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On the Lukacs property for free random variables

by

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Abstract. The Lukacs property of the free Poisson distribution is studied. We prove that if free $\mathbb{X}$ and $\mathbb{Y}$ are free Poisson distributed with suitable parameters, then $\mathbb{X}+\mathbb{Y}$ and $(\mathbb{X}+\mathbb{Y})^{-1/2}\mathbb{X}(\mathbb{X}+\mathbb{Y})^{-1/2}$ are free. As an auxiliary result we compute the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$ for free Poisson distributed $\mathbb{X}$. We also study the Lukacs property of the free Gamma distribution.

1. Introduction. The celebrated Lukacs theorem [17] in the classical probability theory gives a characterization of independent random variables X and Y which have Gamma distributions $G(a,p)$ and $G(a,q)$ respectively, by independence of $V = X+Y$ and $U = \frac{X}{X+Y}$. By the Gamma distribution $G(a,p)$ we mean here a probability distribution given by the probability density function

$$\frac{a^p}{\Gamma(p)}x^{p-1}e^{-ax}I_{(0,\infty)}(x), \qquad a,p>0.$$

This result was generalized in many directions. An important direction was to relax the assumption of independence of U and V. The same characterization holds true when instead of assuming independence, constancy of the first and second conditional moment of U given by V is assumed; see [5] or [14] for more general so called Laha–Lukacs regressions. One can also consider other powers of U (see e.g. [28]).

It is easy to see that U defined as above has a Beta distribution of the first kind $\beta_I(p,q)$, where by a Beta distribution we understand a probability distribution with density

$$\frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)}I_{(0,1)}(x), \qquad p,q>0.$$

Another way of generalizing the Lukacs theorem is by means of the so called dual Lukacs regressions introduced in [4], where it is proved that if U and V

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are independent and such that the ith and jth conditional moments of $Y = V(1-U)$ given $X = UV$ are constant for $(i,j) \in \{(-1,-2),(-1,1),(1,2)\}$ then U has a Beta distribution and V has a Gamma distribution.

The Lukacs property was also studied in the context of random matrices, where it turns out that this property characterizes the Wishart distribution (see [9, 3, 6, 20, 21, 15, 16]).

In the present paper we study analogues of the Lukacs property in free probability. Free probability and the notion of free independence of non-commutative random variables were introduced by Voiculescu [25]. One of the links between classical and free probability is the so called asymptotic freeness, which roughly speaking says that large independent random matrices under the state which is the expectation of a normalized trace are close to free random variables (see [26, 22]). Another relation between these two theories is given by Bercovici–Pata bijections (see [1]) between infinitely divisible measures under classical and free convolution. Also an important link is given in [22, 19] where it is proved that free cumulants can be defined using the lattice of non-crossing partitions while classical cumulants are defined using the lattice of all partitions.

It was also noticed in the literature that characterizations of independent and free random variables are closely related. However, this relationship is not completely understood. There are many examples of classical characterizations which have free counterparts. A basic example is the Bernstein theorem [2] which characterizes the normal distribution by independence of $X+Y$ and $X-Y$ for independent X and Y. Its free analogue was proved in [18] and says that if $\mathbb{X}$ and $\mathbb{Y}$ are free then $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}-\mathbb{Y}$ are free if and only if $\mathbb{X}$ and $\mathbb{Y}$ have a semicircular distribution.

The Lukacs theorem was also studied in free probability. It turns out that in the context of characterizations the role of the Gamma distribution in free probability is played by the Marchenko–Pastur distribution, also known as the free Poisson distribution. In [7] the authors proved a free analogue of Laha–Lukacs regressions, they described the family of free Meixner distributions under some assumptions on the first two conditional moments of $\mathbb{X}$ given $\mathbb{X}+\mathbb{Y}$; one of the cases of Laha–Lukacs regressions is a free analogue of Lukacs regressions. Laha–Lukacs regressions and related characterizations were also studied in [10, 11, 12]. Dual Lukacs regressions in free probability were studied in our previous works [24, 23].

The Lukacs property in free probability is well studied, but there has been a significant gap in the study of the free Lukacs property, namely there has been no proof that for free $\mathbb{X}$ and $\mathbb{Y}$ free Poisson distributed, the random variables

$$\mathbb{U} = (\mathbb{X} + \mathbb{Y})^{-1/2}\mathbb{X}(\mathbb{X} + \mathbb{Y})^{-1/2} \quad \text{and} \quad \mathbb{V} = \mathbb{X} + \mathbb{Y}$$

are free. In Section 4 we give a proof of this fact.

We have to note that the closest result in this direction was proved in [8] where the authors proved the asymptotic freeness of

$$\mathbf{U} = (\mathbf{X} + \mathbf{Y})^{-1/2}\mathbf{X}(\mathbf{X} + \mathbf{Y})^{-1/2} \quad \text{and} \quad \mathbf{V} = \mathbf{X} + \mathbf{Y}$$

for $\mathbf{X}, \mathbf{Y}$ independent, complex Wishart distributed. Using results from [8], in [24] a result in the opposite direction is proved: for $\mathbb{U}$ free Binomial distributed and $\mathbb{V}$ free Poisson distributed, the random variables $\mathbb{X} = \mathbb{V}^{1/2}\mathbb{U}\mathbb{V}^{1/2}$ and $\mathbb{Y} = \mathbb{V}^{1/2}(\mathbb{I} - \mathbb{U})\mathbb{V}^{1/2}$ are free.

Here we do not use the asymptotic freeness technique for the proof of the free Lukacs property of the free Poisson distribution. Our approach is based on a direct calculation of joint cumulants of $\mathbb{U}$ and $\mathbb{V}$. The computation involves the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$ for the free Poisson distributed $\mathbb{X}$, for which we find a closed formula. The form of these cumulants also leads to some new characterizations of the free Poisson distribution.

We also prove that if $\mathbb{X}$ is free Poisson distributed then $\mathbb{X}^{-1}$ belongs to the free Gamma family defined in [7]. This observation together with knowledge of the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$ allows us to answer the question raised in [7], whether for free, positive, identically free Gamma distributed $\mathbb{X}$ and $\mathbb{Y}$, the random variables $\mathbb{V} = \mathbb{X} + \mathbb{Y}$ and $\mathbb{U} = (\mathbb{X} + \mathbb{Y})^{-1}\mathbb{X}^2(\mathbb{X} + \mathbb{Y})^{-1}$ are free. The answer turns out to be negative.

The paper is organized as follows. In Section 2 we give the basics of free probability. Section 3 is devoted to finding the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$ for a free Poisson distributed $\mathbb{X}$. We also prove in that section characterizations of the free Poisson distribution related to the formula for the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$. In Section 4 we study the Lukacs property for the free Poisson and free Gamma distributions.

2. Preliminaries. In this section we will give basic definitions and facts necessary for understanding our results; for a more comprehensive introduction to free probability consult [19] or [27].

A *non-commutative $*$-probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\phi(\mathbb{I}) = 1$, where $\mathbb{I}$ is the unit of $\mathcal{A}$, $\varphi(\mathbb{X}^*\mathbb{X}) \geq 0$, and $\varphi(\mathbb{X}^*\mathbb{X}) = 0$ iff $\mathbb{X} = 0$. If $\mathcal{A}$ is a C^*-algebra then $(\mathcal{A}, \varphi)$ is called a *C^*-probability space.* Any element $\mathbb{X}$ of $\mathcal{A}$ is called a (non-commutative) *random variable.*

The *$*$-distribution* μ of a self-adjoint element $\mathbb{X} \in \mathcal{A} \subset \mathcal{B}(H)$ is a probability measure μ on $\mathbb{R}$ such that

$$\varphi(\mathbb{X}^r) = \int_{\mathbb{R}} t^r \mu(dt) \qquad \forall r = 1, 2, \ldots.$$

Unital subalgebras $\mathcal{A}_i \subset \mathcal{A}$, $i \in I$, are said to be *freely independent* if $\varphi(\mathbb{X}_1, \ldots, \mathbb{X}_k) = 0$ for $\mathbb{X}_j \in \mathcal{A}_{i(j)}$, where $i(j) \in I$, such that $\varphi(\mathbb{X}_j) = 0$, $j = 1, \ldots, k$, where neighbouring elements are from different subalgebras, that is, $i(1) \neq i(2) \neq \cdots \neq i(k-1) \neq i(k)$. Similarly, random variables $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are free (freely independent) when the subalgebras generated by $(\mathbb{X}, \mathbb{I})$ and $(\mathbb{Y}, \mathbb{I})$ are freely independent (here $\mathbb{I}$ denotes the identity operator).

Let $\chi = \{B_1, B_2, \ldots\}$ be a partition of the set $\{1, \ldots, k\}$. Then χ is a *crossing partition* if there exist distinct blocks $B_r, B_s \in \chi$ and numbers $i_1, i_2 \in B_r$, $j_1, j_2 \in B_s$ such that $i_1 < j_1 < i_2 < j_2$. Otherwise χ is a *non-crossing partition.* The set of all non-crossing partitions of $\{1, \ldots, k\}$ is denoted by $\mathrm{NC}(k)$.

For any $k = 1, 2, \ldots$, the (*joint*) *cumulants* of order k of non-commutative random variables $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are defined recursively as k-linear maps $\mathcal{R}_k : \mathcal{A}^k \to \mathbb{C}$ through the equations

$$\varphi(\mathbb{X}_1 \ldots \mathbb{X}_m) = \sum_{\chi \in \mathrm{NC}(m)} \prod_{B \in \chi} \mathcal{R}_{|B|}(\mathbb{X}_i,\, i \in B) \tag{2.1}$$

holding for any $m = 1, 2, \ldots$, with $|B|$ denoting the size of the block B.

Freeness can be characterized in terms of the behaviour of the cumulants in the following way. Consider unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of an algebra $\mathcal{A}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$. The subalgebras $(\mathcal{A}_i)_{i \in I}$ are *freely independent* iff for any $n = 2, 3, \ldots$ and for any $\mathbb{X}_j \in \mathcal{A}_{i(j)}$ with $i(j) \in I$, $j = 1, \ldots, n$, any n-cumulant satisfies

$$\mathcal{R}_n(\mathbb{X}_1, \ldots, \mathbb{X}_n) = 0$$

if there exists a pair $k, l \in \{1, \ldots, n\}$ such that $i(k) \neq i(l)$.

For a non-commutative random variable $\mathbb{X}$ its *r-transform* is defined as

$$r_{\mathbb{X}}(z) = \sum_{n=0}^{\infty} \mathcal{R}_{n+1}(\mathbb{X}) z^n, \tag{2.2}$$

where $\mathcal{R}_n(\mathbb{X}) = \mathcal{R}_n(\mathbb{X}, \ldots, \mathbb{X})$. The r-transform of $\mathbb{X}$ uniquely determines the distribution of $\mathbb{X}$.

In particular we will need a result which states that joint cumulants of $\mathbb{I}$ with any other non-commutative random variable vanish (see [19, Proposition 11.15]).

PROPOSITION 2.1. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $\mathbb{X}_1, \ldots, \mathbb{X}_n \in \mathcal{A}$, $n \geq 2$. Then*

$$\mathcal{R}_n(\mathbb{X}_1, \ldots, \mathbb{X}_n) = 0$$

whenever there exists $k \in \{1, \ldots, n\}$ such that $\mathbb{X}_k = \mathbb{I}$.

A non-commutative random variable $\mathbb{X}$ is said to be *free Poisson distributed* if it has Marchenko–Pastur (or free Poisson) distribution $\nu = \nu(\lambda, \alpha)$

defined by the formula

$$\nu = \max\{0, 1-\lambda\}\delta_0 + \lambda\tilde{\nu}, \tag{2.3}$$

where $\lambda \geq 0$ and the measure $\tilde{\nu}$ is absolutely continuous, supported on the interval $(\alpha(1-\sqrt{\lambda})^2, \alpha(1+\sqrt{\lambda})^2)$, $\alpha > 0$, has the density (with respect to the Lebesgue measure)

$$\tilde{\nu}(dx) = \frac{1}{2\pi\alpha x}\sqrt{4\lambda\alpha^2 - (x-\alpha(1+\lambda))^2}\,dx.$$

The parameters λ and α are called the *rate* and the *jump size*, respectively.

It is easy to see that if $\mathbb{X}$ is free Poisson $\nu(\lambda,\alpha)$ then $\mathcal{R}_n(\mathbb{X}) = \alpha^n\lambda$, $n = 1, 2, \ldots$. Therefore the r-transform of $\mathbb{X}$ has the form

$$r_{\nu(\lambda,\alpha)}(z) = \frac{\lambda\alpha}{1-\alpha z}.$$

Assume that $\mathbb{X}$ has a free Poisson distribution with $\lambda > 1$; note that in this case $\mathbb{X}$ is invertible since its spectrum does not contain 0. One can easily find that the probability density function of the distribution of $\mathbb{X}^{-1}$ is given by

$$\begin{aligned} f_{\mathbb{X}^{-1}}(x) &\\ &= \frac{(\lambda-1)\sqrt{\frac{4\lambda}{\alpha^2(\lambda-1)^4} - \left(x - \frac{\lambda+1}{\alpha(\lambda-1)^2}\right)^2}}{2\pi x^2} I_{\left(\frac{1}{\alpha(1+\sqrt{\lambda})^2}, \frac{1}{\alpha(1-\sqrt{\lambda})^2}\right)}(x). \end{aligned} \tag{2.4}$$

The random variable $\mathbb{Y}$ defined by $\mathbb{Y} = \alpha(\lambda-1)^{3/2}\mathbb{X}^{-1} - (\lambda-1)^{1/2}\mathbb{I}$ has the standardized free Gamma law μ_{2a,a^2} defined in [7], where $a = 1/\sqrt{\lambda-1}$. To see that one can compare the probability density function of $\mathbb{Y}$ and the probability density function of the standardized free Gamma distribution [7, p. 62]. The cumulants of the standardized free Gamma law are (see [7, Remark 5.7]) $\mathcal{R}_1(\mu_{2a,a^2}) = 0$, $\mathcal{R}_k(\mu_{2a,a^2}) = C_{k-1}a^{k-2}$ for $k \geq 2$, where C_k, $k \geq 1$, are the Catalan numbers. By multilinearity of cumulants we see that the cumulants of $\mathbb{X}^{-1}$ are equal to

$$\mathcal{R}_k(\mathbb{X}^{-1}) = C_{k-1}\frac{1}{\alpha^k(\lambda-1)^{2k-1}}, \qquad k \geq 1. \tag{2.5}$$

The Catalan numbers are defined by

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \qquad k \geq 0. \tag{2.6}$$

They can also be equivalently defined by recurrence: $C_0 = 1$ and

$$C_k = \sum_{i=1}^{k} C_{i-1}C_{k-i}. \tag{2.7}$$

One can easily prove the following lemma.

LEMMA 2.2. *Let $(\mathcal{A},\varphi)$ be a $*$-probability space. If φ is tracial, then for any $n\in\mathbb{N}$,*

$$\mathcal{R}_n(\mathbb{X}_1,\dots,\mathbb{X}_n)=\mathcal{R}_n(\mathbb{X}_n,\mathbb{X}_1,\dots,\mathbb{X}_{n-1}).$$

In the next lemma we denote by σ_n the partition of $\{1,\dots,n\}$ which consists of $n-1$ blocks of the form $\{(1,2),(3),\dots,(n)\}$, so the first and the second elements are in the same block, and all other elements are singletons, and by $\mathit{1}_n$ the partition $\{(1,\dots,n)\}$. The following lemma is [19, Theorem 11.12] rewritten for the partition σ_n.

LEMMA 2.3. *Let $(\mathcal{A},\varphi)$ be a $*$-probability space, and $\mathbb{X}_1,\dots,\mathbb{X}_n\in\mathcal{A}$. Then*

$$\begin{aligned}(2.8)\qquad &\mathcal{R}_n(\mathbb{X}_1\cdot\mathbb{X}_2,\mathbb{X}_3,\dots,\mathbb{X}_{n+1})\\ &\qquad=\sum_{\pi\in\mathrm{NC}(n+1):\pi\vee\sigma_{n+1}=\mathit{1}_{n+1}}\mathcal{R}_\pi(\mathbb{X}_1,\mathbb{X}_2,\dots,\mathbb{X}_{n+1}).\end{aligned}$$

It can be rewritten as

$$\begin{aligned}(2.9)\qquad &\mathcal{R}_n(\mathbb{X}_1\cdot\mathbb{X}_2,\mathbb{X}_3,\dots,\mathbb{X}_{n+1})\\ &\qquad=\sum_{i=1}^{n-1}\mathcal{R}_i(\mathbb{X}_2,\mathbb{X}_3,\dots,\mathbb{X}_{i+1})\mathcal{R}_{n+1-i}(\mathbb{X}_1,\mathbb{X}_{i+2},\dots,\mathbb{X}_{n+1})\\ &\qquad\quad+\mathcal{R}_n(\mathbb{X}_2,\dots,\mathbb{X}_{n+1})\mathcal{R}_1(\mathbb{X}_1)\\ &\qquad\quad+\mathcal{R}_{n+1}(\mathbb{X}_1,\mathbb{X}_2,\dots,\mathbb{X}_{n+1}).\end{aligned}$$

3. Joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$.

In this section we will find the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$ for an invertible, free Poisson distributed non-commutative random variable $\mathbb{X}$. This result will be the main tool in the proof of the Lukacs property for the free Poisson distribution in Section 4. Additionally we will also prove two characterizations of the free Poisson distribution related to the formula for the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$.

PROPOSITION 3.1. *Let $\mathbb{X}$ have free Poisson distribution with $\lambda>1$, and $\alpha=1$. If $m\geq 1$ then*

$$\begin{aligned}(3.1)\quad &\mathcal{R}_{i_1+\dots+i_m+m}(\mathbb{X}^{-1},\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i_1},\mathbb{X}^{-1},\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i_2},\mathbb{X}^{-1},\dots,\mathbb{X}^{-1},\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i_m})\\ &\qquad=\begin{cases}0 & \text{if } \exists k\in\{1,\dots,m\},\ i_k>1,\\ (-1)^{i_1+\dots+i_m}\mathcal{R}_m(\mathbb{X}^{-1}) & \text{if } \forall k\in\{1,\dots,m\},\ i_k\leq 1.\end{cases}\end{aligned}$$

Proof. Let n be the length of cumulant, i.e. for the above cumulant, $n=i_1+\dots+i_m+m$. We will argue by induction on n. For a cumulant of length 1 the assertion is clear. Since the inductive step will work for cumulants of length greater than or equal to 3, we have to check whether the assertion holds true for cumulants of length 2. For $\mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}^{-1})$ the assertion is obviously true. It is enough to check whether $\mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}) = -\mathcal{R}_1(\mathbb{X}^{-1})$. By Lemma 2.3 we obtain

$$\begin{aligned} 1 = \mathcal{R}_1(\mathbb{I}) = \mathcal{R}_1(\mathbb{X}^{-1}\mathbb{X}) &= \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}) + \mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{X}) \\ &= \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}) + \frac{\lambda}{\lambda - 1}. \end{aligned}$$

From the above equation we see that

$$\mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}) = -\frac{1}{\lambda - 1} = -\mathcal{R}_1(\mathbb{X}^{-1}).$$

Assume that the assertion holds true for $l \le n-1$. Note that by Lemma 2.2 the joint cumulant of $\mathbb{X}$ and $\mathbb{X}^{-1}$ can be transformed to the form where the first two variables are $\mathbb{X}^{-1}, \mathbb{X}$. This is impossible only in the case when only $\mathbb{X}^{-1}$ appears in the cumulant. In that case the assertion is obviously true.

We will consider two cases.

Case 1. We will prove that a cumulant in which there is at least one pair of neighbouring $\mathbb{X}$'s is zero. We may assume that the last (reading from the left) pair of neighbouring $\mathbb{X}$'s is at positions $k, k+1$, where $k = 2, \ldots, n-1$. We will prove that

$$\mathcal{R}_n\Big(\mathbb{X}^{-1}, \mathbb{X}, \ldots, \underbrace{\mathbb{X}, \mathbb{X}}_{k,k+1}, \ldots\Big) = 0.$$

From Proposition 2.1 we get

$$\begin{aligned} 0 = \mathcal{R}_{n-1}(\mathbb{I}, \ldots, \mathbb{X}, \mathbb{X}, \ldots) &= \mathcal{R}_{n-1}(\mathbb{X}^{-1}\mathbb{X}, \ldots, \mathbb{X}, \mathbb{X}, \ldots) \\ &= \mathcal{R}_{n-1}(\mathbb{Y}_1 \cdot \mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_n). \end{aligned}$$

Now we expand the right hand side using Lemma 2.3:

$$\begin{aligned} 0 = &\sum_{i=1}^{n-2} \mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_{i+1})\mathcal{R}_{n-i}(\mathbb{Y}_1, \mathbb{Y}_{i+2}, \ldots, \mathbb{Y}_n) \\ &+ \mathcal{R}_{n-1}(\mathbb{Y}_2, \ldots, \mathbb{Y}_n)\mathcal{R}_1(\mathbb{Y}_1) + \mathcal{R}_n(\mathbb{Y}_1, \ldots, \mathbb{Y}_n). \end{aligned}$$

Note that for $i \le k-2$, the cumulant $\mathcal{R}_{n-i}(\mathbb{Y}_1, \mathbb{Y}_{i+2}, \ldots, \mathbb{Y}_n)$ contains $\mathbb{Y}_1 = \mathbb{X}^{-1}$, $\mathbb{Y}_k = \mathbb{X}$, $\mathbb{Y}_{k+1} = \mathbb{X}$, so by the inductive assumption it is zero.

It remains to prove that

$$\begin{aligned} &\mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_n) \\ &\quad = -\sum_{i=k-1}^{n-2} \mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_{i+1})\mathcal{R}_{n-i}(\mathbb{Y}_1, \mathbb{Y}_{i+2}, \ldots, \mathbb{Y}_n) \\ &\qquad - \mathcal{R}_{n-1}(\mathbb{Y}_2, \ldots, \mathbb{Y}_n)\mathcal{R}_1(\mathbb{Y}_1) = 0. \end{aligned} \tag{3.2}$$

We will consider two subcases:

Subcase a: *There is* $j \in \{3, \dots, k-1\}$ *such that* $\mathbb{Y}_j = \mathbb{X}^{-1}$. Then the cumulant $\mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{i+1})$ for $i = k-1$ contains $\mathbb{Y}_2 = \mathbb{X}$, $\mathbb{Y}_j = \mathbb{X}^{-1}$ and $\mathbb{Y}_k = \mathbb{X}$. So by Lemma 2.2 and the inductive assumption this cumulant is zero. For $i \in \{k, k+1, \dots, n-1\}$ the cumulant $\mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{i+1})$ contains $\mathbb{Y}_j = \mathbb{X}^{-1}$, $\mathbb{Y}_k = \mathbb{X}$, $\mathbb{Y}_{k+1} = \mathbb{X}$, so by the inductive assumption it is zero, which completes the proof in Subcase a.

Subcase b: *For all* $j \in \{3, \dots, k-1\}$ *we have* $\mathbb{Y}_j = \mathbb{X}$. If $n > k+1$ (which means that the last pair of neighbouring $\mathbb{X}$'s is not at positions $n-1, n$) then we have $\mathbb{Y}_{k+2} = \mathbb{X}^{-1}$, otherwise the chosen pair of $\mathbb{X}$'s would not be the last pair of neighbouring $\mathbb{X}$'s. Hence for $i \geq k+1$ the cumulant $\mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{i+1})$ contains $\mathbb{Y}_k = \mathbb{X}$, $\mathbb{Y}_{k+1} = \mathbb{X}$ and $\mathbb{Y}_{k+2} = \mathbb{X}^{-1}$, so by the inductive assumption and Lemma 2.2 it is zero.

It remains to prove that in the case $n > k+1$ we have

$$\mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_n) = -\sum_{i=k-1}^{k} \mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{i+1})\mathcal{R}_{n-i}(\mathbb{Y}_1, \mathbb{Y}_{i+2}, \dots, \mathbb{Y}_n) = 0,$$

and in the case $n = k+1$,

$$\begin{aligned}\mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_n) = &-\mathcal{R}_{n-2}(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{n-1})\mathcal{R}_2(\mathbb{Y}_1, \mathbb{Y}_n) \\ &-\mathcal{R}_{n-1}(\mathbb{Y}_2, \dots, \mathbb{Y}_n)\mathcal{R}_1(\mathbb{Y}_1) = 0.\end{aligned}$$

Both cases can be proved similarly. For all $j \in \{2, \dots, k+1\}$ we have $\mathbb{Y}_j = \mathbb{X}$ and by assumption $\mathbb{X}$ has a free Poisson distribution with parameters $(\lambda, 1)$, so cumulants of $\mathbb{X}$ are constant, which means $\mathcal{R}_{k-1}(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_k) = \mathcal{R}_k(\mathbb{Y}_2, \mathbb{Y}_3, \dots, \mathbb{Y}_{k+1})$. Moreover note that $\mathbb{Y}_1 = \mathbb{X}^{-1}$, $\mathbb{Y}_{k+1} = \mathbb{X}$, and at positions $k+1, \dots, n$ there are no neighbouring $\mathbb{X}$'s, so by the inductive assumption $\mathcal{R}_{n-k+1}(\mathbb{Y}_1, \mathbb{Y}_{k+1}, \dots, \mathbb{Y}_n) = -\mathcal{R}_{n-k}(\mathbb{Y}_1, \mathbb{Y}_{k+2}, \dots, \mathbb{Y}_n)$ (for $n = k+1$ the right hand side equals $-\mathcal{R}_1(\mathbb{X}^{-1})$). Hence the above sum is zero, which completes the proof in Case 1.

Case 2. We now handle the case when there are no neighbouring $\mathbb{X}$'s in the cumulant (of course there might be neighbouring $\mathbb{X}^{-1}$'s). Without loss of generality we may assume that apart from position 2, $\mathbb{X}$ appears in the cumulant exactly $k \leq \lfloor n/2 \rfloor - 1$ times, at positions $i_1, \dots, i_k$ where $4 \leq i_1 < \dots < i_k \leq n$, and for all $j \in \{1, \dots, k-1\}$ we have $i_{j+1} - i_j > 1$.

Taking into account formula (2.5) which gives cumulants of $\mathbb{X}^{-1}$, we have to prove that

$$\mathcal{R}_n(\mathbb{X}^{-1}, \mathbb{X}, \mathbb{X}^{-1}, \dots) = \frac{(-1)^{k+1}}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2}.$$

For $i \in \{1, \dots, n\}$ we denote by $\mathbb{Y}_i$ the variable at the ith position. We proceed similarly to the previous case:

$$\begin{aligned}
0 &= \mathcal{R}_{n-1}(\mathbb{I}, \mathbb{X}^{-1} \ldots,) = \mathcal{R}_{n-1}(\mathbb{X}^{-1}\mathbb{X}, \mathbb{X}^{-1} \ldots,) = \mathcal{R}_{n-1}(\mathbb{Y}_1\mathbb{Y}_2, \ldots, \mathbb{Y}_n) \\
&= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + \mathcal{R}_1(\mathbb{Y}_2)\mathcal{R}_{n-1}(\mathbb{Y}_1, \mathbb{Y}_3, \ldots, \mathbb{Y}_n) \\
&\quad + \sum_{i=2}^{n-2} \mathcal{R}_i(\mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_{i+1})\mathcal{R}_{n-i}(\mathbb{Y}_1, \mathbb{Y}_{i+2}, \mathbb{Y}_{i+3}, \ldots, \mathbb{Y}_n) \\
&\quad + \mathcal{R}_{n-1}(\mathbb{Y}_2, \ldots, \mathbb{Y}_n)\mathcal{R}_1(\mathbb{Y}_1).
\end{aligned}$$

Note that $\mathbb{Y}_2 = \mathbb{X}$ and $\mathbb{Y}_j = \mathbb{X}$ for $j \in \{i_1, \ldots, i_k\}$. Since in the initial cumulant there were no neighbouring $\mathbb{X}$'s, and by the inductive assumption, we find that all terms of the above sum with $i+1 \in \{i_1, \ldots, i_k\}$ are zero, and similarly $\mathcal{R}_{n-1}(\mathbb{Y}_2, \ldots, \mathbb{Y}_n)\mathcal{R}_1(\mathbb{Y}_1)$ is zero when $\mathbb{Y}_n = \mathbb{X}$. All other terms are non-zero.

Note that $k+1$ variables from among $\mathbb{Y}_1, \ldots, \mathbb{Y}_n$ are equal to $\mathbb{X}$, and $n-k-1$ are equal to $\mathbb{X}^{-1}$.

From the above remarks and the inductive assumption we get

$$\begin{aligned}
0 &= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + (-1)^k \mathcal{R}_1(\mathbb{X})\mathcal{R}_{n-k-1}(\mathbb{X}^{-1}) \\
&\quad + \sum_{i=1}^{n-k-2} (-1)^{k+1} \mathcal{R}_i(\mathbb{X}^{-1})\mathcal{R}_{n-k-1-i}(\mathbb{X}^{-1}) \\
&= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + \frac{(-1)^k \lambda}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2} \\
&\quad + \sum_{i=1}^{n-k-2} (-1)^{k+1} \frac{C_{i-1}C_{n-k-i-2}}{(\lambda-1)^{2i-1+2(n-k-1-i)-1}} \\
&= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + \frac{(-1)^k \lambda}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2} \\
&\quad + \frac{(-1)^{k+1}}{(\lambda-1)^{2(n-k-1)-2}} \sum_{i=1}^{n-k-2} C_{i-1}C_{n-k-2-i} \\
&= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + \frac{(-1)^k \lambda}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2} \\
&\quad + \frac{(-1)^{k+1}}{(\lambda-1)^{2(n-k-1)-2}} C_{n-k-2} \\
&= \mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) + \frac{(-1)^k}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2}.
\end{aligned}$$

In the penultimate equality we have used the recurrence (2.7) for Catalan numbers.

From the above equation we see that

$$\mathcal{R}_n(\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n) = \frac{(-1)^{k+1}}{(\lambda-1)^{2(n-k-1)-1}} C_{n-k-2},$$

which completes the proof of the lemma. ∎

The following remarks give characterizations of an invertible, free Poisson distributed random variable in the language of the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$.

REMARK 3.2. If $\mathbb{X}$ is invertible and such that for $n \geq 1$ we have

$$\mathcal{R}_{n+1}(\mathbb{X}, \mathbb{X}^{-1}, \mathbb{X}^{-1}, \ldots, \mathbb{X}^{-1}) = -\mathcal{R}_n(\mathbb{X}^{-1}), \tag{3.3}$$

then

$$\mathcal{R}_n(\mathbb{X}^{-1}) = \frac{1}{(\mathcal{R}_1(\mathbb{X})-1)^{2n-1}} C_{n-1} \quad \text{for } n \geq 1.$$

In particular $\mathbb{X}^{-1}$ has a free Gamma distribution and $\mathbb{X}$ has a free Poisson distribution.

Proof. We will prove the remark inductively. Using the asumption and Lemma 2.3 we obtain

$$\begin{aligned} 1 = \mathcal{R}_1(\mathbb{I}) &= \mathcal{R}_1(\mathbb{X}\mathbb{X}^{-1}) = \mathcal{R}_1(\mathbb{X})\mathcal{R}(\mathbb{X}^{-1}) + \mathcal{R}_2(\mathbb{X}, \mathbb{X}^{-1}) \\ &= \mathcal{R}(\mathbb{X}^{-1})(\mathcal{R}_1(\mathbb{X}) - 1), \end{aligned}$$

hence

$$\mathcal{R}_1(\mathbb{X}^{-1}) = \frac{1}{\mathcal{R}_1(\mathbb{X}) - 1}.$$

Assume that the remark holds true for $k \leq n-1$. Using consecutively Lemma 2.3, the assumption, the inductive assumption and the recurrence (2.7) for the Catalan numbers, we obtain

$$\begin{aligned} 0 &= \mathcal{R}_n(\mathbb{X}\mathbb{X}^{-1}, \mathbb{X}^{-1}, \ldots, \mathbb{X}^{-1}) \\ &= \mathcal{R}_{n+1}(\mathbb{X}, \mathbb{X}^{-1}, \ldots, \mathbb{X}^{-1}) + \mathcal{R}_1(\mathbb{X})\mathcal{R}_n(\mathbb{X}^{-1}) \\ &\quad + \sum_{i=1}^{n-1} \mathcal{R}_i(\mathbb{X}^{-1})\mathcal{R}_{n+1-i}(\mathbb{X}, \mathbb{X}^{-1}, \ldots, \mathbb{X}^{-1}) \\ &= (\mathcal{R}_1(\mathbb{X}) - 1)\mathcal{R}_n(\mathbb{X}^{-1}) - \sum_{i=1}^{n-1} \mathcal{R}_i(\mathbb{X}^{-1})\mathcal{R}_{n-i}(\mathbb{X}^{-1}) \\ &= (\mathcal{R}_1(\mathbb{X}) - 1)\mathcal{R}_n(\mathbb{X}^{-1}) - \frac{1}{(\mathcal{R}_1(\mathbb{X})-1)^{2n-2}} \sum_{i=1}^{n-1} C_{i-1}C_{n-1-i} \\ &= (\mathcal{R}_1(\mathbb{X}) - 1)\mathcal{R}_n(\mathbb{X}^{-1}) - \frac{1}{(\mathcal{R}_1(\mathbb{X})-1)^{2n-2}} C_{n-1}, \end{aligned}$$

which means that $\mathcal{R}_n(\mathbb{X}^{-1}) = \frac{1}{(\mathcal{R}_1(\mathbb{X})-1)^{2n-1}} C_{n-1}$. ∎

REMARK 3.3. Let $\mathbb{X}$ be an invertible random variable such that for all $n \geq 3$,

$$\mathcal{R}_n(\mathbb{X}^{-1}, \underbrace{\mathbb{X}, \ldots, \mathbb{X}}_{n}) = 0. \tag{3.4}$$

Then $\mathbb{X}$ has a free Poisson distribution with parameters

$$\lambda = \frac{\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{X})}{\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{X}) - 1} \quad \text{and} \quad \alpha = \frac{\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{X}) - 1}{\mathcal{R}_1(\mathbb{X}^{-1})}.$$

Proof. Lemma 3.3 from [23] implies that

$$C(z) = \sum_{n=1}^{\infty} \mathcal{R}_n(\mathbb{X}^{-1}, \underbrace{\mathbb{X}, \ldots, \mathbb{X}}_{n}) z^{n-1} = \frac{\mathcal{R}_1(\mathbb{X}^{-1}) + z}{1 + zr(z)}, \tag{3.5}$$

where r is the r-transform of $\mathbb{X}$.

The assumption (3.4) means that $C(z) = \mathcal{R}_1(\mathbb{X}^{-1}) + \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X})z$; combining this with (3.5) we obtain

$$\frac{\mathcal{R}_1(\mathbb{X}^{-1}) + z}{1 + zr(z)} = \mathcal{R}_1(\mathbb{X}^{-1}) + \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X})z.$$

This gives

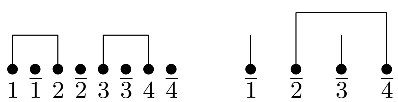
$$r(z) = \frac{1 - \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X})}{\mathcal{R}_1(\mathbb{X}^{-1}) + \mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X})z}.$$

Taking into account that $\mathcal{R}_2(\mathbb{X}^{-1}, \mathbb{X}) = 1 - \mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{X})$, we deduce that the above function is the r-transform of the free Poisson distribution with the stated parameters. ■

4. The Lukacs property in free probability. Before we start the proof of the Lukacs theorem, we need to introduce the so called Kreweras complement of a non-crossing partition and a useful formula for computing cumulants of products of free random variables. The following definition and theorem can be found in [19, Def. 9.21, Thm. 14.4] (see also [13]).

DEFINITION 4.1. Let π be a partition of $\{1, \ldots, n\}$. Consider the set $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$. The *Kreweras complement* of the partition π, denoted by $K(\pi)$, is the biggest partition $\sigma \in \mathrm{NC}(\overline{1}, \ldots, \overline{n})$ such that $\pi \cup \sigma \in \mathrm{NC}(1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n})$.

For example if $\pi = \{(1, 2), (3, 4)\}$, then $K(\pi) = \{(\overline{1}), (\overline{2}, \overline{4}), (\overline{3})\}$. The partitions π and $K(\pi)$ are illustrated below.



THEOREM 4.2. *Let* $\{\mathbb{X}_1,\dots,\mathbb{X}_n\}$ *and* $\{\mathbb{Y}_1,\dots,\mathbb{Y}_n\}$ *be free. Then*

$$\mathcal{R}_n(\mathbb{X}_1\mathbb{Y}_1,\dots,\mathbb{X}_n\mathbb{Y}_n)=\sum_{\pi\in\mathrm{NC}(n)}\mathcal{R}_\pi(\mathbb{X}_1,\dots,\mathbb{X}_n)\mathcal{R}_{K(\pi)}(\mathbb{Y}_1,\dots,\mathbb{Y}_n). \tag{4.1}$$

Now we are ready to prove the following theorem, which is the main result of the paper.

MAIN THEOREM 4.3. *Let* $(\mathcal{A},\varphi)$ *be a* C^**-probability space and* φ *be a faithful, tracial state. Let* $\mathbb{X},\mathbb{Y}\in\mathcal{A}$ *be free and free Poisson distributed with parameters* (λ,α) *and* (κ,α) *respectively, where* $\lambda+\kappa>1$. *Then the random variables* $\mathbb{U}=(\mathbb{X}+\mathbb{Y})^{-1/2}\mathbb{X}(\mathbb{X}+\mathbb{Y})^{-1/2}$ *and* $\mathbb{V}=\mathbb{X}+\mathbb{Y}$ *are free.*

Proof. First we note that it is sufficient to prove that mixed cumulants of $\mathbb{U},\mathbb{V}$ vanish. Since α is a multiplicative constant and the cumulants are multilinear, we can assume that $\alpha=1$.

First we will prove the theorem under the additional assumption that $\mathbb{X}$ is invertible, which means that $\lambda>1$. In this case we can equivalently prove freeness of $\mathbb{U}^{-1}=(\mathbb{X}+\mathbb{Y})^{1/2}\mathbb{X}^{-1}(\mathbb{X}+\mathbb{Y})^{1/2}$ and $\mathbb{V}$. Since φ is tracial, any moment $\varphi(P(\mathbb{U}^{-1},\mathbb{V}))$, where $P(x,y)$ is a non-commutative polynomial in variables x and y, can be rewritten as $\varphi(P(\mathbb{W},\mathbb{V}))$, where $\mathbb{W}=\mathbb{X}^{-1}(\mathbb{X}+\mathbb{Y})$. From this and the definition of free cumulants we deduce that vanishing of joint cumulants of $\mathbb{U}^{-1}=(\mathbb{X}+\mathbb{Y})^{1/2}\mathbb{X}^{-1}(\mathbb{X}+\mathbb{Y})^{1/2}$ and $\mathbb{V}=\mathbb{X}+\mathbb{Y}$ is equivalent to vanishing of mixed cumulants of $\mathbb{X}^{-1}(\mathbb{X}+\mathbb{Y})=\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y}$ and $\mathbb{X}+\mathbb{Y}$. Since any joint cumulant containing $\mathbb{I}$ and any other random variable is zero, to prove the theorem (in the case $\lambda>1$) it is sufficient to prove that all mixed cumulants of the random variables $\widetilde{\mathbb{U}}=\mathbb{X}^{-1}\mathbb{Y}$ and $\mathbb{V}=\mathbb{X}+\mathbb{Y}$ vanish.

Fix $n\geq 2$ and some mixed cumulant of length n of the random variables $\widetilde{\mathbb{U}},\mathbb{V}$. By Lemma 2.2 we can change the order of variables in the cumulant so as to have $\widetilde{\mathbb{U}}$ at the first position. Without loss of generality we may assume that in this cumulant, $\mathbb{V}$ appears $k<n$ times, at positions (after the change of the order) $1\leq j_1<\dots<j_k\leq n$,

$$\begin{aligned}\mathcal{R}_n\Big(\widetilde{\mathbb{U}},\dots,\underbrace{\mathbb{V}}_{j_1},\dots,\underbrace{\mathbb{V}}_{j_k},\dots\Big)\\ =\sum_{Z_{j_1},\dots,\mathbb{Z}_{j_k}\in\{\mathbb{X},\mathbb{Y}\}}\mathcal{R}_n(\widetilde{\mathbb{U}},\dots,\mathbb{Z}_{j_1},\dots,\mathbb{Z}_{j_k},\dots)\end{aligned} \tag{4.2}$$

In the first step of the proof we will find the terms of the above sum which are equal to 0. Note that if we write the variables $\mathbb{X}$ and $\mathbb{Y}$ as $\mathbb{X}\mathbb{I},\mathbb{I}\mathbb{Y}$ respectively, then we can apply Theorem 4.2.

Consider now a cumulant of the form $\mathcal{R}_n(\dots,\mathbb{X}\mathbb{I},\mathbb{I}\mathbb{Y},\dots)$. Expanding it according to Theorem 4.2 we see that either $\mathbb{X}$ and $\mathbb{I}$ are in the same block of the partition π, or $\mathbb{I}$ and $\mathbb{Y}$ are in the same block of the partition $K(\pi)$. In both cases the cumulant is zero by Proposition 2.1.

Fix now $l \geq 2$ and consider a cumulant of length $n > l+1$ which contains $\mathbb{X}^{-1}\mathbb{Y}$ and a sequence of random variables consisting of l neighbouring $\mathbb{X}$'s and maybe other sequences of $\mathbb{X}\mathbb{I}, \mathbb{I}\mathbb{Y}, \mathbb{X}^{-1}\mathbb{Y}$. From the previous considerations, to be non-zero this cumulant must be of the form

$$\mathcal{R}_n\Big(\ldots, \underbrace{\mathbb{X}\mathbb{I}}_{1\overline{1}}, \ldots, \underbrace{\mathbb{X}\mathbb{I}}_{l\overline{l}}, \underbrace{\mathbb{X}^{-1}\mathbb{Y}}_{l+1\overline{l+1}}, \ldots\Big).$$

Again we will expand this cumulant according to Theorem 4.2. Since a joint cumulant of $\mathbb{I}$ and any other random variable equals 0, if the elements $\overline{1}, \ldots, \overline{l}$ are not singletons in the partition $K(\pi)$ then the above cumulant is zero. Assume now that all $\overline{1}, \ldots, \overline{l}$ are singletons in $K(\pi)$. Then in the partition π positions $1, \ldots, l+1$ are in the same block, hence one of the cumulants related to π contains at least two neighbouring $\mathbb{X}$'s and $\mathbb{X}^{-1}$, so by Proposition 3.1 it is zero.

From the above remarks we conclude that the only non-zero cumulants in the sum (4.2) are the ones which consist only of $\mathbb{Y}$ and $\mathbb{X}^{-1}\mathbb{Y}$ or the ones that contain $\mathbb{X}$ at position $j \in \{2, \ldots, n-1\}$ (after using Lemma 2.2); then $\mathbb{X}^{-1}\mathbb{Y}$ is at position $j+1$, and $\mathbb{X}$ cannot be at position $j-1$.

Recall that in equation (4.2) the variable $\mathbb{X}^{-1}\mathbb{Y}$ appears $n-k$ times. Some of $\mathbb{X}^{-1}\mathbb{Y}$ may have as left neighbour another $\mathbb{X}^{-1}\mathbb{Y}$; we may assume that $m \leq n-k$ of them do not have $\mathbb{X}^{-1}\mathbb{Y}$ as left neighbour. The above remarks imply that if $\mathbb{Z}_j$ does not have $\mathbb{X}^{-1}\mathbb{Y}$ as right neighbour, then $\mathbb{Z}_j = \mathbb{Y}$, otherwise the cumulant is zero. This means that $\mathbb{X}$ can appear only at m positions which are the left neighbours of $\mathbb{X}^{-1}\mathbb{Y}$. So we can rewrite the right hand side of (4.2) as

(4.3)
$$\sum_{\mathbb{Z}_{i_1}, \ldots, \mathbb{Z}_{i_m} \in \{\mathbb{X}\mathbb{I}, \mathbb{I}\mathbb{Y}\}} \mathcal{R}_n(\mathbb{X}^{-1}\mathbb{Y}, \mathbb{I}\mathbb{Y}, \ldots, \mathbb{I}\mathbb{Y}, \mathbb{Z}_{i_1}, \mathbb{X}^{-1}\mathbb{Y}, \mathbb{I}\mathbb{Y}, \ldots, \mathbb{I}\mathbb{Y}, \mathbb{Z}_{i_j}, \mathbb{X}^{-1}\mathbb{Y}, \ldots, \mathbb{I}\mathbb{Y}, \mathbb{Z}_{i_m}).$$

Fix one term of the above sum. Without loss of generality we can assume that j elements of $\{\mathbb{Z}_{i_1}, \ldots, \mathbb{Z}_{i_m}\}$ are equal to $\mathbb{X}\mathbb{I}$. We expand this term according to Theorem 4.2:

$$\text{(4.4)} \quad \mathcal{R}_n(\mathbb{X}^{-1}\mathbb{Y}, \ldots, \mathbb{Z}_{i_1}, \mathbb{X}^{-1}\mathbb{Y}, \ldots, \mathbb{Z}_{i_j}, \mathbb{X}^{-1}\mathbb{Y}, \ldots, \mathbb{Z}_{i_m}) = \sum_{\pi \in \mathrm{NC}(n)} \mathcal{R}_\pi(\mathbb{X}^{-1}, \ldots)\mathcal{R}_{K(\pi)}(\mathbb{Y}, \ldots).$$

Note that if $\mathbb{Z}_{i_j} = \mathbb{X}\mathbb{I}$ then $\overline{i_j}$ is a singleton in $K(\pi)$ or this cumulant vanishes by Lemma 2.1. From this we see that i_j and i_{j+1} are in the same block of the partition π, which means that the number indicating the position of $\mathbb{X}$ is in the same block with the number indicating the position of $\mathbb{X}^{-1}$. Moreover, by the previous steps, in the cumulant under study there are no neighbouring $\mathbb{X}$'s.

Similarly if in the fixed term of the sum (4.4) $\mathbb{Z}_{i_j} = \mathbb{IY}$, then in π the element i_j is a singleton or this cumulant is zero.

The above remarks lead to the conclusion that the partitions π for which the right hand side of (4.4) is not zero can be identified with the partitions on $n-k$ elements given by the positions of $\mathbb{X}^{-1}$. The other elements are singletons or are in the same block with one of elements indicating the position of $\mathbb{X}^{-1}$. The mapping defined below formalizes this observation.

Recall that $\mathbb{X}^{-1}\mathbb{Y}$ appears $n-k$ times in the cumulant on the left hand side of (4.4), so in $\mathcal{R}_\pi(\mathbb{X}^{-1},\ldots)$ the variable $\mathbb{X}^{-1}$ also appears $n-k$ times. Let us denote the positions of $\mathbb{X}^{-1}\mathbb{Y}$ by $l_1,\ldots,l_{n-k}$.

For any partition $\pi \in \mathrm{NC}(n)$ we define $\widetilde{\pi} \in \mathrm{NC}(n-k)$ by restriction of π to the numbers $l_1,\ldots,l_{n-k}$. This means that $s,t \in \{1,\ldots,n-k\}$ are in the same block of $\widetilde{\pi}$ if and only of l_s and l_t are in the same block of π.

Let $A = \{\pi \in \mathrm{NC}(n) : \mathcal{R}_\pi(\mathbb{X}^{-1},\ldots) \neq 0 \text{ and } \mathcal{R}_{K(\pi)}(\mathbb{Y},\ldots) \neq 0\}$. From the previous remarks it follows that the mapping $\pi \mapsto \widetilde{\pi}$ is a bijection between A and $\mathrm{NC}(n-k)$. We will prove that

$$\begin{aligned}(4.5)\qquad &\sum_{\pi\in A} \mathcal{R}_\pi(\mathbb{X}^{-1},\ldots)\mathcal{R}_{K(\pi)}(\mathbb{Y},\ldots)\\ &\qquad= \sum_{\widetilde{\pi}\in \mathrm{NC}(n-k)} (-1)^j \mathcal{R}_{\widetilde{\pi}}(\underbrace{\mathbb{X}^{-1},\ldots,\mathbb{X}^{-1}}_{n-k})\mathcal{R}_{K(\widetilde{\pi})}(\underbrace{\mathbb{Y},\ldots,\mathbb{Y}}_{n-k}).\end{aligned}$$

If $\pi \in A$ and in $\mathcal{R}_\pi(\mathbb{X}^{-1},\ldots)$ the variable $\mathbb{X}$ appears j times then from Proposition 3.1 we get

$$\mathcal{R}_\pi(\mathbb{X}^{-1},\ldots) = (-1)^j \mathcal{R}_{\widetilde{\pi}}(\underbrace{\mathbb{X}^{-1},\ldots,\mathbb{X}^{-1}}_{n-k}).$$

Recall that in the sum (4.3) some of $\mathbb{Z}_j$'s were fixed to be $\mathbb{IY}$. From this we infer that after applying Lemma 2.2 every maximal sequence of neighbouring $\mathbb{IY}$'s has $\mathbb{X}^{-1}\mathbb{Y}$ as left neighbour. In particular if on the left hand side of (4.3) there is a sequence of l neighbouring $\mathbb{IY}$'s then it must be of the form

$$\mathcal{R}_n\Big(\ldots,\mathbb{X}^{-1}\mathbb{Y},\underbrace{\mathbb{IY},\ldots,\mathbb{IY}}_{l},\ldots\Big).$$

It is clear that for $\pi \in A$ the numbers indicating the positions of $\mathbb{I}$ from $\mathbb{IY}$ are singletons; it follows that in $K(\pi)$ the positions of $\mathbb{Y}$ from $\mathbb{IY}$ are in the same block with some $\mathbb{Y}$ coming from the neighbouring $\mathbb{X}^{-1}\mathbb{Y}$; this implies that the number of blocks in $K(\pi)$ is equal to the number of blocks of $K(\widetilde{\pi})$. Since $\mathbb{Y}$ has a free Poisson distribution with $\alpha = 1$, the cumulants of $\mathbb{Y}$

are constant. This means that $\mathcal{R}_{K(\pi)}(\mathbb{Y},\dots)$ depends only on the number of blocks in $K(\pi)$, showing that

$$\mathcal{R}_{K(\pi)}(\mathbb{Y},\dots) = \mathcal{R}_{K(\widetilde{\pi})}(\underbrace{\mathbb{Y},\dots,\mathbb{Y}}_{n-k}).$$

This proves equation (4.5).

Let us return to the sum (4.2). We can rewrite it by summing over possible appearances of $\mathbb{X}\mathbb{I}$, which gives

$$\sum_{j=0}^{m}\sum_{\widetilde{\pi}\in\mathrm{NC}(n-k)}\binom{m}{j}(-1)^j\mathcal{R}_{\widetilde{\pi}}(\mathbb{X}^{-1},\dots,\mathbb{X}^{-1})\mathcal{R}_{K(\widetilde{\pi})}(\mathbb{Y},\dots,\mathbb{Y}).$$

After changing the order of summation we obtain

$$\sum_{\widetilde{\pi}\in\mathrm{NC}(n-k)}\mathcal{R}_{\widetilde{\pi}}(\mathbb{X}^{-1},\dots,\mathbb{X}^{-1})\mathcal{R}_{K(\widetilde{\pi})}(\mathbb{Y},\dots,\mathbb{Y})\sum_{j=0}^{m}\binom{m}{j}(-1)^j = 0,$$

which completes the proof in the case $\lambda > 1$ and $\kappa > 0$.

Assume now that $\lambda+\kappa > 1$ and $\alpha > 0$.

Note that equation (2.1) defines the cumulants recursively by moments. Any moment of the random variables $\mathbb{V}$ and $\mathbb{U}$ can be expressed as a moment of $\mathbb{X}+\mathbb{Y}$, $\mathbb{X}$, $(\mathbb{X}+\mathbb{Y})^{-1}$ by traciality of φ. Since $\mathbb{X}+\mathbb{Y}$ has a free Poisson distribution with parameters $(\lambda+\kappa,\alpha)$, and $\lambda+\kappa>1$, the support of the distribution of $\mathbb{X}+\mathbb{Y}$ is $[\alpha(1-\sqrt{\lambda+\kappa})^2,\alpha(1+\sqrt{\lambda+\kappa})^2]$. As pointed out at the beginning of the proof, it is sufficient to prove freeness of $\mathbb{U}$ and $\mathbb{V}$ for some fixed α; then freeness for other values of α follows from multilinearity of free cumulants. We fix $\alpha>0$ such that the support of the distribution of the random variable $\mathbb{I}-(\mathbb{X}+\mathbb{Y})$ is contained in $(-1,1)$. Since the support of a random variable is equal to the spectrum, and the spectral norm is equal to the norm, we have $\|\mathbb{I}-(\mathbb{X}+\mathbb{Y})\|<1$ and $(\mathbb{X}+\mathbb{Y})^{-1}=\sum_{n=0}^{\infty}(\mathbb{I}-(\mathbb{X}+\mathbb{Y}))^n$. This means that any joint moment of $\mathbb{X}+\mathbb{Y}$, $\mathbb{X}$, $(\mathbb{X}+\mathbb{Y})^{-1}$ can be expressed as a series of moments of $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}$. Moreover by freeness of $\mathbb{X}$ and $\mathbb{Y}$, any joint moment of $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}$ is a polynomial of moments of $\mathbb{X}$ and moments of $\mathbb{Y}$. Taking into account that moments of a free Poisson distribution with parameters (λ,α) are the same polynomials in λ,α for $\lambda\le 1$ and for $\lambda>1$, we conclude that the cumulants of $\mathbb{U}$ and $\mathbb{V}$ are power series of λ,α,κ, so they are the same analytic function of λ,κ for $\lambda\le 1$ and for $\lambda>1$. This completes the proof. ∎

Recall that Proposition 3.1 gives the joint cumulants of $\mathbb{X}$ and $\mathbb{X}^{-1}$, where $\mathbb{X}$ has a free Poisson distribution. As shown for a positive $\mathbb{X}$ with a free Poisson distribution, $\mathbb{X}^{-1}$ has a free Gamma distribution with density (2.4). Thus we can also use Proposition 3.1 (with the roles of $\mathbb{X}$ and $\mathbb{X}^{-1}$ interchanged) to prove the following result.

Proposition 4.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be free and identically distributed. Assume that the distribution of $\mathbb{X}$ is free Gamma. Then the pair $\mathbb{X}+\mathbb{Y}$, $(\mathbb{X}+\mathbb{Y})^{-1}\mathbb{X}^2(\mathbb{X}+\mathbb{Y})^{-1}$ of random variables is not free.*

Remark 4.5. The above proposition gives a negative answer to the question stated in [7, after Proposition 3.6]. Moreover, the above result, together with [7, Proposition 3.6], implies that there is no pair of free, identically distributed random variables such that $\mathbb{X}+\mathbb{Y}$ and $(\mathbb{X}+\mathbb{Y})^{-1}\mathbb{X}^2(\mathbb{X}+\mathbb{Y})^{-1}$ are free.

Proof of Proposition 4.4. As in the proof of Theorem 4.3, we can equivalently prove that the pair $\mathbb{X}+\mathbb{Y}$, $(\mathbb{X}+\mathbb{Y})\mathbb{X}^{-2}(\mathbb{X}+\mathbb{Y})$ is not free. Note that we have $(\mathbb{X}+\mathbb{Y})\mathbb{X}^{-2}(\mathbb{X}+\mathbb{Y}) = (\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1})(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y})$. We will prove that $\mathcal{R}_2((\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1})(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y}),\mathbb{X}+\mathbb{Y}) \neq 0$.

From Lemma 2.3 we can write

$$\begin{aligned}\mathcal{R}_2((\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1})(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y}),\mathbb{X}+\mathbb{Y}) &= \mathcal{R}_3(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1},\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}+\mathbb{Y})\\ &\quad+\mathcal{R}_2(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}+\mathbb{Y})\mathcal{R}_1(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y})\\ &\quad+\mathcal{R}_2(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}+\mathbb{Y})\mathcal{R}_1(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1}).\end{aligned}$$

Lemma 2.1 implies

$$\begin{aligned}\mathcal{R}_3(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1},\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}+\mathbb{Y}) &= \mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X})\\ &\quad+\mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{Y}).\end{aligned}$$

Using Lemma 2.3 we get

$$\begin{aligned}\mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}) &= \mathcal{R}_4(\mathbb{Y},\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X})\\ &\quad+\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_3(\mathbb{Y},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X})\\ &\quad+\mathcal{R}_2(\mathbb{Y},\mathbb{X})\mathcal{R}_2(\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y})\\ &\quad+\mathcal{R}_1(\mathbb{Y})\mathcal{R}_3(\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}).\end{aligned}$$

We can use Lemma 2.3 once again and by freeness of $\mathbb{X}$ and $\mathbb{Y}$ we obtain

$$\mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}) = \mathcal{R}_1^2(\mathbb{Y})\mathcal{R}_3(\mathbb{X}^{-1},\mathbb{X}^{-1},\mathbb{X}).$$

By Lemma 3.1 (with the roles of $\mathbb{X}$ and $\mathbb{X}^{-1}$ interchanged) we get

$$\mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}) = 0.$$

Similarly

$$\begin{aligned}\mathcal{R}_3(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}^{-1}\mathbb{Y},\mathbb{Y}) &= \mathcal{R}_3(\mathbb{Y})\mathcal{R}_1(\mathbb{X}^{-1})^2\\ &\quad+\mathcal{R}_3(\mathbb{Y})\mathcal{R}_2(\mathbb{X}^{-1},\mathbb{X}^{-1})\\ &\quad+\mathcal{R}_1(\mathbb{Y})\mathcal{R}_2(\mathbb{X}^{-1},\mathbb{X}^{-1})\mathcal{R}_2(\mathbb{Y})\\ &\quad+\mathcal{R}_2(\mathbb{Y})\mathcal{R}_2(\mathbb{X}^{-1},\mathbb{X}^{-1})\mathbb{R}_1(\mathbb{Y})\\ &= \frac{2\lambda^2}{(\lambda-1)^5}+\frac{2\lambda}{(\lambda-1)^5}+\frac{\lambda}{(\lambda-1)^4}+\frac{\lambda}{(\lambda-1)^4}\\ &= \frac{4\lambda^2}{(\lambda-1)^5},\end{aligned}$$

which gives

$$\mathcal{R}_3(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1},\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}+\mathbb{Y})=\frac{4\lambda^2}{(\lambda-1)^5}.$$

One can also check that

$$\begin{aligned}\mathcal{R}_2(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y},\mathbb{X}+\mathbb{Y})\mathcal{R}_1(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1}) &= \mathcal{R}_2(\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1},\mathbb{X}+\mathbb{Y})\mathcal{R}_1(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y})\\ &= (\mathcal{R}_2(\mathbb{Y}\mathbb{X}^{-1},\mathbb{X})+\mathcal{R}_2(\mathbb{Y}\mathbb{X}^{-1},\mathbb{Y}))(1+\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{Y}))\\ &= (-\mathcal{R}_1(\mathbb{Y})\mathcal{R}_1(\mathbb{X})+\mathcal{R}_1(\mathbb{X}^1)\mathcal{R}_2(\mathbb{Y}))\big(1+\mathcal{R}_1(\mathbb{X}^{-1})\mathcal{R}_1(\mathbb{Y})\big)=\frac{2\lambda-1}{(\lambda-1)^4}.\end{aligned}$$

So the cumulant $\mathcal{R}_2((\mathbb{I}+\mathbb{Y}\mathbb{X}^{-1})(\mathbb{I}+\mathbb{X}^{-1}\mathbb{Y}),\mathbb{X}+\mathbb{Y})$ is not zero. ∎

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