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Characterization of associate spaces of generalized weighted weak-Lorentz spaces and embeddings

by

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Abstract. We characterize associate spaces of generalized weighted weak-Lorentz spaces and use this characterization to study embeddings between these spaces.

1. Introduction and main results. Let $(\mathcal{R},\mu)$ be a totally σ-finite non-atomic measure space with $b := \mu(\mathcal{R}) \in (0,\infty]$. We denote by $\mathfrak{M}(\mathcal{R},\mu)$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty,\infty]$. By $\mathfrak{M}^+(\mathcal{R},\mu)$ we denote the set of all non-negative functions from $\mathfrak{M}(\mathcal{R},\mu)$. For $f \in \mathfrak{M}(\mathcal{R},\mu)$, the *non-increasing rearrangement* of f is defined by

$$f^*(t) = \inf\{s > 0;\ \mu(\{x \in \mathcal{R};\ |f(x)| > s\}) \le t\}, \qquad t \in [0,\mu(\mathcal{R})).$$

We use here the convention $\inf \emptyset = \infty$. Thus, if $\mu(\{x \in \mathcal{R};\ |f(x)| > s\}) > t$ for all $s \in [0,\mu(\mathcal{R}))$, then $f^*(t) = \infty$.

We shall assume throughout that $\mu(\mathcal{R}) = \infty$. Moreover, u, v, w, φ and ψ will always denote *weights*, that is, locally integrable non-negative functions on $(0,\infty)$. We set, once and for all,

$$U_p(t) = \|u\|_{p;(0,t)}, \qquad V_p(t) = \|v\|_{p;(0,t)}, \qquad W_p(t) = \|w\|_{p;(0,t)}, \qquad 0 < p \le \infty,$$

where $\|\cdot\|_{p;(c,d)}$, $p \in (0,\infty]$, denotes the usual L^p-(quasi-)norm on the interval $(c,d) \subseteq \mathbb{R}$, defined by

$$\|f\|_{p;(c,d)} := \begin{cases} \Big(\int_c^d |f(y)|^p\,dy\Big)^{1/p} & \text{if } p < \infty,\\ \operatorname*{ess\,sup}_{y\in(c,d)} |f(y)| & \text{if } p = \infty. \end{cases}$$

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When $p = 1$, we will omit the subscript p. We assume that $U(t) > 0$ for every $t \in (0, \infty)$. We then denote

$$f_u^{**}(t) = \frac{1}{U(t)} \int_0^t f^*(s) u(s)\, ds, \qquad t \in (0, \infty).$$

When $u \equiv 1$ (hence $U(t) = t$), we will omit the subscript u.

The function "space" $G\Gamma(p, m, w)(\mathcal{R})$, introduced and studied in [FR] and [FRZ], is defined as the collection of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that

$$\|f\|_{G\Gamma(p,m,w)} = \Big(\int_0^b w(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{m/p} dt \Big)^{1/m} < \infty,$$

where $m, p \in (0, \infty)$ and w is a weight on $(0, b)$. In [GPS] the associate space of $G\Gamma(p, m, w)(\mathcal{R})$ was characterized.

We note that in general $G\Gamma(p, m, w)(\mathcal{R})$ is not a linear space. There are a number of natural and important function spaces which are not normable and even not linear: see for example [CKMP].

The spaces $G\Gamma(p, m, w)$ cover several types of important function spaces and have plenty of applications. For example, when $b = \infty$, $p = 1$, $m > 1$ and $w(t) = t^{-m} v(t)$, where v is another weight on $(0, b)$, then $G\Gamma(p, m, w)$ reduces to the space $\Gamma^m(v)$, whose norm reads

$$\|g\|_{\Gamma^m(v)} = \Big(\int_0^\infty g^{**}(t)^m w(t)\, dt \Big)^{1/m}.$$

This space was introduced by Sawyer [S] who used it to describe the behavior of classical operators on Lorentz spaces and observed, among other things, that, under certain restrictions on the parameters involved, this space is the associate space of the space $\Lambda^{m'}(\tilde{v})$, introduced by Lorentz [L], where $m' = m/(m-1)$, $\tilde{v}$ is an appropriate weight, and the norm in $\Lambda^{m'}(\tilde{v})$ is given by

$$\|g\|_{\Lambda^{m'}(\tilde{v})} = \Big(\int_0^\infty g^*(t)^{m'} \tilde{v}(t)\, dt \Big)^{1/m'}.$$

Spaces of type Λ and Γ have been extensively investigated during the last 25 years under the common label of *classical Lorentz spaces*, and an avalanche of papers by many authors devoted to their detailed study is available.

In this paper we introduce a weak variant of $G\Gamma(p, m, w)(\mathcal{R})$ corresponding to $m = \infty$.

First, we have to recall some definitions and basic facts.

DEFINITION 1.1. Let θ be a continuous strictly increasing function on $[0,\infty)$ such that $\theta(0)=0$ and $\lim_{t\to\infty}\theta(t)=\infty$. Then we say θ is *admissible*.

Let θ be an admissible function. We say that a function h is θ-*quasiconcave* if h is equivalent to a non-decreasing function on $[0,\infty)$ and h/θ is equivalent to a non-increasing function on $(0,\infty)$. We say that a θ-quasiconcave function h is *non-degenerate* if

$$\lim_{t\to 0^+} h(t) = \lim_{t\to\infty}\frac{1}{h(t)} = \lim_{t\to\infty}\frac{h(t)}{\theta(t)} = \lim_{t\to 0^+}\frac{\theta(t)}{h(t)} = 0.$$

The family of non-degenerate θ-quasiconcave functions will be denoted by Ω_θ.

DEFINITION 1.2. Let $0<p\le\infty$ and let $\varphi, v\in\mathfrak{M}^+(0,\infty)$ be such that V_p is admissible function and $\varphi V_p\in\Omega_{V_p}$. We denote by $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})$ the *generalized weighted weak-Lorentz space*, consisting of all measurable functions $f\in\mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{G\Gamma(p,\infty;\varphi,v)} := \operatorname*{ess\,sup}_{r>0}\varphi(r)\|v(\cdot)f^*(\cdot)\|_{p;(0,r)}<\infty.$$

The spaces $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})$ cover several types of important function spaces and have plenty of applications. For example, if $p=1$ and $\varphi(t)=w(t)/V(t)$, $t\in(0,\infty)$, where w is another weight on $(0,\infty)$, then $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})$ reduces to the space $\Gamma_v^\infty(w)$, introduced in [GP2], whose norm is

$$\|f\|_{\Gamma_v^\infty(w)} = \operatorname*{ess\,sup}_{t\in(0,\infty)} w(t)f_v^{**}(t).$$

Carro and Soria [CS1] introduced weak classical Lorentz spaces $\Gamma^{p,\infty}(v)$, with $p\in(0,\infty)$, determined by the norm

$$\|f\|_{\Gamma^{p,\infty}(v)} := \sup_{t\in(0,\infty)} f^{**}(t)V(t)^{1/p}<\infty.$$

In our notation, $\Gamma^{p,\infty}(v) = G\Gamma(1,\infty;V(t)^{1/p}/t,1)$. Weak Lorentz spaces were further investigated e.g. in [CS2], [CS3], [CGS] and [CSPS].

It is also easy to see that

$$\|f\|_{G\Gamma(\infty,\infty;\varphi,v)} = \operatorname*{ess\,sup}_{t>0} f^*(t)v(t)\operatorname*{ess\,sup}_{s\in(t,\infty)}\varphi(s) = \Lambda^\infty(w),$$

where $w(t)=v(t)\operatorname{ess\,sup}_{s\in(t,\infty)}\varphi(s)$.

Our restriction $\varphi V_p\in\Omega_{V_p}$ excludes the case $\varphi\equiv 1$. In this case the space $G\Gamma(p,\infty;1,v)(\mathcal{R})$ reduces to the classical Lorentz space $\Lambda^q(v^q)(\mathcal{R})$.

We note that in general $G\Gamma(p,\infty,\varphi,v)(\mathcal{R})$ need not be a linear space. It will be of interest to give necessary and sufficient conditions for $G\Gamma(p,\infty,\varphi,v)(\mathcal{R})$ to be a linear space.

Our main goal is to give a precise and easily-computable characterization of the norm in the associate space (sometimes also called the Köthe dual) of $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})$. The *associate space* $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})'$ of $G\Gamma(p,\infty;\varphi,v)(\mathcal{R})$ is defined as the collection of all $f \in \mathfrak{M}(\mathcal{R})$ such that

$$\|f\|_{G\Gamma(p,\infty;\varphi,v)(\mathcal{R})'} = \sup_{\|g\|_{G\Gamma(p,\infty;\varphi,v)}\le 1} \int_0^\infty f^*(t)g^*(t)\,dt < \infty. \tag{1.1}$$

Using this result we obtain a characterization of embeddings

$$G\Gamma(p,\infty;\varphi,v)(\mathcal{R}) \hookrightarrow G\Gamma(q,\infty;\psi,w)(\mathcal{R}).$$

Such embeddings have not yet been characterized in a satisfactory way; only a characterization of embeddings

$$G\Gamma(1,\infty;\varphi,v)(\mathcal{R}) \hookrightarrow \Lambda^q(w)(\mathcal{R})$$

was obtained in [GP2].

Thus, given $p,q \in (0,\infty]$, our aim will be to establish necessary and sufficient conditions on the weight functions φ, ψ, w, v for the inequality

$$\|f\|_{G\Gamma(q,\infty;\psi,w)} \le C\|f\|_{G\Gamma(p,\infty;\varphi,v)} \tag{1.2}$$

to hold for every $f \in \mathfrak{M}(\mathcal{R},\mu)$.

Throughout the paper, we write $A \lesssim B$ if there exists a positive constant C, independent of appropriate quantities such as functions, satisfying $A \le CB$. We write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$. As usual, for $p \in (1,\infty)$, we write $p' = p/(p-1)$.

Lemma 1.3. *Let $0<p<\infty$, and let φ and v be weights. Define*

$$\sigma(t) := \operatorname*{ess\,sup}_{s\in(0,t)} V_p(s) \operatorname*{ess\,sup}_{t\in(s,\infty)} \varphi(t), \quad t\in(0,\infty). \tag{1.3}$$

Then σ is the least V_p-quasiconcave majorant of φ, and

$$G\Gamma(p,\infty;\varphi,v) = G\Gamma(p,\infty;\sigma/V_p,v) \tag{1.4}$$

with identical norms. Further, for $t\in(0,\infty)$,

$$\sigma(t) \approx \operatorname*{ess\,sup}_{s\in(0,\infty)} v(s)\frac{V_p(t)}{V_p(s)+V_p(t)}.$$

This is a generalization of [GP2, Lemma 1.5]; the proof is similar and we omit it.

It follows from Lemma 1.3 that in Definition 1.2 the assumption that φV_p is V_p-quasiconcave is not a real restriction. On the other hand, we have not been able to avoid the assumption that φV_p is non-degenerate.

Definition 1.4. Let θ be an admissible function and let ν be a non-negative Borel measure on $[0,\infty)$. We say that function h defined as

$$h(t) := \theta(t) \int\limits_{[0,\infty)} \frac{d\nu(s)}{\theta(s)+\theta(t)}, \qquad t \in (0,\infty),$$

is the *fundamental function* of the measure ν with respect to θ. We will also say that ν is a *representation measure* of h with respect to θ.

We say that the measure ν is *non-degenerate* if for every $t \in (0,\infty)$,

$$\int\limits_{[0,\infty)} \frac{d\nu(s)}{\theta(s)+\theta(t)} < \infty, \qquad \int\limits_{[0,1]} \frac{d\nu(s)}{\theta(s)} = \int\limits_{[1,\infty)} d\nu(s) = \infty.$$

Let us recall [GP1, Remark 2.10(ii)] that if $h \in \Omega_\theta$, then there always exists a representation measure ν of h with respect to θ.

Remark 1.5. It will be useful to note that if $\varphi V_p \in \Omega_{V_p}$, then also $\varphi^{-q} \in \Omega_{V_p^q}$ for all $0<q<\infty$. Let ν be a representation measure of φ^{-q} with respect to V_p^q, i.e.

$$\begin{aligned}\frac{1}{\varphi(t)^q} &\approx V_p(t)^q \int\limits_{[0,\infty)} \frac{d\nu(s)}{V_p(s)^q+V_p(t)^q} \\ &\approx \int\limits_{[0,t]} d\nu(s) + V_p(t)^q \int\limits_{[t,\infty]} \frac{d\nu(s)}{V_p(s)^q}, \qquad t \in (0,\infty).\end{aligned}$$

We shall now formulate our main results.

Theorem 1.6. *Let $0<p,q<\infty$ and let v, w, φ be weights. Assume that v is such that V_p is an admissible function and $\varphi V_p \in \Omega_{V_p}$, and ν is a representation measure of φ^{-q} with respect to V_p^q. Then the inequality*

$$\|f\|_{\Lambda^q(w)} \le C\|f\|_{G\Gamma(p,\infty;\varphi,v)} \tag{1.5}$$

holds for all f if and only if one of the following conditions holds:

(i) $0<p\le q<\infty$ *and*

$$A(1) = \left(\int\limits_0^\infty \left(\sup_{s\in(t,\infty)} \frac{W_q(s)}{V_p(s)} \right)^q d\nu(t) \right)^{1/q} < \infty;$$

(ii) $0<q<p<\infty$ *and*

$$A(2) = \left(\int\limits_0^\infty \left(\int\limits_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p} \right)^{\frac{p}{p-q}} v(s)^p\, ds \right)^{\frac{p-q}{p}} d\nu(t) \right)^{1/q} < \infty.$$

Moreover, the best constant C in (1.5) *satisfies $C \approx A(1)$ in case* (i) *and $C \approx A(2)$ in case* (ii).

THEOREM 1.7. *Let $0 < p < \infty$ and let v, φ be weights. Assume that v is such that V_p is an admissible function and $\varphi V_p \in \Omega_{V_p}$, and ν is a representation measure of φ^{-q} with respect to V_p^q. Then the associate space of $G\Gamma(p,\infty;\varphi,v)$ can be described as follows:*

(i) *If $0 < p \le 1$, then*

$$\|f\|_{G\Gamma(p,\infty;\varphi,v)'} \approx \int_0^\infty \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{sf^{**}(s)}{V_p(s)}\, d\nu(t).$$

(ii) *If $1 < p < \infty$, then*

$$\|f\|_{G\Gamma(p,\infty;\varphi,v)'} \approx \int_0^\infty \bigg(\int_t^\infty \bigg(\frac{sf^{**}(s)}{V_p(s)^p}\bigg)^{p'} v(s)^p\, ds\bigg)^{1/p'} d\nu(t).$$

THEOREM 1.8. *Let $0 < p, q < \infty$ and let v, w, φ, ψ be weights. Assume that v and w are such that V_p and W_q are admissible functions, $\varphi V_p \in \Omega_{V_p}$ and $\psi W_q \in \Omega_{W_q}$. Let ν be a representation measure of φ^{-q} with respect to V_p^q. Then the inequality* (1.2) *holds for some $C > 0$ and all f if and only if one of the following conditions holds:*

(i) $0 < p \le q < \infty$ *and*

$$A(3) := \sup_{s>0} W_q(s)\psi(s)\bigg(\int_s^\infty V_p(t)^{-q}\, d\nu(t)\bigg)^{1/q} < \infty,$$

$$A(4) := \sup_{s>0} V_p(s)^{-1} W_q(s)\psi(s)\bigg(\int_0^s d\nu(t)\bigg)^{1/q} < \infty,$$

$$A(5) := \sup_{s>0} \psi(s)\bigg(\int_0^s \sup_{t\le t<s} V_p(t)^{-q} W_q(t)^q\, d\nu(t)\bigg)^{1/q} < \infty.$$

Moreover, the best constant C in (1.2) *satisfies $C \approx A(3) + A(4) + A(5)$.*

(ii) $0 < q < p < \infty$ *and $A(3) + A(4) + A(6) < \infty$, where*

$$A(6) := \sup_{s>0} \psi(s)\bigg(\int_0^s \bigg(\int_t^s \bigg(\frac{W_q(t)^q}{V_p(t)^p}\bigg)^{\frac{p}{p-q}} v(t)^p\, dt\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q}.$$

Moreover, the best constant C in (1.2) *satisfies $C \approx A(3) + A(4) + A(6)$.*

2. Proofs

Proof of Theorem 1.6. Our aim is to estimate the quantity

$$C = \sup_{\substack{f\in\mathfrak{M}(\mathcal{R})\\ f\ne 0}} \frac{\|f\|_{\Lambda^q(w)}}{\|f\|_{G\Gamma(p,\infty;\varphi,v)}}. \tag{2.1}$$

Rewriting the norm in (2.1) in a more convenient way and setting $h^* = (f^*)^p$, we get

$$\begin{aligned} C &= \sup_{f\neq 0} \frac{\left(\int_0^\infty f^*(t)^q w(t)^q\,dt\right)^{1/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r)\left(\int_0^r f^*(t)^p v(t)^p\,dt\right)^{1/p}} \\ &= \left[\sup_{h\neq 0} \frac{\left(\int_0^\infty h^*(t)^{q/p} w(t)^q\,dt\right)^{p/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r)^p \int_0^r h^*(t) v(t)^p\,dt}\right]^{1/p} \\ &= \left[\sup_{h\neq 0} \frac{\left(\int_0^\infty h^*(t)^{q/p} w(t)^q\,dt\right)^{p/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r)^p V_p(r)^p h^{**}_{v^p}(r)}\right]^{1/p}. \end{aligned}$$

Raising this to the power p, we arrive at

$$C^p = \sup_{h\neq 0} \frac{\left(\int_0^\infty h^*(t)^{q/p} w(t)^q\,dt\right)^{p/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r)^p V_p(r)^p h^{**}_{v^p}(r)}.$$

(i) Let $0 < p \leq q$. If we take $q := q/p$, $w := w^q$, $u := v^p$, $v := \varphi^p V_p^p$, and let ν be the representation measure of φ^{-q} with respect to V_p^q, then $1 \leq q < \infty$ and [GP2, Theorem 1.8(i)] yields

$$C^p \approx \left(\int_0^\infty \sup_{s\in(t,\infty)} \frac{W_q(s)^q}{V_p(s)^q}\,d\nu(t)\right)^{p/q}.$$

Taking the pth root, we get

$$C \approx \left(\int_0^\infty \left(\sup_{s\in(t,\infty)} \frac{W_q(s)}{V_p(s)}\right)^q d\nu(t)\right)^{1/q}.$$

(ii) Let $0 < q < p < \infty$. If we take $q := q/p$, $w(t) := w^q(t)$, $u(t) := v^p$, $v(t) := \varphi(t)^p V_p(t)^p$, and let ν be the representation measure of φ^{-q} with respect to V_p^q, then using [GP2, Theorem 1.8(ii)], we get

$$C^p \approx \left(\int_0^\infty \frac{\zeta(t)}{V_p(t)^q}\,d\nu(t)\right)^{p/q},$$

where

$$\zeta(t) = W_q(t)^q + V_p(t)^q \left(\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{q}{p-q}} w(s)^q\,ds\right)^{1-q/p}, \qquad t\in(0,\infty).$$

Furthermore, we have

$$\zeta(t) \approx V_p(t)^q \left(\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{p}{p-q}} v(s)^p\,ds\right)^{\frac{p-q}{p}} =: \zeta_1(t). \tag{2.2}$$

Indeed,

$$W_q(t)^q = \left(\frac{p}{q}-1\right)^{\frac{p-q}{p}} W_q(t)^q V_p(t)^q \left(\int_t^\infty \frac{v(s)^p}{V_p(s)^{\frac{p^2}{p-q}}}\,ds\right)^{\frac{p-q}{p}} \tag{2.3}$$

$$\lesssim V_p(t)^q \left(\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{p}{p-q}} v(s)^p\,ds\right)^{\frac{p-q}{p}} = \zeta_1(t).$$

Also, integrating by parts, we get

$$\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{q}{p-q}} w(s)^q\,ds \tag{2.4}$$

$$= \frac{p-q}{qp}\int_t^\infty V_p(s)^{-\frac{qp}{p-q}}\,d(W_q(s)^{\frac{qp}{p-q}})$$

$$= \frac{p-q}{qp}\,\frac{W_q(s)^{\frac{qp}{p-q}}}{V_p(s)^{\frac{qp}{p-q}}}\bigg|_t^\infty - \frac{p-q}{qp}\int_t^\infty W_q(s)^{\frac{qp}{p-q}}\,d(V_p(s)^{-\frac{qp}{p-q}})$$

$$= -\frac{p-q}{qp}\,\frac{W_q(t)^{\frac{qp}{p-q}}}{V_p(t)^{\frac{qp}{p-q}}} + \frac{1}{p}\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{p}{p-q}} v(s)^p\,ds.$$

Then from (2.3) and (2.4) we get

$$\zeta(t) \lesssim \zeta_1(t). \tag{2.5}$$

Conversely, from (2.4) we also have

$$\zeta_1(t) \lesssim V_p(t)^q \left(\frac{W_q(t)^{\frac{qp}{p-q}}}{V_p(t)^{\frac{qp}{p-q}}} + \int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{q}{p-q}} w(s)^q\,ds\right)^{\frac{p-q}{p}}$$

$$\lesssim V_p(t)^q W_q(t)^q V_p(t)^{-q} + V_p(t)^q \left(\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{q}{p-q}} w(s)^q\,ds\right)^{1-q/p}$$

$$= \zeta(t).$$

Therefore, together with (2.5) we get (2.2), and consequently

$$C \approx \left(\int_0^\infty \left(\int_t^\infty \left(\frac{W_q(s)^q}{V_p(s)^p}\right)^{\frac{p}{p-q}} v(s)^p\,ds\right)^{\frac{p-q}{p}} d\nu(t)\right)^{1/q}. \ \blacksquare$$

Proof of Theorem 1.7. By the definition of the associate norm,

$$\|f\|_{G\Gamma(p,\infty;\varphi,v)(\mathcal{R})'} = \sup_{\|g\|_{G\Gamma(p,\infty;\varphi,v)}\le 1} \int_0^\infty f^*(t)g^*(t)\,dt$$

$$= \sup_{g\ne 0} \frac{\int_0^\infty f^*(t)g^*(t)\,dt}{\|g\|_{G\Gamma(p,\infty;\varphi,v)}}.$$

This supremum is equal to the optimal constant in (1.5) with $q = 1$ and $w = f^*$. By Theorem 1.6, this optimal constant is comparable to the corresponding value of $A(1)$ if $0 < p \le 1$ and $A(2)$ if $1 < p < \infty$. Therefore, we obtain the required results from Theorem 1.6. ∎

Proof of Theorem 1.8. Our ultimate aim is to estimate the quantity

$$C = \sup_{\substack{f \in \mathfrak{M}(\mathcal{R}) \\ f \neq 0}} \frac{\|f\|_{G\Gamma(q,\infty;\psi,w)}}{\|f\|_{G\Gamma(p,\infty;\varphi,v)}}. \tag{2.6}$$

(i) Assume that $0 < p \le \infty$. Then we can rewrite C in the form

$$\begin{aligned} C &= \sup_{f \neq 0} \frac{\operatorname{ess\,sup}_{r>0} \psi(r) \big(\int_0^r f^*(t)^q w(t)^q \, dt\big)^{1/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r) \big(\int_0^r f^*(t)^p v(t)^p \, dt\big)^{1/p}} \\ &= \operatorname*{ess\,sup}_{r>0} \psi(r) \sup_{f \neq 0} \frac{\big(\int_0^\infty f^*(t)^q w(t)^q \chi_{(0,r)}(t) \, dt\big)^{1/q}}{\operatorname{ess\,sup}_{r>0} \varphi(r) \big(\int_0^r f^*(t)^p v(t)^p \, dt\big)^{1/p}}. \end{aligned}$$

Using Theorem 1.6(i), we get

$$\begin{aligned} C &\approx \operatorname*{ess\,sup}_{r>0} \psi(r) \bigg(\int_0^\infty \bigg(\sup_{s \in (t,\infty)} \frac{W_q(\min(s,r))}{V_p(s)} \bigg)^q d\nu(t) \bigg)^{1/q} \\ &\approx \operatorname*{ess\,sup}_{r>0} \psi(r) \bigg(\int_r^\infty \bigg(\sup_{s \in (t,\infty)} \frac{W_q(\min(s,r))}{V_p(s)} \bigg)^q d\nu(t) \bigg)^{1/q} \\ &\quad + \operatorname*{ess\,sup}_{r>0} \psi(r) \bigg(\int_0^r \bigg(\sup_{s \in (r,\infty)} \frac{W_q(\min(s,r))}{V_p(s)} \bigg)^q d\nu(t) \bigg)^{1/q} \\ &\quad + \operatorname*{ess\,sup}_{r>0} \psi(r) \bigg(\int_0^r \bigg(\sup_{s \in (t,r)} \frac{W_q(\min(s,r))}{V_p(s)} \bigg)^q d\nu(t) \bigg)^{1/q} \\ &= \operatorname*{ess\,sup}_{r>0} \psi(r) W_q(r) \Big(\int_r^\infty V_p(t)^{-q} \, d\nu(t) \Big)^{1/q} \\ &\quad + \operatorname*{ess\,sup}_{r>0} \psi(r) W_q(r) V_p(r)^{-1} \Big(\int_0^r d\nu(t) \Big)^{1/q} \\ &\quad + \operatorname*{ess\,sup}_{r>0} \psi(r) \bigg(\int_0^r \bigg(\sup_{s \in (t,r)} \frac{W_q(\min(s,r))}{V_p(s)} \bigg)^q d\nu(t) \bigg)^{1/q} \\ &= A(3) + A(4) + A(5). \end{aligned}$$

This establishes the assertion in case (i).

(ii) Now assume that $0 < q < p < \infty$. Then using Theorem 1.6(ii) we obtain

$$
\begin{aligned}
C &\approx \operatorname*{ess\,sup}_{r>0} \psi(r)\bigg(\int_0^\infty \bigg(\int_t^\infty \bigg(\frac{W_q(\min(s,r)^q)}{V_p(s)^p}\bigg)^{\frac{p}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\approx \operatorname*{ess\,sup}_{r>0} \psi(r)\bigg(\int_r^\infty \bigg(\int_t^\infty \bigg(\frac{W_q(\min(s,r)^q)}{V_p(s)^p}\bigg)^{\frac{p}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\quad + \operatorname*{ess\,sup}_{r>0} \psi(r)\bigg(\int_0^r \bigg(\int_r^\infty \bigg(\frac{W_q(\min(s,r)^q)}{V_p(s)^p}\bigg)^{\frac{p}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\quad + \operatorname*{ess\,sup}_{r>0} \psi(r)\bigg(\int_0^r \bigg(\int_t^r \bigg(\frac{W_q(\min(s,r)^q)}{V_p(s)^p}\bigg)^{\frac{p}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\approx \operatorname*{ess\,sup}_{r>0} \psi(r)W_q(r)\bigg(\int_r^\infty \bigg(\int_t^\infty V_p(s)^{\frac{p^2}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\quad + \operatorname*{ess\,sup}_{r>0} \psi(r)W_q(r)\bigg(\int_r^\infty V_p(s)^{\frac{p^2}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{pq}} \bigg(\int_0^r d\nu(t)\bigg)^{1/q} \\
&\quad + \operatorname*{ess\,sup}_{r>0} \psi(r)\bigg(\int_0^r \bigg(\int_t^r \bigg(\frac{W_q(s)^q}{V_p(s)^p}\bigg)^{\frac{p}{p-q}} v(s)^p\,ds\bigg)^{\frac{p-q}{p}} d\nu(t)\bigg)^{1/q} \\
&\approx A(3) + A(4) + A(6).
\end{aligned}
$$

The proof is complete. ■

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