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Sharp estimates of the Green function of hyperbolic Brownian motion

by

Kamil Bogus (Wrocław),
Tomasz Byczkowski (Warszawa and Wrocław) and
Jacek Małecki (Wrocław)

Abstract. The main objective of the work is to provide sharp two-sided estimates of the λ-Green function, $\lambda \geq 0$, of the hyperbolic Brownian motion of a half-space. We rely on the recent results obtained by K. Bogus and J. Małecki (2015), regarding precise estimates of the Bessel heat kernel for half-lines. We also substantially use the results of H. Matsumoto and M. Yor (2005) on distributions of exponential functionals of Brownian motion.

1. Introduction. For a given subdomain $D \subseteq \mathbb{R}^n$ the Poisson kernel and the Green function G_D of D are one of the most important analytical objects on D as they provide solutions (under suitable regularity assumptions on D and boundary conditions) of the Dirichlet problem on D. One of the features of probabilistic potential theory is that we can construct these objects using the Brownian motion killed at the first exit time from D. In particular, $G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt$, where $p_D(t,x,y)$ is the transition probability density function of the process killed when leaving D (the heat kernel of D). The Green function of D is very useful when considering the Schrödinger operator $S = \Delta + q$ based on the Laplace operator Δ and examining for which potentials q we get a feasible potential theory of S on D (see e.g. [10]). It is also useful when considering the so-called conditional process and its potential theory [10].

The λ-Green function of D is in turn defined by the formula $G_D^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_D(t,x,y)\,dt$ and it is a more complicated object to deal with; but, in principle, it contains all the information about the heat equation considered, in particular pertaining to the heat kernel itself. Unfortunately, this information is not easy to recover, even if we have an explicit representation of the

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λ-Green function for $\lambda \geq 0$. Traditions of providing estimates of the Green function are long and go back to [9] for the case of the Brownian motion and a ball in $\mathbb{R}^n$, even though the explicit formula for the Green function is then very transparent. For another stochastic process, the so-called isotropic stable process, see [17] and [8]. The estimates are also invaluable when we deal with more general domains, e.g. domains with smooth boundaries, since then explicit formulas are no longer available.

Several people carried out a similar program for various processes and domains, including the hyperbolic Brownian motion (HBM) (see e.g. [18], [22], [24], [14]). HBM is the canonical diffusion on the hyperbolic space, with half the Laplace–Beltrami operator as generator. It is related to many important objects, mainly to Bessel processes and also to some functionals important in mathematical finance [11], [16]. Investigations of the hyperbolic Brownian motion have long traditions; for a comprehensive survey, see e.g. [20] or [4], [7], [6], [15], [19], [23], [25]. In recent years there has been a renewed interest in studying processes exiting a given domain.

The core result for the case of HBM and half-spaces is contained in [4] (see also [7]). It provides a closed formula for the Poisson kernel of a half-space, which enables computing its precise asymptotics. Unfortunately, since the formula is rather complex, neither a reasonable formula nor estimates for the Green function were within reach. To overcome this difficulty, we first established estimates for the density function of the distribution of the hitting times of Bessel processes (see [5]). Then in [2] we gave rather precise estimates for the Bessel heat kernel on a half-line. More specifically, these are estimates for the transition density function $p_a^{(\nu)}(u,x,y)$ of the Bessel process $BES^{(\nu)}$, starting from a point $x > 0$ and killed at the first hitting time of another point $a > 0$, with $x > a > 0$. We briefly call $p_a^{(\nu)}(u,x,y)$ the *Bessel heat kernel*.

Our first result (Theorem 3.5) brings a suitable representation of the λ-Green function G^λ of the hyperbolic Brownian motion on the half-space $\{x = (x_1, \ldots, x_n) : x_n > a\}$, $a > 0$. Even for $\lambda = 0$ this result alone is interesting from the theoretical point of view, as it establishes a somewhat unexpected, and to our knowledge new, kind of relationship between two rather different analytical objects: the Green function of a half-space for HBM, which is a complex multi-dimensional diffusion, and the Bessel heat kernel for a half-line, which concerns a thoroughly investigated, one-dimensional diffusion.

In the second result (Theorem 4.2) we employ the above-mentioned representation and the above mentioned result of [2] (see also [3]) to provide precise two-sided estimates for the λ-Green function of HBM for half-spaces. Such estimates have not been available in the context of hyperbolic spaces until now, even for $\lambda = 0$.

We first introduce some notation. If $n > 2$ denotes the dimension of the hyperbolic space, and $\lambda \geq 0$, we set $\mu = (n-1)/2$ and $\nu = \sqrt{2\lambda + \mu^2}$. Furthermore, $x = (\tilde{x}, x_n)$, $y = (\tilde{y}, y_n)$, $x_n, y_n > a$, where $\tilde{x} = (x_1, \dots x_{n-1}) \in \mathbb{R}^{n-1}$.

The main results of the paper are the following:

- *Representation of G^λ:*

$$G^\lambda(x,y) = (x_n y_n)^{\mu-\nu} \int_0^\infty \frac{1}{(2\pi u)^{(n-1)/2}} \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2u}\right) p_a^{(-\nu)}(u, x_n, y_n)\, du,$$

where $p_a^{(-\nu)}(u, x_n, y_n)$ is the appropriate Bessel heat kernel.

- *Estimates of G^λ:* For every $n > 2$, $\lambda \geq 0$ and $a \geq 0$ we have

$$G^\lambda(x,y) \overset{\lambda,n}{\approx} \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu-1/2} \left(1 \wedge \frac{2(x_n-a)(y_n-a)}{|x-y|^2}\right) \left(1 \wedge \frac{2x_n y_n}{|x-y|^2}\right)^{\nu-1/2}$$

whenever $x_n, y_n > a$, where $\mu = (n-1)/2$. Here $\overset{\lambda,n}{\approx}$ means that the ratio of the two sides is bounded from below and above by positive constants depending only on λ and n.

The main difficulty in studying potential theory on the set $D_a = \{x \in \mathbb{H}^n : x_n > a\}$ is that D_a is unbounded in $\mathbb{H}^n$ with the hyperbolic geometry. Note that the boundedness of a set in $\mathbb{H}^n$ means that it is bounded as a subset of $\mathbb{R}^n$ and is bounded away from the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$. It is also well-known that the Laplace–Beltrami operator on bounded sets in $\mathbb{H}^n$ is strongly elliptic, i.e. the appropriate coefficients are bounded away from zero and infinity. The general theory then implies that the potential theory for such an operator is comparable to the one of the classical Laplace operator. In particular, the λ-Green functions are then comparable. Obviously, this is not the case for D_a (unbounded), as we can see in the above estimate. More precisely, recall the formula and estimates of the classical Green function of the half-space D_a:

$$\begin{aligned} G^0_{\text{gauss}}(x,y) &= C(n)\left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y^*|^{n-2}}\right] \\ &\overset{n}{\approx} \left(\frac{1}{|x-y|^2}\right)^{\mu-1/2} \left(1 \wedge \frac{2(x_n-a)(y_n-a)}{|x-y|^2}\right), \end{aligned}$$

where $y^* = (y_1, \dots, y_{n-1}, 2a - y_n)$ and $\mu = (n-1)/2$ as previously. Thus the difference in the behaviour of the classical and hyperbolic Green function is described by

$$\frac{G^0(x,y)}{G^0_{\text{gauss}}(x,y)} \overset{n}{\approx} (x_n y_n)^{(n-2)/2} \left(1 \wedge \frac{2x_n y_n}{|x-y|^2}\right)^{(n-2)/2}.$$

This relation enables us to see how the estimate of the hyperbolic Green function works. For example, assume that $x_n y_n / |x-y|^2$ is large (tends to infinity), which means that the hyperbolic distance between x and y is small (tends to zero)—see (2.1). If additionally x_n and y_n are bounded, then $G^0(x, y)$ and $G^0_{\text{gauss}}(x, y)$ are comparable. But if x_n or y_n tends to infinity, then the hyperbolic Green function is much larger than the classical one. This reflects the geometry of the space and the nature of HBM: it is easier for HBM to get from x to y without touching $\{x \in \mathbb{H}^n : x_n = a\}$ in this case, since there is much more room in $\mathbb{H}^n$ than in $\mathbb{R}^n$ whenever $x_n \to \infty$. However, if we consider the case when the hyperbolic distance between points is large, then we have comparability of the Green functions if $x_n y_n \approx |x - y|$; the domination of the hyperbolic Green function when $x_n y_n \gg |x - y|$; and the opposite situation for $x_n y_n \ll |x - y|$. Here we can observe the consequences of the fact that HBM is forced to move to the boundary $\{x \in \mathbb{R}^n : x_n = 0\}$ (the last coordinate of HBM is geometric Brownian motion which tends to zero as $t \to \infty$, see Figures 1 and 2).

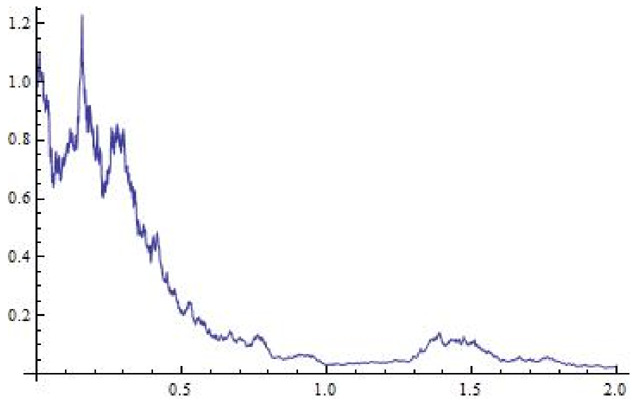

Fig. 1. A plot of a geometric Brownian motion trajectory with $\nu = 2$

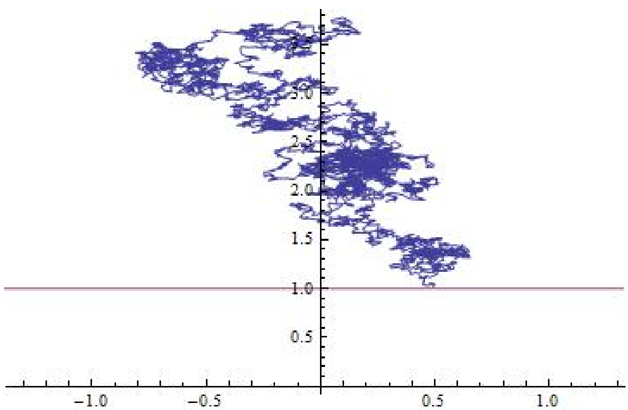

Fig. 2. A trace of a trajectory of HBM in $\mathbb{H}^2$ starting from $(0, 3)$ and killed at σ_1

Thus, if $x_n y_n$ is not large enough, it is more difficult for HBM to move from x to y without exiting D_a, because of the drift pushing it to the boundary.

Throughout the paper we strongly rely on the representation of the joint density of an integral functional $A_t^{(-\nu)} = \int_0^t \exp 2(B_s^{(-\nu)})\,ds$ and the geometric Brownian motion with drift $\exp(B_t^{(-\nu)}) = \exp(B_t - \nu t)$, given in [20]. An important point in our construction is the application of Lamperti's representation stating that the last coordinate of the hyperbolic Brownian motion, that is, the geometric Brownian motion $\exp(B_t^{(-\nu)})$, can be identified with the Bessel process $BES^{(-\nu)}$ with time changed according to the functional $A_t^{(-\nu)}$ (see below).

For the reader's convenience we list the symbols used throughout the paper.

Symbol	Description
$d_{\mathbb{H}^n}(x,y)$	hyperbolic distance between $x, y \in \mathbb{H}^n$
dV_n	hyperbolic volume element
$\Delta_{\mathbb{H}^n}$	Laplace–Beltrami operator in $\mathbb{H}^n$
Δ_ν	Laplace–Beltrami operator with drift
$X(t)$, $X^\nu(t)$	HBM and HBM with drift ν in $\mathbb{H}^n$
$U^\lambda(x,y)$, $U_\nu^\lambda(x,y)$	λ-potential kernel for HBM and for HBM with drift ν
$U(x,y)$, $U_\nu(x,y)$	potential kernel for HBM and for HBM with drift ν
σ_a	first exit time from $D_a = \{x \in \mathbb{H}^n : x_n > a\}$, $a > 0$
$G^\lambda(x,y)$, $G_\nu^\lambda(x,y)$	λ-Green function for HBM and for HBM with drift ν killed at σ_a
$B_t^{(-\mu)}$	Brownian motion with drift $-\mu t$
$A_t^{(\mu)}$	integral functional $A_t^{(\mu)} = \int_0^t \exp(2B_s^{-(\mu)})\,ds$
$\mathcal{Q}_\mu^\lambda(x,y;t)$	λ-potential of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$
$\mathcal{Q}_\mu(x,y;t)$	potential of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$
τ_a	first hitting time of $a > 0$ by the process $\exp(-B_t^{(-\mu)})$
$q_\mu^{x,a}(t) = q_\mu^x(t)$	density function of the r.v. $A_{\tau_a}^{(-\mu)}$ with $\exp(-B_0^{(-\mu)}) = x$
$\mathcal{G}_\mu^\lambda(x,y;t)$	λ-Green function of $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ killed at τ_a
$\mathcal{G}_\mu(x,y;t)$	Green function of $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ killed at τ_a
R_t	Bessel process
$p^{(\mu)}(t,x,y)$	transition pdf of Bessel process with index μ starting from x
$\mathbf{P}_x^{(\mu)}$, $\mathbf{E}_x^{(\mu)}$	prob. law and expected value related to $p^{(\mu)}(t,x,y)$
T_a	first hitting time of a level a by Bessel process
$p_a^{(-\nu)}(t,x,y)$	transition pdf of Bessel process with index $-\nu$ killed at T_a

2. Hyperbolic Brownian motion in $\mathbb{H}^n$. We consider the half-space model of the n-dimensional real hyperbolic space

$$\mathbb{H}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}, \qquad n \geq 2,$$

with the Riemannian metric

$$ds^2 = \frac{dx_1^2 + \dots + dx_{n-1}^2 + dx_n^2}{x_n^2}.$$

The metric induces the hyperbolic distance on $\mathbb{H}^n$ described by

$$\cosh(d_{\mathbb{H}^n}(x,y)) = 1 + \frac{|x-y|^2}{2x_n y_n}, \qquad x,y \in \mathbb{H}^n \tag{2.1}$$

and the corresponding canonical (hyperbolic) volume element

$$dV_n = \frac{dx_1 \dots dx_{n-1} dx_n}{x_n^n},$$

where $dx_1 \dots dx_{n-1} dx_n$ denotes the Lebesgue measure in $\mathbb{R}^n$. The Laplace–Beltrami operator then takes the form

$$\Delta_{\mathbb{H}^n} = x_n^2 \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} - (n-2)x_n \frac{\partial}{\partial x_n}.$$

Consider the following system of stochastic differential equations:

$$\begin{cases} dX_1 = X_n dB_1, \\ \quad \dots \\ dX_{n-1} = X_n dB_{n-1}, \\ dX_n = X_n dB_n - \dfrac{n-2}{2} X_n dt, \end{cases} \tag{2.2}$$

where $B = (B_1, \dots, B_{n-1}, B_n)$ denotes the standard Brownian motion in $\mathbb{R}^n$. The unique solution $X = (X_1, \dots, X_n)$ of the above-given system of SDEs with the initial condition $X(0) \in \mathbb{H}^n$ is called the *hyperbolic Brownian motion* (HBM). Using Itô's formula, one can easily check that the generator of X defined above is just $\frac{1}{2}\Delta_{\mathbb{H}^n}$. Since the Laplace–Beltrami operator is an analogue of the classical Laplacian in $\mathbb{R}^n$, HBM plays the same rôle in $\mathbb{H}^n$ as the Brownian motion in Euclidean space. If we replace the constant $(n-2)/2$ in the drift part in the last equation of (2.2) by $\nu - 1/2$, $\nu > 0$, then the solution $X^\nu = (X_1^\nu, \dots, X_n^\nu)$ is called the *hyperbolic Brownian motion with drift*. Once again using Itô's formula, we find that the generator of X^ν is half the operator

$$\Delta_\nu = x_n^2 \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} - (2\nu-1)x_n \frac{\partial}{\partial x_n}$$

and obviously for $\nu = (n-1)/2$ we have $\Delta_\nu = \Delta_{\mathbb{H}^n}$ and we recover HBM, i.e. $X^\nu = X$. The stochastic description of HBM with drift yields a useful representation of the process by the classical Brownian motion with time changed by means of an integral functional of geometric Brownian motion, and the geometric Brownian motion itself. Indeed, the equation defining the last coordinate of X^ν,

$$dX_n^\nu = X_n^\nu dB_n - \left(\nu - \frac{1}{2}\right) X_n^\nu dt,$$

is the one describing the geometric Brownian motion, i.e.

$$X_n^\nu(t) = \exp(B_n^{(-\nu)}(t)), \quad \text{where} \quad B_n^{(-\nu)}(t) = B_n(t) - \nu t.$$

Moreover, the first $n-1$ coordinates of X^ν is just the Brownian motion $\tilde{B} = (B_1, \ldots, B_{n-1})$ (which is obviously independent of B_n) with time changed by means of the integral functional

$$A_t^{(-\nu)} = \int_0^t \exp(2B_n^{(-\nu)}(s))\, ds,$$

which depends only on B_n. This shows that for $X^\nu(0) = (\tilde{x}, x_n)$, $x_n > 0$, we have

$$X^\nu(t) = (\tilde{X}^\nu(t), X_n^\nu(t)) = (\tilde{B}(A_t^{(-\nu)}), \exp(B_n^{(-\nu)}(t))),$$

where $(\tilde{B}(0), B_n(0)) = (\tilde{x}, \log x_n) \in \mathbb{R}^n$.

3. Representations of the hyperbolic Green function of a horocycle. The above mentioned representation of HBM with drift in terms of the geometric Brownian motion and its integral functional focus our attention on studying the λ-Green functions of the two-dimensional process $(A^{(-\mu)}, \exp(B^{(-\mu)}))$, where

$$A_t^{(-\nu)} = \int_0^t \exp(2B^{(-\nu)}(s))\, ds, \qquad B^{(-\mu)} = B(t) - \mu t;$$

here $B = (B(t))_{t \ge 0}$ is the one-dimensional Brownian motion. The connections between the λ-Green functions of a horocycle for HBM with drift and the ones studied below are explained in Theorem 3.4.

3.1. Green function of $(A^{(-\mu)}, \exp(B^{(-\mu)}))$**.** We begin by providing a formula for the λ-potential of the process $(A^{(-\mu)}, \exp(B^{(-\mu)}))$ together with its Laplace transform. We have not been able to find this result in the literature, so for the completeness of exposition and for the reader's convenience we include it together with the proof in the following proposition.

PROPOSITION 3.1. *The λ-potential of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ is*

$$\mathcal{Q}_\mu^\lambda(x, y; u) = \frac{1}{y}\left(\frac{x}{y}\right)^\mu \exp\left(-\frac{x^2 + y^2}{2u}\right) I_\nu\left(\frac{xy}{u}\right)\frac{1}{u} = \left(\frac{x}{y}\right)^{\mu-\nu} \mathcal{Q}_\nu^0(x, y; u),$$

where $\nu = \sqrt{2\lambda + \mu^2}$. The Laplace transform of $\mathcal{Q}_\mu^\lambda$ is

$$\mathcal{L}\mathcal{Q}_\mu^\lambda(x, y; \cdot)(r^2/2) = 2(xy)^\mu I_\nu(rx) K_\nu(ry) m^{(-\mu)}(y), \tag{3.1}$$

where $m^{(-\mu)}(y) = y^{-2\mu-1}$ denotes the density of the reference measure $dm^{(-\mu)}$.

Proof. Taking into account formulas (5.6) and (5.4) of the Appendix, the λ-potential $\tilde{\mathcal{Q}}_\mu^\lambda(x,y;u)$ of $(A_t^{(-\mu)}, B_t^{(-\mu)})$ is given by

$$\int_0^\infty e^{-\lambda t}\mathbf{P}^x(A_t^{(-\mu)}\in du, B_t^{(-\mu)}\in dy)\,dt = \frac{e^{-\mu(y-x)}}{u}\exp\Big(-\frac{e^{2x}+e^{2y}}{2u}\Big)I_\nu\Big(\frac{e^xe^y}{u}\Big), \tag{3.2}$$

where $\nu=\sqrt{2\lambda+\mu^2}$.

Using the formula for the product of modified Bessel functions [12, p. 64, (37)] we can compute the Laplace transform of $\tilde{\mathcal{Q}}_\mu^\lambda(x,y;u)$ as a function of u in the following way:

$$\begin{aligned}\mathcal{L}\tilde{\mathcal{Q}}_\mu^\lambda(x,y;\cdot)(r^2/2) &= \int_0^\infty e^{-r^2u/2}\int_0^\infty e^{-\lambda t}P^x(A_t^{(-\mu)}\in du, B_t^{(-\mu)}\in dy)\,dt\,du\\ &= \Big(\frac{e^x}{e^y}\Big)^\mu\int_0^\infty e^{-r^2u/2}\exp\Big(-\frac{e^{2x}+e^{2y}}{2u}\Big)I_\nu\Big(\frac{e^xe^y}{u}\Big)\frac{1}{u}\,du\\ &= \Big(\frac{e^x}{e^y}\Big)^\mu\int_0^\infty e^{-\xi/2}\exp\Big(-\frac{(re^x)^2+(re^y)^2}{2\xi}\Big)I_\nu\Big(\frac{re^xre^y}{\xi}\Big)\frac{d\xi}{\xi}\\ &= 2\Big(\frac{e^x}{e^y}\Big)^\mu I_\nu(re^x)K_\nu(re^y).\end{aligned}$$

Changing variables, we compute the λ-potential $\mathcal{Q}_\mu^\lambda(x,y;u)$ of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$. For $\exp(B_t^{(-\mu)})$ starting from x we have

$$\begin{aligned}\mathbf{P}^x(A_t^{(-\mu)}\in du, \exp(B_t^{(-\mu)})\in dy) &= \frac{1}{y}\mathbf{P}^{\log x}(A_t^{(-\mu)}\in du, B_t^{(-\mu)}\in d\log y)\\ &= e^{-\mu^2t/2}\frac{1}{y}\Big(\frac{x}{y}\Big)^\mu\exp\Big(-\frac{x^2+y^2}{2u}\Big)\theta\Big(\frac{xy}{u},t\Big)\frac{1}{u}\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_\mu^\lambda(x,y;u) &= \int_0^\infty e^{-\lambda t}\mathbf{P}^x(A_t^{(-\mu)}\in du, \exp(B_t^{(-\mu)})\in dy)\,dt\\ &= \frac{1}{y}\Big(\frac{x}{y}\Big)^\mu\exp\Big(-\frac{x^2+y^2}{2u}\Big)I_\nu\Big(\frac{xy}{u}\Big)\frac{1}{u}.\end{aligned}$$

The function $\mathcal{Q}_\nu^\lambda(\cdot,\cdot;u)$ is symmetric with respect to the reference measure $dm^{(-\mu)}$ with density

$$m^{(-\mu)}(y)=\frac{1}{y^{2\mu+1}}.$$

When $\lambda=0$, we write $\mathcal{Q}_\mu(x,y;u)$ instead of $\mathcal{Q}_\mu^\lambda(x,y;u)$. Thus, taking into account the previous formula we have

$$\mathcal{Q}^{\lambda}_{\mu}(x,y;u) = \left(\frac{x}{y}\right)^{\mu-\nu} \mathcal{Q}_{\nu}(x,y;u).$$

The Laplace transform of $u \mapsto \mathcal{Q}^{\lambda}_{\mu}(x,y;u)$ is

$$\begin{aligned}
\mathcal{L}\mathcal{Q}^{\lambda}_{\mu}(x,y;\cdot)(r^2/2) &= \int_0^{\infty} e^{-r^2u/2}\mathcal{Q}^{\lambda}_{\mu}(x,y;u)\,du \\
&= \int_0^{\infty} e^{-r^2u/2}\frac{1}{y}\left(\frac{x}{y}\right)^{\mu} \exp\left(-\frac{x^2+y^2}{2u}\right) I_{\nu}\left(\frac{xy}{u}\right)\frac{du}{u} \\
&= \frac{2}{y}\left(\frac{x}{y}\right)^{\mu} I_{\nu}(rx)K_{\nu}(ry) \\
&= 2(xy)^{\mu} I_{\nu}(rx)K_{\nu}(ry)m^{(-\mu)}(y). \ \blacksquare
\end{aligned}$$

To obtain formulas for the Green function we have to deal with τ_a, the first hitting time of the level a by the process $\exp(B^{(-\mu)}_t)$ starting from $x > a$. More precisely, we rather need to consider the density function $q^{x,a}_{\mu}(s)$ (where x is the starting point of $\exp(B^{(-\mu)}_t)$) of the stopped integral functional

$$A^{(-\mu)}_{\tau_a} = \int_0^{\tau_a} \exp(2B^{(-\mu)}_s)\,ds.$$

We have

$$q^{x,a}_{\mu}(t) = \frac{1}{2}\tilde{q}^{x,a}_{\mu}(t/2),$$

where $\tilde{q}^{x,a}_{\mu}(t)$ denotes the density investigated in [7]. The density $q^{x,a}_{\mu}(t)$ has the scaling property

$$q^{x,a}_{\mu}(u) = \frac{1}{a^2} q^{x/a,1}_{\mu}(u/a^2).$$

Considering an appropriate Schrödinger equation, we can write the Laplace transform of $A^{(-\mu)}_{\tau_a}$ as follows:

$$\mathbf{E}^x\left[\exp\left(-\frac{r^2}{2}A^{(-\mu)}_{\tau_a}\right)\right] = \left(\frac{x}{a}\right)^{\mu}\frac{K_{\mu}(rx)}{K_{\mu}(ra)} = \int_0^{\infty} \exp\left(-\frac{r^2}{2}s\right) q^x_{\mu}(s)\,ds.$$

Moreover, for every $\lambda \geq 0$ we get in a similar way

$$\mathbf{E}^x\left[\exp(-\lambda\tau_a)\exp\left(-\frac{r^2}{2}A^{(-\mu)}_{\tau_a}\right)\right] = \left(\frac{x}{a}\right)^{\mu}\frac{K_{\nu}(rx)}{K_{\nu}(ra)}, \tag{3.3}$$

where $\nu = \sqrt{2\lambda+\mu^2}$. Further on we assume that the point a in the definition of $q_\mu^{x,a}(t)$ is fixed so we omit this superscript. Taking into account (3.3) we obtain, for any positive Borel function f,

$$\begin{aligned}\mathbf{E}^x[\exp(-\lambda\tau_a); f(A_{\tau_a}^{(-\mu)})] &= \left(\frac{x}{a}\right)^{\mu-\nu}\mathbf{E}^x[f(A_{\tau_a}^{(-\nu)})] \\ &= \left(\frac{x}{a}\right)^{\mu-\nu}\int_0^\infty f(s)q_\nu^x(s)\,ds.\end{aligned}$$

Theorem 3.2. *The λ-Green function of the two-dimensional process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ killed at the first hitting time τ_a of the level a has the form (for $a < x < y$)*

$$\mathcal{G}_\mu^\lambda(x,y;u) = \left(\frac{x}{y}\right)^{\mu-\nu}[\mathcal{Q}_\nu(x,y;u) - \mathcal{Q}_\nu * q_\nu^x(a,y;u)]$$

with Laplace transform

$$\int_0^\infty e^{-r^2u/2}\mathcal{G}_\mu^\lambda(x,y;u)\,du = 2(xy)^\mu\frac{K_\nu(ry)}{K_\nu(ra)}S_\nu(rx,ra)m^{(-\mu)}(y),$$

where $S_\nu(\alpha,\beta) = I_\nu(\alpha)K_\nu(\beta) - K_\nu(\alpha)I_\nu(\beta)$ for $0 < \beta < \alpha$.

Proof. We start from the formula for the λ-harmonic compensator of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ with respect to τ_a:

$$\begin{aligned}&\mathbf{E}^x[\exp(-\lambda\tau_a); \mathcal{Q}_\mu^\lambda(\exp(B_{\tau_a}^{(-\mu)}), y; A_{\tau_a}^{(-\mu)} + u)] \\ &\qquad = \left(\frac{a}{y}\right)^{\mu-\nu}\mathbf{E}^x[\exp(-\lambda\tau_a); \mathcal{Q}_\nu(a,y;A_{\tau_a}^{(-\mu)} + u)] \\ &\qquad = \left(\frac{x}{a}\right)^{\mu-\nu}\left(\frac{a}{y}\right)^{\mu-\nu}\int_0^\infty \mathcal{Q}_\nu(a,y;s+u)q_\nu^x(s)\,ds \\ &\qquad = \left(\frac{x}{y}\right)^{\mu-\nu}\mathcal{Q}_\nu * q_\nu^x(a,y;u).\end{aligned}$$

Next we compute the Laplace transform of the compensator with respect to u:

$$\begin{aligned}\mathcal{L}\mathcal{Q}_\nu * q_\nu^x(a,y;\cdot)(r^2/2) &= \mathcal{L}\mathcal{Q}_\nu(a,y;\cdot)(r^2/2)\cdot\mathcal{L}q_\nu^x(r^2/2) \\ &= \frac{2}{y}\left(\frac{a}{y}\right)^\nu I_\nu(ra)K_\nu(ry)\cdot\left(\frac{x}{a}\right)^\nu\frac{K_\nu(rx)}{K_\nu(ra)} \\ &= \frac{2}{y}\left(\frac{x}{y}\right)^\nu K_\nu(rx)K_\nu(ry)\frac{I_\nu(ra)}{K_\nu(ra)} \\ &= 2(xy)^\nu K_\nu(rx)K_\nu(ry)\frac{I_\nu(ra)}{K_\nu(ra)}m^{(-\mu)}(y).\end{aligned}$$

The λ-Green function of the process $(A_t^{(-\mu)}, \exp(B_t^{(-\mu)}))$ killed at τ_a has the form (for $a < x < y$)

$$\mathcal{G}_\mu^\lambda(x,y;u) = \left(\frac{x}{y}\right)^{\mu-\nu} [\mathcal{Q}_\nu(x,y;u) - \mathcal{Q}_\nu * q_\nu^x(a,y;u)]$$
$$= \left(\frac{x}{y}\right)^{\mu-\nu} \int_0^\infty [\mathcal{Q}_\nu(x,y;u) - \mathcal{Q}_\nu(a,y;u-s)\mathbf{1}_{[0,u)}(s)]q_\nu^x(s)\,ds.$$

Its Laplace transform $\int_0^\infty e^{-r^2u/2}\mathcal{G}_\mu^\lambda(x,y;u)\,du$ is equal to

$$\frac{2}{y}\left(\frac{x}{y}\right)^{\mu} I_\nu(rx)K_\nu(ry) - \frac{2}{y}\left(\frac{x}{y}\right)^{\mu} K_\nu(rx)K_\nu(ry)\frac{I_\nu(ra)}{K_\nu(ra)}$$
$$= 2(xy)^\mu\left((I_\nu(rx)K_\nu(ra) - K_\nu(rx)I_\nu(ra))\frac{K_\nu(ry)}{K_\nu(ra)}\right)m^{(-\mu)}(y)$$
$$= 2(xy)^\mu \frac{K_\nu(ry)}{K_\nu(ra)} S_\nu(rx,ra)m^{(-\mu)}(y),$$

with $S_\nu(\alpha,\beta) = I_\nu(\alpha)K_\nu(\beta) - K_\nu(\alpha)I_\nu(\beta)$ for $0 < \beta < \alpha$. ∎

The above result immediately implies

Corollary 3.3.

$$\mathcal{G}_\mu^\lambda(x,y;u) = \left(\frac{x}{y}\right)^{\mu-\nu} \mathcal{G}_\nu(x,y;u), \quad \nu = \sqrt{2\lambda+\mu^2}.$$

3.2. Green functions of a horocycle. Our main objective is to study the properties of the Green function and the λ-Green function of a half-space (or equivalently the interior of a horocycle), i.e. the set

$$D_a = \{x \in \mathbb{H}^n : x_n > a\},$$

where $a > 0$. We denote by σ_a the first exit time of X^ν from D_a,

$$\sigma_a = \inf\{t > 0 : X^\nu(t) \notin D_a\} = \inf\{t > 0 : X_n^\nu(t) = a\}.$$

Since the last coordinate of the hyperbolic Brownian motion with drift has the same law as the corresponding geometric Brownian motion, we can easily deduce that σ_a is finite almost surely whenever $a > 0$. For every $\lambda \ge 0$ we define the λ-Green function $G_\nu^\lambda(x,y)$ of D_a for the hyperbolic Brownian motion with drift ν as the integral kernel of the Green operator

$$G_\nu^\lambda f(x) = \int_0^\infty e^{-\lambda t}\mathbf{E}[t < \sigma_a, f(X^\nu(t))]\,dt = \int_{D_a} f(y)G_\nu^\lambda(x,y)\,dV_n(y),$$

for $x \in D_a$ and any Borel function f which is non-negative or bounded. We do not indicate the value of a in the notation of the corresponding objects, i.e. a is assumed to be a fixed positive number. We also recall the reference measure $dV_n(y) = y_n^{-n} d\tilde{y} dy_n = m^{(-\mu)}(dy_n) d\tilde{y}$, where $\mu = (n-1)/2$. The λ-potential operator U_ν^λ and the λ-potential kernel $U_\nu^\lambda(x,y)$ are defined by

$$U_\nu^\lambda f(x) = \int_0^\infty e^{-\lambda t} \mathbf{E}^x[f(X^\nu(t))]\, dt = \int_{D_a} f(y) U_\nu^\lambda(x,y)\, dV_n(y), \qquad x, y \in \mathbb{H}^n.$$

For $\lambda = 0$ we obtain the Green function of D_a and the potential kernel, which will be simply denoted by $G_\nu(x,y)$ and $U_\nu(x,y)$ respectively. Finally, if $\nu = \mu = (n-1)/2$ we will omit the subscript ν in the notation, i.e. $G^\lambda(x,y)$ and $G(x,y)$ are the λ-Green function and the Green function of the set D_a for the hyperbolic Brownian motion.

Now we go back to the crucial representation of HBM introduced previously, i.e. there exists the standard Brownian motion $B = (B(t)) = (\tilde{B}(t), B_n(t))$ in $\mathbb{R}^n$ starting from $x = (\tilde{x}, \log x_n)$, $x_n > 0$, such that

$$X(t) = (\tilde{X}(t), X_n(t)) = (\tilde{B}(A_t^{(-\mu)}), \exp(B_n^{(-\mu)}(t))),$$

where $B_n^{(-\mu)}(t) = B_n(t) - \mu t$, $A_t^{(-\mu)} = \int_0^t \exp(2B_n^{(-\mu)}(s))\, ds$ and $\mu = (n-1)/2$. Since the processes $\tilde{B}(t)$ and $(A_t^{(-\mu)}, \exp(B_n^{(-\mu)}(t)))$ are independent and the first exit time σ_a depends only on the last coordinate of the process, the above representation implies a relation between the hyperbolic Green function of the horocycle and the λ-Green function of $(A_t^{(-\mu)}, \exp(B_n^{(-\mu)}(t)))$. More precisely, the value of the λ-potential operator for the HBM X on a non-negative Borel function f,

$$U^\lambda f(x) = \int_0^\infty e^{-\lambda t} \mathbf{E}^x f(X(t))\, dt,$$

can be written as

$$\int_0^\infty e^{-\lambda t} \int_0^\infty \int_0^\infty \mathbf{E}^{\tilde{x}} f(\tilde{B}(w), y_n) \mathbf{P}^{x_n}(A_t^{(-\mu)} \in dw, \exp(B_n^{(-\mu)}(t)) \in dy_n)\, dt.$$

Since the integrand is non-negative, we can change the order of integration to show that $U^\lambda f(x)$ is equal to

$$\int_{\mathbb{R}^n} \int_0^\infty \frac{f(y)}{(2\pi w)^{(n-1)/2}} \exp\left(-\frac{|\tilde{x} - \tilde{y}|^2}{2w}\right) \mathcal{Q}_\mu^\lambda(x_n, y_n; w)\, dw\, dy.$$

Consequently, using the relation

$$\mathcal{Q}_\mu^\lambda(x_n, y_n; w) = \left(\frac{x_n}{y_n}\right)^{\mu-\nu} \mathcal{Q}_\nu(x_n, y_n; w), \qquad \nu = \sqrt{2\lambda + \mu^2},$$

and the formula for the hyperbolic volume element, we obtain the corresponding formula for the λ-potential kernel:

$$U^\lambda(x,y) = y_n^{2\mu+1}\left(\frac{x_n}{y_n}\right)^{\mu-\nu}\int_0^\infty g(w,\tilde{x},\tilde{y})\mathcal{Q}_\nu(x_n,y_n;w)\,dw, \tag{3.4}$$

where

$$g(w,\tilde{x},\tilde{y}) = \frac{1}{(2\pi w)^{(n-1)/2}}\exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2w}\right), \qquad \tilde{x},\tilde{y}\in\mathbb{R}^{n-1},\ w>0. \tag{3.5}$$

A similar computation gives a formula for the λ-harmonic compensator. Here we use the fact that σ_a and τ_a defined for $\exp(B_t^{(-\mu)})$ have the same distribution. We obtain

$$\begin{aligned}
&\mathbf{E}^x[e^{-\lambda\sigma_a}U^\lambda(X(\sigma_a),y)]\\
&\quad = y_n^{2\mu+1}\left(\frac{a}{y_n}\right)^{\mu-\nu}\mathbf{E}^x\left[\int_0^\infty e^{-\lambda\sigma_a}g(w,\tilde{B}(A_{\sigma_a}^{(-\mu)}),\tilde{y})\mathcal{Q}_\nu(a,y_n;w)\,dw\right]\\
&\quad = y_n^{2\mu+1}\left(\frac{x_n}{y_n}\right)^{\mu-\nu}\mathbf{E}^{\tilde{x}}\left[\int_0^\infty\int_0^\infty g(w,\tilde{B}(A_{\sigma_a}^{(-\mu)}),\tilde{y})\mathcal{Q}_\nu(a,y_n;w)q_\nu^{x_n}(s)\,dw\,ds\right].
\end{aligned}$$

Using the fact that

$$\mathbf{E}^{\tilde{x}}\left[\frac{1}{(2\pi w)^{(n-1)/2}}\exp\left(-\frac{|\tilde{B}(s)-\tilde{y}|^2}{2w}\right)\right] = \frac{1}{(2\pi(w+s))^{(n-1)/2}}\exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2(w+s)}\right)$$

and changing variables by setting $u=w+s$ we arrive at

$$y_n^{2\mu+1}\left(\frac{x_n}{y_n}\right)^{\mu-\nu}\int_0^\infty\frac{1}{(2\pi u)^{(n-1)/2}}\exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2u}\right)\mathcal{Q}_\nu * q_\nu^{x_n}(a,y_n;u)\,du.$$

The above relations yield the following representation formulas for the λ-Green function $G^\lambda(x,y)$ of a half-space for the hyperbolic Brownian motion:

Theorem 3.4. *The λ-Green function of $D_a=\{x\in\mathbb{H}^n : x_n>a\}$ is given by*

$$G^\lambda(x,y) = y_n^{2\mu+1}\int_0^\infty\frac{1}{(2\pi w)^{(n-1)/2}}\exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2w}\right)\mathcal{G}_\mu^\lambda(x_n,y_n;w)\,dw, \tag{3.6}$$

where $\mathcal{G}_\mu^\lambda(x_n,y_n;w)$ is the λ-Green function for the two-dimensional process $(A_t^{(-\mu)},\exp(B_t^{(-\mu)}))$.

This formula together with the relation between a Bessel process and a geometric Brownian motion established by Lamperti's relation lead to the following result, which will be crucial in estimating the λ-Green function.

THEOREM 3.5.

$$G^\lambda(x,y) = (x_n y_n)^{\mu-\nu} \int_0^\infty \frac{1}{(2\pi u)^{(n-1)/2}} \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2u}\right) p_a^{(-\nu)}(u,x_n,y_n)\,du, \tag{3.7}$$

where $p_a^{(-\nu)}(u,x_n,y_n)$ is the density function (with respect to the speed measure $m^{(-\nu)}(dy_n) = y_n^{-2\nu+1}dy_n$) of the transition probability of the Bessel process $BES^{(-\nu)}(x_n)$ killed at the first hitting time T_a.

Proof. From (3.6) and the fact that

$$\mathcal{G}_\mu^\lambda(x_n,y_n;w) = \left(\frac{x_n}{y_n}\right)^{\mu-\nu} \mathcal{G}_\nu(x_n,y_n;w)$$

we obtain

$$G^\lambda(x,y) = \left(\frac{x_n}{y_n}\right)^{\mu-\nu} G_\nu(x,y), \qquad x,y \in D,$$

where $\nu = \sqrt{2\lambda+\mu^2}$ and $\mu = (n-1)/2$. We now use the following fact: after changing variables in the geometric Brownian motion according to the formula $t = \alpha_u$ where $\alpha_u = \inf\{s > 0 : A_s^{(-\nu)} > u\}$ we find that $\exp(B_{\alpha_u}^{(-\nu)}) = BES^{(-\nu)}$ is a Bessel process with index $-\nu$, with $A_s^{(-\nu)}$ as before. Applying this relation we deduce that $T_a = \inf\{u > 0 : \exp(B_{\alpha_u}^{(-\nu)}) = a\}$ is the first hitting time of the level a by the Bessel process $BES^{(-\nu)}$, and $\tau_a = \alpha_{T_a}$. Applying these relations we obtain, for a positive Borel function f,

$$\begin{aligned}
G_\nu f(x) &= \mathbf{E}^x \int_0^\infty \mathbf{1}_{\{t<\tau_a\}} f(\tilde{B}_{A_t^{(-\nu)}}, \exp(B_n^{(-\nu)}(t)))\,dt \\
&= \mathbf{E}^{\tilde{x}} \mathbf{E}_{x_n}^{(-\nu)} \int_0^\infty \mathbf{1}_{\{u<T_a\}} f(\tilde{B}_u, R_u(t))\,\frac{du}{R_u^2} \\
&= \int_0^\infty \int_0^\infty \mathbf{E}^{\tilde{x}}[f(\tilde{B}_u,y_n)] p_a^{(-\nu)}(u,x_n,y_n)\,\frac{dy_n}{y_n^2 y_n^{2\nu-1}}\,du \\
&= \int_{\mathbb{R}^n} \int_0^\infty g(u,\tilde{x},\tilde{y}) p_a^{(-\nu)}(u,x_n,y_n) y_n^{2\mu-2\nu} f(\tilde{y},y_n)\,du\,dV_n(y),
\end{aligned}$$

where g is the density of the $(n-1)$-dimensional Brownian motion given in (3.5). Collecting all together provides the result. ∎

4. Green function estimates

4.1. Sharp estimates of Green functions. We begin by providing two-sided uniform estimates of λ-potential kernels.

PROPOSITION 4.1. *For $n \geq 3$ we have*

$$U^\lambda(x,y) \approx \left(\frac{2x_ny_n}{|x-y|^2}\right)^{\mu-1/2}\left(1\wedge\frac{2x_ny_n}{|x-y|^2}\right)^{\nu+1/2}, \qquad x,y\in\mathbb{H}^n,$$

where $\mu = (n-1)/2$ and $\nu = \sqrt{2\lambda+\mu^2}$. Moreover, if $n=2$ we have

$$U^\lambda(x,y) \approx \left(1\wedge\frac{2x_ny_n}{|x-y|^2}\right)^{\nu+1/2}, \qquad x,y\in\mathbb{H}^2.$$

Proof. Using (3.4) together with the asymptotic description of $\mathcal{Q}_\mu^\lambda(x_n,y_n;t)$, which is an immediate consequence of the relations

$$I_\mu(z) = \left(\frac{z}{2}\right)^\mu \frac{1}{\Gamma(\mu+1)} + O(z^{\mu+2}), \qquad z\to 0^+$$

(see [1, 9.6.7 and 9.6.12]) and

$$I_\mu(z) \sim \frac{e^z}{\sqrt{2\pi z}}(1+O(1/z)), \qquad z\to\infty$$

(see [12, 7.13.1, (5)]), we infer that $U^\lambda(x,y)$ is comparable to

$$(x_ny_n)^{\mu-1/2}\int_0^{x_ny_n}\frac{1}{u^{n/2}}e^{-\frac{|x-y|^2}{2u}}\,du + (x_ny_n)^{\mu+\nu}\int_{x_ny_n}^\infty \frac{1}{u^{\nu+\mu+1}}e^{-\frac{|x-y|^2}{2u}}\,du.$$

Making the substitution $\frac{|x-y|^2}{2u} = s$ in both integrals, we arrive at

$$\left(\frac{2x_ny_n}{|x-y|^2}\right)^{\mu-1/2}\int_{\frac{|x-y|^2}{2x_ny_n}}^\infty s^{n/2-2}e^{-s}ds + \left(\frac{2x_ny_n}{|x-y|^2}\right)^{\mu+\nu}\int_0^{\frac{|x-y|^2}{2x_ny_n}} s^{\mu+\nu-1}e^{-s}ds.$$

Using the asymptotics of incomplete gamma functions (see [5, Lemma 12])

$$\int_0^b s^\alpha e^{-s}\,ds \approx (1\wedge b)^{\alpha+1}, \qquad b>0,$$

$$\int_a^\infty s^\alpha e^{-s}\,ds \approx (a+1)^\alpha e^{-a}, \qquad a>0,$$

we find that for $\frac{|x-y|^2}{2x_ny_n} \geq 1$,

$$U^\lambda(x,y) \approx \frac{2x_ny_n}{|x-y|^2}e^{-\frac{|x-y|^2}{2x_ny_n}} + \left(\frac{2x_ny_n}{|x-y|^2}\right)^{\mu+\nu} \approx \left(\frac{2x_ny_n}{|x-y|^2}\right)^{\mu+\nu}.$$

Finally, whenever $\frac{|x-y|^2}{2x_n y_n} \le 1$ we have

$$U^\lambda(x,y) \approx \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu-1/2} + 1.$$

Note that if $n \ge 3$ then the potential behaves as the first component, and for $n = 2$ as the other one. This ends the proof. ∎

THEOREM 4.2. *For every $n > 2$, $\lambda \ge 0$ and $a \ge 0$ we have*

$$G^\lambda(x,y) \stackrel{\lambda,n}{\approx} \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu-1/2} \left(1 \wedge \frac{2(x_n-a)(y_n-a)}{|x-y|^2}\right)\left(1 \wedge \frac{2x_n y_n}{|x-y|^2}\right)^{\nu-1/2}$$

where $\mu = (n-1)/2$, $x = (\tilde{x}, x_n)$, $y = (\tilde{y}, y_n)$, $x_n, y_n > a$ and $\nu = \sqrt{2\lambda + \mu^2}$.

Proof. The scaling property of HBM gives

$$G^\lambda(ax, ay) = \tilde{G}^\lambda(x,y), \qquad x_n, y_n > 1,$$

where $\tilde{G}^\lambda$ denotes the λ-Green function for the set $D_1 = \{x \in \mathbb{H}^n : x_n > 1\}$. Consequently, we will further consider $a = 1$ and omit tildes. We recall a recent result giving sharp estimates of the transition density of the Bessel process killed when leaving the half-line (a, ∞): it was shown in [2] that for every $\nu \ne 0$ we have

$$p_1^{(-\nu)}(t, x_n, y_n) \stackrel{\nu}{\approx} \left(1 \wedge \frac{(x_n-1)(y_n-1)}{t}\right) p^{(-\nu)}(t, x_n, y_n)$$

whenever $t > 0$ and $x_n, y_n > 1$. Observe that

$$p^{(-\nu)}(t, x_n, y_n)) \stackrel{\nu}{\approx} \left(1 \wedge \frac{x_n y_n}{t}\right)^{\nu-1/2} (x_n y_n)^{\nu-1/2} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x_n-y_n)^2}{2t}\right).$$

Note that in [2] the density was considered with respect to the Lebesgue measure and here the reference measure is $m(dy_n) = y_n^{-2\mu+1}\, dy_n$. Splitting the integral (3.7) representing $G_\mu^\lambda(x,y)$ into three parts we get

$$\left(\int_0^{(x_n-1)(y_n-1)} + \int_{(x_n-1)(y_n-1)}^{x_n y_n} + \int_{x_n y_n}^{\infty}\right) g(t, \tilde{x}, \tilde{y}) p^{(-\nu)}(t, x_n, y_n)\, dt,$$

and consequently

$$G^\lambda(x,y) \approx J_1(x,y) + J_2(x,y) + J_3(x,y),$$

where by substituting $w = |x-y|^2/(2t)$ we get

$$J_1(x,y) = (x_n y_n)^{\mu-1/2} \int_0^{(x_n-1)(y_n-1)} t^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right) dt = \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \int_{\frac{|x-y|^2}{2(x_n-1)(y_n-1)}}^{\infty} w^{n/2-2} e^{-w}\, dw \tag{4.1}$$

and similarly

$$J_2(x,y) = \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \int_{\frac{|x-y|^2}{2x_n y_n}}^{\frac{|x-y|^2}{2(x_n-1)(y_n-1)}} w^{n/2-1} e^{-w}\, dw, \tag{4.2}$$

$$J_3(x,y) = \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu+\nu-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \int_0^{\frac{|x-y|^2}{2x_n y_n}} w^{\mu+\nu-1} e^{-w}\, dw. \tag{4.3}$$

Note that if $|x-y|^2 < 2(x_n-1)(y_n-1)$ then the integral in (4.1) behaves like a constant and

$$J_1(x,y) \approx \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1}.$$

Moreover, since $|x-y|^2 < 2(x_n-1)(y_n-1) < 2x_n y_n$ we have

$$J_2(x,y) \approx \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \int_0^{\frac{|x-y|^2}{2(x_n-1)(y_n-1)}} w^{n/2-1}\, dw$$

$$\approx \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \left(\frac{|x-y|^2}{2(x_n-1)(y_n-1)}\right)^{n/2-1} < \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1}$$

and

$$J_3(x,y) < \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu+\nu-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \int_0^{\frac{|x-y|^2}{2x_n y_n}} w^{\mu+\nu-1}\, dw$$

$$< \frac{(x_n-1)(y_n-1)}{x_n y_n}$$

$$< \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \left(\frac{|x-y|^2}{2(x_n-1)(y_n-1)}\right)^{n/2-1} < \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1},$$

which ends the proof in this case.

In the second case, i.e. when $2(x_n-1)(y_n-1) \le |x-y|^2 \le x_n y_n$, the integral in (4.2) is comparable to a constant and consequently $J_2(x,y)$ dominates the other parts. To see that, recall the asymptotic formula for the incomplete gamma function

$$\int_z^\infty w^{\alpha-1}e^{-w}\,dw \approx z^{\alpha-1}e^{-z}, \qquad z\to\infty, \tag{4.4}$$

which shows that the integral in (4.1) decays exponentially as $|x-y|^2/(2(x_n-1)(y_n-1))$ grows to infinity and consequently $J_1(x,y)/J_2(x,y)$ is bounded from above in the region considered. Moreover, the above estimates of $J_3(x,y)$ and $J_2(x,y)$ give

$$J_3(x,y) < \frac{(x_n-1)(y_n-1)}{x_n y_n} \le 1 \approx J_2(x,y).$$

The final part of the proof concerns the case when $2(x_n-1)(y_n-1)\le |x-y|^2$ and $x_n y_n \le |x-y|^2$. Then the integral in (4.3) behaves like a constant and to finish the proof it is enough to show that $J_3(x,y)$ dominates $J_1(x,y)$ and $J_2(x,y)$ in that case. Indeed, using (4.4) we get

$$\begin{aligned}
J_2(x,y) &< \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \int_{\frac{|x-y|^2}{2x_n y_n}}^\infty w^{n/2-1}e^{-w}\,dw \\
&\approx \left(\frac{2x_n y_n}{|x-y|^2}\right)^{n-2} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \left(\frac{|x-y|^2}{2x_n y_n}\right)^{n-2} \exp\left(-\frac{|x-y|^2}{2x_n y_n}\right) \\
&\le \left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu+\nu-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \left(\frac{|x-y|^2}{2x_n y_n}\right)^{n-2} \exp\left(-\frac{|x-y|^2}{2x_n y_n}\right) \\
&< c_1(n) J_3(x,y),
\end{aligned}$$

where $c_1(n) = (\int_0^{1/2} w^{n-2}e^{-w}\,dw)^{-1}\cdot \sup_{x\ge 1/2} x^{n-2}e^{-x}$. Finally, $J_1(x,y)$ is bounded from above by

$$\left(\frac{2x_n y_n}{|x-y|^2}\right)^{n/2-1} \left(\frac{|x-y|^2}{2(x_n-1)(y_n-1)}\right)^{n/2-2} \exp\left(-\frac{|x-y|^2}{2(x_n-1)(y_n-1)}\right)$$

and the product of the first two factors is clearly bounded by

$$\left(\frac{2x_n y_n}{|x-y|^2}\right)^{\mu+\nu-1} \frac{2(x_n-1)(y_n-1)}{|x-y|^2} \left(\frac{|x-y|^2}{2(x_n-1)(y_n-1)}\right)^{n-2}.$$

This implies that

$$J_1(x,y) \le c_1(n) J_3(x,y),$$

and the proof is complete. ■

The above result can be stated in terms of the hyperbolic distance:

COROLLARY 4.3. *For every $n > 2$ and $\lambda > 0$ we have*

$$G^\lambda(x,y) \overset{\lambda,n}{\approx} \frac{1 \wedge \frac{(1\wedge\delta_a(x))(1\wedge\delta_a(y))}{1\wedge d^2_{\mathbb{H}^n}(x,y)}}{\sinh^{2\mu-1}(d_{\mathbb{H}^n}(x,y)/2)\cosh^{\nu+1/2}(d_{\mathbb{H}^n}(x,y))}, \qquad x,y \in D,$$

where $\delta_a(x)$ denotes the (hyperbolic) distance of x to the boundary of D, or equivalently

$$G^\lambda(x,y) \overset{\lambda,n}{\approx} \left(1 \wedge \frac{(1\wedge\delta_a(x))(1\wedge\delta_a(y))}{1\wedge d^2_{\mathbb{H}^n}(x,y)}\right) U^\lambda(x,y), \qquad x,y \in D.$$

Proof. First observe that by (2.1) we simply have

$$1 \wedge \frac{2x_n y_n}{|x-y|^2} \approx \left(1 + \frac{|x-y|^2}{2x_n y_n}\right)^{-1} = \cosh^{-1}(d_{\mathbb{H}^n}(x,y)).$$

Moreover, (2.1) also implies that

$$\frac{2x_n y_n}{|x-y|^2} = \left(\cosh(d_{\mathbb{H}^n}(x,y)) - 1\right)^{-1} = \frac{1}{2}\sinh^{-2}(d_{\mathbb{H}^n}(x,y)/2).$$

Since the lines perpendicular to the boundary of D are geodesics of the hyperbolic space, the distance of $x = (\tilde{x}, x_n)$ to the boundary of D is realized as the distance between x and $(\tilde{x}, a)$. Thus we have

$$\cosh\delta_a(x) = 1 + \frac{(x_n - a)^2}{2x_n a} = \frac{1}{2}\left(\frac{x_n}{a} + \frac{a}{x_n}\right), \qquad \delta_a(x) = \ln(x_n/a).$$

Consequently,

$$\frac{x_n}{a} - 1 = e^{\delta_a(x)} - 1 \approx \sinh\delta_a(x), \qquad \frac{x_n}{a} = e^{\delta_a(x)} \approx \cosh\delta_a(x).$$

Using this we can write

$$\begin{aligned}\frac{(x_n-a)(y_n-a)}{|x-y|^2} &= \frac{(x_n/a-1)(y_n/a-1)}{2x_n y_n/a^2}\,\frac{2x_n y_n}{|x-y|^2}\\ &\approx \frac{\sinh\delta_a(x)\sinh\delta_a(y)}{\cosh\delta_a(x)\cosh\delta_a(y)}\,\frac{1}{2\sinh^2(d_{\mathbb{H}^n}(x,y)/2)}.\end{aligned}$$

To finish the proof, note that $\tanh\delta_a(x) \approx 1 \wedge \delta_a(x)$ and

$$\left(1 \wedge \frac{(1\wedge\delta_a(x))(1\wedge\delta_a(y))}{\sinh^2(d_{\mathbb{H}^n}(x,y)/2)}\right)\cosh(d_{\mathbb{H}^n}(x,y)) \approx (1\wedge\delta_a(x))(1\wedge\delta_a(y))$$

whenever $d_{\mathbb{H}^n}(x,y) \ge 1$, and

$$\left(1 \wedge \frac{(1\wedge\delta_a(x))(1\wedge\delta_a(y))}{\sinh^2(d_{\mathbb{H}^n}(x,y)/2)}\right)\cosh(d_{\mathbb{H}^n}(x,y)) \approx 1 \wedge \frac{(1\wedge\delta_a(x))(1\wedge\delta_a(y))}{d^2_{\mathbb{H}^n}(x,y)}$$

if $d_{\mathbb{H}^n}(x,y) < 1$. Collecting all together, we get the result. ∎

We end with the following conjecture stating that the form of the above estimates should be the same for more general smooth subsets of $\mathbb{H}^n$. Obviously, the conjecture is true if we assume that the set D under study is bounded in the hyperbolic metric. This follows from the fact that the potential theory on bounded subsets of $\mathbb{H}^n$ is comparable to Euclidean potential theory, since the Laplace–Beltrami operator is strongly elliptic on such domains.

CONJECTURE 4.4. *For every $C^{1,1}$ domain $D \subset \mathbb{H}^n$ we have*

$$G_D(x,y) \overset{n,D}{\approx} \left(1 \wedge \frac{(1 \wedge \delta_a(x))(1 \wedge \delta_a(y))}{1 \wedge d^2_{\mathbb{H}^n}(x,y)}\right) U(x,y), \qquad x,y \in D.$$

5. Appendix

5.1. Bessel processes. The basic material concerning Bessel processes is taken from [21, Ch. XI]. We begin with the definition of a *squared Bessel process* $BESQ^\delta(x)$ started at $x \geq 0$. It is defined as the unique strong solution of the equation

$$dZ(t) = 2\sqrt{|Z(t)|}\,d\beta(t) + \delta\,dt, \qquad Z(0) = x, \tag{5.1}$$

where $\beta(t)$ denotes a one-dimensional Brownian motion. Here $\delta \in \mathbb{R}$ is called the *dimension* of $BESQ^\delta$. For $\delta \geq 0$ the process $Z(t)$ is non-negative and the square root in (5.1) can be omitted. It is known that for $0 < \delta < 2$ the point 0 is *reflecting*; for $\delta = 0$ it is *absorbing*. When $\delta < 0$ the situation gets trickier: the process $Z(t)$ starting from $x > 0$ attains 0 in finite time and then becomes negative (and behaves like the process $-Z(t)$ with positive dimension $-\delta$ (see [13]). In this paper, however, we impose the killing condition at 0 and again call the resulting process the *Bessel process* (of negative dimension). Thus, in our setting, the process $Z(t)$ is always non-negative so we are able to take the square root.

The square root of $BESQ^\delta(x^2)$, $x \geq 0$, is called the *Bessel process* of dimension δ started at x and is denoted by $BES^\delta(x)$. We also introduce the *index* $\mu = \delta/2 - 1$ of the corresponding process, and write $BES^{(\mu)}$ (instead of BES^δ) if we want to use μ (instead of δ).

For $\mu \geq 0$ the probability density function of the $BES^{(\mu)}(x)$ semigroup is

$$p^{(\mu)}(t,x,y) = \frac{y}{t}\left(\frac{y}{x}\right)^{\mu} e^{-(x^2+y^2)/2t} I_\mu(xy/t) \quad \text{for } x > 0, \tag{5.2}$$

and

$$p^{(\mu)}(t,0,y) = 2(2t)^{-\mu-1} t^{-(\mu+1)} \Gamma(\mu+1)^{-1} y^{2\mu+1} e^{-y^2/2t}.$$

Note that the above formulas describe the densites considered with respect to the Lebesgue measure. However, sometimes the symmetric version of the density can be more convenient. Then we have to deal with the reference measure $m^{(\mu)}(dy) = m^{(\mu)}(y)dy$, where $m^{(\mu)}(y) = y^{2\mu+1}$. We have switched to the symmetric version in Section 4.1.

We denote by $\mathbf{P}_x^{(\mu)}$ and $\mathbf{E}_x^{(\mu)}$ the probability law and the corresponding expected value of a Bessel process $BES^{(\mu)}(x)$ on the canonical path space with starting point $x \geq 0$. Let $\mathcal{F}_t = \sigma\{R_s, s \leq t\}$ be the filtration of the coordinate process R_t. Denote the first hitting time of the level $a \geq 0$ by

$$T_a = \inf\{t > 0 : R_t = a\}.$$

We have the absolute continuity property for the laws of the Bessel processes with different indices:

$$\text{(5.3)} \qquad \frac{d\mathbf{P}_x^{(\mu)}}{d\mathbf{P}_x^{(\nu)}}\bigg|_{\mathcal{F}_t} = \left(\frac{R_t}{x}\right)^{\mu-\nu} \exp\left(-\frac{\mu^2-\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) \quad \mathbf{P}_x^{(\nu)}\text{-a.s. on } \{T_0 > t\}.$$

If $\nu \geq 0$ then the condition $\{T_0 > t\}$ can be omitted. In this work we are primarily interested in the case of negative drift, which we indicate by writing the index as $-\mu$, thus assuming in what follows $\mu \geq 0$. Observe that (5.3) implies that in the case of $-\mu$ the density function of $BES^{(-\mu)}(x)$, $x > 0$, is still of the form (5.2) with $\nu = -\mu$.

5.2. Exponential functionals. We now briefly describe the Matsumoto–Yor approach [20], which is a starting point of our construction. The important point is to compute the functional

$$\phi(y) = \mathbf{E}_x^{(0)}\left[\exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) \,\Big|\, R_t = y\right].$$

To do so, we apply (5.3) to obtain

$$\begin{aligned}
\mathbf{E}_x^{(0)}&\left[e^{-\alpha R_t}\left(\frac{R_t}{x}\right)^{-\mu}\mathbf{E}_x^{(0)}\left[\exp\left(-\frac{\mu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)\,\Big|\, R_t\right]\right] \\
&= \int_0^\infty e^{-\alpha y}\frac{y}{t}\left(\frac{y}{x}\right)^{-\mu}\exp\left(-\frac{x^2+y^2}{2t}\right) I_0\left(\frac{xy}{t}\right)\phi(y)\,dy \\
&= \mathbf{E}_x^{(-\mu)}[e^{-\alpha R_t^{(-\mu)}}] = \int_0^\infty e^{-\alpha y}\frac{y}{t}\left(\frac{y}{x}\right)^{-\mu}\exp\left(-\frac{x^2+y^2}{2t}\right) I_{|\mu|}\left(\frac{xy}{t}\right) dy.
\end{aligned}$$

This equality provides the desired formula

$$\phi(y) = \mathbf{E}_x^{(0)}\left[\exp\left(-\frac{\mu^2}{2}\int_0^t\frac{ds}{(R_s)^2}\right)\,\Big|\, R_t = y\right] = \frac{I_{|\mu|}(xy/t)}{I_0(xy/t)}.$$

Observe that the formula defines the so-called Hartman–Watson distribution η_r with parameter $r = xy/t$ described in terms of the Laplace transform:

$$\int_0^\infty e^{-\mu^2 t/2}\,\eta_r(dt) = \frac{I_{|\mu|}(r)}{I_0(r)}.$$

Below, we need a related density function, denoted by $\theta(r,t)$, defined by

$$\int_0^\infty e^{-\mu^2 t/2}\theta(r,t)\,dt = I_{|\mu|}(r), \qquad r > 0. \tag{5.4}$$

5.3. Joint distribution of $(A_t^{(-\mu)}, B_t^{(-\mu)})$. Let $(B_t)_{t\ge 0}$ be the Brownian motion in $\mathbb{R}$ starting from x. We denote by $\mathbf{P}^x$ and $\mathbf{E}^x$ the corresponding probability law and expected value. Note that the starting point is now indicated in the superscript to distinguish it from the probability law and expected value for the Bessel process with index $\mu \in \mathbb{R}$ introduced previously. For $\mu \ge 0$ we denote by $B_t^{(-\mu)} = B_t - \mu t$ the Brownian motion with negative drift. We define

$$A_t^{(-\mu)} = \int_0^t \exp(2B_s^{(-\mu)})\,ds, \qquad t \ge 0.$$

For $\mu = 0$ we will use the shortened notation $A_t := A_t^{(0)}$.

To find the joint distribution of $(A_t^{(-\mu)}, B_t^{(-\mu)})$ we proceed as follows. The crucial point is to determine the formula for the Green function

$$G(x,y,\alpha^2/2) = (H_\lambda + \alpha^2/2)^{-1},$$

which is defined as a formal inverse, with H_λ being the Schrödinger operator

$$H_\lambda = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\lambda^2 e^{2x}. \tag{5.5}$$

We first assume that $\mu = 0$. For $x \le y$ we obtain

$$G(x,y,\alpha^2/2) = 2I_\alpha(\lambda e^x)K_\alpha(\lambda e^y).$$

It is easily seen that the Feynman–Kac semigroup generated by the operator (5.5) is of the form

$$\begin{aligned} e^{-H_\lambda t} f(x) &= \mathbf{E}^x[e^{-(\lambda^2/2)\int_0^t \exp(2B_s)\,ds} f(B_t)] \\ &= \mathbf{E}^x\big[\mathbf{E}^x[e^{-(\lambda^2/2)\int_0^t \exp(2B_s)\,ds} \mid f(B_t)]\big] \\ &= \int_0^\infty \mathbf{E}^0[e^{-(\lambda^2/2)e^{2x}A_t} \mid B_t = y - x] f(y) \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}\,dy. \end{aligned}$$

This provides the following formula for the transition density of the above semigroup:

$$
\begin{aligned}
g_\lambda(t;x,y) &= \mathbf{E}^0[e^{-(\lambda^2/2)e^{2x}A_t} \mid B_t = y-x]\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} \\
&= \int_0^\infty e^{-(\lambda^2/2)e^{2x}u}\mathbf{P}^0(A_t \in du \mid B_t = y-x)\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}.
\end{aligned}
$$

Integrating with respect to $t > 0$ we find that

$$
\int_0^\infty e^{-\alpha^2 t/2} g_\lambda(t;x,y)\,dt = G(x,y,\alpha^2/2) = 2I_\alpha(\lambda e^x)K_\alpha(\lambda e^y).
$$

Taking into account the product formula for Bessel functions and the definition of $\theta(r,t)$ we obtain, for $x=0$,

$$
\begin{aligned}
G(0,y,\alpha^2/2) &= \int_0^\infty e^{-u/2}e^{-\lambda^2/2u}e^{-\lambda^2 e^{2y}/2u}\int_0^\infty e^{-\alpha^2 t/2}\theta\Big(\frac{\lambda^2 e^y}{u},t\Big)\,dt\,\frac{du}{u} \\
&= \int_0^\infty e^{-\lambda^2/2\xi}e^{-1/2\xi}e^{-e^{2y}/2\xi}\int_0^\infty e^{-\alpha^2 t/2}\theta\Big(\frac{e^y}{\xi},t\Big)\,dt\,\frac{d\xi}{\xi} \\
&= \int_0^\infty e^{-\alpha^2 t/2}\int_0^\infty e^{-\lambda^2/2u}e^{-(1+e^{2y})/2u}\theta\Big(\frac{e^y}{u},t\Big)\,dt\,\frac{du}{u} \\
&= \int_0^\infty e^{-\alpha^2 t/2}\int_0^\infty e^{-\lambda^2/2u}\mathbf{P}^0(A_t \in du \mid B_t = y)\frac{e^{-y^2/2t}}{\sqrt{2\pi t}}\,dt.
\end{aligned}
$$

By comparing we obtain, for $x=0$,

$$
\mathbf{P}^0(A_t \in du \mid B_t = y)\frac{e^{-y^2/2t}}{\sqrt{2\pi t}} = \exp\Big(-\frac{1+e^{2y}}{2u}\Big)\theta\Big(\frac{e^y}{u},t\Big)\,\frac{du}{u},
$$

and for arbitrary x,

$$
\mathbf{P}^x(A_t \in du, B_t = y)\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} = \exp\Big(-\frac{e^{2x}+e^{2y}}{2u}\Big)\theta\Big(\frac{e^{x+y}}{u},t\Big)\,\frac{du}{u};
$$

which after taking into account the (Girsanov) drift formula gives the following formula for the joint density of the variables $(A_t^{(-\mu)}, B_t^{(-\mu)})$:

$$
\begin{aligned}
(5.6)\qquad & \mathbf{P}^x(A_t^{(-\mu)} \in du, B_t^{(-\mu)} \in dy) \\
&= e^{-\mu^2 t/2}e^{-\mu(y-x)}\exp\Big(-\frac{e^{2x}+e^{2y}}{2u}\Big)\frac{\theta\big(\frac{e^{x+y}}{u},t\big)}{u}.
\end{aligned}
$$

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Kamil Bogus, Jacek Małecki
Faculty of Pure and Applied Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: kamil.bogus@pwr.edu.pl
jacek.malecki@pwr.edu.pl

Tomasz Byczkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
and
Faculty of Pure and Applied Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: tomasz.byczkowski@gmail.com
tomasz.byczkowski@pwr.edu.pl

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