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Ergodic theorems in fully symmetric spaces of τ-measurable operators

by

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Abstract. Junge and Xu (2007), employing the technique of noncommutative interpolation, established a maximal ergodic theorem in noncommutative L_p-spaces, $1 < p < \infty$, and derived corresponding maximal ergodic inequalities and individual ergodic theorems. In this article, we derive maximal ergodic inequalities in noncommutative L_p-spaces directly from the results of Yeadon (1977) and apply them to prove corresponding individual and Besicovitch weighted ergodic theorems. Then we extend these results to noncommutative fully symmetric Banach spaces with the Fatou property and nontrivial Boyd indices, in particular, to noncommutative Lorentz spaces $L_{p,q}$. Norm convergence of ergodic averages in noncommutative fully symmetric Banach spaces is also studied.

1. Preliminaries and introduction. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$, $B(\mathcal{H})$ the algebra of all bounded linear operators in $\mathcal{H}$, $\|\cdot\|_\infty$ the uniform norm in $B(\mathcal{H})$, and $\mathbb{I}$ the identity in $B(\mathcal{H})$. If $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra, we denote by $\mathcal{P}(\mathcal{M}) = \{e \in \mathcal{M} : e = e^2 = e^*\}$ the complete lattice of all projections in $\mathcal{M}$. For every $e \in \mathcal{P}(\mathcal{M})$ we write $e^\perp = \mathbb{I} - e$. If $\{e_i\}_{i\in I} \subset \mathcal{P}(\mathcal{M})$, the projection on the subspace $\bigcap_{i\in I} e_i(\mathcal{H})$ is denoted by $\bigwedge_{i\in I} e_i$.

A linear operator $x : \mathcal{D}_x \to \mathcal{H}$, where the domain $\mathcal{D}_x$ of x is a linear subspace of $\mathcal{H}$, is said to be *affiliated with the algebra* $\mathcal{M}$ if $yx \subseteq xy$ for every y from the commutant of $\mathcal{M}$.

Assume now that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. A densely-defined closed linear operator x affiliated with $\mathcal{M}$ is called *τ-measurable* if for each $\epsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ with $\tau(e^\perp) \le \epsilon$ such that $e(\mathcal{H}) \subset \mathcal{D}_x$. Let $L_0(\mathcal{M}, \tau)$ denote the set of all τ-measurable operators.

It is well-known [22] that if $x, y \in L_0(\mathcal{M}, \tau)$, then the operators $x + y$ and xy are densely-defined and preclosed. Moreover, the closures $\overline{x+y}$ (the

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strong sum) and $\overline{xy}$ (the strong product) and x^* are also τ-measurable and, equipped with these operations, $L_0(\mathcal{M}, \tau)$ is a unital $*$-algebra over $\mathbb{C}$.

For every subset $X \subset L_0(\mathcal{M}, \tau)$ the set of all self-adjoint [positive] operators in X is denoted by X^h $[X^+]$. The partial order $\leq$ in $L_0^h(\mathcal{M}, \tau)$ is defined by the cone $L_0^+(\mathcal{M}, \tau)$.

The topology defined in $L_0 = L_0(\mathcal{M}, \tau)$ by the family

$$V(\epsilon, \delta) = \{x \in L_0 : \|xe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon\}$$

$$[W(\epsilon, \delta) = \{x \in L_0 : \|exe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon\}],$$

$\epsilon > 0, \delta > 0$, of (closed) neighborhoods of zero is called the *measure topology* [the *bilaterally measure topology*]. It is said that a sequence $\{x_n\} \subset L_0(\mathcal{M}, \tau)$ converges to $x \in L_0(\mathcal{M}, \tau)$ in measure [bilaterally in measure] if this sequence converges to x in measure topology [in bilaterally measure topology]. It is known [3, Theorem 2.2] that $x_n \to x$ in measure if and only if $x_n \to x$ bilaterally in measure. For basic properties of the measure topology in $L_0(\mathcal{M}, \tau)$, see [19].

A sequence $\{x_n\} \subset L_0(\mathcal{M}, \tau)$ is said to converge to $x \in L_0(\mathcal{M}, \tau)$ *almost uniformly (a.u.)* [*bilaterally almost uniformly* (*b.a.u.*)] if for every $\epsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e^\perp) \leq \epsilon$ and $\|(x - x_n)e\|_\infty \to 0$ $[\|e(x - x_n)e\|_\infty \to 0]$. It is clear that every sequence in $L_0(\mathcal{M}, \tau)$ a.u. convergent [b.a.u. convergent] to x converges to x in measure [bilaterally in measure, hence in measure].

For a positive self-adjoint operator $x = \int_0^\infty \lambda \, de_\lambda$ affiliated with $\mathcal{M}$ one can define

$$\tau(x) = \sup_n \tau\Big(\int_0^n \lambda \, de_\lambda\Big) = \int_0^\infty \lambda \, d\tau(e_\lambda).$$

If $1 \leq p < \infty$, then the *noncommutative L_p-space associated with* $(\mathcal{M}, \tau)$ is defined as

$$L_p = (L_p(\mathcal{M}, \tau), \|\cdot\|_p) = \{x \in L_0(\mathcal{M}, \tau) : \|x\|_p = (\tau(|x|^p))^{1/p} < \infty\},$$

where $|x| = (x^*x)^{1/2}$, the absolute value of x (see [24]). Naturally, $L_\infty = (\mathcal{M}, \|\cdot\|_\infty)$. If $x_n, x \in L_p$ and $\|x - x_n\|_p \to 0$, then $x_n \to x$ in measure [11, Theorem 3.7]. Moreover, utilizing the spectral decomposition of $x \in L_p^+$, it is possible to find a sequence $\{x_n\} \subset L_p^+ \cap \mathcal{M}$ such that $0 \leq x_n \leq x$ for each n and $x_n \uparrow x$; in particular, $\|x_n\|_p \leq \|x\|_p$ for all n and $\|x - x_n\|_p \to 0$.

Let $T : L_1 \cap \mathcal{M} \to L_1 \cap \mathcal{M}$ be a positive linear map that satisfies the conditions of [25]:

(Y) $\quad T(x) \leq \mathbb{I} \quad \text{and} \quad \tau(T(x)) \leq \tau(x) \quad \forall x \in L_1 \cap \mathcal{M}$ with $0 \leq x \leq \mathbb{I}$.

It is known [25, Proposition 1] that such a T admits a unique positive ultraweakly continuous linear extension $T : \mathcal{M} \to \mathcal{M}$. In fact, T contracts $\mathcal{M}$:

PROPOSITION 1.1. *Let T be the extension to $\mathcal{M}$ of a positive linear map $T : L_1 \cap \mathcal{M} \to L_1 \cap \mathcal{M}$ satisfying condition* (Y). *Then $\|T(x)\|_\infty \le \|x\|_\infty$ for every $x \in \mathcal{M}$.*

Proof. Since the trace τ is semifinite, there exists a net $\{p_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{P}(\mathcal{M})$, where Λ is a base of neighborhoods of zero of the ultraweak topology ordered by inclusion, such that $0 < \tau(p_\alpha) < \infty$ for every α and $p_\alpha \to \mathbb{I}$ ultraweakly. Then $T(x_\alpha) \to T(\mathbb{I})$ ultraweakly. Since $\|T(p_\alpha)\|_\infty \le 1$, and the unit ball of $\mathcal{M}$ is closed in the ultraweak topology, we conclude that $\|T(\mathbb{I})\|_\infty \le 1$. Therefore, by [20, Corollary 2.9],

$$\|T\|_{\mathcal{M}\to\mathcal{M}} = \|T(\mathbb{I})\|_\infty \le 1. \ \blacksquare$$

In [13, Theorem 4.1], a maximal ergodic theorem in noncommutative L_p-spaces, $1 < p < \infty$, was established for the class of positive linear maps $T : \mathcal{M} \to \mathcal{M}$ satisfying the condition

(JX) $\quad \|T(x)\|_\infty \le \|x\|_\infty \quad \forall x \in \mathcal{M} \quad$ and $\quad \tau(T(x)) \le \tau(x) \quad \forall x \in L_1 \cap \mathcal{M}^+$.

REMARK 1.2. Due to Proposition 1.1, (JX) $\Leftrightarrow$ (Y).

Moreover, by [13, Lemma 1.1], a positive linear map $T : \mathcal{M} \to \mathcal{M}$ that satisfies (JX) uniquely extends to a positive linear contraction T in L_p, $1 < p < \infty$.

We shall write $T \in \mathrm{DS}^+ = \mathrm{DS}^+(\mathcal{M}, \tau)$ to indicate that the map $T : L_1 + \mathcal{M} \to L_1 + \mathcal{M}$ is the unique positive linear extension of a positive linear map $T : \mathcal{M} \to \mathcal{M}$ satisfying condition (JX). Such a T is often called a *positive Dunford–Schwartz transformation* (see, for example, [26]).

Assume that $T \in \mathrm{DS}^+$ and form its ergodic averages:

$$M_n = M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k, \qquad n = 1, 2, \ldots. \tag{1}$$

The following fundamental result provides a maximal ergodic inequality in L_1 for the averages (1).

THEOREM 1.3 ([25]). *If $T \in \mathrm{DS}^+$, then for every $x \in L_1^+$ and $\epsilon > 0$, there is $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(e^\perp) \le \frac{\|x\|_1}{\epsilon} \quad \textit{and} \quad \sup_n \|eM_n(x)e\|_\infty \le \epsilon.$$

Here is a corollary of Theorem 1.3, a noncommutative individual ergodic theorem of Yeadon:

THEOREM 1.4 ([25]). *If $T \in \mathrm{DS}^+$, then for every $x \in L_1$ the averages $M_n(x)$ converge b.a.u. to some $\widehat{x} \in L_1$.*

The next result, an extension of Theorem 1.4, was established in [13].

Theorem 1.5 ([13, Corollary 6.4]). *Let $T \in \mathrm{DS}^+$, $1 < p < \infty$, and $x \in L_p$. Then the averages $M_n(x)$ converge b.a.u. to some $\widehat{x} \in L_p$. If $p \geq 2$, these averages converge also a.u.*

The proof of Theorem 1.5 in [13] is based on an application of a weak type (p, p) maximal inequality for the averages (1), an L_p-version of Theorem 1.3. Note that the proof of this inequality itself relies on Theorem 1.3 and essentialy involves an intricate technique of noncommutative interpolation. Below (Theorem 2.1) we provide a simple proof of such a maximal inequality, based only on Theorem 1.3.

As an application of Theorem 2.1, we prove a Besicovitch weighted noncommutative ergodic theorem in L_p, $1 < p < \infty$ (Theorem 3.5), which contains Theorem 1.5 as a particular case. Theorem 3.5 is an extension of the corresponding result for L_1 in [3]. Note that, in [18], Theorem 1.5 was derived from Theorem 1.3 by utilizing the notion of uniform equicontinuity at zero of a family of additive maps into $L_0(\mathcal{M}, \tau)$.

Having the Besicovitch weighted ergodic theorem for noncommutative L_p-spaces with $1 \leq p < \infty$ allows us to establish its validity for a wide class of noncommutative fully symmetric spaces with Fatou property. As a consequence, we obtain an individual ergodic theorem in noncommutative Lorentz spaces $L_{p,q}$.

The last section of the article is devoted to a study of the mean ergodic theorem in noncommutative fully symmetric spaces for $T \in \mathrm{DS}(\mathcal{M}, \tau)$.

2. Maximal ergodic inequalities in noncommutative L_p-spaces. Everywhere in this section $T \in \mathrm{DS}^+$. Assume that a sequence $\{\beta_k\}_{k=0}^{\infty}$ of complex numbers is such that $|\beta_k| \leq C$ for every k. Let us denote

$$M_{\beta,n} = M_{\beta,n}(T) = \frac{1}{n+1} \sum_{k=0}^{n} \beta_k T^k. \tag{2}$$

Theorem 2.1. *If $1 \leq p < \infty$, then for every $x \in L_p$ and $\epsilon > 0$ there is $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(e^{\perp}) \leq 4\left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|eM_{\beta,n}(x)e\|_\infty \leq 48C\epsilon. \tag{3}$$

Proof. Let first $\beta_k \equiv 1$. In this case, $M_{\beta,n} = M_n$. Fix $\epsilon > 0$. Assume that $x \in L_p^+$, and let $x = \int_0^\infty \lambda \, de_\lambda$ be its spectral decomposition. Since $\lambda \geq \epsilon$ implies $\lambda \leq \epsilon^{1-p}\lambda^p$, we have

$$\int_\epsilon^\infty \lambda \, de_\lambda \leq \epsilon^{1-p} \int_\epsilon^\infty \lambda^p \, de_\lambda \leq \epsilon^{1-p} x^p.$$

Then we can write

$$x = \int_0^\epsilon \lambda\, de_\lambda + \int_\epsilon^\infty \lambda\, de_\lambda \le x_\epsilon + \epsilon^{1-p} x^p, \tag{4}$$

where $x_\epsilon = \int_0^\epsilon \lambda\, de_\lambda$.

As $x^p \in L_1$, Theorem 1.3 entails that there exists $e \in \mathcal{P}(\mathcal{M})$ satisfying

$$\tau(e^\perp) \le \frac{\|x^p\|_1}{\epsilon^p} = \left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|eM_n(x^p)e\|_\infty \le \epsilon^p.$$

It follows from (4) that

$$0 \le M_n(x) \le M_n(x_\epsilon) + \epsilon^{1-p} M_n(x^p),$$
$$0 \le eM_n(x)e \le eM_n(x_\epsilon)e + \epsilon^{1-p} eM_n(x^p)e$$

for every n.

Since $x_\epsilon \in \mathcal{M}$, the inequality

$$\|T(x_\epsilon)\|_\infty \le \|x_\epsilon\|_\infty \le \epsilon$$

holds, and we conclude that

$$\sup_n \|eM_n(x)e\|_\infty \le \epsilon + \epsilon = 2\epsilon.$$

If $x \in L_p$, then $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x = \frac{1}{2}(x + x^*) \in L_p$ and $\operatorname{Im} x = \frac{1}{2i}(x + x^*) \in L_p$ so that $\|\operatorname{Re} x\|_p \le \frac{1}{2}(\|x\|_p + \|x^*\|_p) = \|x\|_p$ and $\|\operatorname{Im} x\|_p \le \|x\|_p$. Moreover, $\operatorname{Re} x = (\operatorname{Re} x)_+ - (\operatorname{Re} x)_-$, $\operatorname{Im} x = (\operatorname{Im} x)_+ - (\operatorname{Im} x)_-$, where $(\operatorname{Re} x)_+ \in L_p$ and $(\operatorname{Re} x)_- \in L_p$ are positive and negative parts of $\operatorname{Re} x$. Since $|(\operatorname{Re} x)_+| \le |\operatorname{Re} x|$ and $|(\operatorname{Re} x)_-| \le |\operatorname{Re} x|$, we have $\|(\operatorname{Re} x)_+\|_p \le \|\operatorname{Re} x\|_p \le \|x\|_p$ and $\|(\operatorname{Re} x)_-\|_p \le \|x\|_p$. Thus $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_j \in L_p^+$ and $\|x_j\|_p \le \|x\|_p$ for every $j = 1, \ldots, 4$. As we have shown, there exists $e_j \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_j^\perp) \le \left(\frac{\|x_j\|_p}{\epsilon}\right)^p \le \left(\frac{\|x\|_p}{\epsilon}\right)^p, \qquad \sup_n \|e_j M_n(x_j) e_j\|_\infty \le 2\epsilon, \tag{5}$$

$j = 1, \ldots, 4$.

Now, let $\{\beta_k\}_{k=0}^\infty \subset \mathbb{C}$ satisfy $|\beta_k| \le C$ for every k. As $0 \le \operatorname{Re} \beta_k + C \le 2C$ and $0 \le \operatorname{Im} \beta_k + C \le 2C$, it follows from the decomposition

$$M_{\beta,n} = \frac{1}{n+1} \sum_{k=0}^n (\operatorname{Re} \beta_k + C) T^k + \frac{i}{n+1} \sum_{k=0}^n (\operatorname{Im} \beta_k + C) T^k - C(1+i) M_n \tag{6}$$

and (5) that

$$\sup_n \|e_j M_{\beta,n}(x_j) e_j\|_\infty \le 6C \sup_n \|e_j M_n(x_j) e_j\|_\infty \le 12C\epsilon, \qquad j = 1, \ldots, 4.$$

Finally, letting $e = \bigwedge_{j=1}^4 e_j$, we arrive at (3). ∎

Remark 2.2. Note that (5) provides the following extension of the maximal ergodic inequality given in Theorem 1.3 for $p=1$: for every $x \in L_p^+$ and $\epsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e^{\perp}) \leq \left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \leq 2\epsilon.$$

To refine Theorem 2.1 when $p \geq 2$ we turn to the fundamental result of Kadison [14]:

Theorem 2.3 (Kadison's inequality). *Let $S : \mathcal{M} \to \mathcal{M}$ be a positive linear map such that $S(\mathbb{I}) \leq \mathbb{I}$. Then $S(x)^2 \leq S(x^2)$ for every $x \in \mathcal{M}^h$.*

We will need the following technical lemma; see the proof of [3, Theorem 2.7] or [18, Theorem 3.1]. The proof of this lemma involves a simple diagonal argument.

Lemma 2.4. *Let $\{a_{mn}\}_{m,n=1}^\infty \subset L_0(\mathcal{M},\tau)$ be such that for any n the sequence $\{a_{mn}\}_{m=1}^\infty$ converges in measure to some $a_n \in L_0(\mathcal{M},\tau)$. Then there exists $\{a_{m_k n}\}_{k,n=1}^\infty$ such that for any n we have $a_{m_k n} \to a_n$ a.u. as $k \to \infty$.*

Proposition 2.5 (cf. [13, proof of Remark 6.5]). *If $2 \leq p < \infty$ and $T \in \mathrm{DS}^+$, then for every $x \in L_p^h$ and $\epsilon > 0$, there exists $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e^{\perp}) \leq \epsilon$ and*

$$\|gM_n(x)^2 g\|_\infty \leq \|gM_n(x^2)g\|_\infty < \infty$$

for all $g \in \mathcal{P}(\mathcal{M})$ with $g \leq e$ and $n = 1, 2, \ldots$.

Proof. Let $x = \int_{-\infty}^{\infty} \lambda\, de_\lambda$ be the spectral decomposition of $x \in L_p^h$, and let $x_m = \int_{-m}^{m} \lambda\, de_\lambda$. Then, since $x \in L_p$, we clearly have $\|x - x_m\|_p \to 0$. Moreover, $\|x^2 - x_m^2\|_{p/2} \to 0$, so $\|M_n(x^2) - M_n(x_m^2)\|_{p/2} \to 0$ for every n, which implies that

$$M_n(x_m^2) \to M_n(x^2) \quad \text{in measure}, \; n = 1, 2, \ldots.$$

Also $\|M_n(x) - M_n(x_m)\|_p \to 0$ for every n, hence $M_n(x_m) \to M_n(x)$ in measure and

$$M_n(x_m)^2 \to M_n(x)^2 \quad \text{in measure}, \; n = 1, 2, \ldots.$$

In view of Lemma 2.4, it is possible to find a subsequence $\{x_{m_k}\} \subset \{x_m\}$ such that

$$M_n(x_{m_k}^2) \to M_n(x^2) \quad \text{and} \quad M_n(x_{m_k})^2 \to M_n(x)^2 \quad \text{a.u.}, \qquad n = 1, 2, \ldots.$$

Then one can construct such $e_1 \in \mathcal{P}(\mathcal{M})$ that $\tau(e_1^{\perp}) \leq \epsilon/2$ and

$$\|e_1(M_n(x_{m_k}^2) - M_n(x^2))e_1\|_\infty \to 0, \; \|e_1(M_n(x_{m_k})^2 - M_n(x)^2)e_1\|_\infty \to 0 \tag{7}$$

for every n.

Since $M_n(x^2)$ and $M_n(x)^2$ are measurable operators, there exists $e_2 \in \mathcal{P}(\mathcal{M})$ such that $\tau(e_2^\perp) \le \epsilon/2$ and $M_n(x^2)e_2, M_n(x)^2 e_2 \in \mathcal{M}$ for all n. Let $e = e_1 \wedge e_2$. It is clear that $\tau(e^\perp) \le \epsilon$, and, in view of (7), for every $g \in \mathcal{P}(\mathcal{M})$ with $g \le e$ we have

$$\|gM_n(x_{m_k}^2)g - gM_n(x^2)g\|_\infty \to 0 \quad \text{and} \quad \|gM_n(x_{m_k})^2 g - gM_n(x)^2 g\|_\infty \to 0.$$

Hence

$$\|gM_n(x_{m_k}^2)g\|_\infty \to \|gM_n(x^2)g\|_\infty < \infty \quad \text{and}$$
$$\|gM_n(x_{m_k})^2 g\|_\infty \to \|gM_n(x)^2 g\|_\infty < \infty.$$

Since, by Kadison's inequality,

$$\|gM_n(x_{m_k})^2 g\|_\infty \le \|gM_n(x_{m_k}^2)g\|_\infty, \qquad k, n = 1, 2, \ldots,$$

the result follows. ∎

THEOREM 2.6. *If $2 \le p < \infty$, then for every $x \in L_p$ and $\epsilon > 0$ there is $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(e^\perp) \le 6\left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \textit{and} \quad \sup_n \|M_{\beta,n}(x)e\|_\infty \le 4C(2\sqrt{2}+1)\epsilon. \tag{8}$$

Proof. Pick $x \in L_p^h$. Since $x^2 \in L_{p/2}^+$, referring to (5), we can find $e_1(x) \in \mathcal{P}(\mathcal{M})$ such that

$$\begin{gathered} \tau(e_1(x)^\perp) \le \left(\frac{\|x^2\|_{p/2}}{\epsilon^2}\right)^{p/2} = \left(\frac{\|x\|_p}{\epsilon}\right)^p, \\ \sup_n \|e_1(x)M_n(x^2)e_1(x)\|_\infty \le 2\epsilon^2. \end{gathered} \tag{9}$$

By Proposition 2.5, there is $e_2(x) \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_2(x)^\perp) \le \left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|gM_n(x)^2 g\|_\infty \le \sup_n \|gM_n(x^2)g\|_\infty$$

for all $g \in \mathcal{P}(\mathcal{M})$ with $g \le e_2(x)$. Then, letting $g(x) = e_1(x) \wedge e_2(x)$, we obtain $\tau(g(x)^\perp) \le 2(\|x\|_p/\epsilon)^p$ and (see Proposition 2.5)

$$\begin{aligned} \sup_n \|M_n(x)g(x)\|_\infty &= \Big(\sup_n \|M_n(x)g(x)\|_\infty^2\Big)^{1/2} \\ &= \Big(\sup_n \|g(x)M_n(x)^2 g(x)\|_\infty\Big)^{1/2} \le \Big(\sup_n \|g(x)M_n(x^2)g(x)\|_\infty\Big)^{1/2} \le \sqrt{2}\,\epsilon. \end{aligned}$$

If $\{\beta_k\}_{k=0}^\infty \subset \mathbb{C}$, $|\beta_k| \le C$, in accordance with the decomposition (6), we denote

$$M_{\beta,n}^{(\mathrm{R})} = \frac{1}{n+1}\sum_{k=0}^n (\operatorname{Re}\beta_k + C)T^k, \qquad M_{\beta,n}^{(\mathrm{I})} = \frac{1}{n+1}\sum_{k=0}^n (\operatorname{Im}\beta_k + C)T^k.$$

Let $x = x_1 + ix_2 \in L_p$, where $x_j \in L_p^h$ and $\|x_j\|_p \leq \|x\|_p$, $j = 1, 2$. Since $x_1^2 \in L_{p/2}^+$, it follows from (9) that there is $e_1(x_1) \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_1(x_1)^\perp) \leq \left(\frac{\|x_1\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|e_1(x_1)M_n(x_1^2)e_1(x_1)\|_\infty \leq 2\epsilon^2.$$

Since $0 \leq \operatorname{Re}\beta_k + C \leq 2C$ and $0 \leq \operatorname{Im}\beta_k + C \leq 2C$, we have

$$\sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{R})}(x_1^2)e_1(x_1)\|_\infty \leq 4C\epsilon^2,$$
$$\sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{I})}(x_1^2)e_1(x_1)\|_\infty \leq 4C\epsilon^2.$$

In addition, $(2C)^{-1}M_{\beta,n}^{(\mathrm{R})} : \mathcal{M} \to \mathcal{M}$ and $(2C)^{-1}M_{\beta,n}^{(\mathrm{I})} : \mathcal{M} \to \mathcal{M}$ are positive linear maps satisfying $(2C)^{-1}M_{\beta,n}^{(\mathrm{R})}(\mathbb{I}) \leq \mathbb{I}$ and $(2C)^{-1}M_{\beta,n}^{(\mathrm{I})}(\mathbb{I}) \leq \mathbb{I}$ for every n. Then, applying Kadison's inequality, we obtain

$$(2C)^{-2}M_{\beta,n}^{(\mathrm{R})}(x_1)^2 \leq (2C)^{-1}M_{\beta,n}^{(\mathrm{R})}(x_1^2),$$
$$(2C)^{-2}M_{\beta,n}^{(\mathrm{I})}(x_1)^2 \leq (2C)^{-1}M_{\beta,n}^{(\mathrm{I})}(x_1^2).$$

This in turn entails

$$\sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{R})}(x_1)^2 e_1(x_1)\|_\infty \leq 2C \sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{R})}(x_1^2)e_1(x_1)\|_\infty,$$
$$\sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{I})}(x_1)^2 e_1(x_1)\|_\infty \leq 2C \sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{I})}(x_1^2)e_1(x_1)\|_\infty.$$

Therefore

$$\begin{aligned}\sup_n \|M_{\beta,n}^{(\mathrm{R})}(x_1)e_1(x_1)\|_\infty^2 &= \sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{R})}(x_1)^2 e_1(x_1)\|_\infty \\ &\leq 2C \sup_n \|e_1(x_1)M_{\beta,n}^{(\mathrm{R})}(x_1^2)e_1(x_1)\|_\infty \leq 8C^2\epsilon^2,\end{aligned}$$

and similarly

$$\sup_n \|M_{\beta,n}^{(\mathrm{I})}(x_1)e_1(x_1)\|_\infty^2 \leq 8C^2\epsilon^2.$$

Then, letting $g_1 = g(x_1) \wedge e_1(x_1)$, we derive that $\tau(g_1^\perp) \leq 3(\|x\|_p/\epsilon)^p$ and

$$\sup_n \|M_{\beta,n}(x_1)g_1\|_\infty \leq 2C(2\sqrt{2} + 1)\epsilon.$$

Similarly, one can find $g_2 \in \mathcal{P}(\mathcal{M})$ with $\tau(g_2^\perp) \leq 3(\|x_2\|_p/\epsilon)^p$ such that

$$\sup_n \|M_{\beta,n}(x_2)g_2\|_\infty \leq 2C(2\sqrt{2} + 1)\epsilon.$$

Finally, we conclude that $e = g_1 \wedge g_2 \in \mathcal{P}(\mathcal{M})$ satisfies (8). ∎

Remark 2.7. Beginning of the proof of Theorem 2.6 contains the following maximal ergodic inequality for the ergodic averages (1): if $2 \leq p < \infty$, given $x \in L_p^h$ and $\epsilon > 0$, there exists $e \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(e^\perp) \le 2\left(\frac{\|x\|_p}{\epsilon}\right)^p \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \le \sqrt{2}\,\epsilon.$$

3. Besicovitch weighted ergodic theorem in noncommutative L_p-spaces. In this section, using the maximal ergodic inequalities of Theorems 2.1 and 2.6, we prove a Besicovitch weighted ergodic theorem in noncommutative L_p-spaces, $1 < p < \infty$. As was already mentioned, this extends the corresponding result for $p = 1$ from [3]. Everywhere in this section $T \in \mathrm{DS}^+$.

We will need the following technical lemma.

LEMMA 3.1 (see [2, Lemma 1.6]). *Let X be a linear space, and let $S_n : X \to L_0(\mathcal{M}, \tau)$ be a sequence of additive maps. Assume that $x \in X$ is such that for every $\epsilon > 0$ there exists a sequence $\{x_k\} \subset X$ and a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying the following conditions:*

(i) *for each k, the sequence $\{S_n(x + x_k)\}$ converges a.u. [b.a.u.] as $n \to \infty$;*
(ii) *$\tau(e^\perp) \le \epsilon$;*
(iii) *$\sup_n \|S_n(x_k)e\|_\infty \to 0$ [$\sup_n \|eS_n(x_k)e\|_\infty \to 0$] as $k \to \infty$.*

Then the sequence $\{S_n(x)\}$ also converges a.u. [b.a.u.]

Using Theorems 2.1 and 2.6, we obtain a corollary:

COROLLARY 3.2. *Let $1 \le p < \infty$ [$2 \le p < \infty$]. Then the set*

$$\{x \in L_p : \{M_{\beta,n}(x)\} \text{ converges b.a.u.}\}$$
$$[\{x \in L_p : \{M_{\beta,n}(x)\} \text{ converges a.u.}\}]$$

is closed in L_p.

Proof. Denote $A = \{x \in L_p : \{M_{\beta,n}(x)\}$ converges b.a.u.$\}$. Fix $\epsilon > 0$. Theorem 2.1 implies that for every given $k \in \mathbb{N}$ there is $\gamma_k > 0$ such that for every $x \in L_p$ with $\|x\|_p < \gamma_k$ it is possible to find $e_{k,x} \in \mathcal{P}(\mathcal{M})$ for which

$$\tau(e_{k,x}^\perp) \le \frac{\epsilon}{2^k} \quad \text{and} \quad \sup_n \|e_{k,x}M_{\beta,n}(x)e_{k,x}\|_\infty \le \frac{1}{k}\ .$$

Let x be in the closure of A in L_p. Given k, let $y_k \in A$ satisfy $\|y_k - x\|_p < \gamma_k$. Denoting $y_k - x = x_k$, choose a sequence $\{e_k\} \subset \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_k^\perp) \le \frac{\epsilon}{2^k} \quad \text{and} \quad \sup_n \|e_kM_{\beta,n}(x_k)e_k\|_\infty \le \frac{1}{k}, \qquad k = 1, 2, \ldots.$$

Then $x + x_k = y_k \in A$ for every k. Also, letting $e = \bigwedge_{k\ge1} e_k$, we have

$$\tau(e^\perp) \le \epsilon \quad \text{and} \quad \sup_n \|eM_{\beta,n}(x_k)e\|_\infty \le \frac{1}{k}.$$

Consequently, Lemma 3.1 yields $x \in A$.

Analogously, applying Theorem 2.6 instead of Theorem 2.1, we obtain the remaining part of the statement. ∎

Corollary 3.2, in the particular case where $\beta_k \equiv 1$, allows us to present a new, direct proof of Theorem 1.5.

Proof of Theorem 1.5. Assume first that $p \geq 2$. Since the map T generates a contraction in the real Hilbert space $(L_2^h, (\cdot,\cdot)_\tau)$ [25, Proposition 1], where $(x,y)_\tau = \tau(xy)$, $x, y \in L_2^h$, it is easy to verify that the set

$$\mathcal{H}_0 = \{x \in L_2^h : T(x) = x\} + \{x - T(x) : x \in L_2^h\}$$

is dense in $(L_2^h, \|\cdot\|_2)$ (see for example [10, Ch. VIII, §5]). Therefore, because the set $L_2^h \cap \mathcal{M}$ is dense in L_2^h and T contracts L_p^h, we conclude that the set

$$\mathcal{H}_1 = \{x \in L_2^h : T(x) = x\} + \{x - T(x) : x \in L_h^2 \cap \mathcal{M}\}$$

is also dense in $(L_2^h, \|\cdot\|_2)$. Moreover, if $y = x - T(x)$, $x \in L_2^h \cap \mathcal{M}$, then the sequence $M_n(y) = (n+1)^{-1}(x - T^{n+1}(x))$ converges to zero with respect to the norm $\|\cdot\|_\infty$, hence a.u. Therefore $\mathcal{H}_1 + i\mathcal{H}_1$ is a dense (in L_2) subset on which the averages M_n converge a.u. This, by Corollary 3.2, implies that $\{M_n(x)\}$ converges a.u. for all $x \in L_2$. Further, since the set $L_p \cap L_2$ is dense in L_p, Corollary 3.2 implies that the sequence $\{M_n(x)\}$ converges a.u. for each $x \in L_p$ (to some $\widehat{x} \in L_0(\mathcal{M}, \tau)$). Then $\{M_n(x)\}$ converges to $\widehat{x}$ in measure. Since $M_n(x) \in L_p$ and $\|M_n(x)\|_p \leq 1$, $n = 1, 2, \ldots$, by [3, Theorem 1.2] we conclude that $\widehat{x} \in L_p$.

Let now $1 < p < \infty$. By the first part of the proof, the sequence $\{M_n(x)\}$ converges b.a.u. for all $x \in L_2$. But $L_p \cap L_2$ is dense in L_p, and Corollary 3.2 entails b.a.u. convergence of the averages $M_n(x)$ for all $x \in L_p$. Remembering that b.a.u. convergence implies convergence in measure (see Section 1), we conclude, as before, that $M_n(x) \to \widehat{x} \in L_p$ b.a.u. ∎

Let $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. A function $P : \mathbb{Z} \to \mathbb{C}$ is said to be a *trigonometric polynomial* if $P(k) = \sum_{j=1}^{s} z_j \lambda_j^k$, $k \in \mathbb{Z}$, for some $s \in \mathbb{N}$, $\{z_j\}_1^s \subset \mathbb{C}$, and $\{\lambda_j\}_{j=1}^s \subset \mathbb{C}_1$. A sequence $\{\beta_k\}_{k=0}^\infty \subset \mathbb{C}$ is called a *bounded Besicovitch sequence* if

(i) $|\beta_k| \leq C < \infty$ for all k;
(ii) for every $\epsilon > 0$ there exists a trigonometric polynomial P such that

$$\limsup_n \frac{1}{n+1} \sum_{k=0}^{n} |\beta_k - P(k)| < \epsilon.$$

Assume now that $\mathcal{M}$ has a separable predual. The reason for this assumption is that our argument essentially relies on [21, Theorem 1.22.13].

Since $L_1 \cap \mathcal{M} \subset L_2$, using Theorem 1.5 for $p = 2$ (or [5, Theorem 3.1]) and repeating the steps of the proof of [3, Lemma 4.2], we arrive at the following.

Proposition 3.3. *For any trigonometric polynomial P and $x \in L_1 \cap \mathcal{M}$, the averages*

$$\frac{1}{n+1} \sum_{k=0}^{n} P(k) T^k(x)$$

converge a.u.

Next, it is easy to verify the following (see the proof of [3, Theorem 4.4]).

Proposition 3.4. *If $\{\beta_k\}$ is a bounded Besicovitch sequence, then the averages* (2) *converge a.u. for every $x \in L_1 \cap \mathcal{M}$.*

Here is an extension of [3, Theorem 4.6] to L_p-spaces, $1 < p < \infty$.

Theorem 3.5. *Assume that $\mathcal{M}$ has a separable predual. Let $1 < p < \infty$, and let $\{\beta_k\}$ be a bounded Besicovitch sequence. Then for every $x \in L_p$ the averages* (2) *converge b.a.u. to some $\widehat{x} \in L_p$. If $p \geq 2$, these averages converge a.u.*

Proof. In view of Proposition 3.4 and Corollary 3.2, we only need to recall that the set $L_1 \cap \mathcal{M}$ is dense in L_p. The inclusion $\widehat{x} \in L_p$ follows as in the proof of Theorem 1.5. ∎

4. Individual ergodic theorems in noncommutative fully symmetric spaces. Let $x \in L_0(\mathcal{M}, \tau)$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x|$ of x. If $t > 0$, then the tth *generalized singular number* of x (see [11]) is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^{\perp}) \leq t\}.$$

A Banach space $(E, \|\cdot\|_E) \subset L_0(\mathcal{M}, \tau)$ is called *fully symmetric* if the conditions

$$x \in E, \quad y \in L_0(\mathcal{M}, \tau), \quad \int_0^s \mu_t(y)\, dt \leq \int_0^s \mu_t(x)\, dt \quad \text{for all } s > 0$$

imply that $y \in E$ and $\|y\|_E \leq \|x\|_E$. It is known [6] that if $(E, \|\cdot\|_E)$ is a fully symmetric space, $x_n, x \in E$, and $\|x - x_n\|_E \to 0$, then $x_n \to x$ in measure. A fully symmetric space $(E, \|\cdot\|_E)$ is said to have the *Fatou property* if the conditions

$$x_\alpha \in E^+, \quad x_\alpha \leq x_\beta \quad \text{for } \alpha \leq \beta, \quad \text{and} \quad \sup_\alpha \|x_\alpha\|_E < \infty$$

imply that $x = \sup_\alpha x_\alpha \in E$ exists and $\|x\|_E = \sup_\alpha \|x_\alpha\|_E$. The space $(E, \|\cdot\|_E)$ is said to have *order continuous norm* if $\|x_\alpha\|_E \downarrow 0$ whenever $x_\alpha \in E$ and $x_\alpha \downarrow 0$.

Let $L_0(0, \infty)$ be the linear space of all (equivalence classes of) almost everywhere finite complex-valued Lebesgue measurable functions on the interval $(0, \infty)$. We identify $L_\infty(0, \infty)$ with the commutative von Neumann algebra acting on the Hilbert space $L_2(0,\infty)$ via multiplication by the elements of $L_\infty(0,\infty)$ with the trace given by integration with respect to Lebesgue measure. A fully symmetric space $E\subset L_0(\mathcal{M},\tau)$, where $\mathcal{M}=L_\infty(0,\infty)$ and τ is given by the Lebesgue integral, is called a *fully symmetric function space* on $(0,\infty)$.

Let $E=(E(0,\infty),\|\cdot\|_E)$ be a fully symmetric function space. For each $s>0$ let $D_s: E(0,\infty)\to E(0,\infty)$ be the bounded linear operator given by $D_s(f)(t)=f(t/s),\ t>0$. The *Boyd indices* p_E and q_E are defined as

$$p_E=\lim_{s\to\infty}\frac{\log s}{\log\|D_s\|_E},\qquad q_E=\lim_{s\to+0}\frac{\log s}{\log\|D_s\|_E}.$$

It is known that $1\le p_E\le q_E\le\infty$ [17, II, Ch. 2, Proposition 2.b.2]. A fully symmetric function space is said to have *nontrivial Boyd indices* if $1<p_E$ and $q_E<\infty$. For example, the spaces $L_p(0,\infty)$, $1<p<\infty$, have nontrivial Boyd indices:

$$p_{L_p(0,\infty)}=q_{L_p(0,\infty)}=p$$

(see [1, Ch. 4, §4, Theorem 4.3]).

If $E(0,\infty)$ is a fully symmetric function space, define

$$E(\mathcal{M})=E(\mathcal{M},\tau)=\{x\in L_0(\mathcal{M},\tau):\mu_t(x)\in E\}$$

and set

$$\|x\|_{E(\mathcal{M})}=\|\mu_t(x)\|_E,\qquad x\in E(\mathcal{M}).$$

It is shown in [6] that $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ is a fully symmetric space. If $1\le p<\infty$ and $E=L_p(0,\infty)$, the space $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ coincides with the noncommutative L_p-space $(L_p(\mathcal{M},\tau),\|\cdot\|_p)$ because

$$\|x\|_p=\Big(\int_0^\infty\mu_t^p(x)\,dt\Big)^{1/p}=\|x\|_{E(\mathcal{M})}$$

(see [24, Proposition 2.4]).

It was shown in [4, Proposition 2.2] that if $\mathcal{M}$ is nonatomic, then every noncommutative fully symmetric $(E,\|\cdot\|_E)\subset L_0(\mathcal{M},\tau)$ is of the form $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ for a suitable fully symmetric function space $E(0,\infty)$.

Let $L_{p,q}(0,\infty)$, $1\le p,q<\infty$, be the classical Lorentz function space, that is, the space of all functions $f\in L_0(0,\infty)$ such that

$$\|f\|_{p,q}=\Big(\int_0^\infty(t^{1/p}\mu_t(f))^q\,\frac{dt}{t}\Big)^{1/q}<\infty.$$

It is known that for $q\le p$ the space $(L_{p,q}(0,\infty),\|\cdot\|_{p,q})$ is a fully symmetric function space with the Fatou property and order continuous norm. In addition, $L_{p,p}=L_p$. In the case $1<p<q$, $\|\cdot\|_{p,q}$ is a quasi-norm on $L_{p,q}(0,\infty)$, but there exists a norm $\|\cdot\|_{(p,q)}$ on $L_{p,q}(0,\infty)$ that is equivalent to the norm $\|\cdot\|_{p,q}$ and such that $(L_{p,q}(0,\infty),\|\cdot\|_{(p,q)})$ is a fully symmetric function

space with the Fatou property and order continuous norm [1, Ch. 4, §4]. In addition, if $1 \le q \le p < \infty$ $[1 < p < \infty,\ 1 \le q < \infty]$, then

$$p_{(L_{p,q}(0,\infty),\|\cdot\|_{p,q})} = q_{(L_{p,q}(0,\infty),\|\cdot\|_{p,q})} = p \qquad \text{([1, Ch. 4, §4, Theorem 4.3])}$$
$$[p_{(L_{p,q}(0,\infty),\|\cdot\|_{(p,q)})} = q_{(L_{p,q}(0,\infty),\|\cdot\|_{(p,q)})} = p \qquad \text{([1, Ch. 4, §4, Theorem 4.5])}].$$

Using the Lorentz function space $(L_{p,q}(0,\infty), \|\cdot\|_{p,q})$ $[(L_{p,q}(0,\infty), \|\cdot\|_{(p,q)})]$, one can define the *noncommutative Lorentz space*

$$L_{p,q}(\mathcal{M},\tau) = \left\{ x \in L_0(\mathcal{M},\tau) : \|x\|_{p,q} = \left(\int_0^\infty (t^{1/p}\mu_t(x))^q \, \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

that is fully symmetric with respect to the norm $\|\cdot\|_{p,q}$ for $1 \le q \le p$ $[\|\cdot\|_{(p,q)}$ for $q > p > 1]$. In addition, the norm $\|\cdot\|_{p,q}$ $[\|\cdot\|_{(p,q)}]$ is order continuous [7, Proposition 3.6] and satisfies the Fatou property [8, Theorem 4.1]. These spaces were first introduced in [15].

Following [16], a *Banach couple* (X, Y) is a pair of Banach spaces, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, which are algebraically and topologically embedded in a Hausdorff topological space. With any Banach couple (X, Y) the following Banach spaces are associated:

(i) the space $X \cap Y$ equipped with the norm

$$\|x\|_{X\cap Y} = \max\{\|x\|_X, \|x\|_Y\}, \qquad x \in X \cap Y;$$

(ii) the space $X + Y$ equipped with the norm

$$\|x\|_{X+Y} = \inf\{\|y\|_X + \|z\|_Y : x = y + z,\ y \in X,\ z \in Y\},\ x \in X + Y.$$

Let (X, Y) be a Banach couple. A linear map $T : X + Y \to X + Y$ is called a *bounded operator for the couple* (X, Y) if both $T : X \to X$ and $T : Y \to Y$ are bounded. Denote by $\mathcal{B}(X, Y)$ the linear space of all bounded linear operators for the couple (X, Y). Equipped with the norm

$$\|T\|_{\mathcal{B}(X,Y)} = \max\{\|T\|_{X\to X}, \|T\|_{Y\to Y}\},$$

it is a Banach space. A Banach space Z is said to be *intermediate for the Banach couple* (X, Y) if

$$X \cap Y \subset Z \subset X + Y$$

with continuous inclusions. If Z is intermediate for the Banach couple (X, Y), then it is called an *interpolation space for* (X, Y) if every bounded linear operator for the couple (X, Y) acts boundedly from Z to Z.

If Z is an interpolation space for a Banach couple (X, Y), then there exists a constant $C > 0$ such that $\|T\|_{Z\to Z} \le C\|T\|_{\mathcal{B}(X,Y)}$ for all $T \in \mathcal{B}(X, Y)$. An interpolation space Z for a Banach couple (X, Y) is called an *exact interpolation space* if $\|T\|_{Z\to Z} \le \|T\|_{\mathcal{B}(X,Y)}$ for all $T \in \mathcal{B}(X, Y)$.

Every fully symmetric function space $E = E(0,\infty)$ is an exact interpolation space for the Banach couple $(L_1(0,\infty), L_\infty(0,\infty))$ [16, Ch. II, §4, Theorem 4.3].

We need the following noncommutative interpolation result for the spaces $E(\mathcal{M})$.

THEOREM 4.1 ([6, Theorem 3.4]). *Let E, E_1, E_2 be fully symmetric function spaces on $(0,\infty)$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful semifinite normal trace. If (E_1, E_2) is a Banach couple and E is an exact interpolation space for (E_1, E_2), then $E(\mathcal{M})$ is an exact interpolation space for the Banach couple $(E_1(\mathcal{M}), E_2(\mathcal{M}))$.*

It follows now from [16, Ch. II, Theorem 4.3] and Theorem 4.1 that every noncommutative fully symmetric space $E(\mathcal{M})$, where $E = E(0,\infty)$ is a fully symmetric function space, is an exact interpolation space for the Banach couple $(L_1(\mathcal{M}), \mathcal{M})$.

Let $T \in \mathrm{DS}^+(\mathcal{M},\tau)$. Let $E(0,\infty)$ be a fully symmetric function space. Since the noncommutative fully symmetric space $E(\mathcal{M})$ is an exact interpolation space for the Banach couple $(L_1(\mathcal{M},\tau), \mathcal{M})$, we conclude that $T(E(\mathcal{M})) \subset E(\mathcal{M})$ and T is a positive linear contraction on $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$. Thus

$$M_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} T^k(x) \in E(\mathcal{M})$$

for each $x \in E(\mathcal{M})$ and all n. Moreover, the inequalities

$$\|T(x)\|_1 \le \|x\|_1, \quad x \in L_1, \qquad \|T(x)\|_\infty \le \|x\|_\infty, \quad x \in \mathcal{M},$$

imply that

$$\sup_{n\ge1} \|M_n\|_{L_1\to L_1} \le 1 \quad \text{and} \quad \sup_{n\ge1} \|M_n\|_{\mathcal{M}\to\mathcal{M}} \le 1.$$

Since the noncommutative fully symmetric space $E(\mathcal{M})$ is an exact interpolation space for the Banach couple $(L_1(\mathcal{M},\tau), \mathcal{M})$, we have

$$\sup_{n\ge1} \|M_n\|_{E(\mathcal{M})\to E(\mathcal{M})} \le 1. \tag{10}$$

Now, let $\{\beta_k\}_{k=0}^\infty \subset \mathbb{C}$ satisfy $|\beta_k| \le C$, $k = 1,2,\ldots$. As $0 \le \operatorname{Re}\beta_k + C \le 2C$ and $0 \le \operatorname{Im}\beta_k + C \le 2C$, it follows from (6) that

$$\sup_{n\ge1} \|M_{\beta,n}\|_{L_1\to L_1} \le 6C \quad \text{and} \quad \sup_{n\ge1} \|M_{\beta,n}\|_{\mathcal{M}\to\mathcal{M}} \le 6C.$$

Since the noncommutative fully symmetric space $E(\mathcal{M})$ is an exact interpolation space for the Banach couple $(L_1(\mathcal{M},\tau), \mathcal{M})$, we obtain

$$\sup_{n\ge1} \|M_{\beta,n}\|_{E(\mathcal{M})\to E(\mathcal{M})} \le 6C. \tag{11}$$

The following theorem is a version of Theorem 1.5 for noncommutative fully symmetric Banach spaces with nontrivial Boyd indices.

THEOREM 4.2. *Let $E(0,\infty)$ be a fully symmetric function space with the Fatou property and nontrivial Boyd indices. If $T \in \mathrm{DS}^+(\mathcal{M},\tau)$, then for any given $x \in E(\mathcal{M},\tau)$ the averages $M_n(x)$ converge b.a.u. to some $\widehat{x} \in E(\mathcal{M},\tau)$. If $p_{E(0,\infty)} > 2$, these averages converge a.u.*

Proof. Since $E(0,\infty)$ has nontrivial Boyd indices, according to [17, II, Ch. 2, Proposition 2.b.3], there exist $1 < p, q < \infty$ such that $E(0,\infty)$ is an intermediate space for the Banach couple $(L_p(0,\infty), L_q(0,\infty))$. Since

$$(L_p + L_q)(\mathcal{M},\tau) = L_p(\mathcal{M},\tau) + L_q(\mathcal{M},\tau)$$

(see [6, Proposition 3.1]), we have

$$E(\mathcal{M},\tau) \subset L_p(\mathcal{M},\tau) + L_q(\mathcal{M},\tau).$$

Then $x = x_1 + x_2$, where $x_1 \in L_p(\mathcal{M},\tau)$, $x_2 \in L_q(\mathcal{M},\tau)$, and, by Theorem 1.5, there exist $\widehat{x}_1 \in L_p(\mathcal{M},\tau)$ and $\widehat{x}_1 \in L_q(\mathcal{M},\tau)$ such that $M_n(x_j)$ converge b.a.u. to $\widehat{x}_j$, $j = 1, 2$. Therefore

$$M_n(x) \to \widehat{x} = \widehat{x}_1 + \widehat{x}_2 \in L_p(\mathcal{M},\tau) + L_q(\mathcal{M},\tau) \subset L_0(\mathcal{M},\tau)$$

b.a.u., hence $M_n(x) \to \widehat{x}$ in measure. Since $E(\mathcal{M})$ has the Fatou property, the unit ball of $E(\mathcal{M})$ is closed in the measure topology [8, Theorem 4.1], and (10) implies that $\widehat{x} \in E(\mathcal{M})$.

If $p_{E(0,\infty)} > 2$, then the numbers p and q can be chosen such that $2 < p, q < \infty$. Utilizing Theorem 1.5 and repeating the argument above, we conclude that the averages $M_n(x)$ converge to $\widehat{x}$ a.u. ∎

Following the proof of Theorem 4.2, we obtain its extended version:

THEOREM 4.3. *Let $E(0,\infty)$ be a fully symmetric function space with the Fatou property. If $T \in \mathrm{DS}^+$ and $x \in E(\mathcal{M},\tau)$ is such that $x = x_1 + \cdots + x_{n(x)}$, where $x_j \in L_{p_j(x)}(\mathcal{M},\tau)$ and $p_j(x) \geq 1$ for $j = 1, \ldots, n(x)$, then the averages $M_n(x)$ converge b.a.u. to some $\widehat{x} \in E(\mathcal{M},\tau)$. If $p_j(x) \geq 2$ for all $j = 1, \ldots, n(x)$, these averages converge a.u.*

Since any Lorentz function space $E = L_{p,q}(0,\infty)$ with $1 < p < \infty$ and $1 \leq q < \infty$ has nontrivial Boyd indices $p_E = q_E = p$, we have the following corollary of Theorem 4.2.

THEOREM 4.4. *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then, given $x \in L_{p,q}(\mathcal{M},\tau)$, the averages $M_n(x)$ converge b.a.u. to some $\widehat{x} \in L_{p,q}(\mathcal{M},\tau)$. If $p > 2$, these averages converge a.u.*

REMARK 4.5. If $1 \leq q \leq p$, then $L_{p,q}(\mathcal{M},\tau) \subset L_{p,p}(\mathcal{M},\tau) = L_p(\mathcal{M},\tau)$ (see [15] and [12, Lemma 1.6]). Then it follows directly from Theorem 1.5 along with the ending of the proof of the first part of Theorem 4.2 that for every $x \in L_{p,q}(\mathcal{M}, \tau)$ the averages $M_n(x)$ converge to some $\widehat{x} \in L_{p,q}(\mathcal{M}, \tau)$ b.a.u. (a.u. for $p \geq 2$).

The next theorem is a version of the Besicovitch weighted ergodic theorem for a noncommutative fully symmetric space $E(\mathcal{M}, \tau)$.

THEOREM 4.6. *Assume that $\mathcal{M}$ has a separable predual. Let $E(0, \infty)$ be a fully symmetric function space with the Fatou property and nontrivial Boyd indices. Let $\{\beta_k\}$ be a bounded Besicovitch sequence. If $T \in \mathrm{DS}^+(\mathcal{M}, \tau)$, then for any given $x \in E(\mathcal{M}, \tau)$ the averages $M_{\beta,n}(x)$ converge b.a.u. to some $\widehat{x} \in E(\mathcal{M}, \tau)$. If $p_{E(0,\infty)} > 2$, these averages converge a.u.*

The proof of Theorem 4.6 uses Theorem 3.5 and the inequality (11) and is analogous to the proof of Theorem 4.2.

Immediately from Theorem 4.6 we obtain the following individual ergodic theorem for the Lorentz spaces $L_{p,q}(\mathcal{M}, \tau)$ (cf. Theorem 4.4).

THEOREM 4.7. *Let $\mathcal{M}$ have a separable predual. If $1 < p < \infty$ and $1 \leq q < \infty$, then for any $x \in L_{p,q}(\mathcal{M}, \tau)$ the averages $M_{\beta,n}$ converge b.a.u. to some $\widehat{x} \in L_{p,q}(\mathcal{M}, \tau)$. If $p > 2$, these averages converge a.u.*

REMARK 4.8. If $1 \leq q \leq p$, then $L_{p,q}(\mathcal{M}, \tau) \subset L_p(\mathcal{M}, \tau)$, and it follows directly from Theorem 3.5 along with the ending of the proof of the first part of Theorem 4.2 that for every $x \in L_{p,q}(\mathcal{M}, \tau)$ the averages $M_{\beta,n}$ converge to some $\widehat{x} \in L_{p,q}(\mathcal{M}, \tau)$ b.a.u. (a.u. for $p \geq 2$).

5. Mean ergodic theorems in noncommutative fully symmetric spaces. Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. In [24] the following mean ergodic theorem for noncommutative fully symmetric spaces was proven.

THEOREM 5.1. *Let $E(\mathcal{M})$ be a noncommutative fully symmetric space such that:*

(i) *$L_1 \cap \mathcal{M}$ is dense in $E(\mathcal{M})$;*
(ii) *$\|e_n\|_{E(\mathcal{M})} \to 0$ for any sequence of projections $\{e_n\} \subset L_1 \cap \mathcal{M}$ with $e_n \downarrow 0$;*
(iii) *$\|e_n\|_{E(\mathcal{M})}/\tau(e_n) \to 0$ for any increasing sequence $\{e_n\} \subset L_1 \cap \mathcal{M}$ of projections with $\tau(e_n) \to \infty$.*

Then, given $x \in E(\mathcal{M})$ and $T \in \mathrm{DS}^+(\mathcal{M}, \tau)$, there exists $\widehat{x} \in E(\mathcal{M})$ such that $\|\widehat{x} - M_n(x)\|_{E(\mathcal{M})} \to 0$.

It is clear that any noncommutative fully symmetric space $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$ with order continuous norm satisfies conditions (i) and (ii) of Theorem 5.1. Moreover, in the case of a noncommutative Lorentz space $L_{p,q}(\mathcal{M}, \tau)$, the inequality $p > 1$ together with

$$\|e\|_{p,q} = \left(\frac{p}{q}\right)^{1/q} \tau(e)^{1/p}, \qquad e \in L_1 \cap \mathcal{P}(\mathcal{M}),$$

implies that condition (iii) is also satisfied. Therefore Theorem 5.1 entails the following.

COROLLARY 5.2. *Let $1 < p < \infty$, $1 \le q < \infty$, $T \in \mathrm{DS}^+$, and $x \in L_{p,q}(\mathcal{M},\tau)$. Then there exists $\widehat{x} \in L_{p,q}(\mathcal{M},\tau)$ such that $\|\widehat{x} - M_n(x)\|_{p,q} \to 0$.*

The next theorem asserts convergence in the norm $\|\cdot\|_{E(\mathcal{M})}$ of the averages $M_n(x)$ for any noncommutative fully symmetric space $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$ with order continuous norm, under the assumption that $\tau(\mathbb{I}) < \infty$.

THEOREM 5.3. *Let τ be finite, and let $E(\mathcal{M},\tau)$ be a noncommutative fully symmetric space with order continuous norm. Then for any $x \in E(\mathcal{M})$ and $T \in \mathrm{DS}^+$ there exists $\widehat{x} \in E(\mathcal{M})$ such that $\|\widehat{x} - M_n(x)\|_{E(\mathcal{M})} \to 0$.*

Proof. Since the trace τ is finite, we have $\mathcal{M} \subset E(\mathcal{M},\tau)$. As the norm $\|\cdot\|_{E(\mathcal{M})}$ is order continuous, applying the spectral theorem for selfadjoint operators in $E(\mathcal{M},\tau)$, we conclude that $\mathcal{M}$ is dense in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$. Therefore $\mathcal{M}^+$ is a fundamental subset of $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$, that is, the linear span of $\mathcal{M}^+$ is dense in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$.

We will now show that the sequence $\{M_n(x)\}$ is relatively weakly sequentially compact for every $x \in \mathcal{M}^+$. Without loss of generality, assume that $0 \le x \le \mathbb{I}$. Since $T \in \mathrm{DS}^+$, we have $0 \le M_n(x) \le M_n(\mathbb{I}) \le \mathbb{I}$ for any n. By [9, Proposition 4.3], given $y \in E^+(\mathcal{M},\tau)$, the set $\{a \in E(\mathcal{M},\tau) : 0 \le a \le y\}$ is weakly compact in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$, which implies that the sequence $\{M_n(x)\}$ is relatively weakly sequentially compact in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$.

Since $\sup_{n\ge 1} \|M_n\|_{E(\mathcal{M})\to E(\mathcal{M})} \le 1$ (see (10)) and

$$0 \le \left\|\frac{T^n(x)}{n}\right\|_{E(\mathcal{M})} \le \frac{\|x\|_{E(\mathcal{M})}}{n} \to 0$$

whenever $x \in \mathcal{M}^+$, the result follows by [10, Ch. VIII, §5, Corollary 3]. ∎

REMARK 5.4. In the commutative case, Theorem 5.3 was established in [23]. It was also shown that if $\mathcal{M} = L_\infty(0,1)$, then for every fully symmetric Banach function space $E(0,1)$ with the norm that is not order continuous there exist $T \in \mathrm{DS}^+$ and $x \in E(\mathcal{M})$ such that the averages $M_n(x)$ do not converge in $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$.

The following proposition is a version of Theorem 5.1 for a noncommutative fully symmetric space with order continuous norm with condition (iii) being replaced by nontriviality of the Boyd indices of $E(0,\infty)$. Note that we do not require T to be positive.

PROPOSITION 5.5. *Let $E(0,\infty)$ be a fully symmetric function space with nontrivial Boyd indices and order continuous norm. Then for any*

$x \in E(\mathcal{M},\tau)$ and $T \in \mathrm{DS}(\mathcal{M},\tau)$ there exists $\widehat{x} \in E(\mathcal{M},\tau)$ such that $\|\widehat{x} - M_n(x)\|_{E(\mathcal{M})} \to 0$.

Proof. By [17, Theorem 2.b.3], it is possible to find $1 < p, q < \infty$ such that

$$L_p(0,\infty) \cap L_q(0,\infty) \subset E(0,\infty) \subset L_p(0,\infty) + L_q(0,\infty)$$

with continuous inclusion maps. In particular,

$$\|f\|_{E(0,\infty)} \leq C\|f\|_{L_p(0,\infty)\cap L_q(0,\infty)}$$

for all $f \in L_p(0,\infty) \cap L_q(0,\infty)$ and some $C > 0$. Hence

$$\|x\|_{E(\mathcal{M},\tau)} \leq C\|x\|_{L_p(\mathcal{M},\tau)\cap L_q(\mathcal{M},\tau)}$$

for all $x \in \mathcal{L} := L_p(\mathcal{M},\tau) \cap L_q(\mathcal{M},\tau)$. Therefore the space $\mathcal{L}$ is continuously embedded in $E(\mathcal{M},\tau)$. Furthermore, it follows as in Theorem 5.3 that $\mathcal{L}$ is a fundamental subset of $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$.

We will show that for every $x \in \mathcal{L}$ the sequence $\{M_n(x)\}$ is relatively weakly sequentially compact in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$. Since $p, q > 1$, the spaces $L_p(\mathcal{M},\tau)$ and $L_q(\mathcal{M},\tau)$ are reflexive. As $T \in \mathrm{DS}$ and $x \in L_p(\mathcal{M},\tau) \cap L_q(\mathcal{M},\tau)$, we conclude that the averages $\{M_n(x)\}$ converge in $(L_p(\mathcal{M},\tau), \|\cdot\|_p)$ and in $(L_q(\mathcal{M},\tau), \|\cdot\|_q)$ to $\widehat{x}_1 \in L_p(\mathcal{M},\tau)$ and to $\widehat{x}_2 \in L_q(\mathcal{M},\tau)$, respectively [10, Ch.VIII, §5, Corollary 4]. This implies that the sequence $\{M_n(x)\}$ converges to $\widehat{x}_1$ and to $\widehat{x}_2$ in measure, hence $\widehat{x}_1 = \widehat{x}_2 := \widehat{x}$. Since $\mathcal{L}$ is continuously embedded in $E(\mathcal{M},\tau)$, the sequence $\{M_n(x)\}$ converges to $\widehat{x}$ with respect to the norm $\|\cdot\|_{E(\mathcal{M})}$, thus, it is relatively weakly sequentially compact in $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M})})$.

Now we can proceed as in the ending of the proof of Theorem 5.3. ∎

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