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On some dilation theorems for positive definite operator valued functions

by

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Abstract. The aim of this paper is to prove dilation theorems for operators from a linear complex space to its Z-anti-dual space. The main result is that a bounded positive definite function from a $*$-semigroup Γ into the space of all continuous linear maps from a topological vector space X to its Z-anti-dual can be dilated to a $*$-representation of Γ on a Z-Loynes space. There is also an algebraic counterpart of this result.

1. Introduction. It is well known that a function on a $*$-semigroup Γ into the C^*-algebra of all bounded linear operators on a given Hilbert space, that is positive definite, can be dilated to a $*$-representation of Γ on a larger Hilbert space (see the Principal Theorem of [SN]).

Probability theory on Banach spaces triggered the development of dilation theory of operator functions in non-Hilbert spaces [GW2]. The close connection between dilation theory and the theory of second order stochastic processes was exhibited in [W]. In 1976, J. Górniak and A. Weron [GW1] proved an analogue of the Principal Theorem of Sz.-Nagy for functions with values in the space of all anti-linear bounded operators from a complex normed space to its topological dual. In the same paper, an algebraic version of this result was also given. Similar approaches and applications were presented in [GW2], [It], [L], [GL] and [K].

Another analogue of the above mentioned dilation theorem was given by R. M. Loynes [Lo1] for operators acting on a VH-space, along with many important results on the same issue [Lo2], [Lo3]. Later on, Cobanjan and Weron [CW] proved that the space $\overline{\mathcal{L}}(B, \mathcal{H})$ endowed with the inner product $[\cdot, \cdot]$ is a Loynes space (for more examples see [Is] and [S]).

The results of [CW] are a variation of the original Aronszajn construction [A], considering the Aronszajn kernel $K : (S \times \mathcal{A}) \times (S \times \mathcal{A}) \to \mathcal{B}$, where

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S is just a set and $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, given by

$$K((t,a),(s,a')) = \mathbb{K}(t,s)[a' * a].$$

In 2005, D. Gaşpar and P. Gaşpar [GP] extended the reproducing kernel Hilbert space technique of [A] to more general structures such as Loynes spaces and $\mathcal{D}_2$-normal $\mathcal{B}(\mathcal{X})$-modules.

The results of our paper, partly announced in [BPL], are also variations of the original Aronszajn construction in the case of a kernel $K : (X \times \Gamma) \times (X \times \Gamma) \to Z$, where X is a linear space or a topological linear space, Γ is a $*$-semigroup and Z is an admissible space in the sense of Loynes. Our paper extends the fundamental theorem of Loynes [Lo1, Section 3, Theorem 3] to the case where the set of continuous linear operators in a Loynes Z-space is replaced by $\mathcal{C}(X, X_Z^*)$, the set of continuous linear maps from a topological space X to its Z-anti-dual. In the proof we use a version of the Cauchy–Schwarz inequality for seminorms in a Loynes space, which is significantly different from the Loynes space case [Lo1].

The main result of the article may be applied to the characterization of spectral bi-measures and to the stationary dilation of q-dimensional V-bounded processes (see [T] and [W]).

2. Preliminaries. In this section we mention some notation and known notions and results from [GP].

Recall first that a complete locally convex space Z is called *admissible in the sense of Loynes* if there exist a closed convex cone Z_+ in Z with $Z_+ \cap (-Z_+) = \{0\}$ and an involution "$^\lozenge$" on Z (conjugate linear and idempotent) such that each element of Z_+ is self-adjoint, the topology of Z is compatible with the partial order in Z induced by Z_+, and decreasing sequences in Z_+ are convergent [Lo1, pp. 11].

In the following, Z will be an admissible space in the sense of Loynes.

It is known that the topology of Z can be defined by a sufficient and directed family, say $\mathcal{P}_Z$, of monotone Minkowski seminorms.

For any given set Λ, a function $K : \Lambda \times \Lambda \to Z$ is said to be a *Z-valued kernel on Λ*.

A Z-valued kernel on Λ will be called *weakly positive definite* [GP] if for each $n \in \mathbb{N}^*$, $\{c_1, \dots, c_n\} \subset \mathbb{C}$ and $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j K(\lambda_i, \lambda_j) \in Z_+. \tag{1}$$

A locally convex space $\mathcal{H}$ is called a *pre-Loynes Z-space* if it is endowed with a Z-valued inner product (called *Gramian*)

$$\mathcal{H} \times \mathcal{H} \ni (h,k) \mapsto [h,k] \in Z,$$

which has the properties

(G_1) $[h,h] \geq 0, \quad [h,h] = 0$ implies $h = 0,$

(G_2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h],$

(G_3) $[\lambda h, k] = \lambda [h, k],$

(G_4) $[h,k]^{\Diamond} = [k,h],$

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ (where the positivity in Z is considered) and the topology in $\mathcal{H}$ is the weakest one for which the mapping $\mathcal{H} \ni h \mapsto [h,h] \in Z$ is continuous.

If $\mathcal{H}$ is complete in this topology, it will be called a *Loynes Z-space* [Lo1].

A pre-Loynes Z-space $\mathcal{H}$ consisting of Z-valued functions on Λ *admits a reproducing kernel* or *is a reproducing kernel pre-Loynes Z-space* if there exists a Z-valued kernel K satisfying the conditions

(IP) $K(\lambda, \cdot) \in \mathcal{H}$ for any $\lambda \in \Lambda$,

(RP) $h(\lambda) = [h, K(\lambda, \cdot)]$ for all $\lambda \in \Lambda$ and $h \in \mathcal{H}$.

The kernel K is called a *reproducing kernel* for $\mathcal{H}$, and (IP), (RP) are called the *inclusion property* and the *reproducing property*, respectively (see [GP]).

Now, let X and Y be complex linear spaces. Then $\mathcal{L}(X,Y)$ denotes the class of all linear operators from X to Y. For a complex linear space X, the *algebraic Z-anti-dual* X'_Z is the set of all anti-linear operators from X to Z. For an operator $A \in \mathcal{L}(X,F)$, where F is a pre-Loynes Z-space, its *Z-algebraic adjoint operator* $A' \in \mathcal{L}(F, X'_Z)$ is defined by

$$(A'f)(x) = [f, Ax]_F, \qquad f \in F,\, x \in X,$$

where $[\cdot,\cdot]_F$ is the Gramian of F.

If F is a Loynes space and $A \in \mathcal{L}(F,F)$, then an operator $B \in \mathcal{L}(F,F)$ with the property

$$[Af_1, f_2]_F = [f_1, Bf_2]_F$$

is called the *adjoint* of A and will be denoted by A^*.

An operator $U \in \mathcal{L}(F_1, F_2)$, where F_1, F_2 are pre-Loynes Z-spaces, is said to be *unitary* if $U(F_1) = F_2$ and

$$[Uf, Ug]_{F_2} = [f,g]_{F_1}$$

for all $f, g \in F_1$.

If X is a complex topological linear space, then its *topological Z-anti-dual* X^*_Z is the set of all continuous anti-linear operators from X to Z.

On X^*_Z the uniform convergence topology is considered, that is, a net $(T_\alpha)_{\alpha \in \mathcal{A}}$ of operators from X^*_Z converges uniformly to the null-operator 0 iff for any 0-neighborhood V in Z there exists $\alpha_0 \in \mathcal{A}$ such that, for each $\alpha \geq \alpha_0, T_\alpha x \in V$ for all $x \in X$.

If X and Y are topological linear spaces, we denote by $\mathcal{C}(X,Y)$ the space of all continuous linear operators from X to Y.

Let Γ be a $*$-*semigroup*, that is, a semigroup with unit e and involution "$*$" satisfying $e^*=e$, $s^{**}=s$, and $(st)^*=t^*s^*$ for all $s,t\in\Gamma$.

Following [GW1], let $\mathcal{X}$ be the set of all functions $x=(x_s):\Gamma\to X$ with finite support. A family $\{T_s\}_{s\in\Gamma}$ of functions from $\mathcal{L}(X,X'_Z)$ indexed by the $*$-semigroup Γ is called *positive definite* if

$$\sum_{s,t\in\Gamma}(T_{s^*t}x_t)(x_s)\ge 0 \tag{2}$$

for all $(x_s)_{s\in\Gamma}\in\mathcal{X}$.

We recall a version of the classical Cauchy–Schwarz inequality, in terms of seminorms, in a pre-Loynes space.

If $\mathcal{H}$ is a pre-Loynes Z-space and $\mathcal{P}_Z$ is a sufficient directed set of monotone seminorms defining the topology of Z, then

$$p([h,k])\le 2(p([h,h]))^{1/2}(p([k,k]))^{1/2}$$

for any $h,k\in\mathcal{H}$ and any $p\in\mathcal{P}_Z$.

3. A characterization of $\mathcal{L}(X,X'_Z)$-valued positive definite families. The theorem below is an algebraic analogue of Górniak and Weron's result [GW1].

THEOREM 3.1. *Let X be a complex linear space with algebraic Z-antidual space X'_Z. If $\{T_s\}_{s\in\Gamma}\subset\mathcal{L}(X,X'_Z)$ is a family indexed by a $*$-semigroup Γ satisfying*

(i) $(T_sx)(y)=(T_{s^*}y)(x)^{\diamond}$ *for all $x,y\in X$ and $s\in\Gamma$,*
(ii) $\{T_s\}_{s\in\Gamma}$ *is positive definite,*

then there exist a pre-Loynes Z-space F and a function $D:\Gamma\to\mathcal{L}(F,F)$ with the following properties:

$$D_e=I,\qquad D_{st}=D_sD_t,\qquad D_s^*=D_{s^*},\qquad s,t\in\Gamma; \tag{$\bullet$}$$

there exists an operator $A\in\mathcal{L}(X,F)$ such that

$$T_s=A'D_sA,\qquad s\in\Gamma; \tag{$\bullet\bullet$}$$

and the space F is minimal in the sense that it is generated by elements of the form D_sAx for $x\in X$ and $s\in\Gamma$.

Moreover:

1°. *The space F is uniquely determined up to unitary equivalence, i.e. if $T_s=A'_1D^1_sA_1$, $s\in\Gamma$, where $D^1:\Gamma\to\mathcal{L}(F_1,F_1)$ satisfies ($\bullet$), F_1 is a minimal pre-Loynes Z-space and $A_1\in\mathcal{L}(X,F_1)$, then there exists a unitary operator $U:F_1\to F$ such that*

$$A=UA_1,\qquad UD^1_s=D_sU,\qquad s\in\Gamma.$$

2°. *If* $T_{s\alpha t} = T_{s\beta t} + T_{s\gamma t}$ *for some fixed* α, β, γ, *and all* s, t *in* Γ, *then* $D_\alpha = D_\beta + D_\gamma$.

Proof. The argument is like that used to prove Sz.-Nagy's original theorem [SN].

Let $\Lambda = \Gamma \times X$. We define $K_T : \Lambda \times \Lambda \to Z$ by

$$K_T(\lambda, \mu) = (T_{t^*s}x)(y)$$

where $\lambda = (s, x)$, $\mu = (t, y)$, $s, t \in \Gamma$, $x, y \in X$.

First we will show that K_T is a weak Z-valued positive definite kernel. Indeed, let $c_1, \ldots, c_n \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_n \in \Lambda$, $\lambda_i = (s_i, x_i)$, $i = \overline{1, n}$. We have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j K_T(\lambda_i, \lambda_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j (T_{s_j^* s_i}x_i)(x_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (T_{s_j^* s_i}c_i x_i)(c_j x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} (T_{s_j^* s_i}k_i)(k_j) \geq 0\end{aligned}$$

from the positivity of T.

Next, define

$$F = \Big\{\sum_{l=1}^{n} c_l K_T(\lambda_l, \cdot) : n \in \mathbb{N}^*,\ c_l \in \mathbb{C},\ \lambda_l \in \Lambda,\ l = \overline{1, n}\Big\}.$$

We will prove that F is a pre-Loynes Z-space with Gramian

$$[f_1, f_2]_F = \sum_{j,l=1}^{n} c_j^1 \bar{c}_l^2 K_T(\lambda_j^1, \lambda_l^2)$$

for $f_1 = \sum_{j=1}^{n} c_j^1 K_T(\lambda_j^1, \cdot)$, $f_2 = \sum_{l=1}^{n} c_l^2 K_T(\lambda_l^2, \cdot)$.

Obviously F is a complex linear space with the usual operations.

The first part of (G_1) follows from the fact that K_T is a weak Z-valued positive definite kernel and from the definition of $[\cdot, \cdot]_F$. Conditions (G_2) and (G_3) easily result from the definition of $[\cdot, \cdot]_F$. We prove (G_4):

$$\begin{aligned}[f_2, f_1]_F^{\lozenge} &= \Big(\sum_{j,l=1}^{n} c_l^2 \bar{c}_j^1 K_T(\lambda_l^2, \lambda_j^1)\Big)^{\lozenge} = \sum_{j,l=1}^{n} c_j^1 \bar{c}_l^2 [(T_{s_j^* s_l}x_l)(x_j)]^{\lozenge}\\ &= \sum_{j,l=1}^{n} c_j^1 \bar{c}_l^2 (T_{s_l^* s_j}x_j)(x_l) = \sum_{j,l=1}^{n} c_j^1 \bar{c}_l^2 K_T(\lambda_j^1, \lambda_l^2) = [f_1, f_2]_F.\end{aligned}$$

Let $\mathcal{P}_Z$ be a sufficient set of monotone seminorms that generates the topology in Z. We will verify that F has K_T as reproducing kernel. Condition (IP) comes from the definition of the kernel. To show (RP), let

$$h = \sum_{j=1}^{n} c_j K_T(\lambda_j, \cdot) \in F, \qquad \lambda_j \in \Lambda,\ c_j \in \mathbb{C},\ n \in \mathbb{N}^*.$$

Then

$$[h, K_T(\lambda,\cdot)]_F = \Big[\sum_{j=1}^{n} c_j K_T(\lambda_j,\cdot), K_T(\lambda,\cdot)\Big]_F = \sum_{j=1}^{n} c_j K_T(\lambda_j,\lambda) = h(\lambda).$$

We now prove the second part of (G_1), using the reproducing property. Assume $[h,h]_F = 0$. From

$$p(h(\lambda)) \le 2(p[h,h]_F)^{1/2} \cdot (p[K_T(\lambda,\cdot), K_T(\lambda,\cdot)]_F)^{1/2}, \qquad \lambda \in \varLambda,$$

since $\mathcal{P}_Z$ is a sufficient set of seminorms in Z, it follows that $h(\lambda) = 0$ for all $\lambda \in \varLambda$, i.e. $h = 0$.

We have shown so far that $[\cdot,\cdot]_F$ is a Gramian, so F is a pre-Loynes Z-space.

Define $A : X \to F$ by

$$Ax = K_T(\lambda_x,\cdot) \in F, \qquad \lambda_x = (e,x) \in \varLambda.$$

Let $\mu = (t,y) \in \varLambda$. We prove that A is linear:

$$\begin{aligned}[A(c_1x_1 + c_2x_2)](\mu) &= K_T(\lambda_{c_1x_1+c_2x_2},\mu) = [T_{t^*e}(c_1x_1 + c_2x_2)](y)\\ &= c_1(T_{t^*e}x_1)(y) + c_2(T_{t^*e}x_2)(y)\\ &= c_1K_T(\lambda_{x_1},\mu) + c_2K_T(\lambda_{x_2},\mu) = (c_1Ax_1 + c_2Ax_2)(\mu)\end{aligned}$$

for all $c_1, c_2 \in \mathbb{C}$, $x_1, x_2 \in X$.

For the existence of A', let $k \in F$ and

$$k = \sum_{j=1}^{n} c_j K_T(\lambda_j,\cdot), \qquad c_j \in \mathbb{C},\ \lambda_j = (s_j,x_j) \in \varLambda,\ j = \overline{1,n}, n \in \mathbb{N}^*.$$

Let $x \in X$. We get

$$\begin{aligned}(A'k)(x) = [k, Ax]_F &= \Big[\sum_{j=1}^{n} c_j K_T(\lambda_j,\cdot), K_T(\lambda_x,\cdot)\Big]_F = \sum_{j=1}^{n} c_j K_T(\lambda_j,\lambda_x)\\ &= \sum_{j=1}^{n} c_j (T_{e^*s_j}x_j)(x) = \sum_{j=1}^{n} c_j (T_{s_j}x_j)(x).\end{aligned}$$

Therefore $A'k = \sum_{j=1}^{n} c_j T_{s_j} x_j$.

We define a representation $D : \varGamma \to \mathcal{L}(F,F)$ by $D(s) = D_s$ with

$$D_s\Big(\sum_{j=1}^{n} c_j K_T(\lambda_j,\cdot)\Big) = \sum_{j=1}^{n} c_j K_T(\lambda_j^s,\cdot), \qquad s \in \varGamma,$$

where $\lambda_j = (s_j,x_j) \in \varLambda$, and $\lambda_j^s = (ss_j,x_j)$.

It is obvious that $D_s \in \mathcal{L}(F,F)$. Set $k_\nu = \sum_{j=1}^{n} c_j^\nu K_T(\lambda_j^\nu,\cdot)$, $\lambda_j^\nu = (s_j^\nu, x_j^\nu)$, $\nu = \overline{1,2}$, $j = \overline{1,n}$, $n \in \mathbb{N}^*$. We obtain

$$\begin{aligned}[D_s k_1, k_2]_F &= \Big[\sum_{j=1}^n c_j^1 K_T((\lambda_j^1)^s, \cdot), \sum_{l=1}^n c_l^2 K_T(\lambda_l^2, \cdot)\Big]_F \\ &= \sum_{j,l=1}^n c_j^1 \bar{c}_l^2 K_T((\lambda_j^1)^s, \lambda_l^2) = \sum_{j,l=1}^n c_j^1 \bar{c}_l^2 (T_{(s_l^2)^* s s_j^1} x_j^1)(x_l^2) \\ &= \sum_{j,l=1}^n c_j^1 \bar{c}_l^2 K_T(\lambda_j^1, (\lambda_l^2)^{s^*}) = [k_1, k_2^*]_F\end{aligned}$$

where $k_2^* = \sum_{l=1}^n c_l^2 K_T((\lambda_l^2)^{s^*}, \cdot)$. This implies that $D_s^* k_2 = k_2^* = D_{s^*} k_2$. In the same manner, setting $s = e$, we obtain $D_e = I$.

Let once again $k = \sum_{j=1}^n c_j K_T(\lambda_j, \cdot) \in F$, $c_j \in \mathbb{C}$, $\lambda_j = (s_j, x_j) \in \Lambda$, $j = \overline{1, n}$, $n \in \mathbb{N}^*$, $\mu = (t, y) \in \Lambda$. For $s, t \in \Gamma$, we have

$$(D_s D_t)(k) = D_s\Big(\sum_{j=1}^n c_j K_T(\lambda_j^t, \cdot)\Big) = \sum_{j=1}^n c_j K_T(\lambda_j^{st}, \cdot) = D_{st}(k).$$

Thus $D_{st} = D_s D_t$ for all $s, t \in \Gamma$.

We prove (••):

$$(A' D_s A)(x) = A' D_s K_T(\lambda_x, \cdot) = A' K_T(\lambda_x^s, \cdot) = T_s x.$$

Since

$$\sum_{j=1}^n c_j K_T(\lambda_j, \cdot) = \sum_{j=1}^n c_j K_T(\lambda_{x_j}^{s_j}, \cdot) = \sum_{j=1}^n c_j D_{s_j} A x_j, \tag{3}$$

the minimality condition is verified.

For 1°, let

$$D : \Gamma \to \mathcal{L}(F, F) \quad \text{and} \quad D^1 : \Gamma \to \mathcal{L}(F_1, F_1)$$

satisfy condition (•), and let $A \in \mathcal{L}(X, F)$ and $A_1 \in \mathcal{L}(X, F_1)$ be such that

$$A' D_s A = T_s = A_1' D_s^1 A_1,$$

where the linear Z-valued spaces F, F_1 are minimal, i.e. F and F_1 are generated by elements of the form $D_s A x$ and $D_s^1 A_1 x$ respectively. If $f_1 \in F_1$, then $f_1 = \sum_{j=1}^n c_j D_{s_j}^1 A_1 x_j$ and we set

$$U f_1 = \sum_{j=1}^n c_j D_{s_j} A x_j. \tag{4}$$

Thus $U \in \mathcal{L}(F_1, F)$ and $U(F_1) = F$.

Taking $g_1 = \sum_{l=1}^n d_l D_{t_l} A_1 y_l \in F_1$, $\mu_l = (t_l, y_l) \in \Lambda$ and using (3) and the properties of D, we obtain

$$
\begin{aligned}
[Uf_1, Ug_1]_F &= \Big[\sum_{j=1}^{n} c_j D_{s_j} A x_j, \sum_{l=1}^{n} d_l D_{t_l} A y_l\Big]_F \\
&= \Big[\sum_{j=1}^{n} c_j K_T(\lambda_j, \cdot), \sum_{l=1}^{n} d_l K_T(\mu_l, \cdot)\Big]_F \\
&= \sum_{j,l=1}^{n} c_j \bar{d}_l K_T(\lambda_j, \mu_l) = \sum_{j,l=1}^{n} c_j \bar{d}_l (T_{t_l^* s_j} x_j)(y_l) \\
&= \sum_{j,l=1}^{n} c_j \bar{d}_l (A_1' D^1_{t_l^* s_j} A_1 x_j)(y_l) = \sum_{j,l=1}^{n} c_j \bar{d}_l [D^1_{t_l^* s_j} A_1 x_j, A_1 y_l]_{F_1} \\
&= \sum_{j,l=1}^{n} c_j \bar{d}_l \big[(D^1_{t_l})^* D^1_{s_j} A_1 x_j, A_1 y_l\big]_{F_1} \\
&= \Big[\sum_{j=1}^{n} c_j D^1_{s_j} A_1 x_j, \sum_{l=1}^{n} d_l D^1_{t_l} A_1 y_l\Big]_{F_1} = [f_1, g_1]_{F_1}.
\end{aligned}
$$

Hence U is unitary. Moreover, by (4),

$$
UA_1 x = U(D^1_e A_1 x) = D_e A x = Ax, \qquad x \in X,
$$

and

$$
\begin{aligned}
UD^1_s f_1 &= UD^1_s\Big(\sum_{j=1}^{n} c_j D^1_{s_j} A_1 x_j\Big) = U\Big(\sum_{j=1}^{n} c_j D^1_{s s_j} A_1 x_j\Big) \\
&= \sum_{j=1}^{n} c_j D_{s s_j} A x_j = D_s\Big(\sum_{j=1}^{n} c_j D_{s_j} A x_j\Big) = D_s U f_1, \qquad f_1 \in F_1.
\end{aligned}
$$

In order to show 2°, suppose that $T_{s\alpha t} = T_{s\beta t} + T_{s\gamma t}$ for some $\alpha, \beta, \gamma \in \Gamma$ fixed and for all $s, t \in \Gamma$. Take $k \in F$, $k = \sum_{j=1}^{n} c_j K_T(\lambda_j, \cdot)$, $\lambda_j = (s_j, x_j)$, $j = \overline{1, n}$ and $\mu = (t, y) \in \Lambda$. We have

$$
\begin{aligned}
[(D_\beta + D_\gamma)(k)](\mu) = (D_\beta k + D_\gamma k)(\mu) &= \sum_{j=1}^{n} c_j [K_T(\lambda_j^\beta, \mu) + K_T(\lambda_j^\gamma, \mu)] \\
&= \sum_{j=1}^{n} c_j (T_{t^* \beta s_j} x_j)(y) + \sum_{j=1}^{n} c_j (T_{t^* \gamma s_j} x_j)(y) \\
&= \sum_{j=1}^{n} c_j (T_{t^* \alpha s_j} x_j)(y) = \sum_{j=1}^{n} c_j K_T(\lambda_j^\alpha, \mu) = (D_\alpha k)(\mu).
\end{aligned}
$$

Thus 2° is verified and the proof of Theorem 3.1 is complete. ∎

The converse of Theorem 3.1 is also valid.

THEOREM 3.2. *Let Γ be a $*$-semigroup, let X be a complex linear space and let Z be an admissible space in Loynes sense. If there exist a linear Z-valued space F, a function $D:\Gamma\to\mathcal{L}(F,F)$ which satisfies* ($\bullet$) *and an operator $A\in\mathcal{L}(X,F)$ such that $T_s=A'D_sA$, $s\in\Gamma$, then the family $\{T_s\}_{s\in\Gamma}$ has properties* (i) *and* (ii) *from Theorem* 3.1.

Proof. Let $x,y\in X$ and $s\in\Gamma$. Using the definition of A' and the third of relations ($\bullet$), we have

$$\begin{aligned}(T_{s^*}y)(x)^{\Diamond}=(A'D_{s^*}Ay)(x)^{\Diamond}&=[D_{s^*}Ay,Ax]_F^{\Diamond}=[Ay,D_sAx]_F^{\Diamond}\\&=[D_sAx,Ay]_F=(A'D_sAx)(y)=(T_sx)(y),\end{aligned}$$

proving (i).

Now let $(x_s)_{s\in\Gamma}\in\mathcal{X}$. Using the second and the third of relations ($\bullet$), we obtain

$$\begin{aligned}\sum_{s,t\in\Gamma}(T_{s^*t}x_t)(x_s)&=\sum_{s,t\in\Gamma}(A'D_{s^*t}Ax_t)(x_s)=\sum_{s,t\in\Gamma}(A'D_s^*D_tAx_t)(x_s)\\&=\sum_{s,t\in\Gamma}[D_s^*D_tAx_t,Ax_s]_F=\sum_{s,t\in\Gamma}[D_tAx_t,D_sAx_s]_F\\&=\Big[\sum_{t\in\Gamma}D_tAx_t,\sum_{t\in\Gamma}D_tAx_t\Big]_F\geq 0,\end{aligned}$$

whence (ii) holds. ∎

REMARK 3.3. In the particular case when $Z=\mathbb{C}$, one obtains Górniak and Weron's result [GW1, Theorem 1].

4. A topological characterization of positive definite $\mathcal{C}(X,X_Z^*)$-valued families.

As an analogue of the Sz.-Nagy dilation theorem [SN], we give a topological version of Theorem 3.1 under some boundedness conditions.

Let (X,τ) be a linear topological space with topological Z-anti-dual X_Z^*, and $\{T_s\}_{s\in\Gamma}\subset\mathcal{C}(X,X_Z^*)$ be a family of functions indexed by a $*$-semigroup Γ.

DEFINITION 4.1. Consider the following properties of the family $\{T_s\}_{s\in\Gamma}$:

(iii)$_1$ for every $u\in\Gamma$ there exists $c_u\geq 0$ such that

$$\sum_{s,t\in\Gamma}(T_{s^*u^*ut}x_t)(x_s)\leq c_u\sum_{s,t\in\Gamma}(T_{s^*t}x_t)(x_s)$$

for each $(x_s)_{s\in\Gamma}\in\mathcal{X}$;

(iii)$_2$ for every $u\in\Gamma$ and $p\in\mathcal{P}_Z$ there exists $c_{p,u}\geq 0$ such that

$$p\Big[\sum_{s,t\in\Gamma}(T_{s^*u^*ut}x_t)(x_s)\Big]\leq c_{p,u}p\Big[\sum_{s,t\in\Gamma}(T_{s^*t}x_t)(x_s)\Big]$$

for each $(x_s)_{s\in\Gamma}\in\mathcal{X}$;

(iii)$_3$ for any $u \in \varGamma$ there exists $c_u \geq 0$ such that

$$p\Big[\sum_{s,t\in\varGamma}(T_{s^*u^*ut}x_t)(x_s)\Big] \leq c_u p\Big[\sum_{s,t\in\varGamma}(T_{s^*t}x_t)(x_s)\Big]$$

for every $p \in \mathcal{P}_Z$ and $(x_s)_{s\in\varGamma} \in \mathcal{X}$;

(iii)$_4$ for any $u \in \varGamma$ and $p \in \mathcal{P}_Z$ there exist $c_{u,p} \geq 0$ and $q_{u,p} \in \mathcal{P}_Z$ such that

$$p\Big[\sum_{s,t\in\varGamma}(T_{s^*u^*ut}x_t)(x_s)\Big] \leq c_{u,p}q_{u,p}\Big[\sum_{s,t\in\varGamma}(T_{s^*t}x_t)(x_s)\Big]$$

for each $(x_s)_{s\in\varGamma} \in \mathcal{X}$.

We say that a function $D : \varGamma \to \mathcal{C}(\mathcal{H},\mathcal{H})$ is a *representation* of the $*$-semigroup $\varGamma$ in the Loynes space $\mathcal{H}$ if D satisfies ($\bullet$).

Theorem 4.2. *If the family $\{T_s\}_{s\in\varGamma}$ satisfies conditions* (i) *and* (ii) *from Theorem* 3.1 *and property* (iii)$_1$*, then there exist a Loynes Z-space $\mathcal{H}$, a representation D of the $*$-semigroup $\varGamma$ in $\mathcal{H}$ and an operator $A \in \mathcal{C}(X,\mathcal{H})$ such that for any $s \in \varGamma$, $T_s = A^*D_sA$ and there exists $c_s \geq 0$ such that*

$$[D_sk, D_sk]_{\mathcal{H}} \leq c_s[k,k]_{\mathcal{H}} \tag{4.1}$$

for all $k \in \mathcal{H}$.

Also, if $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is a net in $\varGamma$ with $\sup_{\alpha\in\mathcal{A}} c_{u_\alpha} < \infty$ and $(T_{su_\alpha t})_{\alpha\in\mathcal{A}}$ converges to T_{sut} for any $s,t \in \varGamma$ in the weak topology of $\mathcal{C}(X,X_Z^*)$, then $(D_{u_\alpha})_{\alpha\in\mathcal{A}}$ converges to D_u in the weak topology of $\mathcal{C}(\mathcal{H},\mathcal{H})$.

Proof. From Theorem 3.1, there exist a pre-Loynes space F, $A \in \mathcal{L}(X,F)$ and $D : \varGamma \to \mathcal{L}(F,F)$ such that

$$T_s = A'D_sA.$$

Now, let $\mathcal{H}$ be the completion of the pre-Loynes Z-space F which admits K_T as a reproducing kernel. We also mention that the Gramian on F can be extended on $\mathcal{H}$. Evaluating $[Ax,Ax]_F$ we get

$$[Ax,Ax]_F = [K_T(\lambda_x,\cdot),K_T(\lambda_x,\cdot)]_F = K_T(\lambda_x,\lambda_x) = (T_ex)(x)$$

for all $x \in X$ and $\lambda_x = (e,x)$.

In order to verify the continuity of A, consider a net $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset X$ with $x_\alpha \xrightarrow[\alpha\in\mathcal{A}]{X} 0$. Now let V be a 0-neighborhood in Z. Since T_e is continuous, we have $T_ex_\alpha \xrightarrow[\alpha\in\mathcal{A}]{\mathcal{C}(X,X_Z^*)} 0$, that is, there exists $\alpha_0 \in \mathcal{A}$ such that for any $\alpha \geq \alpha_0$, $(T_ex_\alpha)(x) \in V$ for all $x \in X$. Thus $(T_ex_\alpha)(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$, whence $(T_ex_\alpha)(x_\alpha) \xrightarrow[\alpha\in\mathcal{A}]{Z} 0$. Hence $[Ax_\alpha,Ax_\alpha]_F \xrightarrow[\alpha\in\mathcal{A}]{F} 0$. Equivalently, $p([Ax_\alpha,Ax_\alpha]) \xrightarrow[\alpha\in\mathcal{A}]{} 0$ for all $p \in \mathcal{P}_Z$. Using the Cauchy–Schwarz inequality

$$p([Ax_\alpha,y]_F) \leq 2p([Ax_\alpha,Ax_\alpha]_F)^{1/2}p([y,y]_F)^{1/2},$$

for all $p \in \mathcal{P}_Z$ and $y \in F$, we deduce that $p([Ax_\alpha, y]_F) \to 0$, which implies $Ax_\alpha \xrightarrow{F} 0$, so A is continuous.

By evaluating $[D_s k, D_s k]_F$ for $s \in \Gamma$ and $k \in F$, we have

$$\begin{aligned}[D_s k, D_s k]_F &= \Big[\sum_{j=1}^{n} c_j K_T((\lambda_j)^s, \cdot), \sum_{l=1}^{n} c_l K_T((\lambda_l)^s, \cdot)\Big]_F \\ &= \sum_{l=1}^{n}\sum_{j=1}^{n} c_j \overline{c_l} K_T((\lambda_j)^s, (\lambda_l)^s) = \sum_{l,j=1}^{n} c_j \overline{c_l} (T_{(ss_l)^*(ss_j)} x_j)(x_l) \\ &= \sum_{l,j=1}^{n} c_j \overline{c_l} (T_{s_l^* s^* s s_j} x_j)(x_l),\end{aligned}$$

and

$$[k, k]_F = \sum_{l,j=1}^{n} c_j \overline{c_l} (\lambda_j, \lambda_l) = \sum_{l,j=1}^{n} c_j \overline{c_l} (T_{s_l^* s_j} x_j)(x_l),$$

where $k = \sum_{j=1}^{n} c_j K_T(\lambda_j, \cdot)$ and

$$\lambda_j = (s_j, x_j), \quad \lambda_l = (s_l, x_l), \quad (\lambda_j)^s = (ss_j, x_j), \quad (\lambda_l)^s = (ss_l, x_l).$$

Under the boundedness hypothesis (iii)$_1$ we obtain

$$[D_u k, D_u k]_F \le c_u [k, k]_F.$$

So, for each $u \in \Gamma$, D_u is a bounded operator on F, and consequently we may extend it (following the same technique as in the last part of the proof of [Lo1, Theorem 3]) to a bounded linear operator D_s from $\mathcal{H}$ into $\mathcal{H}$, whence D is a representation of Γ in $\mathcal{H}$. Now, if $T_{su_\alpha t} \to T_{sut}$ in the weak topology of $\mathcal{C}(X, X_Z^*)$, that is, $(T_{su_\alpha t} x)(x') \xrightarrow[\alpha\in\mathcal{A}]{} (T_{sut} x)(x')$ for all $x, x' \in X$ and all $s, t \in \Gamma$, then for $f, f' \in F$, we have

$$\begin{aligned}[D_{u_\alpha} f, f']_F &= \Big[\sum_{j=1}^{n} c_j^1 K_T((\lambda_j^1)^{u_\alpha}, \cdot), \sum_{l=1}^{n} c_l^2 K_T(\lambda_l^2, \cdot)\Big]_F \\ &= \sum_{l,j=1}^{n} c_j^1 \overline{c_l^2} K_T((\lambda_j^1)^{u_\alpha}, \lambda_l^2) = \sum_{l,j=1}^{n} c_j^1 \overline{c_l^2} (T_{(s_l)^* u_\alpha t_j} x_j)(x_l'),\end{aligned}$$

which for $\alpha \in \mathcal{A}$ converges to

$$\sum_{l,j=1}^{n} c_j^1 \overline{c_l^2} (T_{(s_l)^* u t_j} x_j)(x_l') = [D_u f, f']_F.$$

As F is dense in $\mathcal{H}$, we have $[D_{u_\alpha} k, D_{u_\alpha} k]_{\mathcal{H}} \le c_{u_\alpha} [k, k]_{\mathcal{H}}$ and $\sup_{\alpha\in\mathcal{A}} c_{u_\alpha} < \infty$, implying that $\{D_{u_\alpha} k\}_{\alpha\in\mathcal{A}}$ is bounded for each $k \in \mathcal{H}$.

We now prove that D_{u_α} converges to D_u in the weak operator topology of $\mathcal{H}$, the completion of F. Take $f, f' \in F$ such that $[D_{u_\alpha} f, f'] \to [D_u f, f']$. Suppose that g and g' are fixed elements of $\mathcal{H}$ and that $g = f + \delta$, $g' = f' + \delta'$, δ and δ' being chosen in a suitable neighborhood of the origin (possible since F is dense in $\mathcal{H}$). Denote $R_\alpha = D_{u_\alpha} - D_u$. We obtain

$$p([R_\alpha g, g'] - [R_\alpha f, f']) \leq p([R_\alpha g, \delta']) + p([R_\alpha \delta, g']) + p([R_\alpha \delta, \delta'])$$

for any $p \in \mathcal{P}_Z$. Since δ and δ' can be chosen sufficiently small to make each of the three terms on the right-hand side small uniformly in α (by a standard reasoning), we conclude that D_{u_α} converges to D_u in the weak operator topology of $\mathcal{C}(\mathcal{H}, \mathcal{H})$. ∎

Remark 4.3. If in Theorem 4.2, instead of $(\mathrm{iii})_1$, the family $\{T_s\}_{s\in\Gamma}$ satisfies property $(\mathrm{iii})_2$ from Definition 4.1, then for all $s \in \Gamma$ and all $p \in \mathcal{P}_Z$ there exists $c_{p,s} \geq 0$ such that

$$p([D_s k, D_s k]_{\mathcal{H}}) \leq c_{p,s} p([k,k]_{\mathcal{H}})$$

for any $k \in \mathcal{H}$.

Also, if $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is a net in Γ with $\sup_{\alpha\in\mathcal{A}} c_{p,u_\alpha} < \infty$ for all $p \in \mathcal{P}_Z$, and $(T_{su_\alpha t})_{\alpha\in\mathcal{A}}$ converges to T_{sut} for any $s, t \in \Gamma$ in the weak topology of $\mathcal{C}(X, X_Z^*)$, then $(D_{u_\alpha})_{\alpha\in\mathcal{A}}$ converges to D_u in the weak topology of $\mathcal{C}(\mathcal{H}, \mathcal{H})$.

Remark 4.4. If in Theorem 4.2, instead of $(\mathrm{iii})_1$, the family $\{T_s\}_{s\in\Gamma}$ satisfies $(\mathrm{iii})_3$, then for all $s \in \Gamma$ there exists $c_s > 0$ such that

$$p([D_s k, D_s k]_{\mathcal{H}}) \leq c_s p([k,k]_{\mathcal{H}})$$

for any $p \in \mathcal{P}_Z$ and any $k \in \mathcal{H}$.

Also, if $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is a net in Γ with $\sup_{\alpha\in\mathcal{A}} c_{u_\alpha} < \infty$, and $(T_{su_\alpha t})_{\alpha\in\mathcal{A}}$ converges to T_{sut} for any $s, t \in \Gamma$ in the weak topology of $\mathcal{C}(X, X_Z^*)$, then $(D_{u_\alpha})_{\alpha\in\mathcal{A}}$ converges to D_u in the weak topology of $\mathcal{C}(\mathcal{H}, \mathcal{H})$.

Remark 4.5. If in Theorem 4.2, instead of $(\mathrm{iii})_1$, the family $\{T_s\}_{s\in\Gamma}$ satisfies $(\mathrm{iii})_4$, then for all $s \in \Gamma$ and all $p \in \mathcal{P}_Z$ there exist $c_{s,p} \geq 0$ and $q_{s,p} \in \mathcal{P}_Z$ such that

$$p([D_s k, D_s k]_{\mathcal{H}}) \leq c_{s,p} q_{s,p}([k,k]_{\mathcal{H}})$$

for any $k \in \mathcal{H}$.

Also, if $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is a net in Γ with the property that there exists $p_0 \in \mathcal{P}_Z$ such that $\sup_{\alpha\in\mathcal{A}} c_{p,u_\alpha} < \infty$ and $q_{u_\alpha,p}(z) \leq p_0(z)$ for all $z \in Z$ and $p \in \mathcal{P}_Z$, and $(T_{su_\alpha t})_{\alpha\in\mathcal{A}}$ converges to T_{sut} for any $s, t \in \Gamma$ in the weak topology of $\mathcal{C}(X, X_Z^*)$, then $(D_{u_\alpha})_{\alpha\in\mathcal{A}}$ converges to D_u in the weak topology of $\mathcal{C}(\mathcal{H}, \mathcal{H})$.

One can prove these remarks by standard evaluations similar to those in the proof of Theorem 4.2.

REMARK 4.6. If $X = \mathcal{K}$ is a Hilbert space and $T_e = I$, then Theorem 4.2 will lead to the classical Sz.-Nagy dilation theorem. Precisely, A is an "injection" of $\mathcal{K}$ in the space $\mathcal{H}$ which contains it, and "A^*" is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Consequently, $T_\xi = \operatorname{Proj}_K D_\xi$.

REMARK 4.7. If in Theorem 4.2 we replace X with a normed space with topological dual X^*, we obtain Theorem 2 from [GW2].

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