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Locally convex quasi C*-algebras and noncommutative integration

by

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Abstract. We continue the analysis undertaken in a series of previous papers on structures arising as completions of C*-algebras under topologies coarser that their norm topology and we focus our attention on the so-called *locally convex quasi C*-algebras*. We show, in particular, that any strongly *-semisimple locally convex quasi C*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ can be represented in a class of noncommutative local L^2-spaces.

1. Introduction. The completion $\mathfrak{X}$ of a C*-algebra $\mathfrak{A}_0$ with respect to a norm weaker than the C*-norm provides a mathematical framework for discussing certain quantum physical systems for which the usual algebraic approach in terms of C*-algebras turned out to be insufficient.

First of all, $\mathfrak{X}$ is a Banach $\mathfrak{A}_0$-module and becomes a quasi *-algebra if $\mathfrak{X}$ carries an involution which extends the involution $*$ of $\mathfrak{A}_0$. This structure has been called a *proper CQ*-algebra* in a series of papers [4]–[10], [21]–[22] to which we refer for a detailed analysis. On the other hand, if $\mathfrak{X}$ is endowed with an isometric involution different from that of $\mathfrak{A}_0$, then the structure becomes more involved.

CQ*-algebras are examples of more general structures called *locally convex quasi C*-algebras* [3]. They are obtained by completing a C*-algebra with respect to a new locally convex topology τ on $\mathfrak{A}_0$ compatible with the corresponding $\|\cdot\|$-topology. Under certain conditions on τ, a quasi *-subalgebra $\mathfrak{A}$ of the completion $\widetilde{\mathfrak{A}_0}[\tau]$ is a locally convex quasi *-algebra which is named a locally convex quasi C*-algebra.

In [9] quasi *-algebras of measurable and/or integrable operators (in the sense of Segal [19], [27] and Nelson [17]) were examined in detail and it was proved that any *-semisimple CQ*-algebra can be realized as a CQ*-algebra of measurable operators, with the help of a particular class of positive bounded sesquilinear forms on $\mathfrak{X}$.

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In this paper, after a short overview of the main results obtained on this subject, we continue our study of locally convex quasi C*-algebras and we generalize to these structures the results obtained in [9] for proper CQ*-algebras.

The main question we pose in the present paper is the following: given a *-semisimple locally convex quasi C*-algebras $(\mathfrak{X}, \mathfrak{A}_0)$ and the universal *-representation of $\mathfrak{A}_0$, defined via the Gelfand–Naimark theorem, can $\mathfrak{X}$ be realized as a locally convex quasi C*-algebra of operators of type L^2?

The paper is organized as follows. We begin with a short overview of noncommutative L^p-spaces (constructed starting from a von Neumann algebra $\mathfrak{M}$ and a normal, semifinite, faithful trace φ on $\mathfrak{M}$), considered as CQ*-algebras. We also introduce the noncommutative $L^p_{\rm loc}$-space constructed on a von Neumann algebra possessing a family of mutually orthogonal central projections whose sum is the identity operator. We show that $(L^p_{\rm loc}(\varphi), \mathfrak{M})$ is a locally convex quasi C*-algebra.

Finally we give some results on the structure of locally convex quasi C*-algebras: we prove that any locally convex quasi C*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be continuously embedded into a locally convex quasi C*-algebra of measurable operators of the type $(L^2_{\rm loc}(\varphi), \mathfrak{M})$.

1.1. Definitions and results on noncommutative measures. The following basic definitions and results on noncommutative measure theory and integration are needed in what follows. Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$.

Set

$$\mathcal{J} = \{X \in \mathfrak{M} : \varphi(|X|) < \infty\}.$$

Then $\mathcal{J}$ is a *-ideal of $\mathfrak{M}$.

Let $P \in \mathrm{Proj}(\mathfrak{M})$, the lattice of projections of $\mathfrak{M}$. Two projections $P, Q \in \mathrm{Proj}(\mathfrak{M})$ are called *equivalent*, written $P \sim Q$, if there is a $U \in \mathfrak{M}$ with $U^*U = P$ and $UU^* = Q$. We write $P \prec Q$ when P is equivalent to a subprojection of Q.

A projection P of a von Neumann algebra $\mathfrak{M}$ is said to be *finite* if $P \sim Q \leq P$ implies $P = Q$, and *purely infinite* if there is no nonzero finite projection $Q \preceq P$ in $\mathfrak{M}$. A von Neumann algebra $\mathfrak{M}$ is said to be *finite* (respectively, *purely infinite*) if the identity operator $\mathbb{I}$ is finite (respectively, purely infinite).

We say that P is *φ-finite* if $P \in \mathcal{J}$. Any φ-finite projection is finite.

We will need the following result (see [15, Vol. IV, Ex. 6.9.12]).

Lemma 1.1. *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. There is an or-*

thogonal family $\{Q_j : j \in J\}$ of nonzero central projections in $\mathfrak{M}$ such that $\bigvee_{j\in J} Q_j = \mathbb{I}$ and each Q_j is the sum of an orthogonal family of mutually equivalent finite projections in $\mathfrak{M}$.

A vector subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be *strongly dense* (resp., *strongly φ-dense*) if

- $U'\mathcal{D} \subset \mathcal{D}$ for any unitary U' in $\mathfrak{M}'$;
- there exists a sequence $P_n \in \mathrm{Proj}(\mathfrak{M})$ such that $P_n\mathcal{H} \subset \mathcal{D}$, $P_n^\perp \downarrow 0$ and $P_n^\perp$ is a finite projection (resp., $\varphi(P_n^\perp) < \infty$).

Clearly, every strongly φ-dense domain is strongly dense.

Throughout this paper, when we say that an operator T is *affiliated with the von Neumann algebra* $\mathfrak{M}$, written $T\,\eta\,\mathfrak{M}$, we always mean that T is closed, densely defined on $\mathcal{H}$, and $TU \supseteq UT$ for every unitary operator $U \in \mathfrak{M}'$.

An operator $T\,\eta\,\mathfrak{M}$ is called

- *measurable* (with respect to $\mathfrak{M}$) if its domain $D(T)$ is strongly dense;
- *φ-measurable* if $D(T)$ is strongly φ-dense.

From the definition itself it follows that if T is φ-measurable, then there exists $P \in \mathrm{Proj}(\mathfrak{M})$ such that TP is bounded and $\varphi(P^\perp) < \infty$.

We recall that any operator affiliated with a finite von Neumann algebra is measurable [19, Cor. 4.1] but not necessarily φ-measurable.

Remark 1.2. The following statements will be used later:

(i) Let $T\,\eta\,\mathfrak{M}$ and $Q \in \mathfrak{M}$. If $D(TQ) = \{\xi \in \mathcal{H} : Q\xi \in D(T)\}$ is dense in $\mathcal{H}$, then $TQ\,\eta\,\mathfrak{M}$.

(ii) If $Q \in \mathrm{Proj}(\mathfrak{M})$, then $Q\mathfrak{M}Q = \{QXQ\restriction_{Q\mathcal{H}} : X \in \mathfrak{M}\}$ is a von Neumann algebra on the Hilbert space $Q\mathcal{H}$; moreover $(Q\mathfrak{M}Q)' = Q\mathfrak{M}'Q$. If $T\,\eta\,\mathfrak{M}$ and $Q \in \mathfrak{M}$ and $D(TQ) = \{\xi \in \mathcal{H} : Q\xi \in D(T)\}$ is dense in $\mathcal{H}$, then $QTQ\,\eta\,Q\mathfrak{M}Q$.

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. For each $p \geq 1$, let

$$\mathcal{J}_p = \{X \in \mathfrak{M} : \varphi(|X|^p) < \infty\}.$$

Then $\mathcal{J}_p$ is a *-ideal of $\mathfrak{M}$. Following [17], we denote by $L^p(\varphi)$ the Banach space completion of $\mathcal{J}_p$ with respect to the norm

$$\|X\|_{p,\varphi} := \varphi(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.$$

One usually defines $L^\infty(\varphi) := \mathfrak{M}$. Thus, if φ is a finite trace, then $L^\infty(\varphi) \subset L^p(\varphi)$ for every $p \geq 1$. As shown in [17], if $X \in L^p(\varphi)$, then X is a measurable operator.

If A is a measurable operator and $A \geq 0$, one defines the *integral* of A by

$$\mu(A) = \sup\{\varphi(X) : 0 \leq X \leq A,\ X \in \mathcal{J}_1\}.$$

Then the space $L^p(\varphi)$ can also be defined [17] as the space of all measurable operators A such that $\mu(|A|^p) < \infty$.

The integral of an element $A \in L^p(\varphi)$ can be defined, in the obvious way, taking into account that any measurable operator A can be decomposed as $A = B_+ - B_- + iC_+ - iC_-$, where $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$ and B_+, B_- (resp. C_+, C_-) are the positive and negative parts of B (resp. C).

1.2. Locally convex quasi C*-algebras. In what follows we recall some definitions and facts.

Definition 1.3. Let $\mathfrak{X}$ be a complex vector space and $\mathfrak{A}_0$ a *-algebra contained in $\mathfrak{X}$. Then $\mathfrak{X}$ is said a *quasi *-algebra with distinguished *-algebra* $\mathfrak{A}_0$ (or simply over $\mathfrak{A}_0$) if

(i) the multiplication of $\mathfrak{A}_0$ is extended on $\mathfrak{X}$ as follows: the correspondences

$$\mathfrak{X} \times \mathfrak{A}_0 \to \mathfrak{A} : (a, x) \mapsto ax \text{ (left multiplication of } x \text{ by } a) \text{ and}$$
$$\mathfrak{A}_0 \times \mathfrak{X} \to \mathfrak{A} : (x, a) \mapsto xa \text{ (right multiplication of } x \text{ by } a)$$

are always defined and are bilinear;

(ii) $x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for all $x_1, x_2 \in \mathfrak{A}_0$ and $a \in \mathfrak{X}$;

(iii) the involution * of $\mathfrak{A}_0$ is extended on $\mathfrak{X}$, denoted also by *, and satisfies $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$, for all $x \in \mathfrak{A}_0$ and $a \in \mathfrak{X}$.

Thus a *quasi *-algebra* [18] is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$, and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. The *unit* of $(\mathfrak{X}, \mathfrak{A}_0)$ is an element $e \in \mathfrak{A}_0$ such that $xe = ex = x$ for every $x \in \mathfrak{X}$.

A quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is said to be *locally convex* if $\mathfrak{X}$ is endowed with a topology τ which makes $\mathfrak{X}$ a locally convex space such that the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in \mathfrak{A}_0$, are continuous. If τ is a norm topology and the involution is isometric with respect to the norm, we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a *normed quasi *-algebra*, and if it is complete, we say it is a *Banach quasi*-algebra*.

Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a C*-algebra. We shall use the symbol $\|\cdot\|_0$ of the C*-norm to also denote the corresponding topology. Suppose that τ is a topology on $\mathfrak{A}_0$ such that $\mathfrak{A}_0[\tau]$ is a locally convex *-algebra. Then the topologies τ and $\|\cdot\|_0$ on $\mathfrak{A}_0$ are *compatible* whenever each Cauchy net in both topologies that converges with respect to one of them, also converges with respect to the other.

Under certain conditions on τ, a quasi *-subalgebra $\mathfrak{A}$ of the quasi *-algebra $\mathfrak{X} = \widetilde{\mathfrak{A}_0}[\tau]$ over $\mathfrak{A}_0$ is named a *locally convex quasi C^*-algebra.* More precisely, let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a directed family of seminorms defining the topology τ. Suppose that τ is compatible with $\|\cdot\|_0$ and has the following properties:

- (T_1) $\mathfrak{A}_0[\tau]$ is a locally convex *-algebra with separately continuous multiplication.
- (T_2) $\tau \preceq \|\cdot\|_0$.

Then the identity map $\mathfrak{A}_0[\|\cdot\|_0] \to \mathfrak{A}_0[\tau]$ extends to a continuous *-linear map $\mathfrak{A}_0[\|\cdot\|_0] \to \widetilde{\mathfrak{A}_0}[\tau]$. Since τ and $\|\cdot\|_0$ are compatible, the C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$ can be regarded as embedded into $\widetilde{\mathfrak{A}_0}[\tau]$. It is easily shown that $\widetilde{\mathfrak{A}_0}[\tau]$ is a quasi *-algebra over $\mathfrak{A}_0$ (cf. [13, Section 3]).

We denote by $(\mathfrak{A}_0)_+$ the set of all positive elements of the C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$.

Further, we employ the following two extra conditions (T_3), (T_4) for the locally convex topology τ on $\mathfrak{A}_0$:

- (T_3) For each $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda$ such that

$$p_\lambda(xy) \le \|x\|_0 p_{\lambda'}(y) \quad \text{for all } x, y \in \mathfrak{A}_0 \text{ with } xy = yx.$$

- (T_4) The set $\mathcal{U}(\mathfrak{A}_0)_+ := \{x \in (\mathfrak{A}_0)_+ : \|x\|_0 \le 1\}$ is τ-closed.

Definition 1.4. By a *locally convex quasi C^*-algebra* over $\mathfrak{A}_0$ (see [3]), we mean any quasi *-subalgebra $\mathfrak{A}$ of the locally convex quasi *-algebra $\mathfrak{X} = \widetilde{\mathfrak{A}_0}[\tau]$ over $\mathfrak{A}_0$, where $\mathfrak{A}_0[\|\cdot\|_0]$ is a C*-algebra with identity e and τ a locally convex topology on $\mathfrak{A}_0$, defined by a directed family $\{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms satisfying conditions (T_1)–(T_4).

The following examples have been discussed in [3].

Example 1.5 (CQ*-algebras). Let $\mathfrak{A}_0$ be a C*-algebra with norm $\|\cdot\|$ and involution *. Let $\|\cdot\|_1$ be a norm on $\mathfrak{A}_0$, weaker than $\|\cdot\|$ and such that, for every $a, b \in \mathfrak{A}$,

- (i) $\|ab\|_1 \le \|a\|_1 \|b\|$,
- (ii) $\|a^*\|_1 = \|a\|_1$.

Let $\mathfrak{X}$ denote the $\|\cdot\|_1$-completion of $\mathfrak{A}_0$; then ([1]) the couple $(\mathfrak{X}, \mathfrak{A}_0)$ is called a *CQ*-algebra.* Every CQ*-algebra is a locally convex quasi C*-algebra.

([1]) In previous papers this structure was called a *proper* CQ*-algebra. Since this is the sole case we consider here, we will systematically omit the specification *proper.*

Example 1.6. The space $L^p([0,1])$ with $1 \le p < \infty$ is a Banach $L^\infty([0,1])$-bimodule. The couple $(L^p([0,1]), L^\infty([0,1]))$ may be regarded as a CQ*-algebra, thus a locally convex quasi C*-algebra over $L^\infty([0,1])$.

2. Locally convex quasi C*-algebras of measurable operators. Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then, as shown in [9], $(L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi))$ is a *Banach quasi*-algebra*, and if φ is a finite trace, then $(L^p(\varphi), \mathfrak{M})$ is a CQ*-algebra.

In analogy to [9] we consider the following two sets of sesquilinear forms enjoying certain invariance properties.

Definition 2.1. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a locally convex quasi C*-algebra with unit e. We denote by $\mathcal{S}(\mathfrak{X})$ the set of all sesquilinear forms Ω on $\mathfrak{X} \times \mathfrak{X}$ with the following properties:

(i) $\Omega(x,x) \ge 0$ for all $x \in \mathfrak{X}$;
(ii) $\Omega(xa,b) = \Omega(a,x^*b)$ for all $x \in \mathfrak{X}$ and $a,b \in \mathfrak{A}_0$;
(iii) $|\Omega(x,y)| \le p(x)p(y)$ for some τ-continuous seminorm p on $\mathfrak{X}$ and all $x,y \in \mathfrak{X}$;
(iv) $\Omega(e,e) \le 1$.

The locally convex quasi C*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called **-semisimple* if whenever $x \in \mathfrak{X}$ and $\Omega(x,x) = 0$ for every $\Omega \in \mathcal{S}(\mathfrak{X})$, then $x = 0$.

We denote by $\mathcal{T}(\mathfrak{X}) \subseteq \mathcal{S}(\mathfrak{X})$ the set of all sesquilinear forms $\Omega \in \mathcal{S}(\mathfrak{X})$ with the following property:

(v) $\Omega(x,x) = \Omega(x^*,x^*)$ for all $x \in \mathfrak{X}$.

Remark 2.2.

- By (v) of Definition 2.1 and by polarization, we get

$$\Omega(y^*,x^*) = \Omega(x,y) \quad \text{for all } x,y \in \mathfrak{X}.$$

- The set $\mathcal{T}(\mathfrak{X})$ is convex.

Example 2.3. Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then, $(L^p(\varphi), \mathcal{J}_p)$, $p \ge 2$, is a *-semisimple Banach quasi *-algebra. If φ is a finite trace (we assume $\varphi(\mathbb{I}) = 1$), then $(L^p(\varphi), \mathfrak{M})$, with $p \ge 2$, is a *-semisimple locally convex quasi C*-algebra. If $p \ge 2$ then L^p-spaces possess a sufficient family of positive sesquilinear forms. Indeed, in this case, since $|W|^{p-2} \in L^{p/(p-2)}(\varphi)$ for every $W \in L^p(\varphi)$, the sesquilinear form Ω_W defined by

$$\Omega_W(X,Y) := \frac{\varphi[X(Y|W|^{p-2})^*]}{\|W\|_{p,\varphi}^{p-2}}$$

is positive and satisfies conditions (i)–(iv) of Definition 2.1 (see [9], and [24] for more details). Moreover,

$$\Omega_W(W, W) = \|W\|^p_{p,\varphi}.$$

Remark 2.4. The notion of *-semisimplicity of locally convex partial *-algebras has been studied in full generality in [2] and [14].

Definition 2.5. Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. We denote by $L^p_{\text{loc}}(\varphi)$ the set of all measurable operators T such that $TP \in L^p(\varphi)$ for every central φ-finite projection P of $\mathfrak{M}$.

Remark 2.6. The von Neumann algebra $\mathfrak{M}$ is a subset of $L^p_{\text{loc}}(\varphi)$. Indeed, if $X \in \mathfrak{M}$, then for every φ-finite central projection P of $\mathfrak{M}$ the product XP belongs to the *-ideal $\mathcal{J}_p$.

Throughout this section we are given a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$ with a family $\{P_j\}_{j\in J}$ of φ-finite central projections of $\mathfrak{M}$ such that

- if $l, m \in J$, $l \neq m$, then $P_l P_m = 0$ (i.e., the P_j's are orthogonal);
- $\bigvee_{j\in J} P_j = \mathbb{I}$, where $\bigvee_{j\in J} P_j$ denotes the projection onto the subspace generated by $\{P_j\mathcal{H} : j \in J\}$.

These conditions always hold in a von Neumann algebra with a faithful normal semifinite trace (see Lemma 1.1 and [15, 20] for more details).

If φ is a normal faithful semifinite trace on $\mathfrak{M}_+$, we define, for each $X \in \mathfrak{M}$, the seminorms $q_j(X) := \|XP_j\|_{p,\varphi}$, $j \in J$. The translation-invariant locally convex topology defined by the system $\{q_j : j \in J\}$ is denoted by τ_p.

Definition 2.7. Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. We denote by $\widetilde{\mathfrak{M}}^{\tau_p}$ the τ_p-completion of $\mathfrak{M}$.

Proposition 2.8. *Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then $L^p_{\text{loc}}(\varphi) \subseteq \widetilde{\mathfrak{M}}^{\tau_p}$. Moreover, if there exists a family $\{P_j\}_{j\in J}$ as above with all P_j's mutually equivalent, then $L^p_{\text{loc}}(\varphi) = \widetilde{\mathfrak{M}}^{\tau_p}$.*

Proof. From Remark 2.6, $\mathfrak{M} \subseteq L^p_{\text{loc}}(\varphi)$. If $Y \in L^p_{\text{loc}}(\varphi)$, for every $j \in J$ we have $YP_j \in L^p(\varphi)$. Hence, for every $j \in J$, there exist $(X^j_n)^\infty_{n=1} \subseteq \mathcal{J}_p$ such that $\|X^j_n - YP_j\|_{p,\varphi} \to 0$ as $n \to \infty$.

Let $\mathbb{F}_J$ be the family of finite subsets of J ordered by inclusion, and let $F \in \mathbb{F}_J$. We set

$$T_{n,F} := \sum_{j\in F} X^j_n P_j \in \mathfrak{M}.$$

Then the net $(T_{n,F})$ converges to Y with respect to τ_p. Indeed, for every $m \in J$,

$$q_m(T_{n,F} - Y) = \|(T_{n,F} - Y)P_m\|_{p,\varphi} = \|(X_n^m - Y)P_m\|_{p,\varphi}$$

for sufficiently large F. Thus, $\|(X_n^m - Y)P_m\|_{p,\varphi} \le \|X_n^m - YP_m\|_{p,\varphi}$ implies that $q_m(T_{n,F} - Y) \underset{n,F}{\longrightarrow} 0$.

Hence $L^p_{\mathrm{loc}}(\varphi) \subseteq \widetilde{\mathfrak{M}}^{\tau_p}$.

Now, assume that all P_j' s are mutually equivalent. If $Y \in \widetilde{\mathfrak{M}}^{\tau_p}$, there exists a net $(X_\alpha) \subseteq \mathfrak{M}$ such that $X_\alpha \to Y$ with respect to τ_p; hence

$$X_\alpha P_j \to YP_j \in L^p(\varphi) \quad \text{in } \|\cdot\|_{p,\varphi}. \tag{2.1}$$

But for each central φ-finite projection P we have

$$\varphi(P) = \varphi\Big(P \sum_{j\in J} P_j\Big) = \sum_{j\in J} \varphi(PP_j). \tag{2.2}$$

By our assumption, for any $l, m \in J$ we may pick $U \in \mathfrak{M}$ so that $U^*U = P_l$ and $UU^* = P_m$, hence

$$\varphi(PP_l) = \varphi(PU^*U) = \varphi(UPU^*) = \varphi(PUU^*) = \varphi(PP_m).$$

So, all terms on the right hand side of (2.2) are equal, and since the above series converges, only a finite number of them can be nonzero. Thus, for some $s \in \mathbb{N}$ we may write $J = \{1, \dots, s\}$ and then

$$P = P\sum_{j\in J} P_j = P\sum_{j=1}^{s} P_j = \sum_{j=1}^{s} PP_j, \tag{2.3}$$

and hence

$$YP = \sum_{j=1}^{s} YPP_j = \sum_{j=1}^{s} YP_jP \in L^p(\varphi). \tag{2.4}$$

Therefore, if $Y \in \widetilde{\mathfrak{M}}^{\tau_p}$, then for each central φ-finite projection P we have $YP \in L^p(\varphi)$. Hence $L^p_{\mathrm{loc}}(\varphi) \supseteq \widetilde{\mathfrak{M}}^{\tau_p}$. ∎

Remark 2.9. In general, a von Neumann algebra need not have an orthogonal family $\{P_j\}_{j\in J}$ of mutually equivalent finite central projections such that $\bigvee_{j\in J} P_j = \mathbb{I}$, but if this is the case, then $L^p_{\mathrm{loc}}(\varphi) = \widetilde{\mathfrak{M}}^{\tau_p}$.

Theorem 2.10. *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then $(\widetilde{\mathfrak{M}}^{\tau_p}, \mathfrak{M})$ is a locally convex quasi C^*-algebra with respect to τ_p, consisting of measurable operators.*

Proof. The topology τ_p has properties (T_1)–(T_4). We will just prove (T_3)–(T_4) here.

(T_3) For each $\lambda \in J$,

$$q_\lambda(XY) = \|P_\lambda XY\|_{p,\varphi} \le \|X\|\,\|P_\lambda Y\|_{p,\varphi} = \|X\| q_\lambda(Y), \quad \forall X, Y \in \mathfrak{M}.$$

(T_4) The set $\mathcal{U}(\mathfrak{M})_+ := \{X \in (\mathfrak{M})_+ : \|X\| \le 1\}$ is τ_p-closed. To see this, consider a net $\{F_\alpha\}$ in $\mathcal{U}(\mathfrak{M})_+$ with $F_\alpha \to F$ in the topology τ_p. Then for each $j \in J$, $\|(F_\alpha - F)P_j\|_{p,\varphi} \to 0$. By assumption on P_j, the trace φ is a normal faithful finite trace on the von Neumann algebra $P_j\mathfrak{M}_+$. Then (see [9]) $(L^p(\varphi), P_j\mathfrak{M})$ is a CQ*-algebra. Therefore, using (T_4) for $(L^p(\varphi), P_j\mathfrak{M})$, we have $FP_j \in \mathcal{U}(P_j\mathfrak{M})_+$ for each $j \in J$. This, by definition, implies that $F \in \mathfrak{M}$. Indeed, for every

$$h = \sum_{j\in J} P_j h \in \mathcal{H} = \bigoplus_{j\in J} P_j\mathcal{H}$$

we have

$$\|Fh\|_{\mathcal{H}}^2 = \sum_{j\in J} \|FP_jh\|^2 = \sum_{j\in J} \|FP_jP_jh\|^2 \le \sum_{j\in J} \|P_jh\|^2 = \|h\|_{\mathcal{H}}^2.$$

Hence $F \in \mathcal{U}(\mathfrak{M})_+$. ∎

REMARK 2.11. By Proposition 2.8, $(L^p_{\mathrm{loc}}(\varphi), \mathfrak{M})$ itself is a *locally convex quasi C^*-algebra* with respect to τ_p.

3. Representation theorems. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a locally convex quasi C*-algebra with a unit e. For each $\Omega \in \mathcal{T}(\mathfrak{X})$, we define a linear functional ω_Ω on $\mathfrak{A}_0$ by

$$\omega_\Omega(a) := \Omega(a, e), \quad a \in \mathfrak{A}_0.$$

We have

$$\omega_\Omega(a^*a) = \Omega(a^*a, e) = \Omega(a, a) = \Omega(a^*, a^*) = \omega_\Omega(aa^*) \ge 0.$$

This shows at once that ω_Ω is positive and tracial.

By the Gelfand–Naimark theorem each C*-algebra is isometrically *-isomorphic to a C*-algebra of bounded operators in Hilbert space. This isometric *-isomorphism is called the *universal *-representation*. We denote it by π.

For every $\Omega \in \mathcal{T}(\mathfrak{X})$ and $a \in \mathfrak{A}_0$, we set

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a).$$

Then, for each $\Omega \in \mathcal{T}(\mathfrak{X})$, φ_Ω is a positive bounded linear functional on the operator algebra $\pi(\mathfrak{A}_0)$.

Clearly,

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a) = \Omega(a, e).$$

Since $\{p_\lambda\}$ is directed, there exist $\gamma > 0$ and $\lambda \in \Lambda$ such that

$$|\varphi_\Omega(\pi(a))| = |\omega_\Omega(a)| = |\Omega(a, e)| \le \gamma^2 p_\lambda(ae) p_\lambda(e).$$

Then by (T_3), for some $\lambda' \in \Lambda$,

$$|\varphi_\Omega(\pi(a))| \le \gamma^2 \|a\|_0 p_{\lambda'}(e)^2.$$

Thus φ_Ω is continuous on $\pi(\mathfrak{A}_0)$.

By [15, Vol. 2, Proposition 10.1.1], φ_Ω is weakly continuous and so it extends uniquely to $\pi(\mathfrak{A}_0)''$, by the Hahn–Banach theorem. Moreover, since φ_Ω is a trace on $\pi(\mathfrak{A}_0)$, the extension $\widetilde{\varphi_\Omega}$ is also a trace on the von Neumann algebra $\mathfrak{M} := \pi(\mathfrak{A}_0)''$ generated by $\pi(\mathfrak{A}_0)$.

Clearly, the set $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0) = \{\widetilde{\varphi_\Omega} : \Omega \in \mathcal{T}(\mathfrak{X})\}$ is convex.

Definition 3.1. The locally convex quasi C*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is said to be *strongly *-semisimple* if

(a) the equality $\Omega(x,x) = 0$ for every $\Omega \in \mathcal{T}(\mathfrak{X})$ implies $x = 0$;
(b) the set $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ is w^*-closed.

Remark 3.2. If $(\mathfrak{X}, \mathfrak{A}_0)$ is a CQ*-algebra, then by [9, Proposition 4.1], (b) is automatically satisfied.

Example 3.3. Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then, as seen in Example 2.3, if φ is a finite trace, then $(L^p(\varphi), \mathfrak{M})$, with $p \ge 2$, is a *-semisimple locally convex quasi C*-algebra. Conditions (a) and (b) of Definition 3.1 are satisfied. Indeed, in this case, the set $\mathfrak{N}_{\mathcal{T}}(\mathfrak{M})$ is w^*-closed by [9, Proposition 4.1]. Therefore $(L^p(\varphi), \mathfrak{M})$, with φ finite, is a strongly *-semisimple locally convex quasi C*-algebra.

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a locally convex quasi C*-algebra with unit e, π the universal representation of $\mathfrak{A}_0$, and $\mathfrak{M} = \pi(\mathfrak{A}_0)''$. Denote by $\|f\|^\sharp$ the norm of a bounded functional f on $\mathfrak{M}$, and by $\mathfrak{M}^\sharp$ the topological dual of $\mathfrak{M}$. Then the norm $\|\widetilde{\varphi_\Omega}\|^\sharp$ of $\widetilde{\varphi_\Omega}$ as a linear functional on $\mathfrak{M}$ equals the norm of φ_Ω as a functional on $\pi(\mathfrak{A}_0)$.

By (iv) of Definition 2.1, $\|\widetilde{\varphi_\Omega}\|^\sharp = \widetilde{\varphi}_\Omega(\pi(e)) = \Omega(e,e) \le 1$.

Hence, if (b) of Definition 3.1 is satisfied, then the set $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$, being a w^*-closed subset of the unit ball of $\mathfrak{M}^\sharp$, is w^*-compact.

Let $\mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ be the set of extreme points of $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$; then $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ coincides with the w^*-closure of the convex hull of $\mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$.

Thus $\mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ is a family of normal finite traces on the von Neumann algebra $\mathfrak{M}$.

We define $\mathcal{F} := \{\Omega \in \mathcal{T}(\mathfrak{X}) : \widetilde{\varphi_\Omega} \in \mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)\}$ and denote by P_Ω the support projection corresponding to the trace $\widetilde{\varphi_\Omega}$. By [9, Lemma 3.5], $\{P_\Omega\}_{\Omega\in\mathcal{F}}$ consists of mutually orthogonal projections and if $Q := \bigvee_{\Omega\in\mathcal{F}} P_\Omega$ then

$$\mu = \sum_{\widetilde{\varphi_\Omega} \in \mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)} \widetilde{\varphi_\Omega}$$

is a normal faithful semifinite trace defined on the direct sum (see [20] and [26]) of von Neumann algebras

$$Q\mathfrak{M} = \bigoplus_{\Omega\in\mathcal{F}} P_\Omega\mathfrak{M}.$$

THEOREM 3.4. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a strongly *-semisimple locally convex quasi C^*-algebra with unit e. Then there exists a monomorphism*

$$\Phi : \mathfrak{X} \ni x \mapsto \Phi(x) := \widetilde{X} \in \widetilde{Q\mathfrak{M}}^{\tau_2}$$

with the following properties:

(i) *Φ extends the isometry $\pi : \mathfrak{A}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ given by the Gelfand–Naimark theorem;*
(ii) *$\Phi(x^*) = \Phi(x)^*$ for every $x \in \mathfrak{X}$;*
(iii) *$\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathfrak{X}$ such that $x \in \mathfrak{A}_0$ or $y \in \mathfrak{A}_0$.*

Proof. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be, as before, the family of seminorms defining the topology τ of $\mathfrak{X}$. For fixed $x \in \mathfrak{X}$, there exists a net $\{a_\alpha : \alpha \in \Delta\}$ of elements of $\mathfrak{A}_0$ such that $p_\lambda(a_\alpha - x) \to 0$ for each $\lambda \in \Lambda$. We write $X_\alpha = \pi(a_\alpha)$.

By (iii) of Definition 2.1, for every $\Omega \in \mathcal{T}(\mathfrak{X})$, there exist $\gamma > 0$ and $\lambda' \in \Lambda$ such that for each $\alpha, \beta \in \Delta$,

$$\begin{aligned}\|P_\Omega(X_\alpha - X_\beta)\|_{2,\widetilde{\varphi_\Omega}} &= \|P_\Omega(\pi(a_\alpha) - \pi(a_\beta))\|_{2,\widetilde{\varphi_\Omega}} \\ &= [\widetilde{\varphi_\Omega}(|P_\Omega(\pi(a_\alpha) - \pi(a_\beta))|^2)]^{1/2} \\ &= [\Omega\,((a_\alpha - a_\beta)^*(a_\alpha - a_\beta), e)]^{1/2} \\ &= [\Omega(a_\alpha - a_\beta, a_\alpha - a_\beta)]^{1/2} \le \gamma p_{\lambda'}(a_\alpha - a_\beta) \underset{\alpha,\beta}{\longrightarrow} 0.\end{aligned}$$

Let $\widetilde{X_\Omega}$ be the $\|\cdot\|_{2,\widetilde{\varphi_\Omega}}$-limit of the net $(P_\Omega X_\alpha)$ in $L^2(\widetilde{\varphi_\Omega})$. Clearly $\widetilde{X_\Omega} = P_\Omega\widetilde{X_\Omega}$. We define

$$\Phi(x) := \sum_{\Omega\in\mathcal{F}} P_\Omega\widetilde{X_\Omega} =: \widetilde{X}.$$

Clearly $\widetilde{X} \in \widetilde{Q\mathfrak{M}}^{\tau_2}$.

It is easy to see that the map $\mathfrak{X} \ni x \mapsto \widetilde{X} \in \widetilde{Q\mathfrak{M}}^{\tau_2}$ is well defined and injective. Indeed, if $a_\alpha \to 0$, there exist $\gamma > 0$ and $\lambda' \in \Lambda$ such that

$$\begin{aligned}\|P_\Omega X_\alpha\|_{2,\widetilde{\varphi_\Omega}} &= \|P_\Omega\pi(a_\alpha)\|_{2,\widetilde{\varphi_\Omega}} = [\widetilde{\varphi_\Omega}(|P_\Omega(\pi(a_\alpha)|^2)]^{1/2} \\ &= [\Omega(a_\alpha^* a_\alpha, e)]^{1/2} = [\Omega(a_\alpha, a_\alpha)]^{1/2} \le \gamma p_{\lambda'}(a_\alpha) \to 0.\end{aligned}$$

Thus $P_\Omega(X_\alpha) = 0$ for every $\Omega \in \mathcal{T}(\mathfrak{X})$, and so $\widetilde{X} = 0$. Moreover if $P_\Omega\widetilde{X} = 0$ for each $\Omega \in \mathcal{F}$, then $\Omega(x, x) = 0$ for every $\Omega \in \mathcal{F}$. Since every $\Omega \in \mathcal{T}(\mathfrak{X})$ is a w^*-limit of convex combinations of elements of $\mathcal{F}$, we get $\Omega(x, x) = 0$ for every $\Omega \in \mathcal{T}(\mathfrak{X})$. Therefore, by assumption, $x = 0$. ∎

Remark 3.5. In the same way we can prove that:

- If $(\mathfrak{X}, \mathfrak{A}_0)$ is a strongly *-semisimple locally convex quasi C*-algebra and there exists a faithful $\Omega \in \mathcal{T}(\mathfrak{X})$ (i.e., $\Omega(x,x) = 0$ implies $x = 0$) then there exists a monomorphism

$$\Phi : \mathfrak{X} \ni x \to \Phi(x) := \widetilde{X} \in L^2(\widetilde{\varphi_\Omega})$$

with the following properties:

 (i) Φ extends the isometry $\pi : \mathfrak{A}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ given by the Gelfand–Naimark theorem;
 (ii) $\Phi(x^*) = \Phi(x)^*$ for every $x \in \mathfrak{X}$,
 (iii) $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathfrak{X}$ such that $x \in \mathfrak{A}_0$ or $y \in \mathfrak{A}_0$.

- If the semifinite von Neumann algebra $\pi(\mathfrak{A}_0)''$ admits an orthogonal family $\{P'_i : i \in I\}$ of mutually equivalent projections such that $\sum_{i \in I} P'_i = \mathbb{I}$, then it is easy to see that the map $\mathfrak{X} \ni x \mapsto \widetilde{X} \in L^2_{\mathrm{loc}}(\tau)$ is a monomorphism.

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