\sigma)$ be such that $p \in C$. Since $\pi(\sigma) \prec \tau_1 \cup \tau_2$, there is $L \in \tau_1 \cup \tau_2$ such that $C \subseteq L$. If $L \in \tau_1$, then set $\sigma_1(p) = \sigma(p)$. Otherwise set $\sigma_1(p) = p$. Analogously, if $L \in \tau_1$, then set $\sigma_2(p) = p$, and otherwise set $\sigma_2(p) = \sigma(p)$. Obviously σ_1 and σ_2 are L-permutations and $\sigma = \sigma_1 \circ \sigma_2$. ∎

Finally, we prove our main result.

THEOREM 2.7. *Every permutation of ω^2 can be represented as a composition of a finite number of axial permutations, where each axial permutation moves only a finite number of elements on each axis.*

Proof. Let σ be a fixed permutation of ω^2. According to Lemma 2.4, σ is a composition of two permutations σ_1, σ_2, each with all cycles finite. By Lemma 2.6, each σ_i is a composition of two L-permutations, and in view of Corollary 2.3 each L-permutation can be represented as a composition of at most 24 axial permutations which have finite supports on each axis. Altogether we find that every permutation of ω^2 can be represented as a composition of 96 axial permutations, where each axial permutation moves only a finite number of elements on each axis. ∎

Notice that while for any permutation σ of ω^2 it is possible to represent it by four axial permutations, it is not true that four permutations with finite supports are sufficient. We now focus on the construction of a counterexample.

We denote by $\sigma[A]$ the image of the set A under the function σ. We are going to use the following lemma.

LEMMA 2.8 ([G, Proposition 1 and Remark 2, p. 156]). *Let $\sigma\colon \omega^2 \to \omega^2$ be a permutation. The following are equivalent:*

(i) *σ can be represented as a composition $\sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$, where σ_1, σ_3 are horizontal and σ_2, σ_4 are vertical.*
(ii) *For every finite set $A \subseteq \omega$ there is no finite set $B \subseteq \omega$ such that $\sigma[(\omega \setminus A) \times \omega] \subseteq \omega \times B$.*

LEMMA 2.9. *Assume $P \subseteq \omega^2$ intersects each row in a single point. There is no vertical and finitely supported permutation τ of ω^2 such that $\omega^2 \setminus \tau[P]$ intersects each row in a single point.*

Proof. There exists an infinite set $Z \subseteq \omega$ and $n < k$ such that $(Z \times \{n,k\}) \cap P = \emptyset$. Indeed, if $|(\omega \times \{0\}) \cap P| < \aleph_0$ and $|(\omega \times \{1\}) \cap P| < \aleph_0$, then it suffices to take $n = 0$, $k = 1$ and $Z = \{m \in \omega : (m,0) \notin P \wedge (m,1) \notin P\}$. On the other hand, if $|(\omega \times \{i\}) \cap P| = \aleph_0$ for $i = 0$ or $i = 1$, then $(Z \times \{2,3\}) \cap P = \emptyset$, where $Z = \{m \in \omega : (m,i) \in P\}$. In that case, $n = 2$, $k = 3$ are sufficient.

Now suppose there does exist a permutation τ as in the statement. Since $\mathrm{supp}(\tau)$ has finite intersections with each column, there exists an infinite set $Z_1 \subseteq Z$ such that $(Z_1 \times \{n,k\}) \cap (P \cup \mathrm{supp}(\tau)) = \emptyset$. But then τ restricted to $Z_1 \times \{n,k\}$ is the identity, and therefore we obtain $Z_1 \times \{n,k\} \subseteq \tau[Z_1 \times \{n,k\}] \subseteq \tau[\omega^2 \setminus P]$, which yields a contradiction, because $\tau[\omega^2 \setminus P]$ has horizontal intersections which consist of one point. ∎

Example 2.10. Divide ω^2 into the first column, $A = \{(m,0) : m \in \omega\}$, and the rest, $B = \{(m,n) : m \in \omega,\ n > 0\}$. Let $\sigma : \omega^2 \to \omega^2$ be a permutation such that $\sigma[A] = B$, $\sigma[B] = A$.

Using Lemma 2.8 we find that σ is a composition of four axial permutations (one can find both a composition such that σ_1 is vertical and a composition such that it is horizontal). Suppose that $\sigma = \sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$ for some axial permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that the support of each of those is finite. We have two cases:

(1) σ_1, σ_3 are vertical and σ_2, σ_4 are horizontal.
(2) σ_1, σ_3 are horizontal and σ_2, σ_4 are vertical.

In the first case, obviously $\sigma_1[A] = A$ and $\sigma_1[B] = B$, since σ_1 is vertical and A is a column. Denote $P = \sigma_2[\sigma_1[A]]$ and notice that P has exactly one common point with each row (because σ_2 is horizontal). Because $\sigma = \sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$, we have $\sigma_3[P] = \sigma_4^{-1}[B]$. As σ_4 is horizontal, $\omega^2 \setminus \sigma_3[P]$ would intersect each row at a single point. But this is impossible due to Lemma 2.9.

In the second case, we have $\sigma^{-1} = \sigma_1^{-1} \circ \sigma_2^{-1} \circ \sigma_3^{-1} \circ \sigma_4^{-1}$ and $\sigma_1^{-1}, \sigma_3^{-1}$ are horizontal, while $\sigma_2^{-1}, \sigma_4^{-1}$ are vertical. Moreover, $\sigma^{-1}[A] = B$ and $\sigma^{-1}[B] = A$, hence we obtain the first case.

In fact, our example has a slightly stronger property: σ is not a composition $\sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$, where σ_1, σ_3 are vertical (resp. horizontal), σ_2, σ_4 are horizontal (resp. vertical) and σ_3 (resp. σ_2) has a finite support on each axis.

Instead of an axial permutation σ such that $\mathrm{supp}(\sigma)$ is finite on each axis, we may consider a more general variation, i.e. σ such that $\mathrm{supp}(\sigma) \in \mathcal{I}$ on each axis, for some proper ideal $\mathcal{I}$.

Recall that a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an *ideal* if it is closed under taking subsets and finite unions. Additionally, it is *proper* if $\omega \notin \mathcal{I}$ (therefore $\mathcal{I} \neq \mathcal{P}(\omega)$). Intuitively, it is a family of *small* (but not necessarily finite) subsets of ω. Obviously, the family of finite subsets of ω (traditionally denoted as $\mathcal{F}in$) is a proper ideal, and in most cases we assume that $\mathcal{F}in \subseteq \mathcal{I}$, so it makes sense to ask about the generalization of the results concerning finite sets to their ideal counterparts.

LEMMA 2.11. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be a proper ideal. Assume $P \subseteq \omega^2$ intersects each row in a single point. There is no vertical permutation τ of ω^2 such that* $\operatorname{supp}(\tau) \in \mathcal{I}$ *on each axis and $\omega^2 \setminus \tau[P]$ intersects each row in a single point.*

The proof is a minor modification of the proof of Lemma 2.9, obtained by replacing "finite" with "belonging to $\mathcal{I}$".

Let σ denote the permutation from Example 2.10. We obtain the following.

COROLLARY 2.12. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be a proper ideal. It is not possible to represent σ as the composition $\sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$ where σ_1, σ_3 are vertical, σ_2, σ_4 are horizontal and σ_3 is such that its support belongs to $\mathcal{I}$ on each axis.*

Obviously, the counterpart for which σ_1 is horizontal (etc.) also holds.

Notice that in Lemma 2.11 one cannot omit the assumption that the support of σ_3 is in $\mathcal{I}$, i.e. there exists a set P and a vertical permutation σ with the properties as in Lemma 2.11. Indeed, let $\{K_n : n \in \omega\}$ be any partition of ω into infinite sets. Define $P = \bigcup_{n\in\omega} K_n \times \{n\}$. For all $n \in \omega$ let $\sigma_n : \omega \to \omega$ be any permutation such that $\sigma_n[K_n] = \omega \setminus K_n$ and $\sigma_n[\omega \setminus K_n] = K_n$. Set $\sigma(m, n) = (\sigma_n(m), n)$. Then $\sigma[P] = \omega^2 \setminus P$ and the intersection of $\omega^2 \setminus P$ with each line is the whole line except one point.

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