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Operators on the stopping time space

by

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Abstract. Let S^1 be the stopping time space and $\mathcal{B}_1(S^1)$ be the Baire-1 elements of the second dual of S^1. To each element x^{**} in $\mathcal{B}_1(S^1)$ we associate a positive Borel measure $\mu_{x^{**}}$ on the Cantor set. We use the measures $\{\mu_{x^{**}} : x^{**} \in \mathcal{B}_1(S^1)\}$ to characterize the operators $T : X \to S^1$, defined on a space X with an unconditional basis, which preserve a copy of S^1. In particular, if $X = S^1$, we show that T preserves a copy of S^1 if and only if $\{\mu_{T^{**}(x^{**})} : x^{**} \in \mathcal{B}_1(S^1)\}$ is non-separable as a subset of $\mathcal{M}(2^{\mathbb{N}})$.

1. Introduction. The stopping time space S^1 was introduced by H. P. Rosenthal as the unconditional analogue of $L^1(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ denotes the Cantor set and $L^1(2^{\mathbb{N}})$ is the Banach space of equivalence classes of measurable functions on $2^{\mathbb{N}}$ which are absolutely integrable on $2^{\mathbb{N}}$ with respect to the Haar measure.

The space S^1 belongs to the wider class of the spaces S^p, $1 \le p < \infty$, which we are about to define. We denote by $2^{<\mathbb{N}}$ the dyadic tree and by $c_{00}(2^{<\mathbb{N}})$ the vector space of all real valued functions defined on $2^{<\mathbb{N}}$ with finite support. For $1 \le p < \infty$ we define the $\|\cdot\|_{S^p}$ norm on $c_{00}(2^{<\mathbb{N}})$ by setting, for $x \in c_{00}(2^{<\mathbb{N}})$,

$$\|x\|_{S^p} = \sup \Big(\sum_{s \in A} |x(s)|^p \Big)^{1/p}$$

where the supremum is taken over all antichains A of $2^{<\mathbb{N}}$. The space S^p is the completion of $(c_{00}(2^{<\mathbb{N}}), \|\cdot\|_{S^p})$.

The space S^1 has a 1-unconditional basis and G. Schechtman, in an unpublished work, showed that it contains almost all ℓ_p isometrically, $1 \le p < \infty$. This result was extended in [6] to all S^p spaces where it was shown that for every $p \le q$, ℓ_q is embedded into S^p. An excellent and detailed study of the stopping time space S^1, in fact in a more general setting, is included in N. Dew's Ph.D. thesis [7]. The interested reader will also find therein, among other things, a proof of Schechtman's result mentioned above. Also,

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in [3] H. Bang and E. Odell showed that S^1 has the weak Banach–Saks and the Dunford–Pettis properties, as does the space $L^1(2^{\mathbb{N}})$.

In what follows, by an *operator* between Banach spaces, we shall always mean a bounded linear operator. Let X, Y, Z be Banach spaces and $T : X \to Y$ be an operator. We will say that T *preserves a copy of* Z if there is a subspace W of X which is isomorphic to Z and the restriction of T to W is an isomorphism. The study of preservation properties of operators on S^1 is important for the isomorphic classification of the complemented subspaces of S^1. The main results of the present paper go in this direction.

In [8] P. Enflo and T. W. Starbird showed that every subspace Y of L^1, isomorphic to L^1, contains a subspace which is isomorphic to L^1 and complemented in L^1. It follows that if a complemented subspace X of L^1 contains a subspace isomorphic to L^1, then X is isomorphic to L^1. Also, if L^1 is isomorphic to an ℓ_1-sum of a sequence of Banach spaces, then one of those spaces is isomorphic to L^1. We will prove the same results in the case of S^1.

The present paper is motivated by the following problems.

Problem 1.1. Let $T : S^1 \to S^1$ be an operator and X an infinite-dimensional reflexive subspace of S^1 such that the restriction of T to X is an isomorphism. Does T preserve a copy of S^1?

Problem 1.2. Let X be a complemented subspace of S^1 such that ℓ_1 does not embed in X. Is X c_0-saturated?

It is well known that a subspace of a space with an unconditional basis is reflexive if and only if it contains neither c_0 nor ℓ_1 isomorphically. Therefore, an affirmative answer to the first problem yields an affirmative answer to the second one.

To state our main results we need to introduce some notation and terminology. If X is a Banach space, we shall denote by $\mathcal{B}_1(X)$ the space of all Baire-1 elements of X^{**}, that is, all $x^{**} \in X^{**}$ such that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x^{**} = w^*\text{-}\lim_n x_n$, which from [10] is equal to the space of all $x^{**} \in X^{**}$ such that $x^{**}|_{(B_{X^*},w^*)}$ is a Baire-1 function. We denote by $\mathcal{M}(2^{\mathbb{N}})$ the space of all Borel measures on $2^{\mathbb{N}}$ endowed with the norm $\|\mu\| = \sup\{|\mu(B)| : B \text{ is a Borel subset of } 2^{\mathbb{N}}\}$. To each $x^{**} \in \mathcal{B}_1(S^1)$ we will associate a positive Borel measure $\mu_{x^{**}}$ on $2^{\mathbb{N}}$. The definition of $\mu_{x^{**}}$ is related to a corresponding concept defined and studied in [1] where the space V_2^0 is studied. It is worth mentioning that in the case of V_2^0 we have $\mathcal{B}_1(V_2^0) = (V_2^0)^{**}$, i.e. a measure can be assigned to every $x^{**} \in (V_2^0)^{**}$. This is not the case for S^1.

We are ready to state the first main result of the paper.

THEOREM 1.3. *Let X be a Banach space with an unconditional basis, Y a closed subspace of X, and $T : X \to S^1$ an operator. The following assertions are equivalent:*

(i) $\{\mu_{T^{**}(y^{**})} : y^{**} \in \mathcal{B}_1(Y)\}$ *is a non-separable subset of* $\mathcal{M}(2^{\mathbb{N}})$.

(ii) *There exists a subspace Z of Y isomorphic to S^1 such that the restriction of T to Z is an isomorphism.*

(iii) *There exists a subspace Z of Y isomorphic to S^1 such that the restriction of T to Z is an isomorphism and $T[Z]$ is complemented in S^1.*

Moreover, if (iii) *is satisfied then Z is complemented in X.*

Clearly, since the basis of S^1 is 1-unconditional, the above theorem holds in particular for $X = S^1$.

We state some consequences of the above theorem.

COROLLARY 1.4. *Let Y be a closed subspace of S^1. Then $\mathcal{M}_{\mathcal{B}_1(Y)} = \{\mu_{y^{**}} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$ if and only if there exists a subspace Z of Y isomorphic to S^1 and complemented in S^1.*

COROLLARY 1.5. *Let Y be a closed subspace of S^1 isomorphic to S^1. Then Y contains a subspace Z isomorphic to S^1 and complemented in S^1.*

COROLLARY 1.6. *Let Y be a complemented subspace of S^1 such that $\{\mu_{y^{**}} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$. Then Y is isomorphic to S^1.*

In [4] it is shown that if L^1 G_δ-embeds into X, where either X is isomorphic to a dual space, or X embeds into L^1, then L^1 embeds isomorphically into X. We recall that an operator $T : X \to Y$ between Banach spaces is called a *G_δ-embedding* if it is injective and $T[K]$ is a G_δ subset of Y for all closed bounded K.

COROLLARY 1.7. *Suppose S^1 is isomorphic to an ℓ_1-sum of a sequence of Banach spaces X_i. Then there is a j such that X_j is isomorphic to S^1.*

Our second result is the following.

THEOREM 1.8. *Let X be a Banach space. If S^1 G_δ-embeds in X and X G_δ-embeds in S^1, then S^1 complementably embeds in X.*

As a consequence of the above theorem we obtain

COROLLARY 1.9. *Let X be a closed subspace of S^1. If S^1 G_δ-embeds in X, then S^1 complementably embeds in X.*

The paper is organized as follows. In Section 2, we fix some notation and recall the definition of S^1. In Section 3, we give the definition of the measure $\mu_{x^{**}}$, while in Section 4 we give the proofs of Theorems 1.3 and 1.8 and their corollaries.

2. Preliminaries

2.1. The dyadic tree. For every $n \geq 0$, we set $2^n = \{0,1\}^n$ (where $2^0 = \{\emptyset\}$). Hence for $n \geq 1$, every $s \in 2^n$ is of the form $s = (s(1), \dots, s(n))$. For $0 \leq m < n$ and $s \in 2^n$, we set $s|m = (s(1), \dots, s(m))$, where $s|0 = \emptyset$. Also, $2^{\leq n} = \bigcup_{i=0}^{n} 2^i$ and $2^{<\mathbb{N}} = \bigcup_{n=0}^{\infty} 2^n$. The *length* $|s|$ of $s \in 2^{<\mathbb{N}}$ is the unique $n \geq 0$ such that $s \in 2^n$. The initial segment partial ordering on $2^{<\mathbb{N}}$ will be denoted by $\sqsubseteq$ (i.e. $s \sqsubseteq t$ if $m = |s| \leq |t|$ and $s = t|m$). For $s, t \in 2^{<\mathbb{N}}$, $s \perp t$ means that s, t are $\sqsubseteq$-incomparable (that is, neither $s \sqsubseteq t$ nor $t \sqsubseteq s$). For $s \in 2^{<\mathbb{N}}$, $s^\frown 0$ and $s^\frown 1$ denote the two immediate successors of s which end with 0 and 1 respectively.

An *antichain* of $2^{<\mathbb{N}}$ is a subset of $2^{<\mathbb{N}}$ such that $s \perp t$ for all distinct $s, t \in A$. If A, B are subsets of $2^{<\mathbb{N}}$ then we write $A \perp B$ if $s \perp t$ for all $s \in A$ and $t \in B$. We denote by $\mathcal{A}$ the set of all antichains of $2^{<\mathbb{N}}$. A *branch* of $2^{<\mathbb{N}}$ is a maximal totally ordered subset of $2^{<\mathbb{N}}$.

A subset $\mathcal{I}$ of $2^{<\mathbb{N}}$ is called a *segment* if $(\mathcal{I}, \sqsubseteq)$ is linearly ordered by $\sqsubseteq$ and for any $s \sqsubset v \sqsubset t$, v is in $\mathcal{I}$ provided that s, t belong to $\mathcal{I}$. For a finite segment $\mathcal{I}$ of $2^{<\mathbb{N}}$, $\max \mathcal{I}$ denotes the $\sqsubseteq$-greatest element of $\mathcal{I}$.

A segment $\mathcal{I}$ is called *initial* if the empty sequence $\emptyset$ belongs to $\mathcal{I}$. For any $s \in 2^{<\mathbb{N}}$, let $\mathcal{I}(s) = \{t \in 2^{<\mathbb{N}} : t \sqsubseteq s\}$. Then clearly $\mathcal{I}(s)$ is an initial segment of $2^{<\mathbb{N}}$. For $s, t \in 2^{<\mathbb{N}}$, the *infimum* of $\{s, t\}$ is defined by $s \wedge t = \max(\mathcal{I}(s) \cap \mathcal{I}(t))$.

The *lexicographical ordering* of $2^{<\mathbb{N}}$, denoted by $\leq_{\text{lex}}$, is defined as follows. For $s, t \in 2^{<\mathbb{N}}$, $s \leq_{\text{lex}} t$ if either $s \sqsubseteq t$, or $s \perp t$, $w^\frown 0 \sqsubseteq s$ and $w^\frown 1 \sqsubseteq t$ where $w = s \wedge t$. Also we write $s <_{\text{lex}} t$ if $s \leq_{\text{lex}} t$ and $s \neq t$. The lexicographical ordering is a total ordering of $2^{<\mathbb{N}}$. This ordering (which identifies $2^{<\mathbb{N}}$ with $\mathbb{N}$) will be called the *natural ordering* of $2^{<\mathbb{N}}$. According to the above, we can write $2^{<\mathbb{N}}$ as a sequence $(s_n)_{n \in \mathbb{N}}$, where $n < m$ if either $|s_n| < |s_m|$, or $|s_n| = |s_m|$ and $s_n <_{\text{lex}} s_m$.

A *dyadic subtree* is a subset T of $2^{<\mathbb{N}}$ such that there is an order isomorphism $\phi : 2^{<\mathbb{N}} \to T$. In this case T is denoted by $T = (t_s)_{s \in 2^{<\mathbb{N}}}$, where $t_s = \phi(s)$. A dyadic subtree $(t_s)_{s \in 2^{<\mathbb{N}}}$ is said to be *regular* if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\{t_s : s \in 2^n\} \subseteq 2^m$.

Let $2^{\mathbb{N}}$ be the *Cantor space*, i.e., the set of all infinite sequences $\sigma = (\sigma(n))_n$ of elements of $2 = \{0, 1\}$. If $\sigma \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\sigma|n = (\sigma(1), \dots, \sigma(n)) \in 2^n$. We say that $s \in 2^n$ is an *initial segment* of $\sigma \in 2^{\mathbb{N}}$ if $s = \sigma|n$; here $\sigma|0 = \emptyset$. We write $s \sqsubseteq \sigma$ if s is an initial segment of σ.

2.2. The stopping time space S^1. The space S^1 is defined to be the completion of $c_{00}(2^{<\mathbb{N}})$ under the norm

$$\Big\|\sum_s \lambda_s e_s\Big\|_{S^1} = \sup\Big(\sum_{s\in A}|\lambda_s|\Big)$$

where the supremum is taken over all antichains A of $2^{<\mathbb{N}}$; here for every $s \in 2^{<\mathbb{N}}$, we denote by e_s the characteristic function of $\{s\}$. By the definition of the S^1-norm, $\{e_s\}_{s\in 2^{<\mathbb{N}}}$ is a 1-unconditional Schauder basis of S^1. Also, for every infinite chain C of $2^{<\mathbb{N}}$ the subspace of S^1 generated by $\{e_s : s \in C\}$ is isomorphic to c_0, while for every infinite antichain A the corresponding subspace is isomorphic to ℓ_1.

3. Measures associated to elements of Baire-1 space. Our general notation and terminology is standard and can be found, for instance, in [9]. Let X be a Banach space. As mentioned in the introduction, we denote by $\mathcal{B}_1(X)$ the space of all $x^{**} \in X^{**}$ such that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x^{**} = w^*\text{-}\lim_n x_n$. The aim of this section is to introduce and study the fundamental properties of a measure $\mu_{x^{**}}$ corresponding to an element $x^{**} \in \mathcal{B}_1(S^1)$. We start with some general results about Banach spaces with an unconditional basis.

Lemma 3.1. *Let X be a Banach space with an unconditional normalized basis $(e_i)_{i\in\mathbb{N}}$ and let $x^{**} \in \mathcal{B}_1(X)$ be such that $x^{**}(e_i^*) = 0$ for every $i \in \mathbb{N}$. Then $x^{**} = 0$.*

Proof. Since $x^{**} \in \mathcal{B}_1(X)$ there exists a sequence $(x_n)_n$ of elements of X such that $x^{**} = w^*\text{-}\lim_n x_n$ and $\lim_n e_i^*(x_n) = 0$ for every $i \in \mathbb{N}$. By a classical result of Bessaga and Pełczyński, we may assume that $(x_n)_n$ is equivalent to a block basic sequence with respect to $(e_i)_i$. Since $(e_i)_i$ is an unconditional basis, every block sequence of $(e_i)_i$ is unconditional. But then either $(x_n)_n$ has a subsequence which is equivalent to the usual basis of ℓ_1, or $(x_n)_n$ is weakly null. Since the basis of ℓ_1 is not weakly Cauchy, we conclude that $x^{**} = 0$. ∎

Proposition 3.2. *Let X be a Banach space with a 1-unconditional normalized basis $(e_i)_{i=1}^\infty$. Then $\mathcal{B}_1(X)$ can be identified with the space of all sequences $(a_i)_i$ of scalars such that $\sup_n \|\sum_{i=1}^n a_i e_i\| < \infty$. This correspondence is given by $\mathcal{B}_1(X) \ni x^{**} \leftrightarrow (x^{**}(e_i^*))_i$. Moreover, for every $x^{**} \in \mathcal{B}_1(X)$, $x^{**} = w^*\text{-}\lim_n \sum_{i=1}^n x^{**}(e_i^*)e_i$ and*

$$\|x^{**}\| = \sup_n \Big\|\sum_{i=1}^n x^{**}(e_i^*)e_i\Big\|. \tag{1}$$

Proof. Let $x^{**} \in \mathcal{B}_1(X)$ and $x^* \in X^*$. For $n \in \mathbb{N}$, we choose a sequence $(\varepsilon_i)_{i=1}^n$ of signs so that $\sum_{i=1}^n |x^{**}(e_i^*)x^*(e_i)| = \sum_{i=1}^n \varepsilon_i x^{**}(e_i^*)x^*(e_i)$. Since

$(e_i)_i$ is a 1-unconditional normalized basis of X, we have $\|\sum_{i=1}^n \varepsilon_i x^*(e_i)e_i\| \le \|x^*\|$ and

$$\sum_{i=1}^n |x^{**}(e_i^*)x^*(e_i)| = \sum_{i=1}^n \varepsilon_i x^{**}(e_i^*)x^*(e_i) = x^{**}\Big(\sum_{i=1}^n \varepsilon_i x^*(e_i)e_i\Big)$$
$$\le \|x^{**}\|\,\|x^*\|.$$

Hence, the sequence $(\sum_{i=1}^n x^{**}(e_i^*)e_i)_n$ is weakly Cauchy and

$$\Big\|\sum_{i=1}^n x^{**}(e_i^*)e_i\Big\| \le \|x^{**}\| \quad \text{for all } n \in \mathbb{N}.$$

Let $y^{**} = w^*\text{-}\lim_n \sum_{i=1}^n x^{**}(e_i^*)e_i$. Then $(y^{**} - x^{**})(e_i^*) = 0$ for every $i \in \mathbb{N}$ and $y^{**} - x^{**} \in \mathcal{B}_1(X)$. By Lemma 3.1, we have $y^{**} = x^{**}$. Therefore, $x^{**} = w^*\text{-}\lim_n \sum_{i=1}^n x^{**}(e_i^*)e_i$. By the above inequality and since $(e_i)_i$ is a monotone basis of X, from the weak* lower semicontinuity of the second dual norm we derive that

$$\|x^{**}\| = \lim_n \Big\|\sum_{i=1}^n x^{**}(e_i^*)e_i\Big\| = \sup_n \Big\|\sum_{i=1}^n x^{**}(e_i^*)e_i\Big\|.$$

Conversely, let $(a_i)_i$ be such that $C = \sup_n \|\sum_{i=1}^n a_ie_i\| < \infty$. For any $x^* \in X^*$ and $n \in \mathbb{N}$, we choose a sequence $(\varepsilon_i)_{i=1}^n$ of signs such that $\sum_{i=1}^n |a_ix^*(e_i)| = x^*(\sum_{i=1}^n \varepsilon_i a_i e_i)$. As $(e_i)_i$ is a 1-unconditional normalized basis of X, we get $\sum_{i=1}^n |a_ix^*(e_i)| \le C\|x^*\|$. Therefore, $(\sum_{i=1}^n a_ix^*(e_i))_n$ is weakly Cauchy. If $x^{**} = w^*\text{-}\lim_n \sum_{i=1}^n a_ie_i$, by Lemma 3.1, we conclude that x^{**} is the unique $x^{**} \in \mathcal{B}_1(X)$ such that $x^{**}(e_i^*) = a_i$ for every $i \in \mathbb{N}$. ∎

If X is a Banach space and M a subset of X, then the closed linear span of M is denoted by $[M]$.

Proposition 3.3. *Let X be a Banach space with a 1-unconditional normalized basis $(e_i)_{i=1}^\infty$. Then the operator $J : \mathcal{B}_1(X) \to [(e_i^*)_i]^*$ with $J(x^{**}) = x^{**}|_{[(e_i^*)_i]}$ is an isometry and onto.*

Proof. The fact that J is an isometry is an immediate consequence of (1). To show that J is onto, let $f \in [(e_i^*)_i]^*$. Then we easily observe that $\sup_n \|\sum_{i=1}^n f(e_i^*)e_i\| \le \|f\|$. Hence by Proposition 3.2, there is $x^{**} \in \mathcal{B}_1(X)$ such that $x^{**} = w^*\text{-}\lim_n \sum_{i=1}^n f(e_i^*)e_i$. Therefore $x^{**}(e_i^*) = f(e_i^*)$ for all i, and so $J(x^{**}) = x^{**}|_{[(e_i^*)_i]} = f$. ∎

Remark 3.4. By Proposition 3.2, for every $x^{**} \in \mathcal{B}_1(S^1)$,

$$\|x^{**}\| = \sup_{n\ge 0} \Big\|\sum_{|s|\le n} x^{**}(e_s^*)e_s\Big\| = \sup_{A\in\mathcal{A}} \sum_{s\in A} |x^{**}(e_s^*)|.$$

We are now ready to give the definition of $\mu_{x^{**}}$, the measure associated with an element x^{**} in $\mathcal{B}_1(S^1)$. We first introduce some simple notation. For every $x^{**} \in \mathcal{B}_1(S^1)$ and $D \subseteq 2^{<\mathbb{N}}$ we set

$$x^{**}|D(e_s^*) = \begin{cases} x^{**}(e_s^*) & \text{if } s \in D, \\ 0 & \text{if } s \in 2^{<\mathbb{N}} \setminus D. \end{cases}$$

By Remark 3.4, we easily observe that $x^{**}|D \in \mathcal{B}_1(S^1)$ and for any $D_1, D_2 \subseteq 2^{<\mathbb{N}}$ with $D_1 \subseteq D_2$,

$$\|x^{**}|D_1\| \le \|x^{**}|D_2\| \le \|x^{**}\|, \tag{2}$$

and if $D_1 \perp D_2$, then

$$\|x^{**}|D_1 \cup D_2\| = \|x^{**}|D_1\| + \|x^{**}|D_2\|. \tag{3}$$

For every $t \in 2^{<\mathbb{N}}$ we set $V_t = \{\sigma \in 2^{\mathbb{N}} : t \sqsubseteq \sigma\}$. Recall that $\{V_t : t \in 2^{<\mathbb{N}}\}$ is the usual basis of $2^{\mathbb{N}}$, consisting of clopen sets. Also, for $t \in 2^{<\mathbb{N}}$ and $m \ge 0$ we set

$$T_t^m = \{s \in 2^{<\mathbb{N}} : t \sqsubseteq s,\ |s| \ge m\}.$$

By (3), it is easy to see that for every $t \in 2^{<\mathbb{N}}$,

$$\inf_m \|x^{**}|T_t^m\| = \inf_m \|x^{**}|T_{t^\frown 0}^m\| + \inf_m \|x^{**}|T_{t^\frown 1}^m\|.$$

Therefore, by a classical result of Carathéodory, there exists a unique finite positive Borel measure $\mu_{x^{**}}$ on $2^{\mathbb{N}}$ with $\mu_{x^{**}}(V_t) = \inf_m \|x^{**}|T_t^m\|$ for every $t \in 2^{<\mathbb{N}}$.

Definition 3.5. Let $x^{**} \in \mathcal{B}_1(S^1)$. We define $\mu_{x^{**}}$ to be the unique finite positive Borel measure on $2^{\mathbb{N}}$ such that for all $t \in 2^{\mathbb{N}}$,

$$\mu_{x^{**}}(V_t) = \inf_m \|x^{**}|T_t^m\|.$$

Remark 3.6. The technique used to define the measure $\mu_{x^{**}}$ given $x^{**} \in \mathcal{B}_1(S^1)$ is along the same lines as in [1].

Remark 3.7. Let $x^{**} \in \mathcal{B}_1(S^1)$. By (3) we observe that for every finite antichain A of $2^{<\mathbb{N}}$ and $m \ge 0$, $\|x^{**}|\bigcup_{t\in A} T_t^m\| = \sum_{t\in A} \|x^{**}|T_t^m\|$. Therefore, $\mu_{x^{**}}(\bigcup_{t\in A} V_t) = \inf_m \|x^{**}|\bigcup_{t\in A} T_t^m\|$ for every finite antichain A of $2^{<\mathbb{N}}$.

The following proposition is easily established.

Proposition 3.8.

(i) *For every $x^{**} \in \mathcal{B}_1(S^1)$ and $\lambda \in \mathbb{R}$,*

$$\mu_{\lambda x^{**}} = |\lambda| \mu_{x^{**}}.$$

(ii) *For every $x^{**}, y^{**} \in \mathcal{B}_1(S^1)$,*

$$\mu_{x^{**}+y^{**}} \le \mu_{x^{**}} + \mu_{y^{**}}.$$

(iii) *For every $x^{**} \in \mathcal{B}_1(S^1)$,*

$$\mu_{x^{**}}(2^{\mathbb{N}}) = d(x^{**}, S^1) = \inf\{\|x^{**} - x\| : x \in S^1\}.$$

(iv) *For every* $x^{**}, y^{**} \in \mathcal{B}_1(S^1)$,

$$\|\mu_{x^{**}} - \mu_{y^{**}}\| \leq \|x^{**} - y^{**}\|.$$

PROPOSITION 3.9. *Let* $x^{**} \in \mathcal{B}_1(S^1)$. *Then for every antichain* A *of* $2^{<\mathbb{N}}$,

$$\mu_{x^{**}}\Big(\bigcup_{t\in A} V_t\Big) = \inf_m \Big\|x^{**}\Big|\bigcup_{t\in A} T_t^m\Big\|.$$

Proof. Let A be an antichain of $2^{<\mathbb{N}}$. If A is finite, then the conclusion follows by Remark 3.7. Assume that A is infinite and let $A = \{t_i\}_{i=1}^{\infty}$ be an enumeration. Since $\lim_j \mu_{x^{**}}(\bigcup_{i=1}^{j} V_{t_i}) = \mu_{x^{**}}(\bigcup_{t\in A} V_t)$, we have

$$\mu_{x^{**}}\Big(\bigcup_{t\in A} V_t\Big) = \lim_j \inf_m \Big\|x^{**}\Big|\bigcup_{i=1}^{j} T_{t_i}^m\Big\| \leq \inf_m \Big\|x^{**}\Big|\bigcup_{t\in A} T_t^m\Big\|. \tag{4}$$

We claim that $\lim_j \|x^{**}|\bigcup_{i=j}^{\infty} T_{t_i}^0\| = 0$. Indeed, if not, then by Remark 3.4 it is easy to see that there exist $\varepsilon > 0$, a strictly increasing sequence $1 \leq j_1 < j_2 < \cdots$ in $\mathbb{N}$ and a sequence $(A_k)_{k=1}^{\infty}$ of finite antichains of $2^{<\mathbb{N}}$ such that:

(a) $A_k \subseteq \bigcup_{i=j_k+1}^{j_k} T_{t_i}^0$ for every $k \geq 1$.

(b) $\sum_{s\in A_k} |x^{**}(e_s^*)| > \varepsilon$ for every $k \geq 1$.

Clearly the set $B = \bigcup_{k=1}^{\infty} A_k$ is an antichain. Hence by (b),

$$\|x^{**}\| \geq \sum_{s\in B} |x^{**}(e_s^*)| = \sum_{k=1}^{\infty}\sum_{s\in A_k} |x^{**}(e_s^*)| = \infty,$$

a contradiction.

Let $\varepsilon > 0$. By the claim, we choose $j_0 \in \mathbb{N}$ with $\|x^{**}|\bigcup_{i=j_0+1}^{\infty} T_{t_i}^0\| < \varepsilon$. Then

$$\begin{aligned}\inf_m \Big\|x^{**}\Big|\bigcup_{i=1}^{\infty} T_{t_i}^m\Big\| &= \inf_m \Big\|x^{**}\Big|\bigcup_{i=1}^{j_0} T_{t_i}^m\Big\| + \inf_m \Big\|x^{**}\Big|\bigcup_{i=j_0+1}^{\infty} T_{t_i}^m\Big\| \\ &\leq \mu_{x^{**}}\Big(\bigcup_{i=1}^{j_0} V_{t_i}\Big) + \Big\|x^{**}\Big|\bigcup_{i=j_0+1}^{\infty} T_{t_i}^0\Big\| \\ &< \mu_{x^{**}}\Big(\bigcup_{i=1}^{j_0} V_{t_i}\Big) + \varepsilon \leq \mu_{x^{**}}\Big(\bigcup_{i=1}^{\infty} V_{t_i}\Big) + \varepsilon.\end{aligned}$$

Therefore,

$$\inf_m \Big\|x^{**}\Big|\bigcup_{t\in A} T_t^m\Big\| \leq \mu_{x^{**}}\Big(\bigcup_{t\in A} V_{t_i}\Big). \tag{5}$$

By (4) and (5), the conclusion follows. ∎

4. Operators preserving a copy of S^1. This section is devoted to the proof of Theorems 1.3 and 1.8 and their corollaries. The next lemma can be found in [1, Lemma 41, p. 4252].

LEMMA 4.1. *Let $(\alpha_s)_{s\in 2^{<\mathbb{N}}}$ and $(\lambda_s)_{s\in 2^{<\mathbb{N}}}$ be families of non-negative real numbers and let $n \geq 0$. Then there exists a maximal antichain A of $2^{\leq n}$ and a family $(b_t)_{t\in A}$ of branches of $2^{\leq n}$ such that*

$$\sum_{|s|\leq n} \lambda_s \alpha_s \leq \sum_{t\in A} \Big(\sum_{s\in b_t} \alpha_s\Big)\lambda_t$$

and $t \in b_t$ for all $t \in A$. Therefore if $\sum_{n=1}^{\infty} \alpha_{\sigma|n} \leq C$ for all $\sigma \in 2^{\mathbb{N}}$, then for each $n \geq 0$ there is an antichain A of $2^{\leq n}$ such that

$$\sum_{|s|\leq n} \lambda_s \alpha_s \leq C \sum_{s\in A} \lambda_s.$$

PROPOSITION 4.2. *Let X be a Banach space and $(x_s)_{s\in 2^{<\mathbb{N}}}$ be a family of elements of X with the following properties:*

(i) *$(x_s)_{s\in 2^{<\mathbb{N}}}$ is a K-unconditional basic family.*
(ii) *There is a constant $c > 0$ such that for each finite antichain A of $2^{<\mathbb{N}}$ and each family $(\lambda_s)_{s\in A}$ of scalars, we have*

$$\Big\| \sum_{s\in A} \lambda_s x_s \Big\| \geq c \sum_{s\in A} |\lambda_s|.$$

(iii) *There is a constant $C > 0$ such that for every $\sigma \in 2^{\mathbb{N}}$, $n \geq 0$ and every sequence $(a_k)_{k=0}^{n}$ of scalars,*

$$\Big\| \sum_{k=0}^{n} a_k x_{\sigma|k} \Big\| \leq C \max_{0\leq k\leq n} |a_k|.$$

Then the family $(x_s)_{s\in 2^{<\mathbb{N}}}$ is equivalent to the basis $(e_s)_{s\in 2^{<\mathbb{N}}}$ of S^1. In particular for every $n \geq 0$ and every family $(\lambda_s)_{|s|\leq n}$ of scalars,

$$\frac{c}{K}\Big\| \sum_{|s|\leq n} \lambda_s e_s \Big\| \leq \Big\| \sum_{|s|\leq n} \lambda_s x_s \Big\| \leq C \Big\| \sum_{|s|\leq n} \lambda_s e_s \Big\|.$$

Proof. First we show the upper S^1-estimate. Fix a family $(\lambda_s)_{|s|\leq n}$ of scalars. Let $x^* \in (S^1)^*$ with $\|x^*\| = 1$. By Lemma 4.1 (with $|x^*(x_s)|$ in place of α_s), there is an antichain $A \subseteq 2^{\leq n}$ and a family $(b_t)_{t\in A}$ of branches of $2^{\leq n}$ such that

$$\Big| \sum_{|s|\leq n} \lambda_s x^*(x_s) \Big| \leq \sum_{|s|\leq n} |\lambda_s|\,|x^*(x_s)| \leq \sum_{t\in A} \Big(\sum_{s\in b_t} |x^*(x_s)| \Big) |\lambda_t|. \tag{6}$$

By (iii), we have $\sum_{s\in b_t}|x^*(x_s)|\le C$ for every $t\in A$. Hence by (6),

$$\Big|\sum_{|s|\le n}\lambda_s x^*(x_s)\Big|\le C\sum_{t\in A}|\lambda_t|.$$

This yields $\|\sum_{|s|\le n}\lambda_s x_s\|\le C\|\sum_{|s|\le n}\lambda_s e_s\|$.

To show the lower S^1-estimate, let A be an antichain of $2^{\le n}$ such that $\|\sum_{|s|\le n}\lambda_s e_s\|=\sum_{s\in A}|\lambda_s|$. Then by (i) and (ii),

$$\frac{c}{K}\Big\|\sum_{|s|\le n}\lambda_s e_s\Big\|=\frac{c}{K}\sum_{s\in A}|\lambda_s|\le\frac{1}{K}\Big\|\sum_{s\in A}\lambda_s x_s\Big\|\le\Big\|\sum_{|s|\le n}\lambda_s x_s\Big\|. \ ■$$

Remark 4.3. By Proposition 4.2, it is easy to see that if $(x_s)_s$ is a family in X equivalent to the S^1-basis and $(t_s)_s$ is a dyadic subtree of $2^{<\mathbb{N}}$, then $(x_{t_s})_s$ is equivalent to the S^1-basis as well.

Lemma 4.4. *Let $x^{**}\in\mathcal{B}_1(S^1)$ and $(x_n)_n$ be a sequence in S^1 which is weak* convergent to x^{**}. If there exists an antichain B of $2^{<\mathbb{N}}$ such that $\mu_{x^{**}}(\bigcup_{s\in B}V_s)>\rho>0$, then for every $k\in\mathbb{N}$, there exist a finite antichain A of $2^{<\mathbb{N}}$ with $A\subseteq\bigcup_{s\in B}T^0_s$ and $l>k$ such that*

$$\sum_{s\in A}|e^*_s(x_l-x_k)|>\rho.$$

Proof. Let $k\in\mathbb{N}$ and $0<3\varepsilon<\mu_{x^{**}}(\bigcup_{s\in B}V_s)-\rho$. Since $x_k\in S^1$, there is $m\in\mathbb{N}$ such that

$$\|x_k|T^m_\emptyset\|<\varepsilon. \tag{7}$$

Choose a finite antichain A of $2^{<\mathbb{N}}$ with $A\subseteq\bigcup_{s\in B}T^m_s$ such that

$$\mu_{x^{**}}\Big(\bigcup_{s\in B}V_s\Big)-\varepsilon<\sum_{s\in A}|x^{**}(e^*_s)|. \tag{8}$$

Moreover since $(x_n)_n$ converges weak* to x^{**}, there is $l>k$ such that

$$\Big|\sum_{s\in A}|x^{**}(e^*_s)|-\sum_{s\in A}|e^*_s(x_l)|\Big|<\varepsilon. \tag{9}$$

Then by (7)–(9) we have

$$\begin{aligned}\sum_{s\in A}|e^*_s(x_l-x_k)|&\ge\sum_{s\in A}|e^*_s(x_l)|-\sum_{s\in A}|e^*_s(x_k)|>\sum_{s\in A}|x^{**}(e^*_s)|-2\varepsilon\\&>\mu_{x^{**}}\Big(\bigcup_{s\in B}V_s\Big)-3\varepsilon>\rho. \ ■\end{aligned}$$

In the following, if Y is a subspace of the Banach space X, then we identify Y^{**} with the subspace $Y^{\perp\perp}$ in X^{**}.

LEMMA 4.5. *Let X be a Banach space with a 1-unconditional normalized basis $(e_i)_i$, Y a closed subspace of X, and $T : X \to S^1$ an operator. Suppose that there exist $y^{**} \in \mathcal{B}_1(Y)$ and an antichain B of $2^{<\mathbb{N}}$ such that*

$$\mu_{T^{**}(y^{**})}\Big(\bigcup_{s\in B} V_s\Big) > \rho > 0.$$

Then for every $m \in \mathbb{N}$ and $\varepsilon > 0$, there exist $y \in Y$, a finite subset F of $\mathbb{N}$ with $m < \min F$, scalars $\{d_i\}_{i\in F}$ with $0 \le d_i \le 1$ for every $i \in F$, and a finite antichain A of $2^{<\mathbb{N}}$ with $A \subseteq \bigcup_{s\in B} T_s^0$ such that

$$\Big\|y - \sum_{i\in F} d_i y^{**}(e_i^*)e_i\Big\| < \varepsilon \quad \textit{and} \quad \sum_{s\in A} |e_s^*(T(y))| > \rho.$$

Proof. We fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Let $(y_n)_n$ be a sequence in Y which is weak* convergent to y^{**}. By Proposition 3.2, the sequence $(\sum_{i=1}^{j} y^{**}(e_i^*)e_i)_j$ weak* converges to y^{**}. Hence $(y_j - \sum_{i=1}^{j} y^{**}(e_i^*)e_i)_j$ is weakly null. By Mazur's theorem there is a convex block sequence $(\sum_{j\in F_n} r_j(y_j - \sum_{i=1}^{j} y^{**}(e_i^*)e_i))_n$, where $(F_n)_n$ is a sequence of successive finite subsets of $\mathbb{N}$ (i.e., $\max F_n < \min F_{n+1}$ for all $n \in \mathbb{N}$), $\sum_{j\in F_n} r_j = 1$ and $r_j \ge 0$ for all $j \in F_n$, such that

$$\lim_n \Big\| \sum_{j\in F_n} r_j\Big(y_j - \sum_{i=1}^{j} y^{**}(e_i^*)e_i\Big)\Big\| = 0.$$

We choose $n_1 \in \mathbb{N}$ with $m < \min F_{n_1}$ and

$$\Big\|\sum_{j\in F_n} r_j y_j - \sum_{j\in F_n} r_j \sum_{i=1}^{j} y^{**}(e_i^*)e_i\Big\| < \varepsilon/2 \quad \text{for all } n \ge n_1. \tag{10}$$

Since $(\sum_{j\in F_n} r_j y_j)_n$ is weak* convergent to y^{**}, and since T^{**} is weak*-weak* continuous and $T^{**}|X = T$, we find that $(T(\sum_{j\in F_n} r_j y_j))_n$ weak* converges to $T^{**}(y^{**})$. Applying Lemma 4.4 we deduce that there exist a finite antichain A of $2^{<\mathbb{N}}$ with $A \subseteq \bigcup_{s\in B} T_s^0$ and $n_2 > n_1$ such that

$$\sum_{s\in A} \Big| e_s^*\Big(T\Big(\sum_{j\in F_{n_2}} r_j y_j - \sum_{j\in F_{n_1}} r_j y_j\Big)\Big)\Big| > \rho. \tag{11}$$

We set $y_1 = \sum_{j\in F_{n_1}} r_j y_j$, $y_2 = \sum_{j\in F_{n_2}} r_j y_j$ and $y = y_2 - y_1$. Note that

$$\sum_{j\in F_{n_2}} r_j \sum_{i=1}^{j} y^{**}(e_i^*)e_i - \sum_{j\in F_{n_1}} r_j \sum_{i=1}^{j} y^{**}(e_i^*)e_i = \sum_{i\in F} d_i y^{**}(e_i^*)e_i,$$

where F is a finite subset of $\mathbb{N}$, $m < \min F$ and $0 \le d_i \le 1$ for every $i \in F$. Therefore by (10) and (11) the conclusion of the lemma follows. ∎

We recall that $\mathcal{M}^+(2^{\mathbb{N}})$ denotes the positive cone of $\mathcal{M}(2^{\mathbb{N}})$.

LEMMA 4.6. *Let $\{\mu_\xi\}_{\xi<\omega_1}$ be a non-separable subset of $\mathcal{M}^+(2^{\mathbb{N}})$. Then there is an uncountable subset Γ of ω_1 such that for every $\xi \in \Gamma$, $\mu_\xi = \lambda_\xi + \tau_\xi$ where λ_ξ, τ_ξ are positive Borel measures on $2^{\mathbb{N}}$ with the following properties:*

- *For all $\xi \in \Gamma$, $\lambda_\xi \perp \tau_\xi$ and $\|\tau_\xi\| > 0$.*
- *For all $\zeta < \xi$ in Γ, $\mu_\zeta \perp \tau_\xi$.*

In particular, $\tau_\zeta \perp \tau_\xi$ for all $\zeta < \xi$ in Γ.

Proof. We may suppose that for some $\delta > 0$, $\|\mu_\xi - \mu_\zeta\| > \delta$ for all $0 \le \zeta < \xi < \omega_1$. By transfinite induction we construct a strictly increasing sequence $(\xi_\alpha)_{\alpha<\omega_1}$ in ω_1 such that for each $\alpha < \omega_1$, $\mu_{\xi_\alpha} = \lambda_{\xi_\alpha} + \tau_{\xi_\alpha}$ with $\lambda_{\xi_\alpha} \perp \tau_{\xi_\alpha}$, $\|\tau_{\xi_\alpha}\| > 0$ and $\tau_{\xi_\alpha} \perp \mu_{\xi_\beta}$ for all $\beta < \alpha$. The general inductive step of the construction is as follows. Suppose that for some $\alpha < \omega_1$, $(\xi_\beta)_{\beta<\alpha}$ has been defined. Let $(\beta_n)_n$ be an enumeration of α and set

$$\zeta_\alpha = \sup_n \xi_{\beta_n}, \quad \nu_\alpha = \sum_n \mu_{\xi_{\beta_n}}/2^n, \quad N_\alpha = \{\xi < \omega_1 : \zeta_\alpha < \xi \text{ and } \mu_\xi \ll \nu_\alpha\}.$$

By the Radon–Nikodym theorem, $\{\mu_\xi\}_{\xi\in N_\alpha}$ is isometrically contained in $L_1(2^{\mathbb{N}}, \nu_\alpha)$ and therefore it is norm separable. Since we have assumed that $\|\mu_\xi - \mu_\zeta\| > \delta$ for all $0 \le \zeta < \xi < \omega_1$, we infer that N_α is countable. Hence we can choose $\xi_\alpha > \sup N_\alpha$. Let $\mu_{\xi_\alpha} = \lambda_{\xi_\alpha} + \tau_{\xi_\alpha}$ be the Lebesgue analysis of μ_{ξ_α} where $\lambda_{\xi_\alpha} \ll \nu_\alpha$ and $\tau_{\xi_\alpha} \perp \nu_\alpha$. By the definition of ν_α and ξ_α, we have $\|\tau_{\xi_\alpha}\| > 0$, $\tau_{\xi_\alpha} \perp \mu_{\xi_\beta}$ for all $\beta < \alpha$, and the inductive step of the construction has been completed. ∎

LEMMA 4.7. *Let $\{\tau_\xi\}_{\xi<\omega_1}$ be an uncountable family of pairwise singular positive Borel measures on $2^{\mathbb{N}}$. Then for every finite family $(\Gamma_i)_{i=1}^k$ of pairwise disjoint uncountable subsets of ω_1 and every $\varepsilon > 0$ there exist a family $(\Gamma_i')_{i=1}^k$ with Γ_i' an uncountable subset of Γ_i and a family $(O_i)_{i=1}^k$ of open and pairwise disjoint subsets of $2^{\mathbb{N}}$ such that $\tau_\xi(2^{\mathbb{N}} \setminus O_i) < \varepsilon$ for all $1 \le i \le k$ and $\xi \in \Gamma_i'$.*

Proof. For every $\alpha < \omega_1$, we choose $(\xi_i^\alpha)_{i=1}^k \in \prod_{i=1}^k \Gamma_i$ such that $\xi_i^\alpha \ne \xi_i^\beta$ for every $\alpha \ne \beta$ in ω_1 and every $1 \le i \le k$. For each $0 \le \alpha < \omega_1$ the k-tuple $(\tau_{\xi_\alpha^i})_{i=1}^k$ consists of pairwise singular measures and so we may choose a k-tuple $(U_i^\alpha)_{i=1}^k$ of clopen subsets of $2^{\mathbb{N}}$ with $\tau_{\xi_i^\alpha}(2^{\mathbb{N}} \setminus O_i^\alpha) < \varepsilon$ for each i, and $O_i^\alpha \cap O_j^\alpha = \emptyset$ for $i \ne j$.

Since the family of all clopen subsets of $2^{\mathbb{N}}$ is countable, there is a k-tuple $(O_i)_i$ and an uncountable subset Γ of ω_1 such that $U_i^\alpha = O_i$ for all $1 \le i \le k$ and $\alpha \in \Gamma$. For each $1 \le i \le k$, set $\Gamma_i' = \{\xi_i^\alpha : \alpha \in \Gamma\}$. Then $\tau_\xi(2^{\mathbb{N}} \setminus O_i) < \varepsilon$ for all $1 \le i \le k$ and $\xi \in \Gamma_i'$. ∎

The following lemma is the main tool used to prove Theorem 1.3.

LEMMA 4.8. *Let X be a Banach space with a 1-unconditional basis, Y a closed subspace of X and $T : X \to S^1$ an operator. Suppose that $\mathcal{B}_1(Y)$*

contains an uncountable family $\mathcal{G}$ such that $\mathcal{M}_{\mathcal{G}} = \{\mu_{T^{**}(x^{**})} : x^{**} \in \mathcal{G}\}$ is non-separable. Then there is a constant $\rho > 0$ such that for every $\varepsilon > 0$ there exist a family $(y_s)_s$ of elements of Y and a family $(A_s)_s$ of finite antichains of $2^{<\mathbb{N}}$ such that:

(C1) *$A_s \perp A_t$ for all $s \perp t$.*
(C2) *$\sum_{t\in A_s} |e_t^*(T(y_s))| > \rho$ for every $s \in 2^{<\mathbb{N}}$.*
(C3) *For every $\sigma \in 2^{\mathbb{N}}$ and every sequence $(a_k)_{k=0}^\infty$ of scalars,*

$$\Big\| \sum_{k=0}^{n} a_k y_{\sigma|k} \Big\| \le (1+\varepsilon) \max_{0\le k\le n} |a_k| \quad \text{for all } n \ge 0.$$

Proof. Let $\{e_i\}_{i=1}^\infty$ be a 1-unconditional normalized basis of X. Since $\mu_{\lambda x^{**}} = |\lambda|\mu_{x^{**}}$ for all $x^{**} \in \mathcal{B}_1(S^1)$ and $\lambda \in \mathbb{R}$, we may assume that $\mathcal{G} \subseteq \{y^{**} \in \mathcal{B}_1(Y) : \|y^{**}\| = 1\}$. By Lemma 4.6, there is a non-separable subset $\{\mu_{T^{**}(y_\xi^{**})}\}_{\xi<\omega_1}$ of $\mathcal{M}_{\mathcal{G}}$ such that $\mu_{T^{**}(y_\xi^{**})} = \lambda_\xi + \tau_\xi$ for all $0 \le \xi < \omega_1$, and $\tau_\zeta \perp \tau_\xi$ for all $\zeta < \xi$. By passing to a further uncountable subset and relabeling, we may also assume that there is $\rho_0 > 0$ such that $\|\tau_\xi\| > \rho_0$. We fix $\varepsilon > 0$ and a sequence $(\varepsilon_n)_n$ of positive real numbers with $\sum_{n=0}^\infty \varepsilon_n < \varepsilon/2$. We will construct the following objects:

- a Cantor scheme $(\Gamma_s)_s$ of uncountable subsets of ω_1 (that is, for all $s \in 2^{<\mathbb{N}}$, $\Gamma_{s^\frown 0} \cup \Gamma_{s^\frown 1} \subseteq \Gamma_s$ and $\Gamma_{s^\frown 0} \cap \Gamma_{s^\frown 1} = \emptyset$),
- a family $(y_{\xi_s}^{**})_s$ with $\xi_s \in \Gamma_s$ for all $s \in 2^{<\mathbb{N}}$,
- a Cantor scheme $(U_s)_s$ of open subsets of $2^{\mathbb{N}}$, $U_s = \bigcup_{t\in B_s} V_t$, where B_s is an antichain of $2^{<\mathbb{N}}$ for all $s \in 2^{<\mathbb{N}}$,
- a family $(y_s)_s$ in Y,
- a family $(F_s)_s$ of finite subsets of $\mathbb{N}$ and a sequence $(d_i)_{i\in F_s}$ of scalars with $0 \le d_i \le 1$ for all $i \in F_s$, and
- a family $(A_s)_s$ of finite antichains of $2^{<\mathbb{N}}$,

such that for each $s \in 2^{<\mathbb{N}}$ the following conditions are satisfied:

(i) For every $\xi \in \Gamma_s$, $\tau_\xi(U_s) > \rho_0/2$ and $\tau_\xi(2^{\mathbb{N}} \setminus U_s) < \big(\sum_{i=0}^{|s|} 2^{-(i+2)}\big)\rho_0$.
(ii) The element $y_{\xi_s}^{**}$ is a w^*-condensation point of $\{y_\xi^{**}\}_{\xi\in\Gamma_s}$ (by Proposition 3.3, we identify $\mathcal{B}_1(X)$ with $[(e_i^*)_i]^*$ endowed with the weak* topology).
(iii) For all $n \ge 1$, $s \in 2^n$, $\xi \in \Gamma_s$ and $i \in F_{s^-}$,

$$|y_\xi^{**}(e_i^*) - y_{\xi_{s^-}}^{**}(e_i^*)| < \frac{\varepsilon_n}{\#F_{s^-}}, \quad \text{where} \quad s^- = (s(1), \dots, s(n-1)).$$

(iv) $\max F_{\sigma|n} < \min F_{\sigma|n+1}$ for all $\sigma \in 2^{\mathbb{N}}$ and $n \ge 0$.
(v) $A_s \subseteq \bigcup_{t\in B_s} T_t^0$ and $\sum_{t\in A_s} |e_t^*(T(y_s))| > \rho$.

(vi) For every $s \in 2^{<\mathbb{N}}$,

$$\Big\| y_s - \sum_{i \in F_s} d_i y^{**}_{\xi_s}(e_i^*) e_i \Big\| < \varepsilon_{|s|}.$$

Given the above construction, we set $\rho = \rho_0/2$ and we claim that the families $(y_s)_s$ and $(A_s)_s$ satisfy conditions (C1)–(C3). Observe that (C1) follows from the fact that $(U_s)_s$ is a Cantor scheme and from the first part of (v), while (C2) from the second part of (v).

It remains to verify (C3). So let $n \geq 0$ and $\sigma \in 2^{\mathbb{N}}$. Then $\Gamma_{\sigma|k+1} \subseteq \Gamma_{\sigma|k}$ for every $k \geq 0$, and so by (iii) we get

$$\begin{aligned}
(12)\quad & \Big\| \sum_{k=0}^{n} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\| \\
&= \Big\| \sum_{k=0}^{n-1} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i - \sum_{k=0}^{n-1} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|n}}(e_i^*) e_i + \sum_{k=0}^{n} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|n}}(e_i^*) e_i \Big\| \\
&\leq \sum_{k=0}^{n-1} \sum_{i \in F_{\sigma|k}} |y^{**}_{\xi_{\sigma|k}}(e_i^*) - y^{**}_{\xi_{\sigma|n}}(e_i^*)| + \Big\| \sum_{k=0}^{n} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|n}}(e_i^*) e_i \Big\| \\
&< \sum_{k=0}^{n-1} \varepsilon_{k+1} + \|y^{**}_{\xi_{\sigma|n}}\| < \frac{\varepsilon}{2} + 1.
\end{aligned}$$

As $\{e_i\}_i$ is a 1-unconditional normalized basis of X, by (iv) we infer that if $(a_k)_{k=0}^{\infty}$ is a sequence of scalars, then

$$(13)\quad \Big\| \sum_{k=0}^{n} a_k \sum_{i \in F_{\sigma|k}} d_i y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\| \leq \max_{0 \leq k \leq n} |a_k| \, \Big\| \sum_{k=0}^{n} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\|.$$

By (12), (13) and (vi) we obtain

$$\begin{aligned}
\Big\| \sum_{k=0}^{n} a_k y_{\sigma|k} \Big\| &\leq \sum_{k=0}^{n} |a_k| \, \Big\| y_{\sigma|k} - \sum_{i \in F_{\sigma|k}} d_i y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\| \\
&\qquad + \Big\| \sum_{k=0}^{n} a_k \sum_{i \in F_{\sigma|k}} d_i y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\| \\
&\leq \max_{0 \leq k \leq n} |a_k| \Big(\sum_{k=0}^{n} \varepsilon_k \Big) + \max_{0 \leq k \leq n} |a_k| \, \Big\| \sum_{k=0}^{n} \sum_{i \in F_{\sigma|k}} y^{**}_{\xi_{\sigma|k}}(e_i^*) e_i \Big\| \\
&\leq (1+\varepsilon) \max_{0 \leq k \leq n} |a_k|.
\end{aligned}$$

We now present the general inductive step of the construction. Suppose that the construction has been carried out for all $s \in 2^{\leq n}$. For every $s \in 2^n$, we define

$$(14)\qquad \varGamma_s^1=\{\xi\in\varGamma_s:\forall i\in F_s,\ |y_\xi^{**}(e_i^*)-y_{\xi_s}^{**}(e_i^*)|<\varepsilon_{n+1}/\#F_s\}.$$

Since F_s is a finite subset of $\mathbb{N}$, the set $\{y_\xi^{**}:\xi\in\varGamma_s^1\}$ is a relatively weak*-open neighborhood of $y_{\xi_s}^{**}$ in $\{y_\xi^{**}\}_{\xi\in\varGamma_s}$. By our inductive assumption, $y_{\xi_s}^{**}$ is a weak*-condensation point of $\{y_\xi^{**}\}_{\xi\in\varGamma_s}$ and therefore for all $s\in 2^n$ the set $\varGamma_s^1$ is uncountable. For every $s\in 2^n$, we choose uncountable disjoint subsets $\varGamma_{s^\frown 0}^1$ and $\varGamma_{s^\frown 1}^1$ of $\varGamma_s^1$. Applying Lemma 4.7 we obtain a 2^{n+1}-tuple $(O_s)_{s\in 2^{n+1}}$ of pairwise disjoint open subsets of $2^{\mathbb{N}}$ and a family $(\varGamma_s^2)_{s\in 2^{n+1}}$ such that for each $s\in 2^{n+1}$, $\varGamma_s^2$ is an uncountable subset of $\varGamma_s^1$, and for all $\xi\in\varGamma_s^2$,

$$(15)\qquad \tau_\xi(2^{\mathbb{N}}\setminus O_s)<\rho_0/2^{n+3}.$$

For every $s\in 2^{n+1}$ we set $U_s=O_s\cap U_{s^-}$. Notice that $U_s=\bigcup_{t\in B_s}V_t$, where B_s is an antichain of $2^{<\mathbb{N}}$. Since $\varGamma_s^2\subseteq\varGamma_{s^-}^1\subseteq\varGamma_{s^-}$, (i) implies that for all $s\in 2^{n+1}$ and all $\xi\in\varGamma_s^2$,

$$\tau_\xi(2^{\mathbb{N}}\setminus U_{s^-})<\Big(\sum_{i=0}^{n}2^{-(i+2)}\Big)\rho_0,$$

and hence, by (15),

$$(16)\qquad \tau_\xi(2^{\mathbb{N}}\setminus U_s)\le\tau_\xi(2^{\mathbb{N}}\setminus O_s)+\tau_\xi(2^{\mathbb{N}}\setminus U_{s^-})<\Big(\sum_{i=0}^{n+1}2^{-(i+2)}\Big)\rho_0.$$

Moreover as $(\sum_{i=0}^{n+1}2^{-(i+2)})\rho_0<\rho_0/2$ and $\|\tau_\xi\|>\rho_0$, we deduce that for all $\xi\in\varGamma_s^2$,

$$(17)\qquad \tau_\xi(U_s)>\rho_0/2.$$

We set $\varGamma_s=\varGamma_s^2$ and we choose ξ_s in $\varGamma_s$ such that $y_{\xi_s}^{**}$ is a weak*-condensation point of $\{y_\xi^{**}\}_{\xi\in\varGamma_s}$. Since $\mu_{T^{**}(y_\xi^{**})}\ge\tau_\xi$ for all $\xi<\omega_1$ Lemma 4.5 implies that for every $s\in 2^{n+1}$ there exist $y_s\in Y$, a finite subset F_s of $\mathbb{N}$, a sequence $\{d_i\}_{i\in F_s}$ of scalars with $\max F_{s^-}<\min F_s$ and $0\le d_i\le 1$ for every $i\in F_s$, and a finite antichain A_s of $2^{<\mathbb{N}}$ with $A_s\subseteq\bigcup_{t\in B_s}T_t^0$, such that

$$(18)\qquad \Big\|y_s-\sum_{i\in F_s}d_iy_{\xi_s}^{**}(e_i^*)e_i\Big\|<\varepsilon_{n+1}\quad\text{and}\quad\sum_{t\in A_s}|e_t^*(T(y_s))|>\rho_0/2.$$

By (12)–(18), the proof of the inductive step is complete. ∎

Let $(x_s)_{s\in 2^{<\mathbb{N}}}$ be a family in S^1. We will say that $(x_s)_{s\in 2^{<\mathbb{N}}}$ is a *block family* in S^1 if $(x_s)_{s\in 2^{<\mathbb{N}}}$ is a block sequence of $(e_s)_{s\in 2^{<\mathbb{N}}}$ under the natural ordering of $2^{<\mathbb{N}}$.

PROPOSITION 4.9. *Let $(y_s)_{s\in 2^{<\mathbb{N}}}$ be a block family in S^1 and $(x_s^*)_{s\in 2^{<\mathbb{N}}}$ a family in $(S^1)^*$ with the following properties:*

(a) *There is a constant $c>0$ such that $x_s^*(y_s)\ge c$ for all $s\in 2^{<\mathbb{N}}$.*

(b) *For every* $s \in 2^{<\mathbb{N}}$, $x_s^* = \sum_{t \in A_s} \varepsilon_t e_t^*$, *where* A_s *is a finite antichain of* $2^{<\mathbb{N}}$ *with* $A_s \subseteq \operatorname{supp}(y_s)$ *and* $\varepsilon_t \in \{-1, 1\}$ *for all* $t \in A_s$.

(c) $A_s \perp A_{s'}$ *for all* $s \perp s'$.

(d) *There is a constant* $C > 0$ *such that for every* $\sigma \in 2^{\mathbb{N}}$, $n \geq 0$ *and every sequence* $(a_k)_{k=0}^n$ *of scalars,*

$$\Big\| \sum_{k=0}^{n} a_k y_{\sigma|k} \Big\| \leq C \max_{0 \leq k \leq n} |a_k|.$$

Then:

(i) *The family* $(y_s)_{s \in 2^{<\mathbb{N}}}$ *is equivalent to the basis* $(e_s)_{s \in 2^{<\mathbb{N}}}$ *of* S^1.

(ii) *There exists a projection* $P : S^1 \to [(y_s)_{s \in 2^{<\mathbb{N}}}]$ *with* $\|P\| \leq C/c$.

Proof. (i) Since $(y_s)_{s \in 2^{<\mathbb{N}}}$ is a block family in S^1, hence 1-unconditionally basic, we easily observe that $(y_s)_{s \in 2^{<\mathbb{N}}}$ satisfies the assumptions of Proposition 4.2. Therefore (i) holds.

(ii) We define $P : S^1 \to S^1$ by

$$P(x) = \sum_{s \in 2^{<\mathbb{N}}} \frac{x_s^*(x)}{x_s^*(y_s)} y_s, \qquad x \in S^1.$$

First we show that P is well defined. By (b) we have $P(y_s) = y_s$ for every $s \in 2^{<\mathbb{N}}$. Let $x \in S^1$, $n \geq 0$ and $x^* \in (S^1)^*$ with $\|x^*\| = 1$. By Lemma 4.1 (with $|x_s^*(x)|/x_s^*(y_s)$ in place of λ_s and $|x^*(y_s)|$ in place of α_s), there is an antichain A of $2^{\leq n}$ and a family of branches $(b_t)_{t \in A}$ of $2^{\leq n}$ such that

$$\Big| \sum_{|s| \leq n} \frac{x_s^*(x)}{x_s^*(y_s)} x^*(y_s) \Big| \leq \sum_{t \in A} \Big(\sum_{s \in b_t} |x^*(y_s)| \Big) \frac{|x_t^*(x)|}{x_t^*(y_t)}.$$

By (a)–(c), we obtain $\sum_{t \in A} |x_t^*(x)|/x_t^*(y_t) \leq \|x\|/c$, and by (d), we have $\sum_{s \in b_t} |x^*(y_s)| \leq C$ for all $t \in A$. Hence

$$\Big\| \sum_{|s| \leq n} \frac{x_s^*(x)}{x_s^*(y_s)} y_s \Big\| \leq \frac{C}{c} \|x\|.$$

Since this holds for all $x \in S^1$ and $n \geq 0$, it follows that P is well defined and, in addition, it is a bounded projection onto $[(y_s)_{s \in 2^{<\mathbb{N}}}]$ with $\|P\| \leq C/c$. ■

The following lemma is easily proved by using a sliding hump argument.

LEMMA 4.10. *Let* $(x_s)_s$ *be a family in* S^1 *such that for every* $\sigma \in 2^{\mathbb{N}}$, *the sequence* $(x_{\sigma|n})_n$ *is weakly null. Then for every* $\delta > 0$, *there exist a dyadic subtree* $(t_s)_s$ *of* $2^{<\mathbb{N}}$ *and a block family* $(w_s)_s$ *in* S^1 *with the natural ordering of* $2^{<\mathbb{N}}$ *such that*

$$\sum_{s \in 2^{<\mathbb{N}}} \|x_{t_s} - w_s\| < \delta.$$

PROPOSITION 4.11. *Let X be a Banach space with an unconditional basis, Y a closed subspace of X, and $T : X \to S^1$ an operator such that $\{\mu_{T^{**}(y^{**})} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$. Then there exists a subspace Z of Y isomorphic to S^1 such that the restriction of T to Z is an isomorphism and $T[Z]$ is complemented in S^1. Moreover Z is complemented in X.*

Proof. By passing to an equivalent norm, we may assume that the basis of X is 1-unconditional. From Lemma 4.8, there is a constant $\rho > 0$ such that for $0 < \varepsilon < \rho$ there exist a family $(y_s)_s$ of elements of Y and a family $(A_s)_s$ of finite antichains of $2^{<\mathbb{N}}$ such that:

(C1) $A_s \perp A_t$ for all $s \perp t$.
(C2) $\sum_{t\in A_s} |e_t^*(T(y_s))| > \rho$ for every $s \in 2^{<\mathbb{N}}$.
(C3) For every $\sigma \in 2^{\mathbb{N}}$, $n \geq 0$ and every sequence $(a_k)_{k=0}^n$ of scalars,

$$\Big\| \sum_{k=0}^{n} a_k y_{\sigma|k} \Big\| \leq (1+\varepsilon) \max_{0\leq k\leq n} |a_k|.$$

As T is bounded, (C3) implies that for every $\sigma \in 2^{\mathbb{N}}$, the sequence $(T(y_{\sigma|k}))_k$ is weakly null. Therefore by Lemma 4.10, for $0 < \delta < \varepsilon$, there exist a dyadic subtree $(t_s)_s$ of $2^{<\mathbb{N}}$ and a block family $(w_s)_s$ in S^1 such that

$$\sum_{s\in 2^{<\mathbb{N}}} \|T(y_{t_s}) - w_s\| < \delta. \tag{19}$$

First we will show that the family $(w_s)_s$ is equivalent to the basis $(e_s)_{s\in 2^{<\mathbb{N}}}$ of S^1 and the space $[(w_s)_s]$ is complemented in S^1. Indeed, for every $s \in 2^{<\mathbb{N}}$, let $x_s^* = \sum_{v\in A_s} \varepsilon_v e_v^*$, where $(\varepsilon_v)_{v\in A_s}$ is a family of signs such that

$$x_s^*(T(y_{t_s})) = \sum_{v\in A_s} |e_v^*(T(y_{t_s}))| > \rho.$$

Without loss of generality, we may assume that $A_s \subseteq \operatorname{supp}(w_s)$ for every $s \in 2^{<\mathbb{N}}$ (otherwise, we replace A_s by $B_s = A_s \cap \operatorname{supp}(w_s)$). Then by (19), it is easy to see that for every finite antichain A of $2^{<\mathbb{N}}$,

$$x_s^*(w_s) > \rho - \varepsilon \quad \text{for all } s \in 2^{<\mathbb{N}}. \tag{20}$$

Since $(w_s)_s$ is a block family in S^1, by (C1) and (20) we find that for every finite antichain A of $2^{<\mathbb{N}}$ and every family $(\lambda_s)_{s\in A}$ of scalars,

$$\Big\| \sum_{s\in A} \lambda_s w_s \Big\| \geq (\rho - \varepsilon) \sum_{s\in A} |\lambda_s|. \tag{21}$$

On the other hand, by (C3), for every $\sigma \in 2^{\mathbb{N}}$, $n \geq 0$ and every sequence

$(a_k)_{k=0}^n$ of scalars,

$$\Big\| \sum_{k=0}^{n} a_k w_{t_{\sigma|k}} \Big\| \le ((1+\varepsilon)\|T\| + \varepsilon) \max_{0\le k\le n} |a_k|. \tag{22}$$

By (20)–(22), we see that $(w_s)_s$ satisfies the assumptions of Proposition 4.9. Hence $(w_s)_s$ is equivalent to $(e_s)_{s\in 2^{<\mathbb{N}}}$ and the space $[(w_s)_s]$ is complemented in S^1. Therefore by (19), for $\delta > 0$ sufficiently small, $(T(y_{t_s}))_s$ is equivalent to $(e_s)_{s\in 2^{<\mathbb{N}}}$ and $T[Z]$ is complemented in S^1, where $Z = [(y_{t_s})_s]$. Now by (C3), as in the proof of Proposition 4.2, $(y_{t_s})_s$ has an upper S^1-estimate, which implies that $(y_{t_s})_s$ is also equivalent to $(e_s)_{s\in 2^{<\mathbb{N}}}$.

We now proceed to show that Z is complemented in X. Let P be a bounded projection from S^1 onto $T[Z]$. We set $Q = T^{-1}PT$. Then Q is a bounded projection from X onto Z. ∎

LEMMA 4.12. *Let $x^{**} \in \mathcal{B}_1(S^1)$. Then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every finite antichain A of $2^{<\mathbb{N}}$,*

$$\Big\| x^{**} \Big| \bigcup_{s\in A} T_s^n \Big\| < \mu_{x^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \varepsilon.$$

Therefore, for every finite antichain A of $2^{<\mathbb{N}}$ with $\min\{|s| : s\in A\} \ge n$,

$$\sum_{s\in A} |x^{**}(e_s^*)| < \mu_{x^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \varepsilon.$$

Proof. Let $\varepsilon > 0$ and A be a finite antichain of $2^{<\mathbb{N}}$. By the definition of $\mu_{x^{**}}$, there exists $n\in\mathbb{N}$ such that

$$\|x^{**}|T_\emptyset^n\| < \mu_{x^{**}}(2^{\mathbb{N}}) + \varepsilon. \tag{23}$$

We choose an antichain A' of $2^{<\mathbb{N}}$ such that $A \cup A'$ is a finite maximal antichain of $2^{<\mathbb{N}}$ and $A \cap A' = \emptyset$. Then

$$\Big\| x^{**} \Big| \bigcup_{s\in A} T_s^n \Big\| + \Big\| x^{**} \Big| \bigcup_{s\in A'} T_s^n \Big\| = \|x^{**}|T_\emptyset^n\| \tag{24}$$

and

$$\mu_{x^{**}}(2^{\mathbb{N}}) = \mu_{x^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \mu_{x^{**}}\Big(\bigcup_{s\in A'} V_s\Big). \tag{25}$$

By (23)–(25), we conclude that for every $n \ge l$,

$$\begin{aligned}\Big\| x^{**} \Big| \bigcup_{s\in A} T_s^n \Big\| + \Big\| x^{**} \Big| \bigcup_{s\in A'} T_s^n \Big\| &< \mu_{x^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \mu_{x^{**}}\Big(\bigcup_{s\in A'} V_s\Big) + \varepsilon \\ &\le \mu_{x^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \Big\| x^{**} \Big| \bigcup_{s\in A'} T_s^n \Big\| + \varepsilon.\end{aligned}$$

Therefore, $\|x^{**}|\bigcup_{s\in A} T_s^n\| < \mu_{x^{**}}(\bigcup_{s\in A} V_s) + \varepsilon$. ∎

PROPOSITION 4.13. *Let $(y_s)_{s\in 2^{<\mathbb{N}}}$ be a block family in S^1 with the following properties:*

(i) *There is a constant $\rho > 0$ such that $\|\sum_{|s|=n} \lambda_s y_s\| \geq \rho \sum_{|s|=n} |\lambda_s|$ for every $n \geq 0$ and every family $(\lambda_s)_{|s|=n}$ of scalars.*
(ii) *For every $\sigma \in 2^{\mathbb{N}}$, the sequence $(y_{\sigma|k})_{k=0}^{\infty}$ is equivalent to the basis of c_0.*

Then $\{\mu_{y_\sigma^{**}} : \sigma \in 2^{\mathbb{N}}\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$, where $y_\sigma^{**} = w^*\text{-}\lim_n \sum_{k=0}^{n} y_{\sigma|k}$ for every $\sigma \in 2^{\mathbb{N}}$.

Proof. If $\{\mu_{y_\sigma^{**}} : \sigma \in 2^{\mathbb{N}}\}$ is separable, we can choose a norm-condensation point $\mu \in \mathcal{M}(2^{\mathbb{N}})$ of $\{\mu_{y_\sigma^{**}} : \sigma \in 2^{\mathbb{N}}\}$. Also fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Then for uncountably many $\sigma \in 2^{\mathbb{N}}$, we have

$$\|\mu_{y_\sigma^{**}} - \mu\| \leq \varepsilon/m. \tag{26}$$

Let $\sigma_1, \ldots, \sigma_m \in 2^{\mathbb{N}}$ satisfy (26) and let $n_0 \in \mathbb{N}$ be such that $\sigma_i|n \perp \sigma_j|n$ whenever $n \geq n_0$ and $1 \leq i < j \leq m$. By Lemma 4.12, there is $k \geq n_0$ such that

$$\sum_{s\in A} |y_{\sigma_i}^{**}(e_s^*)| < \mu_{y_{\sigma_i}^{**}}\Big(\bigcup_{s\in A} V_s\Big) + \varepsilon/m \tag{27}$$

for every finite antichain A of $2^{<\mathbb{N}}$ with $\min\{|s| : s \in A\} \geq k$ and each $i = 1, \ldots, m$. We choose A such that

$$\Big\|\sum_{i=1}^{m} y_{\sigma_i|k}\Big\| = \sum_{s\in A} \Big|e_s^*\Big(\sum_{i=1}^{m} y_{\sigma_i|k}\Big)\Big|$$

and we set $A_i = A \cap \operatorname{supp}(y_{\sigma_i|k})$. Since $(y_s)_{s\in 2^{<\mathbb{N}}}$ is a block family in S^1, we may assume that $\min\{|s| : s \in A_i\} \geq k$. Then by (i), (26) and (27),

$$\begin{aligned} m\rho &\leq \Big\|\sum_{i=1}^{m} y_{\sigma_i|k}\Big\| = \sum_{s\in A} \Big|e_s^*\Big(\sum_{i=1}^{m} y_{\sigma_i|k}\Big)\Big| \leq \sum_{i=1}^{m} \sum_{s\in A_i} |y_{\sigma_i}^{**}(e_s^*)| \\ &< \sum_{i=1}^{m} \mu_{y_{\sigma_i}^{**}}\Big(\bigcup_{s\in A_i} V_s\Big) + \varepsilon \leq \mu\Big(\bigcup_{s\in \bigcup_{i=1}^{m} A_i} V_s\Big) + 2\varepsilon \leq \|\mu\| + 2\varepsilon, \end{aligned}$$

a contradiction for ε sufficiently small. ∎

LEMMA 4.14. *Let $(x_n)_n$ and $(y_n)_n$ be sequences in S^1 both equivalent to the c_0-basis, and $\sum_n \|x_n - y_n\| < \infty$. Set*

$$x^{**} = w^*\text{-}\lim_n \sum_{k=0}^{n} x_k \quad \text{and} \quad y^{**} = w^*\text{-}\lim_n \sum_{k=0}^{n} y_k.$$

Then $\mu_{x^{**}} = \mu_{y^{**}}$.

Proof. For every $n \in \mathbb{N}$, we set

$$x^{**}_{|\geq n} = w^*\text{-}\lim_m \sum_{k=n}^{m} x_k \quad \text{and} \quad y^{**}_{|\geq n} = w^*\text{-}\lim_m \sum_{k=n}^{m} y_k.$$

By Proposition 3.8(iii), $x^{**} - x^{**}_{|\geq n} \in S^1$ and $y^{**} - y^{**}_{|\geq n} \in S^1$, so $\mu_{x^{**}} = \mu_{x^{**}_{|\geq n}}$ and $\mu_{y^{**}} = \mu_{y^{**}_{|\geq n}}$. Then from the weak* lower semicontinuity of the second dual norm we get

$$\begin{aligned}\|\mu_{x^{**}} - \mu_{y^{**}}\| = \|\mu_{x^{**}_{|\geq n}} - \mu_{y^{**}_{|\geq n}}\| &\leq \|x^{**}_{|\geq n} - y^{**}_{|\geq n}\| \\ &\leq \liminf_m \Big\| \sum_{k=n}^{m} x_k - \sum_{k=n}^{m} y_k \Big\| \leq \sum_{k=n}^{\infty} \|x_k - y_k\|.\end{aligned}$$

Therefore, letting $n \to \infty$ gives $\|\mu_{x^{**}} - \mu_{y^{**}}\| = 0$ and so $\mu_{x^{**}} = \mu_{y^{**}}$. ∎

PROPOSITION 4.15. *Let X be a Banach space with an unconditional basis, and $T : X \to S^1$ be an operator such that there exists a subspace Y of X isomorphic to S^1 such that the restriction of T to Y is an isomorphism. Then $\{\mu_{T^{**}(y^{**})} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$.*

Proof. Since Y is isomorphic to S^1, there is a family $(y_s)_{s \in 2^{\mathbb{N}}}$ in Y equivalent to the S^1-basis with $Y = [(y_s)_s]$. By Lemma 4.10, there exist a dyadic subtree $(t_s)_s$ of $2^{<\mathbb{N}}$ and a block family $(w_s)_s$ in S^1 such that

$$\sum_{s \in 2^{<\mathbb{N}}} \|T(y_{t_s}) - w_s\| < \delta/2 \tag{28}$$

for $\delta > 0$ small enough so that $(w_s)_s$ is equivalent to the canonical basis of S^1. For every $\sigma \in 2^{\mathbb{N}}$, we set $y^{**}_\sigma = w^*\text{-}\lim_n \sum_{k=0}^{n} y_{t_{\sigma|k}}$ and $w^{**}_\sigma = w^*\text{-}\lim_n \sum_{k=0}^{n} w_{\sigma|k}$. Proposition 4.13 shows that $\{\mu_{w^{**}_\sigma} : \sigma \in 2^{\mathbb{N}}\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$. Since $T^{**}(y^{**}_\sigma) = w^*\text{-}\lim_n \sum_{k=0}^{n} T(y_{t_{\sigma|k}})$, by (28) and Lemma 4.14 we see that $\mu_{w^{**}_\sigma} = \mu_{T^{**}(y^{**}_\sigma)}$ for every $\sigma \in 2^{\mathbb{N}}$. Therefore $\{\mu_{T^{**}(y^{**})} : y^{**} \in \mathcal{B}_1(Y)\}$ is non-separable. ∎

Observe that Propositions 4.11 and 4.15 yield Theorem 1.3 from the introduction.

REMARK 4.16. The proof of Theorem 1.3 actually yields a slightly stronger result, in particular the conclusion holds if X is assumed to be a subspace of a space with an unconditional basis.

COROLLARY 4.17. *Let Y be a closed subspace of S^1. Then $\mathcal{M}_{\mathcal{B}_1(Y)} = \{\mu_{y^{**}} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$ if and only if there exists a subspace Z of Y isomorphic to S^1 and complemented in S^1.*

Proof. Let $I : S^1 \to S^1$ be the identity operator. We observe that $I^{**}[\mathcal{B}_1(Y)] = \mathcal{B}_1(Y)$. By Theorem 1.3 the conclusion follows. ∎

The following is an immediate consequence of Corollary 4.17.

COROLLARY 4.18. *Let Y be a closed subspace of S^1 which is isomorphic to S^1. Then Y contains a subspace Z isomorphic to S^1 and complemented in S^1.*

To state one more consequence of the above theorem we will need

LEMMA 4.19. *Let Y be a subspace of S^1. Suppose that S^1 contains a complemented copy of Y and vice versa. Then Y is isomorphic to S^1.*

Proof. Notice that

$$\begin{aligned} S^1 &\approx c_0 \oplus (S^1 \oplus S^1 \oplus \cdots)_{\ell_1} \approx c_0 \oplus (S^1 \oplus S^1 \oplus \cdots)_{\ell_1} \oplus (S^1 \oplus S^1 \oplus \cdots)_{\ell_1} \\ &\approx S^1 \oplus (S^1 \oplus S^1 \oplus \cdots)_{\ell_1} \approx (S^1 \oplus S^1 \oplus \cdots)_{\ell_1}, \end{aligned}$$

and apply the Pełczyński decomposition method [9]. ∎

COROLLARY 4.20. *Let Y be a complemented subspace of S^1 such that $\{\mu_{y^{**}} : y^{**} \in \mathcal{B}_1(Y)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$. Then Y is isomorphic to S^1.*

Proof. Since Y is a subspace of S^1 and $\{\mu_{y^{**}} : y^{**} \in \mathcal{B}_1(Y)\}$ is non-separable, by Corollary 4.17 we know that Y contains a complemented copy of S^1. Since Y is complemented in S^1, Lemma 4.19 shows that Y is isomorphic to S^1. ∎

REMARK 4.21. A standard argument using Rosenthal's lemma [11] shows that if $(X_i)_i$ is a sequence of Banach spaces and $X = (\sum_{i=1}^{\infty} \oplus X_i)_{\ell_1}$ then $\mathcal{B}_1(X) = (\sum_{i=1}^{\infty} \oplus \mathcal{B}_1(X_i))_{\ell_1}$. More precisely, for every $x^{**} \in \mathcal{B}_1(X)$, there exists a unique sequence $(x_i^{**})_i$ with $x_i^{**} \in \mathcal{B}_1(X_i)$ for all $i \in \mathbb{N}$ and $\sum_i \|x_i^{**}\| < \infty$ such that $x^{**} = \sum_{i=1}^{\infty} x_i^{**}$.

COROLLARY 4.22. *Suppose S^1 is isomorphic to an ℓ_1-sum of a sequence of Banach spaces X_i. Then there is a j such that X_j is isomorphic to S^1.*

Proof. Let $X = (\sum_{i=1}^{\infty} \oplus X_i)_{\ell_1}$ and assume that X is isomorphic to S^1. If we identify X with S^1, by Corollary 4.20 it is enough to prove that there is a j such that $\{\mu_{x^{**}} : x^{**} \in \mathcal{B}_1(X_j)\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$.

Assume this is not true. Then for every i, $\{\mu_{x^{**}} : x^{**} \in \mathcal{B}_1(X_i)\}$ is separable. Therefore for every i there is $\mu_i \in \mathcal{M}^+(2^{\mathbb{N}})$ such that $\mu_{x^{**}} \ll \mu_i$ for all $x^{**} \in \mathcal{B}_1(X_i)$. We set

$$\mu = \sum_{i=1}^{\infty} \frac{\mu_i}{2^i(1 + \|\mu_i\|)}.$$

By Proposition 3.8(ii) & (iv), $\mu_{x^{**}} \ll \mu$ for every x^{**} in $(\sum_i \oplus \mathcal{B}_1(X_i))_{\ell_1}$, which by Remark 4.21 is $\mathcal{B}_1(S^1)$. However, by the Radon–Nikodym theorem, the set $\{\mu_{x^{**}} : x^{**} \in \mathcal{B}_1(S^1)\}$ is separable, a contradiction. ∎

Remark 4.23. In [2] H. Bang and E. Odell showed that S^1 is *primary*, that is, whenever $S^1 = X \oplus Y$, then either X or Y is isomorphic to S^1. We note that this fact also follows from Corollary 4.22.

The following is an immediate consequence of the main result from [5], where actually a stronger statement is proven.

Proposition 4.24. *Let D be a subset of $2^{<\mathbb{N}}$ such that*

$$\limsup_{n\to\infty} \frac{\#(D \cap \{s \in 2^{<\mathbb{N}} : |s| = n\})}{2^n} > 0.$$

Then D contains a regular dyadic subtree $(t_s)_s$ of $2^{<\mathbb{N}}$.

The next proposition can be found in [4, Proposition 2.2, p. 161]. Recall that an operator $T : X \to Y$ between Banach spaces is called a G_δ-embedding if it is injective and $T[K]$ is a G_δ subset of Y for all closed bounded K.

Proposition 4.25. *Let X and Y be Banach spaces and $T : X \to Y$ be a G_δ-embedding. If X is isomorphic to c_0, then T is an isomorphism.*

Theorem 4.26. *Let X be a Banach space. If S^1 G_δ-embeds in X, and X G_δ-embeds in S^1, then S^1 complementably embeds in X.*

Proof. Let $W : S^1 \to X$ and $R : X \to S^1$ be G_δ-embeddings and set $T = RW : S^1 \to S^1$. By Lemma 4.10, passing to the dyadic subtree, we may assume that $(T(e_s))_s$ is a block family in S^1. For every $n \in \mathbb{N}$, we set $x_n = \sum_{|s|=n} 2^{-n} e_s$. Notice that $[(x_n)_{n\in\mathbb{N}}]$ is isometric to c_0. Then by Proposition 4.25, the restriction of T to $[(x_n)_{n\in\mathbb{N}}]$ is an isomorphism. Hence there is $\rho > 0$ with $2\|T\| > \rho$ such that $\|T(x_n)\| \geq \rho$ for all $n \in \mathbb{N}$. For each n, we choose a norm one functional $x_n^* \in (S^1)^*$ such that $x_n^*(T(x_n)) = \|T(x_n)\|$ and we set

$$A_n = \{s \in 2^{<\mathbb{N}} : x_n^*(T(e_s)) \geq \rho/2,\ |s| = n\} \quad \text{and} \quad D = \bigcup_{n\in\mathbb{N}} A_n.$$

Then

$$\begin{aligned}
\rho \leq x_n^*(T(x_n)) &= \frac{1}{2^n} \sum_{|s|=n} x_n^*(T(e_s)) \\
&= \frac{1}{2^n} \sum_{s\in A_n} x_n^*(T(e_s)) + \frac{1}{2^n} \sum_{s\in A_n^c} x_n^*(T(e_s)) \\
&\leq \frac{1}{2^n}\left(\#A_n\|T\| + (2^n - \#A_n)\frac{\rho}{2}\right).
\end{aligned}$$

By the above inequality we get $\frac{\#A_n}{2^n} \geq \frac{\rho}{2\|T\|-\rho}$. Hence,

$$\frac{\#(D \cap \{s \in 2^{<\mathbb{N}} : |s| = n\})}{2^n} = \frac{\#A_n}{2^n} \geq \frac{\rho}{2\|T\| - \rho} > 0 \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

By (29) and Proposition 4.24, there is a regular dyadic subtree $(t_s)_s$ of $2^{<\mathbb{N}}$ contained in D. Since $(T(e_s))_s$ is a block family, we easily observe that for every $n \geq 0$ and every family $(\lambda_s)_{|s|=n}$ of scalars,

$$\Big\| \sum_{|s|=n} \lambda_s T(e_{t_s}) \Big\| \geq \frac{\rho}{2} \sum_{|s|=n} |\lambda_s|.$$

Therefore, the assumptions of Proposition 4.13 are fulfilled and so $\{\mu_{T^{**}(\sigma^{**})} : \sigma \in 2^{\mathbb{N}}\}$ is a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$, where $\sigma^{**} = w^*\text{-}\lim_n \sum_{k=0}^{n} e_{t_{\sigma|k}}$ for every $\sigma \in 2^{\mathbb{N}}$. Now by Theorem 1.3 the conclusion follows. ■

As a consequence of Theorem 4.26, we obtain

Corollary 4.27. *Let X be a closed subspace of S^1. If S^1 G_δ-embeds in X, then S^1 complementably embeds in X.*

Remark 4.28. Note that a positive answer to the following question implies that Problem 1.2 has a positive answer as well. Let $T : S^1 \to S^1$ be an operator and X be an infinite-dimensional reflexive subspace of S^1 such that the restriction of T to X is an isomorphism. Is $\{\mu_{T^{**}(x^{**})} : x^{**} \in \mathcal{B}_1(S^1)\}$ a non-separable subset of $\mathcal{M}(2^{\mathbb{N}})$?

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