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The strong Morita equivalence for coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras

by

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Abstract. Following Jansen and Waldmann, and Kajiwara and Watatani, we introduce notions of coactions of a finite-dimensional C^*-Hopf algebra on a Hilbert C^*-bimodule of finite type in the sense of Kajiwara and Watatani and define their crossed product. We investigate their basic properties and show that the strong Morita equivalence for coactions preserves the Rokhlin property for coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras.

1. Introduction. Let A and B be unital C^*-algebras and X a Hilbert A-B-bimodule of finite type in the sense of Kajiwara and Watatani [8]. Let H be a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. In this paper, following Jansen and Waldmann [7], we shall introduce the notion of coactions of H^0 on X and define their crossed product. That is, for coactions ρ and σ of H^0 on A and B, respectively, we introduce a linear map λ from X to $X \otimes H^0$, which is compatible with the coactions ρ, σ and the left A-module action, the right B-module action and the left A-valued and right B-valued inner products. Then we can define the crossed product $X \rtimes_\lambda H$, which is a Hilbert $A \rtimes_\rho H$ - $B \rtimes_\sigma H$-bimodule of finite type. Furthermore, we shall give a duality theorem similar to the ordinary one. The corresponding theorems in the case of countably discrete group actions and of Kac systems are found in Kajiwara and Watatani [9] and Guo and Zhang [5], respectively. The latter result is almost a generalization of our duality theorem. But our approach to coactions of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra is a useful addition, especially the main result on preservation of the Rokhlin property under strong Morita equivalence. So, in Section 5, we give a duality theorem, a version of crossed product duality for coactions of finite-dimensional C^*-Hopf algebras on Hilbert

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C^*-bimodules of finite type. Also, if X is an A-B-equivalence bimodule, we can show that $X \rtimes_\lambda H$ is an $A \rtimes_\rho H$ - $B \rtimes_\sigma H$-equivalence bimodule. Hence $A \rtimes_\rho H$ is strongly Morita equivalent to $B \rtimes_\sigma H$. Finally, if X is an A-B-equivalence bimodule and ρ has the Rokhlin property, then σ has also the Rokhlin property. As an application of this result, we obtain the following: Under a certain condition, if a unital C^*-algebra A has a coaction of H^0 with the Rokhlin property, then any unital C^*-algebra that is strongly Morita equivalent to A also has a coaction of H^0 with the Rokhlin property. In [13, Section 4], we gave an incorrect example of an action of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra with the Rokhlin property. But applying the above result to [11, Section 7], we can give several such examples.

For an algebra A, we denote by 1_A and id_A the unit element in A and the identity map on A, respectively. If no confusion arises, we denote them by 1 and id, respectively. For each $n \in \mathbb{N}$, we denote by $M_n(\mathbb{C})$ the $n \times n$-matrix algebra over $\mathbb{C}$, and I_n denotes the unit element in $M_n(\mathbb{C})$.

For projections p, q in a C^*-algebra A, we write $p \sim q$ in A if p and q are Murray–von Neumann equivalent in A.

2. Preliminaries. Let H be a finite-dimensional C^*-Hopf algebra. We denote its comultiplication, counit and antipode by Δ, ϵ and S, respectively. We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$, which suppresses a possible summation when we write comultiplications. We denote by N the dimension of H. Let H^0 be the dual C^*-Hopf algebra of H. We denote its comultiplication, counit and antipode by Δ^0, ϵ^0 and S^0, respectively. There is a distinguished projection e in H. We note that e is the Haar trace on H^0. Also, there is a distinguished projection τ in H^0 which is the Haar trace on H. Since H is finite-dimensional, $H \cong \bigoplus_{k=1}^{L} M_{f_k}(\mathbb{C})$ and $H^0 \cong \bigoplus_{k=1}^{K} M_{d_k}(\mathbb{C})$ as C^*-algebras. Let $\{v_{ij}^k \mid k = 1, \ldots, L,\, i, j = 1, \ldots, f_k\}$ be a system of matrix units of H. Let $\{w_{ij}^k \mid k = 1, \ldots, K,\, i, j = 1, \ldots, d_k\}$ be a basis of H satisfying Szymański and Peligrad's [17, Theorem 2.2,2], which is called a system of *comatrix units* of H, that is, the dual basis of a system of matrix units of H^0. Also let $\{\phi_{ij}^k \mid k = 1, \ldots, K,\, i, j = 1, \ldots, d_k\}$ and $\{\omega_{ij}^k \mid k = 1, \ldots, L,\, i, j = 1, \ldots, f_k\}$ be systems of matrix units and comatrix units of H^0, respectively.

Let A and B be unital C^*-algebras and X a Hilbert A-B-bimodule of finite type in the sense of [8]. We regard a C^*-Hopf algebra H^0 as an H^0-H^0-equivalence bimodule in the usual way.

Let $X \otimes H^0$ be the exterior tensor product of the Hilbert C^*-bimodules X and H^0, which is a Hilbert $A \otimes H^0$ - $B \otimes H^0$-bimodule.

LEMMA 2.1. *With the above notation, $X \otimes H^0$ is a Hilbert $A \otimes H^0$-$B \otimes H^0$-bimodule of finite type. In particular, if X is an A-B-equivalence bimodule, then $X \otimes H^0$ is an $A \otimes H^0$-$B \otimes H^0$-equivalence bimodule.*

Proof. Since X is of finite type, there is a right B-basis $\{u_i\}_{i=1}^n$ of X. Then for any $x \in X$ and $\phi \in H^0$,

$$\sum_{i=1}^n (u_i \otimes 1^0)\langle u_i \otimes 1^0, x \otimes \phi\rangle_{B\otimes H^0} = \sum_{i=1}^n (u_i \otimes 1^0)(\langle u_i, x\rangle_B \otimes \phi) = x \otimes \phi.$$

Thus the family $\{u_i \otimes 1^0\}_{i=1}^n$ is a right $B \otimes H^0$-basis of $X \otimes H^0$. In the same way, we can see that there is a left $A \otimes H^0$-basis of $X \otimes H^0$. Hence by [8, Proposition 1.12] or [9, Lemma 1.3], $X \otimes H^0$ is a Hilbert $A \otimes H^0$-$B \otimes H^0$-bimodule of finite type. Now suppose that X is an A-B-equivalence bimodule. Since X is full with both-sided inner products, by the definitions of the left and right inner products of $X \otimes H^0$, so is $X \otimes H^0$. Moreover, the associativity condition holds for the left and right inner products of $X \otimes H^0$ since it holds for the left and right inner products of X. Hence $X \otimes H^0$ is an $A \otimes H^0$-$B \otimes H^0$-equivalence bimodule. ∎

Let $\mathrm{Hom}(H, X)$ be the vector space of all linear maps from H to X. Then $X \otimes H^0$ and $\mathrm{Hom}(H, X)$ are isomorphic as vector spaces. Sometimes, we identify them.

3. Coactions of a finite-dimensional C^*-Hopf algebra on a Hilbert C^*-bimodule of finite type and strong Morita equivalence. Let A and B be unital C^* algebras and X a Hilbert A-B-bimodule of finite type. Let H be a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. Let ρ be a weak coaction of H^0 on A, and λ a linear map from X to $X \otimes H^0$. Following [7], [9], we introduce several definitions.

DEFINITION 3.1. With the above notation, we say that $(A, X, \rho, \lambda, H^0)$ is a *weak left covariant system* if:

(1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$ and $x \in X$,
(2) $\rho({}_A\langle x, y\rangle) = {}_{A\otimes H^0}\langle \lambda(x), \lambda(y)\rangle$ for any $x, y \in X$,
(3) $(\mathrm{id}_X \otimes \epsilon^0) \circ \lambda = \mathrm{id}_X$.

We then call λ a *weak left coaction* of H^0 on X with respect to (A, ρ).

We define the weak action of H on A induced by ρ as follows: For any $a \in A$ and $h \in H$,

$$h \cdot_\rho a = (\mathrm{id} \otimes h)(\rho(a)),$$

where we regard H as the dual space of H^0. In the same way as above, we can define the action of H on X induced by λ as follows: For any $x \in X$ and $h \in H$,

$$h \cdot_\lambda x = (\mathrm{id} \otimes h)(\lambda(x)) = \widehat{\lambda(x)}(h),$$

where $\widehat{\lambda(x)}$ is the element in $\mathrm{Hom}(H, X)$ induced by $\lambda(x)$ in $X \otimes H^0$. Then we obtain the following conditions which are equivalent to conditions (1)–(3) in Definition 3.1, respectively:

(1)′ $h \cdot_\lambda ax = [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\lambda x]$ for any $a \in A$, $x \in X$ and $h \in H$,
(2)′ $h \cdot_\rho {}_A\langle x, y\rangle = {}_A\langle [h_{(1)} \cdot_\lambda x], [S(h^*_{(2)}) \cdot_\lambda y]\rangle$ for any $x, y \in X$ and $h \in H$,
(3)′ $1_H \cdot_\lambda x = x$ for any $x \in X$.

Definition 3.2. Let σ be a weak coaction of H^0 on B. With the above notation, we say that $(B, X, \sigma, \lambda, H^0)$ is a *weak right covariant system* if:

(4) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$ and $x \in X$,
(5) $\sigma(\langle x, y\rangle_B) = \langle \lambda(x), \lambda(y)\rangle_{B\otimes H^0}$ for any $x, y \in X$,
(6) $(\mathrm{id}_X \otimes \epsilon^0) \circ \lambda = \mathrm{id}_X$.

We then call λ a *weak right coaction* of H^0 on X with respect to (B, σ). We can also define the weak action of H on X induced by λ satisfying conditions similar to (1)′–(3)′. That is, we have the following conditions which are equivalent to conditions (4)–(6), respectively:

(4)′ $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_\sigma b]$ for any $b \in B$, $x \in X$ and $h \in H$,
(5)′ $h \cdot_\sigma \langle x, y\rangle_B = \langle [S(h^*_{(1)}) \cdot_\lambda x], [h_{(2)} \cdot_\lambda y]\rangle_B$ for any $x, y \in X$ and $h \in H$,
(6)′ $1_H \cdot_\lambda x = x$ for any $x \in X$.

Let ρ and σ be weak coactions of H^0 on A and B, respectively. Let X be a Hilbert A-B-bimodule of finite type.

Definition 3.3. We say that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a *weak covariant system* if:

(1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$ and $x \in X$,
(2) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$ and $x \in X$,
(3) $\rho({}_A\langle x, y\rangle) = {}_{A\otimes H^0}\langle \lambda(x), \lambda(y)\rangle$ for any $x, y \in X$,
(4) $\sigma(\langle x, y\rangle_B) = \langle \lambda(x), \lambda(y)\rangle_{B\otimes H^0}$ for any $x, y \in X$,
(5) $(\mathrm{id}_X \otimes \epsilon^0) \circ \lambda = \mathrm{id}_X$.

We then call λ a *weak coaction* of H^0 on X with respect to (A, B, ρ, σ). We note that the above conditions are equivalent to the following conditions, respectively:

(1)′ $h \cdot_\lambda ax = [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\lambda x]$ for any $a \in A$, $x \in X$ and $h \in H$,
(2)′ $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_\sigma b]$ for any $b \in B$, $x \in X$ and $h \in H$,
(3)′ $h \cdot_\rho {}_A\langle x, y\rangle = {}_A\langle [h_{(1)} \cdot_\lambda x], [S(h^*_{(2)}) \cdot_\lambda y]\rangle$ for any $x, y \in X$ and $h \in H$,
(4)′ $h \cdot_\sigma \langle x, y\rangle_B = \langle [S(h^*_{(1)}) \cdot_\lambda x], [h_{(2)} \cdot_\lambda y]\rangle_B$ for any $x, y \in X$ and $h \in H$,
(5)′ $1_H \cdot_\lambda x = x$ for any $x \in X$.

We extend the above notions to coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras.

DEFINITION 3.4. Let A, B and H, H^0 be as above. Let ρ and σ be coactions of H^0 on A and B, respectively, and let X be a Hilbert A-B-bimodule of finite type.

(i) We say that $(A, X, \rho, \lambda, H^0)$ is a *left covariant system* if it is a weak left covariant system and the weak left coaction λ of H^0 on X with respect to (A, ρ) satisfies

$(*)$ $(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \Delta^0) \circ \lambda$,

which is equivalent to

$(*)'$ $h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x$ for any $x \in X$ and $h, l \in H$.

We then call λ a *left coaction* of H^0 on X with respect to (A, ρ).

(ii) We say that $(B, X, \sigma, \lambda, H^0)$ is a *right covariant system* if it is a weak right covariant system and the weak right coaction λ of H^0 on X with respect to (B, σ) satisfies $(*)$ or $(*)'$. We call λ a *right coaction* of H^0 on X with respect to (B, σ).

(iii) We say that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a *covariant system* if it is the weak covariant system and the weak coaction λ with respect to (A, B, ρ, σ) satisfies $(*)$ or $(*)'$. We then call λ a *coaction* of H^0 on X with respect to (A, B, ρ, σ).

Furthermore, we extend the notion of the covariant system to twisted coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras. We recall the definition of a twisted coaction (ρ, u) of a C^*-Hopf algebra H^0 on a unital C^*-algebra A (see [9], [10]). Let ρ be a weak coaction of H^0 on A and u a unitary element in $A \otimes H^0 \otimes H^0$. Then we say that (ρ, u) is a *twisted coaction* of H^0 on A if:

(1) $(\rho \otimes \mathrm{id}) \circ \rho = \mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta^0) \circ \rho$,
(2) $(u \otimes 1^0)(\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id})(u) = (\rho \otimes \mathrm{id} \otimes \mathrm{id})(u)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^0)(u)$,
(3) $(\mathrm{id} \otimes h \otimes \epsilon^0)(u) = (\mathrm{id} \otimes \epsilon^0 \otimes h)(u) = \epsilon^0(h)1^0$ for any $h \in H$.

The above conditions are respectively equivalent to:

(1)' $h \cdot_\rho [l \cdot_\rho a] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)} l_{(2)} \cdot_\rho a]\widehat{u^*}(h_{(3)}, l_{(3)})$ for any $a \in A$ and $h, l \in H$,
(2)' $\widehat{u}(h_{(1)}, l_{(1)})\widehat{u}(h_{(2)} l_{(2)}, m) = [h_{(1)} \cdot_\rho \widehat{u}(l_{(1)}, m_{(1)})]\widehat{u}(h_{(2)}, l_{(2)} m_{(2)})$ for any $h, l, m \in H$,
(3)' $\widehat{u}(h, 1) = \widehat{u}(1, h) = \epsilon(h)1^0$ for any $h \in H$.

DEFINITION 3.5. Let A, B and H, H^0 be as above. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively, and let X be a Hilbert A-B-bimodule of finite type. We say that $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a *twisted covariant system* if it is a weak covariant system and the weak coaction λ of H^0 with respect to (A, B, ρ, σ) satisfies

$$(**)\quad (\lambda \otimes \mathrm{id})(\lambda(x)) = u(\mathrm{id} \otimes \Delta^0)(\lambda(x))v^* \text{ for any } x \in X,$$

which is equivalent to

$$(**)'\quad h \cdot_\lambda [l \cdot_\lambda x] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)} l_{(2)} \cdot_\lambda x]\widehat{v}^*(h_{(3)}, l_{(3)}) \text{ for any } x \in X \text{ and } h, l \in H,$$

where $\widehat{u}$ and $\widehat{v}$ are the elements in $\mathrm{Hom}(H \times H, A)$ and $\mathrm{Hom}(H \times H, B)$ induced by $u \in A \otimes H^0 \otimes H^0$ and $v \in B \otimes H^0 \otimes H^0$, respectively. We then call λ a *twisted coaction* of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$.

Next, we introduce the notion of strong Morita equivalence for coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras.

Definition 3.6. Let A, B and H, H^0 be as above.

(i) Let ρ and σ be weak coactions of H^0 on A and B, respectively. We say that ρ is *strongly Morita equivalent* to σ if there are an A-B-equivalence bimodule X and a weak coaction λ of H^0 on X such that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a weak covariant system.

(ii) Let ρ and σ be coactions of H^0 on A and B, respectively. We say that ρ is *strongly Morita equivalent* to σ if there are an A-B-equivalence bimodule X and a coaction λ of H^0 on X such that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system.

(iii) Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively. We say that (ρ, u) is *strongly Morita equivalent* to (σ, v) if there are an A-B-equivalence bimodule X and a twisted coaction λ of H^0 on X such that $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system.

We shall show that the above strong Morita equivalences are equivalence relations.

Proposition 3.7. *The strong Morita equivalence of weak coactions of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra is an equivalence relation.*

Proof. It suffices to show transitivity since the other conditions clearly hold. Let A, B, C be unital C^*-algebras and let X and Y be an A-B-equivalence bimodule and a B-C-equivalence bimodule, respectively. Let ρ, σ and γ be weak coactions of H^0 on A, B and C, respectively. We suppose that ρ is strongly Morita equivalent to σ and that σ is strongly Morita equivalent to γ. Let λ and μ be weak coactions of H^0 on X and Y respectively satisfying Definition 3.6(i). Then $X \otimes_B Y$ is an A-C-equivalence bimodule. We define a bilinear map " $\cdot_{\lambda\otimes\mu}$" from $H \times (X \otimes_B Y)$ to $X \otimes_B Y$ as follows:

For any $x \in X$, $y \in Y$ and $h \in H$,

$$h \cdot_{\lambda\otimes\mu} (x \otimes y) = [h_{(1)} \cdot_\lambda x] \otimes [h_{(2)} \cdot_\mu y].$$

Then we can show that the above map "$\cdot_{\lambda\otimes\mu}$" satisfies conditions $(1)'$–$(5)'$ in Definition 3.3 by routine computations. ∎

COROLLARY 3.8. *The strong Morita equivalence of twisted coactions of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra is an equivalence relation.*

Proof. By Proposition 3.7, we have only to prove condition $(**)'$ in Definition 3.5. Let (ρ, u), (σ, v) and (γ, w) be twisted coactions of H^0 on unital C^*-algebras A, B and C, respectively. Let the notation be as in the proof of Proposition 3.7. For any $x \in X$, $y \in Y$ and $h, l \in H$,

$$\begin{aligned} h \cdot_{\lambda\otimes\mu} [l \cdot_{\lambda\otimes\mu} x \otimes y] &= [h_{(1)} \cdot_\lambda [l_{(1)} \cdot_\lambda x]] \otimes [h_{(2)} \cdot_\mu [l_{(2)} \cdot_\mu y]] \\ &= \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)}l_{(2)} \cdot_\lambda x] \otimes [h_{(3)}l_{(3)} \cdot_\mu y]\widehat{w}^*(h_{(4)}, l_{(4)}) \\ &= \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)}l_{(2)} \cdot_{\lambda\otimes\mu} (x \otimes y)]\widehat{w}^*(h_{(3)}, l_{(3)}). \end{aligned}$$

Therefore, we obtain the conclusion. ∎

The notion of strong Morita equivalence of coactions of a finite-dimensional C^*-Hopf algebra on unital C^*-algebras is an extension of that of actions of a finite group on unital C^*-algebras. To see this, let G be a finite group and α an action of G on a unital C^*-algebra A. We consider the coaction of $C(G)$ on A induced by the action α of G on A; we denote it also by α. That is,

$$\alpha : A \to A \otimes C(G), \qquad a \mapsto \sum_{t\in G} \alpha_t(a) \otimes \delta_t$$

for any $a \in A$, where for any $t \in G$, δ_t is the projection in $C(G)$ defined by

$$\delta_t(s) = \begin{cases} 0 & \text{if } s \neq t, \\ 1 & \text{if } s = t. \end{cases}$$

Let B be a unital C^*-algebra and β an action of G on B. We denote by the same symbol β the coaction of $C(G)$ on B induced by β.

PROPOSITION 3.9. *With the above notation, the following conditions are equivalent:*

(1) *The actions α and β of G on A and B are strongly Morita equivalent.*
(2) *The coactions α and β of $C(G)$ on A and B are strongly Morita equivalent.*

Proof. Suppose (1) holds. Then by Raeburn and Williams [15, Definition 7.2], there are an A-B-equivalence bimodule X and an action u of G by linear isomorphisms of X such that

$$\alpha_t({}_A\langle x, y\rangle) = {}_A\langle u_t(x), u_t(y)\rangle, \qquad \beta_t(\langle x, y\rangle_B) = \langle u_t(x), u_t(y)\rangle_B$$

for any $x, y \in X$ and $t \in G$. We note that by [15, Remark 7.3],

$$u_t(ax) = \alpha_t(a)u_t(x), \qquad u_t(xb) = u_t(x)\beta_t(b)$$

for any $a \in A$, $b \in B$, $x \in X$ and $t \in G$. Let λ be a linear map from X to $X \otimes C(G)$ defined by setting, for any $x \in X$,

$$\lambda(x) = \sum_{t \in G} u_t(x) \otimes \delta_t.$$

Then by routine computations, λ is a coaction of $C(G)$ on X with respect to (A, B, α, β). Hence we obtain (2).

Conversely, suppose (2) holds. Then there are an A-B-equivalence bimodule X and a coaction λ of $C(G)$ on X with respect to (A, B, α, β). We regard G as a subset of $C^*(G)$. For any $t \in G$, we define a linear map u_t on X as follows: for any $x \in X$, $u_t(x) = t \cdot_\lambda x$. Then for any $t, s \in G$ and $x \in X$,

$$u_t(u_s(x)) = t \cdot_\lambda [s \cdot_\lambda x] = ts \cdot_\lambda x = u_{ts}(x).$$

Thus u is an action of G by linear isomorphisms of X, which satisfies the desired conditions by easy computations. Hence we obtain (1). ■

Modifying [15, Example 7.4(b)], we shall obtain the following lemma, which can give examples of the strong Morita equivalence of coactions of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra. First, we introduce the following definition:

Definition 3.10. Let ρ and σ be weak coactions of H^0 on A. We say that ρ is *exterior equivalent* to σ if there is a unitary element $w \in A \otimes H^0$ such that

$$\sigma = \mathrm{Ad}(w) \circ \rho, \qquad (\mathrm{id} \otimes \epsilon^0)(w) = 1.$$

Lemma 3.11. *Let ρ and σ be weak coactions of H^0 on A. Then the following conditions are equivalent:*

(1) *ρ and σ are exterior equivalent.*
(2) *ρ and σ are strongly Morita equivalent via a weak coaction λ from an A-A-equivalence bimodule ${}_A A_A$ to an $A \otimes H^0$ - $A \otimes H^0$-equivalence bimodule ${}_{A \otimes H^0} A \otimes H^0{}_{A \otimes H^0}$, where we regard A and $A \otimes H^0$ as an A-A-equivalence bimodule and an $A \otimes H^0$ - $A \otimes H^0$-equivalence bimodule respectively in the usual way.*

Proof. Suppose (1) holds. Then there is a unitary element $w \in A \otimes H^0$ such that $\sigma = \mathrm{Ad}(w) \circ \rho$ and $(\mathrm{id} \otimes \epsilon^0)(w) = 1$. Let λ be a linear map from ${}_A A_A$ to ${}_{A \otimes H^0} A \otimes H^0{}_{A \otimes H^0}$ defined by $\lambda(x) = \rho(x)w^*$ for any $x \in {}_A A_A$. By routine computations, λ is a weak coaction of H^0 on ${}_A A_A$ with respect to (A, A, ρ, σ).

Conversely, suppose (2) holds. We note that λ is a weak coaction of H^0 on ${}_A A_A$ with respect to (A, A, ρ, σ). We identify $A \otimes H^0$ with

$\mathrm{End}_{A\otimes H^0}(A\otimes H^0{}_{A\otimes H^0})$, the C^*-algebra of all right $A\otimes H^0$-module maps on $A\otimes H^0{}_{A\otimes H^0}$. Let $w=\theta_{\lambda(1)^*,1\otimes 1^0}$ be the rank-one operator on $A\otimes H^0{}_{A\otimes H^0}$ induced by $\lambda(1)^*$ and $1\otimes 1^0$. Then w is a unitary element in $\mathrm{End}_{A\otimes H^0}(A\otimes H^0{}_{A\otimes H^0})$. Indeed, for any $x\in A\otimes H^0{}_{A\otimes H^0}$,

$$ww^*(x)=\lambda(1)^*(1\otimes 1^0)\lambda(1)x=\langle\lambda(1),\lambda(1)\rangle_{A\otimes H^0}\,x=\sigma(1)x=x,$$
$$w^*w(x)=\lambda(1)\lambda(1)^*x={}_{A\otimes H^0}\langle\lambda(1),\lambda(1)\rangle x=\rho(1)x=x.$$

Also, for any $a\in A$ and $x\in A\otimes H^0{}_{A\otimes H^0}$,

$$\begin{aligned}(w\rho(a)w^*)(x)&=w(\rho(a)\lambda(1)x)=\lambda(1)^*\lambda(a)x\\&=\langle\lambda(1),\lambda(a)\rangle_{A\otimes H^0}\,x=\sigma(a)x.\end{aligned}$$

Thus w is a unitary element in $A\otimes H^0$ and $\sigma=\mathrm{Ad}(w)\circ\rho$. Let $z=(\mathrm{id}\otimes\epsilon^0)(w)$. Then z is a unitary element in A such that $az=za$ for any $a\in A$ since $\sigma=\mathrm{Ad}(w)\circ\rho$. Let $w_1=w(z^*\otimes 1^0)$. Then w_1 is a unitary element in $A\otimes H^0$ such that $\sigma=\mathrm{Ad}(w_1)\circ\rho$ and $(\mathrm{id}\otimes\epsilon^0)(w_1)=1$. Therefore we obtain (1). ∎

LEMMA 3.12. *Let (ρ,u) and (σ,v) be twisted coactions of H^0 on A. Then the following conditions are equivalent:*

1. *(ρ,u) and (σ,v) are exterior equivalent.*
2. *(ρ,u) and (σ,v) are strongly Morita equivalent via a twisted coaction λ from an A-A-equivalence bimodule ${}_AA_A$ to an $A\otimes H^0$-$A\otimes H^0$-equivalence bimodule ${}_{A\otimes H^0}A\otimes H^0{}_{A\otimes H^0}$, where we regard A and $A\otimes H^0$ as an A-A-equivalence bimodule and an $A\otimes H^0$-$A\otimes H^0$-equivalence bimodule respectively in the usual way.*

Proof. Suppose (1) holds. Then there is a unitary $w\in A\otimes H^0$ such that

$$\sigma=\mathrm{Ad}(w)\circ\rho,\qquad v=(w\otimes 1^0)(\rho\otimes\mathrm{id})(w)u(\mathrm{id}\otimes\Delta^0)(w^*).$$

Let λ be as in the proof of Lemma 3.11. Then for any $x\in{}_AA_A$,

$$\begin{aligned}((\lambda\otimes\mathrm{id})\circ\lambda)(x)&=u(\mathrm{id}\otimes\Delta^0)(\rho(x))u^*(\rho\otimes\mathrm{id})(w^*)(w^*\otimes 1^0)\\&=u(\mathrm{id}\otimes\Delta^0)(\rho(x))(\mathrm{id}\otimes\Delta^0)(w^*)v^*\\&=u(\mathrm{id}\otimes\Delta^0)(\lambda(x))v^*.\end{aligned}$$

Thus by Lemma 3.11, λ is a twisted coaction of H^0 on ${}_AA_A$ with respect to (A,A,ρ,u,σ,v).

Conversely, suppose (2) holds. We note that λ is a twisted coaction of H^0 on ${}_AA_A$ with respect to (A,A,ρ,u,σ,v). We identify $A\otimes H^0$ with $\mathrm{End}_{A\otimes H^0}(A\otimes H^0{}_{A\otimes H^0})$. Let $w=\theta_{\lambda(1)^*,1\otimes 1^0}$ be the rank-one operator on $A\otimes H^0{}_{A\otimes H^0}$ induced by $\lambda(1)^*$ and $1\otimes 1^0$. Then w is a unitary element in $\mathrm{End}_{A\otimes H^0}(A\otimes H^0{}_{A\otimes H^0})$ such that $\sigma=\mathrm{Ad}(w)\circ\rho$ by Lemma 3.11. We note that $w^*={}_{A\otimes H^0}\langle\lambda(1),1\otimes 1^0\rangle$. Indeed, for any $x\in A\otimes H^0{}_{A\otimes H^0}$,

$$w^*x=(1\otimes 1^0)\langle\lambda(1)^*,x\rangle_{A\otimes H^0}=\lambda(1)x={}_{A\otimes H^0}\langle\lambda(1),1\otimes 1^0\rangle x.$$

Hence $w^* = {}_{A\otimes H^0}\langle \lambda(1), 1\otimes 1^0\rangle$. Thus

$$\begin{aligned}(\rho\otimes \mathrm{id})(w^*) &= (\rho\otimes \mathrm{id})({}_{A\otimes H^0}\langle \lambda(1), 1\otimes 1^0\rangle)\\ &= {}_{A\otimes H^0\otimes H^0}\langle ((\lambda\otimes \mathrm{id})\circ\lambda)(1), \lambda(1)\otimes 1^0\rangle\\ &= {}_{A\otimes H^0\otimes H^0}\langle u((\mathrm{id}\otimes\Delta^0)\circ\lambda)(1)v^*, \lambda(1)\otimes 1^0\rangle\\ &= u((\mathrm{id}\otimes\Delta^0)\circ\lambda)(1)v^*(\lambda(1)^*\otimes 1^0).\end{aligned}$$

It follows that

$$\begin{aligned}&(\rho\otimes \mathrm{id})(w^*)(w^*\otimes 1^0)\\ &\quad= u((\mathrm{id}\otimes\Delta^0)\circ\lambda)(1)v^*(\lambda(1)^*\otimes 1^0)\big({}_{A\otimes H^0}\langle \lambda(1), 1\otimes 1^0\rangle\otimes 1^0\big)\\ &\quad= u((\mathrm{id}\otimes\Delta^0)\circ\lambda)(1)v^*\big(\langle \lambda(1), \lambda(1)\rangle_{A\otimes H^0}\otimes 1^0\big)\\ &\quad= u((\mathrm{id}\otimes\Delta^0)\circ\lambda)(1)v^*(\sigma(\langle 1,1\rangle_A)\otimes 1^0)\\ &\quad= u(\mathrm{id}\otimes\Delta^0)(\lambda(1))v^* = u(\mathrm{id}\otimes\Delta^0)({}_{A\otimes H^0}\langle \lambda(1), 1\otimes 1^0\rangle)v^*\\ &\quad= u(\mathrm{id}\otimes\Delta^0)(w^*)v^*.\end{aligned}$$

Thus $v = (w\otimes 1^0)(\rho\otimes\mathrm{id})(w)u(\mathrm{id}\otimes\Delta^0)(w^*)$, proving (1). ∎

Next, we discuss relations between innerness, outerness and strong Morita equivalence. Let $\rho_{H^0}^A$ be the trivial coaction of H^0 on A.

Lemma 3.13. (i) *Let ρ be a weak coaction of H^0 on A. Then the following conditions are equivalent:*

(1) *ρ is inner.*
(2) *ρ is strongly Morita equivalent to $\rho_{H^0}^A$.*

(ii) *Let ρ be a coaction of H^0 on A. Then the following conditions are equivalent:*

(1) *ρ is strongly inner,*
(2) *ρ is strongly Morita equivalent to $\rho_{H^0}^A$.*

Proof. (i) Suppose that ρ is inner. Then there is a unitary $w\in A\otimes H^0$ such that $\rho = \mathrm{Ad}(w)\circ\rho_{H^0}^A$ and $(\mathrm{id}\otimes\epsilon^0)(w) = 1$ (we argue as in the proof of (2)⇒(1) in Lemma 3.11). Thus ρ is exterior equivalent to $\rho_{H^0}^A$. Hence by Lemma 3.11, ρ is strongly Morita equivalent to $\rho_{H^0}^A$.

Conversely, suppose that ρ is strongly Morita equivalent to $\rho_{H^0}^A$. Then there are an A-A-equivalence bimodule X and a weak coaction λ of H^0 on X with respect to $(A, A, \rho, \rho_{H^0}^A)$. We note that for any $a\in A$ and $x\in X$,

$$\lambda(xa) = \lambda(x)\rho_{H^0}^A(a) = \lambda(x)(a\otimes 1^0).$$

For any $h\in H$, let $\widehat{w}(h)$ be a linear map on X defined by setting, for any $x\in X$,

$$\widehat{w}(h)x = h\cdot_\lambda x.$$

Then by the above discussion, $\widehat{w}(h)$ is in $\mathrm{End}_A(X)$, the C^*-algebra of all right A-module maps on X. Since X is an A-A-equivalence bimodule, we can identify $\mathrm{End}_A(X)$ with A and regard $\widehat{w}(h)$ as an element in A for any $h \in H$. Furthermore, since the map $h \mapsto \widehat{w}(h)$ is linear, $\widehat{w} \in \mathrm{Hom}(H, A)$. Let $w \in A \otimes H^0$ be induced by $\widehat{w}$. By the definition, clearly $\widehat{w}(1) = 1$. We show that w is a unitary element in $A \otimes H^0$ such that $\rho = \mathrm{Ad}(w) \circ \rho^A_{H^0}$. For any $x, y \in X$ and $h \in H$,

$$\begin{aligned}\langle (\widehat{w^* w})(h)x, y\rangle_A &= \langle \widehat{w}(h_{(2)})x, \widehat{w}(S(h^*_{(1)}))y\rangle_A \\ &= \langle [h_{(2)} \cdot_\lambda x]\,, [S(h^*_{(1)}) \cdot_\lambda y]\rangle_A = S(h^*) \cdot_{\rho^A_{H^0}} \langle x, y\rangle_A = \langle \epsilon(h)x, y\rangle_A.\end{aligned}$$

Thus $w^* w = 1 \otimes 1^0$. Also, for any $x, y \in X$ and $h \in H$,

$$\begin{aligned}h \cdot_\rho {}_A\langle x, y\rangle &= {}_A\langle [h_{(1)} \cdot_\lambda x], [S(h^*_{(2)}) \cdot_\lambda y]\rangle = {}_A\langle \widehat{w}(h_{(1)})x, \widehat{w}(S(h^*_{(2)}))y\rangle \\ &= \widehat{w}(h_{(1)})_A\langle x, y\rangle \widehat{w^*}(h_{(2)}).\end{aligned}$$

Hence $\rho = \mathrm{Ad}(w) \circ \rho^A_{H^0}$ since X is an A-A-equivalence bimodule. Thus $ww^* = w\rho^A_{H^0}(1)w^* = \rho(1) = 1 \otimes 1^0$. Therefore, the weak coaction ρ is inner.

(ii) Suppose that ρ is strongly inner. Then it is exterior equivalent to $\rho^A_{H^0}$. Hence by Lemma 3.12, ρ is strongly Morita equivalent to $\rho^A_{H^0}$.

Conversely, suppose that ρ is strongly Morita equivalent to $\rho^A_{H^0}$. Then there are an A-A-equivalence bimodule X and a coaction λ of H^0 on X with respect to $(A, A, \rho, \rho^A_{H^0})$. Let w be as in (i). It suffices to show that for any $h, l \in H$ we have $\widehat{w}(hl) = \widehat{w}(h)\widehat{w}(l)$. Indeed, for any $x \in X$ and $h, l \in H$,

$$\widehat{w}(h)\widehat{w}(l)x = h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x = \widehat{w}(hl)x.$$

Therefore, ρ is strongly inner. ∎

Let $\rho^A_{H^0}$ and $\rho^B_{H^0}$ be the trivial coactions of H^0 on A and B, respectively. Suppose that A and B are strongly Morita equivalent and let X be an A-B-equivalence bimodule. Then $\rho^A_{H^0}$ and $\rho^B_{H^0}$ are strongly Morita equivalent. If a linear map $\lambda^X_{H^0}$ from X to $X \otimes H^0$ is defined by $\lambda^X_{H^0}(x) = x \otimes 1^0$ for any $x \in X$, then $\lambda^X_{H^0}$ is a coaction of H^0 on X with respect to $(A, B, \rho^A_{H^0}, \rho^B_{H^0})$.

Corollary 3.14.

(i) *Let ρ and σ be weak coactions of H^0 on A and B, respectively. If ρ is strongly Morita equivalent to σ, then ρ is inner if and only if so is σ.*

(ii) *Let ρ and σ be coactions of H^0 on A and B, respectively. If ρ is strongly Morita equivalent to σ, then ρ is strongly inner if and only if so is σ.*

Proof. (i) Suppose that ρ is inner. Then σ is strongly Morita equivalent to $\rho^B_{H^0}$ by Lemma 3.13(i), Proposition 3.7 and the above discussion. Therefore, σ is inner by Lemma 3.13(i).

(ii) Suppose that ρ is strongly inner. Then σ is strongly Morita equivalent to $\rho^B_{H^0}$ by Lemma 3.13(ii), Corollary 3.8 and the above discussion. Therefore, σ is strongly inner by Lemma 3.13(ii). ∎

PROPOSITION 3.15. *Suppose that H^0 is not trivial. Let ρ and σ be coactions of H^0 on A and B, respectively. If ρ is strongly Morita equivalent to σ, then ρ is outer if and only if so is σ.*

Proof. Suppose that ρ is outer. To show that σ is outer, let π be a surjective C^*-Hopf algebra homomorphism of H^0 onto a non-trivial C^*-Hopf algebra K^0. Suppose that $(\mathrm{id}\otimes\pi)\circ\sigma$ is inner. Then $(\mathrm{id}\otimes\pi)\circ\sigma$ is strongly Morita equivalent to $(\mathrm{id}\otimes\pi)\circ\rho$ by easy computations. Thus by Corollary 3.14(i), $(\mathrm{id}\otimes\pi)\circ\rho$ is inner. This is a contradiction completing the proof. ∎

Furthermore, we also have the following easy lemma:

LEMMA 3.16. *Let (ρ, u) be a twisted coaction of H^0 on A and let $(\rho\otimes\mathrm{id}, u\otimes I_n)$ be a twisted coaction of H^0 on $A\otimes M_n(\mathbb{C})$, where n is a positive integer and we identify $A\otimes H^0\otimes M_n(\mathbb{C})$ with $A\otimes M_n(\mathbb{C})\otimes H^0$. Then (ρ, u) is strongly Morita equivalent to $(\rho\otimes\mathrm{id}, u\otimes I_n)$.*

Proof. Let f be a minimal projection in $M_n(\mathbb{C})$ and define $X = (1\otimes f)(A\otimes M_n(\mathbb{C}))$. We regard it as an A-$A\otimes M_n(\mathbb{C})$-equivalence bimodule in the usual way. Let λ be the linear map from X to $X\otimes H^0$ defined by

$$\lambda((1\otimes f)x) = (1\otimes f\otimes 1^0)(\rho\otimes\mathrm{id})(x)$$

for any $x\in A\otimes M_n(\mathbb{C})$, where we identify $A\otimes H^0\otimes M_n(\mathbb{C})$ with $A\otimes M_n(\mathbb{C})$ $\otimes H^0$. By routine computations, we can see that λ satisfies conditions (1)–(5) in Definition 3.3 and condition $(**)$. ∎

4. Crossed products of Hilbert C^*-bimodules of finite type by finite-dimensional C^*-Hopf algebras. In this section, we extend the notion of crossed products of Hilbert C^*-bimodules of finite type defined in [7], [9] to (twisted) coactions of finite-dimensional C^*-Hopf algebras.

Let H be a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. Let A and B be unital C^*-algebras and X a Hilbert A-B-bimodule of finite type. Let $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ be a twisted covariant system. Under certain conditions, we define $X\rtimes_\lambda H$, a Hilbert $A\rtimes_{\rho,u}H$ - $B\rtimes_{\sigma,v}H$-bimodule of finite type, as follows: $X\rtimes_\lambda H$ is just $X\otimes H$ (the algebraic tensor product) as a vector space; and its left action and right action are given by

$$(a\rtimes_{\rho,u}h)(x\rtimes_\lambda l) = a[h_{(1)}\cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)})\rtimes_\lambda h_{(3)}l_{(2)},$$
$$(x\rtimes_\lambda l)(b\rtimes_{\sigma,v}m) = x[l_{(1)}\cdot_{\sigma,v}b]\widehat{v}(l_{(2)}, m_{(1)})\rtimes_\lambda l_{(3)}m_{(2)}$$

for any $a \in A$, $b \in B$, $x \in X$ and $h, l, m \in H$. Then for any $a_1, a_2 \in A$, $x \in X$ and $h, l, m \in H$,

$$
\begin{aligned}
&\big((a_1 \rtimes_{\rho,u} h)(a_2 \rtimes_{\rho,u} l)\big)(x \rtimes_\lambda m) \\
&\qquad = a_1[h_{(1)} \cdot_{\rho,u} a_2]\widehat{u}(h_{(2)}, l_{(1)})[h_{(3)}l_{(2)} \cdot_\lambda x]\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\
&\qquad = a_1\big[h_{(1)} \cdot_\lambda a_2[l_{(1)} \cdot_\lambda x]\big]\widehat{v}(h_{(2)}, l_{(2)})\widehat{v}(h_{(3)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(4)}l_{(4)}m_{(2)} \\
&\qquad = a_1\big[h_{(1)} \cdot_\lambda a_2[l_{(1)} \cdot_\lambda x]\widehat{v}(l_{(2)}, m_{(1)})\big]\widehat{v}(h_{(2)}, l_{(3)}m_{(2)}) \rtimes_\lambda h_{(3)}l_{(4)}m_{(3)} \\
&\qquad = (a_1 \rtimes_{\rho,u} h)\big((a_2 \rtimes_{\rho,u} l)(x \rtimes_\lambda m)\big).
\end{aligned}
$$

Also, for any $b_1, b_2 \in B$, $x \in X$ and $h, l, m \in H$,

$$
\begin{aligned}
&(x \rtimes_\lambda h)\big((b_1 \rtimes_{\sigma,v} l)(b_2 \rtimes_{\sigma,v} m)\big) \\
&\qquad = x[h_{(1)} \cdot_{\sigma,v} b_1]\big[h_{(2)} \cdot_{\sigma,v} [l_{(1)} \cdot_{\sigma,v} b_2]\big]\widehat{v}(h_{(3)}, l_{(2)})\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \\
&\qquad\quad \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\
&\qquad = x[h_{(1)} \cdot_{\sigma,v} b_1]\widehat{v}(h_{(2)}, l_{(1)})[h_{(3)}l_{(2)} \cdot_{\sigma,v} b_2]\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\
&\qquad = \big((x \rtimes_\lambda h)(b_1 \rtimes_{\sigma,v} l)\big)(b_2 \rtimes_{\sigma,v} m).
\end{aligned}
$$

Thus $X \rtimes_\lambda H$ is a left $A \rtimes_{\rho,u} H$- and right $B \rtimes_{\sigma,v} H$-bimodule. Also, its left $A \rtimes_{\rho,u} H$-valued and right $B \rtimes_{\sigma,v} H$-valued inner products are given by

$$
\begin{aligned}
&{}_{A \rtimes_{\rho,u} H}\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle \\
&\qquad\qquad = {}_A\langle x, [S(h_{(2)}l^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(1)}l^*_{(2)})^*, l_{(1)})\rangle \rtimes_{\rho,u} h_{(3)}l^*_{(4)}, \\
&\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle_{B \rtimes_{\sigma,v} H} \\
&\qquad\qquad = \widehat{v^*}(h^*_{(2)}, S(h_{(1)})^*)[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B]\widehat{v}(h^*_{(4)}, l_{(1)}) \rtimes_{\sigma,v} h^*_{(5)}l_{(2)}
\end{aligned}
$$

for any $x, y \in X$ and $h, l \in H$. We shall show that $X \rtimes_\lambda H$ is a Hilbert $A \rtimes_{\rho,u} H$ - $B \rtimes_{\sigma,v} H$-bimodule of finite type by proving that $X \rtimes_\lambda H$ satisfies conditions (1)–(10) in [9, Lemma 1.3]. Clearly $X \rtimes_\lambda H$ is a left $A \rtimes_\rho H$- and right $B \rtimes_\sigma H$-bimodule. Thus conditions (1), (4) in [9, Lemma 1.3] are satisfied. For any $a, b \in A$, $x, y \in X$ and $h, l, m \in H$,

$$
\begin{aligned}
&(a \rtimes_{\rho,u} h)\, {}_{A \rtimes_{\rho,u} H}\langle x \rtimes_\lambda l, y \rtimes_\lambda m\rangle \\
&= a\big[h_{(1)} \cdot_{\rho,u} {}_A\big\langle x, [S(l_{(2)}m^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(l_{(1)}m^*_{(2)})^*, m_{(1)})\big\rangle\big]\widehat{u}(h_{(2)}, l_{(3)}m^*_{(4)}) \\
&\quad \rtimes_{\rho,u} h_{(3)}l_{(4)}m^*_{(5)} \\
&= a\, {}_A\big\langle [h_{(1)} \cdot_\lambda x], [S(h^*_{(3)} \cdot_\lambda [S(l_{(2)}m^*_{(3)})^* \cdot_\lambda y]\big[S(h^*_{(2)}) \cdot_{\sigma,v} \widehat{v}(S(l_{(1)}m^*_{(2)})^*, m_{(1)})\big]\big\rangle \\
&\quad \times \widehat{u}(h_{(4)}, l_{(3)}m^*_{(4)}) \rtimes_{\rho,u} h_{(5)}l_{(4)}m^*_{(5)} \\
&= a\, {}_A\langle [h_{(1)} \cdot_\lambda x], \big[S(h^*_{(5)}) \cdot_\lambda [S(l_{(4)}m^*_{(4)})^* \cdot_\lambda y]\big]\widehat{v}(S(h^*_{(4)}), S(l_{(3)}m^*_{(3)})^*) \\
&\quad \times \widehat{v}(S(h_{(3)}l_{(2)}m^*_{(2)})^*, m_{(1)})\widehat{v^*}(S(h^*_{(2)}), S(l^*_{(1)}))\rangle\widehat{u}(h_{(6)}, l_{(5)}m^*_{(5)}) \\
&\quad \rtimes_{\rho,u} h_{(7)}l_{(6)}m^*_{(6)}
\end{aligned}
$$

$$
\begin{aligned}
&= a\, {}_A\big\langle [h_{(1)} \cdot_\lambda x], \widehat{u}(S(h^*_{(5)}), S(l_{(4)}m^*_{(4)})^*)[S(h_{(4)}l_{(3)}m^*_{(3)})^* \cdot_\lambda y] \\
&\quad \times \widehat{v}(S(h_{(3)}l_{(2)}m^*_{(2)})^*, m_{(1)})\widehat{v}(h_{(2)}, l_{(1)})^* \big\rangle \widehat{u}(h_{(6)}, l_{(5)}m^*_{(5)}) \rtimes_{\rho,u} h_{(7)}l_{(6)}m^*_{(6)} \\
&= a\, {}_A\big\langle [h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}), [S(h_{(4)}l_{(3)}m^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(3)}l_{(2)}m^*_{(2)})^*, m_{(1)}) \big\rangle \\
&\quad \times \widehat{u^*}(h_{(5)}, l_{(4)}m^*_{(4)})\widehat{u}(h_{(6)}, l_{(5)}m^*_{(5)}) \rtimes_{\rho,u} h_{(7)}l_{(6)}m^*_{(6)} \\
&= a\, {}_A\big\langle [h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}), [S(h_{(4)}l_{(3)}m^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(3)}l_{(2)}m^*_{(2)})^*, m_{(1)}) \big\rangle \\
&\quad \rtimes_{\rho,u} h_{(5)}l_{(4)}m^*_{(4)} \\
&= {}_{A\rtimes_{\rho,u}H}\langle a[h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}) \rtimes_\lambda h_{(3)}l_{(2)}, y \rtimes_\lambda m \rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (a \rtimes_{\rho,u} h)(x \rtimes_\lambda l), y \rtimes_\lambda m \rangle.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\langle x \rtimes_\lambda h, y \rtimes_\lambda l \rangle_{B\rtimes_{\sigma,v}H}(b \rtimes_{\sigma,v} m) \\
&= \widehat{v^*}(h^*_{(2)}, S(h^*_{(1)}))[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B]\widehat{v}(h^*_{(4)}, l_{(1)})[h^*_{(5)}l_{(2)} \cdot_{\sigma,v} b]\widehat{v}(h^*_{(6)}l_{(3)}, m_{(1)}) \\
&\quad \rtimes_{\sigma,v} h^*_{(7)}l_{(4)}m_{(2)} \\
&= \widehat{v^*}(h^*_{(2)}, S(h^*_{(1)}))[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B]\big[h^*_{(4)} \cdot_{\sigma,v} [l_{(1)} \cdot_{\sigma,v} b]\big]\widehat{v}(h^*_{(5)}, l_{(2)})\widehat{v}(h^*_{(6)}l_{(3)}, m_{(1)}) \\
&\quad \rtimes_{\sigma,v} h^*_{(7)}l_{(4)}m_{(2)} \\
&= \widehat{v^*}(h^*_{(2)}S(h^*_{(1)}))\big[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B[l_{(1)} \cdot_{\sigma,v} b]\big]\widehat{v}(h^*_{(4)}, l_{(2)})\widehat{v}(h^*_{(5)}l_{(3)}, m_{(1)}) \\
&\quad \rtimes_{\sigma,v} h^*_{(6)}l_{(4)}m_{(2)} \\
&= \widehat{v^*}(h^*_{(2)}, S(h^*_{(1)}))\big[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B[l_{(1)} \cdot_{\sigma,v} b]\big][h^*_{(4)} \cdot_{\sigma,v} \widehat{v}(l_{(2)}, m_{(1)})]\widehat{v}(h^*_{(5)}, l_{(3)}m_{(2)}) \\
&\quad \rtimes_{\sigma,v} h^*_{(6)}l_{(4)}m_{(3)} \\
&= \widehat{v^*}(h^*_{(2)}, S(h^*_{(1)}))\big[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B[l_{(1)} \cdot_{\sigma,v} b]\widehat{v}(l_{(2)}, m_{(1)})\big]\widehat{v}(h^*_{(4)}, l_{(3)}m_{(2)}) \\
&\quad \rtimes_{\sigma,v} h^*_{(5)}l_{(4)}m_{(3)} \\
&= \langle x \rtimes_\lambda h, (y \rtimes_\lambda l)(b \rtimes_{\sigma,v} m)\rangle_{B\rtimes_{\sigma,v}H}.
\end{aligned}
$$

Thus conditions (3), (6) in [9, Lemma 1.3] are satisfied. For any $x, y \in X$ and $h, l \in H$,

$$
\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle^* \\
&= \widehat{u^*}(l_{(5)}h^*_{(4)}, S(l_{(4)}h^*_{(3)}))\big[l_{(6)}h^*_{(5)} \cdot_{\rho,u} {}_A\langle [S(l_{(3)}h^*_{(2)}) \cdot_\lambda y]\widehat{v}(S(l_{(2)}h^*_{(1)}), l_{(1)}), x\rangle\big] \\
&\quad \rtimes_{\rho,u} l_{(7)}h^*_{(6)} \\
&= \widehat{u^*}(l_{(5)}h^*_{(4)}, S(l_{(4)}h^*_{(3)})) \\
&\quad \times {}_A\big\langle \big[l_{(6)}h^*_{(5)} \cdot_\lambda [S(l_{(3)}h^*_{(2)}) \cdot_\lambda y]\big]\big[l_{(7)}h^*_{(6)} \cdot_{\sigma,v} \widehat{v}(S(l_{(2)}h^*_{(1)}), l_{(1)})\big], [S(l_{(8)}h^*_{(7)})^* \cdot_\lambda x]\big\rangle \\
&\quad \rtimes_{\rho,u} l_{(9)}h^*_{(8)}
\end{aligned}
$$

$$
\begin{aligned}
&= \widehat{u^*}(l_{(6)}h^*_{(6)}, S(l_{(5)}h^*_{(5)})) \\
&\quad \times {}_A\big\langle \big[l_{(7)}h^*_{(7)} \cdot_\lambda [S(l_{(4)}h^*_{(4)}) \cdot_\lambda y]\big]\widehat{v}(l_{(8)}h^*_{(8)}, S(l_{(3)}h^*_{(3)}))\widehat{v}(l_{(9)}h^*_{(9)}S(l_{(2)}h^*_{(2)}), l_{(1)}) \\
&\quad \times \widehat{v^*}(l_{(10)}h^*_{(10)}, S(h^*_{(1)})), [S(l_{(11)}h^*_{(11)})^* \cdot_\lambda x]\big\rangle \rtimes_{\rho,u} l_{(12)}h^*_{(12)} \\
&= \widehat{u^*}(l_{(6)}h^*_{(6)}, S(l_{(5)}h^*_{(5)})) \\
&\quad \times {}_A\big\langle \widehat{u}(l_{(7)}h^*_{(7)}, S(l_{(4)}h^*_{(4)}))[l_{(8)}h^*_{(8)}S(l_{(3)}h^*_{(3)}) \cdot_\lambda y]\widehat{v}(l_{(9)}h^*_{(9)}S(l_{(2)}h^*_{(2)}), l_{(1)}) \\
&\quad \times \widehat{v^*}(l_{(10)}h^*_{(10)}, S(h^*_{(1)})), [S(l_{(11)}h^*_{(11)})^* \cdot_\lambda x]\big\rangle \rtimes_{\rho,u} l_{(8)}h^*_{(8)} \\
&= {}_A\langle y, [S(l_{(2)}h^*_{(3)})^* \cdot_\lambda x]\widehat{v}(S(l_{(1)}h^*_{(2)})^*, h_{(1)})\rangle \rtimes_{\rho,u} l_{(3)}h^*_{(4)} \\
&= {}_{A\rtimes_{\rho,u}H}\langle y \rtimes_\lambda l, x \rtimes_\lambda h\rangle.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle^*_{B\rtimes_{\sigma,v}H} \\
&= \widehat{v^*}(l^*_{(3)}h_{(6)}, S(l^*_{(2)}h_{(5)}))\big[l^*_{(4)}h_{(7)} \cdot_\sigma \widehat{v}(h^*_{(4)}, l_{(1)})^*[S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\widehat{v}(S(h_{(2)}), h_{(1)})\big] \\
&\quad \rtimes_{\sigma,v} l^*_{(5)}h_{(8)} \\
&= \widehat{v^*}(l^*_{(3)}h_{(6)}, S(l_{(2)}h_{(5)}))[l_{(4)^*}h_{(7)} \cdot_{\sigma,v} \widehat{v^*}(S(h_{(4)}), S(l^*_{(1)}))] \\
&\quad \times \big[l^*_{(5)}h_{(8)} \cdot_{\sigma,v} [S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\big][l^*_{(6)}h_{(9)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(7)}h_{(10)} \\
&= \widehat{v}(S(l^*_{(3)}h_{(6)})^*, (l^*_{(2)}h_{(5)})^*)^*[S(h^*_{(7)}l_{(4)}) \cdot_\sigma \widehat{v}(h^*_{(4)}, l_{(1)})]^* \\
&\quad \times \big[l^*_{(5)}h_{(8)} \cdot_{\sigma,v} [S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\big][l^*_{(6)}h_{(9)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(7)}h_{(10)} \\
&= \big[[S(h^*_{(7)}l_{(4)}) \cdot_{\sigma,v} \widehat{v}(h^*_{(4)}, l_{(1)})]\widehat{v}(S(h^*_{(6)}l_{(3)}), h^*_{(5)}l_{(2)})\big]^* \\
&\quad \times \big[l^*_{(5)}h_{(8)} \cdot_{\sigma,v} [S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\big][l^*_{(6)}h_{(9)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(7)}h_{(10)} \\
&= [\widehat{v}(S(h^*_{(7)}l_{(3)}), h^*_{(4)})\widehat{v}(S(h^*_{(6)}l_{(2)})h^*_{(5)}, l_{(1)})]^*\big[l^*_{(4)}h_{(8)} \cdot_{\sigma,v} [S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\big] \\
&\quad \times [l^*_{(5)}h_{(9)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(6)}h_{(10)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^*\widehat{v^*}(l^*_{(3)}h_{(5)}, S(h_{(4)}))\big[l^*_{(4)}h_{(6)} \cdot_{\sigma,v} [S(h_{(3)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\big] \\
&\quad \times [l^*_{(5)}h_{(7)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(6)}h_{(8)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^*[l^*_{(3)}h_{(5)}S(h_{(4)}) \cdot_{\sigma,v} \langle y, x\rangle_B]\widehat{v^*}(l^*_{(4)}h_{(6)}, S(h_{(3)})) \\
&\quad \times [l^*_{(5)}h_{(7)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma,v} l^*_{(6)}h_{(8)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^*[l^*_{(3)} \cdot_{\sigma,v} \langle y, x\rangle_B]\widehat{v^*}(l^*_{(4)}h_{(4)}, S(h_{(3)}))l^*_{(5)}h_{(5)} \cdot_{\sigma,v} \widehat{v}(S(h_{(2)}), h_{(1)})] \\
&\quad \rtimes_{\sigma,v} l^*_{(6)}h_{(6)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^*[l^*_{(3)} \cdot_{\sigma,v} \langle y, x\rangle_B]\widehat{v}(l^*_{(4)}h_{(5)}S(h_{(4)}), h_{(1)})\widehat{v^*}(l^*_{(5)}h_{(6)}, S(h_{(3)})h_{(2)}) \\
&\quad \rtimes_{\sigma,v} l^*_{(6)}h_{(7)} \\
&= \langle y \rtimes_\lambda l, x \rtimes_\lambda h\rangle_{B\rtimes_{\sigma,v}H}.
\end{aligned}
$$

Thus conditions (2), (5) in [9, Lemma 1.3] are satisfied. Moreover, for any $b\in B$, $x,y\in X$ and $l,m\in H$,

$$
\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle x\rtimes_\lambda l,(y\rtimes_\lambda m)(b\rtimes_{\sigma,v}1)^*\rangle \\
&\quad={}_A\big\langle x,[S(l_{(3)}m^*_{(5)})^*\cdot_\lambda y]\widehat{v}(S(l_{(2)}m^*_{(4)})^*,m_{(1)})[S(l_{(1)}m^*_{(3)})^*m_{(2)}\cdot_{\sigma,v}b^*]\big\rangle \\
&\qquad\rtimes_{\rho,u}l_{(4)}m^*_{(6)} \\
&\quad={}_A\langle x[l_{(1)}\cdot_{\sigma,v}b],[S(l_{(3)}m^*_{(3)})^*\cdot_\lambda y]\widehat{v}(S(l_{(2)}m^*_{(2)})^*,m_{(1)})\rangle\rtimes_{\rho,u}l_{(4)}m^*_{(4)} \\
&\quad={}_{A\rtimes_{\rho,u}H}\langle(x\rtimes_\lambda l)(b\rtimes_{\sigma,v}1),y\rtimes_\lambda m\rangle.
\end{aligned}
$$

Also, for any $x,y\in X$ and $h,l,m\in H$,

$$
\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle x\rtimes_\lambda l,(y\rtimes_\lambda m)(1\rtimes_{\sigma,v}h)^*\rangle \\
&\quad={}_{A\rtimes_{\rho,u}H}\big\langle x\rtimes_\lambda l,y[m_{(1)}\cdot_{\sigma,v}\widehat{v}(S(h_{(2)}),h_{(1)})^*]\widehat{v}(m_{(2)},h^*_{(3)})\rtimes_\lambda m_{(3)}h^*_{(4)}\big\rangle \\
&\quad={}_A\big\langle x,\big[S(l_{(2)}h_{(6)}m^*_{(5)})^*\cdot_\lambda y[m_{(1)}\cdot_{\sigma,v}\widehat{v^*}(h^*_{(2)},S(h_{(1)})^*)]\widehat{v}(m_{(2)},h^*_{(3)})\big] \\
&\qquad\times\widehat{v}(S(l_{(1)}h_{(5)}m^*_{(4)})^*,m_{(3)}h^*_{(4)})\big\rangle\rtimes_{\rho,u}l_{(3)}h_{(7)}m^*_{(6)} \\
&\quad={}_A\big\langle x,\big[S(l_{(2)}h_{(7)}m^*_{(5)})^*\cdot_\lambda y\widehat{v}(m_{(1)},h^*_{(3)}S(h^*_{(2)}))\widehat{v^*}(m_{(2)}h^*_{(4)},S(h^*_{(1)}))\big] \\
&\qquad\times\widehat{v}(S(l_{(1)}h_{(6)}m^*_{(4)})^*,m_{(3)}h^*_{(5)})\big\rangle\rtimes_{\rho,u}l_{(3)}h_{(8)}m^*_{(6)} \\
&\quad={}_A\big\langle x,[S(l_{(3)}h_{(6)}m^*_{(5)})^*\cdot_\lambda y]\big[S(l_{(2)}h_{(5)}m^*_{(4)})^*\cdot_{\sigma,v}\widehat{v^*}(m_{(1)}h^*_{(2)},S(h^*_{(1)}))\big] \\
&\qquad\times\widehat{v}(S(l_{(1)}h_{(4)}m^*_{(3)})^*,m_{(2)}h^*_{(3)})\big\rangle\rtimes_{\rho,u}l_{(4)}h_{(7)}m^*_{(6)} \\
&\quad={}_A\big\langle x,[S(l_{(3)}h_{(7)}m^*_{(5)})^*\cdot_\lambda y]\widehat{v}\big(S(l_{(2)}h_{(6)}m^*_{(4)})^*,m_{(1)}h^*_{(3)}S(h^*_{(2)})\big) \\
&\qquad\times\widehat{v^*}\big(S(l_{(1)}h_{(5)}m^*_{(3)})^*m_{(2)}h^*_{(4)},S(h^*_{(1)})\big)\big\rangle\rtimes_{\rho,u}l_{(4)}h_{(8)}m^*_{(6)} \\
&\quad={}_A\big\langle x\widehat{v}(l_{(1)},h_{(1)}),[S(l_{(3)}h_{(3)}m^*_{(3)})^*\cdot_\lambda y]\widehat{v}(S(l_{(2)}h_{(2)}m^*_{(2)})^*,m_{(1)})\big\rangle \\
&\qquad\rtimes_{\rho,u}l_{(4)}h_{(4)}m^*_{(4)} \\
&\quad={}_{A\rtimes_{\rho,u}H}\langle(x\rtimes_\lambda l)(1\rtimes_{\sigma,v}h),y\rtimes_\lambda m\rangle.
\end{aligned}
$$

Thus we see that for any $b\in B$, $x,y\in X$ and $h,l,m\in H$,

$$
{}_{A\rtimes_{\rho,u}H}\langle(x\rtimes_\lambda l)(b\rtimes_{\sigma,v}h),y\rtimes_\lambda m\rangle={}_{A\rtimes_{\rho,u}H}\langle x\rtimes_\lambda l,(y\rtimes_\lambda m)(b\rtimes_{\sigma,v}h)^*\rangle.
$$

We note that for any $a\in A$, $x,y\in X$ and $h,l,m\in H$,

$$
\begin{aligned}
&\langle(a\rtimes_{\rho,u}h)(x\rtimes_\lambda l),y\rtimes_\lambda m\rangle_{B\rtimes_{\sigma,v}H} \\
&\qquad\qquad=(1\rtimes_{\sigma,v}l)^*\langle(a\rtimes_{\rho,u}h)(x\rtimes_\lambda 1),y\rtimes_\lambda 1\rangle_{B\rtimes_{\sigma,v}H}(1\rtimes_{\sigma,v}m).
\end{aligned}
$$

Hence in order to show that for any $a\in A$, $x,y\in X$ and $h,l,m\in H$,

$$
\langle(a\rtimes_{\rho,u}h)(x\rtimes_\lambda l),y\rtimes_\lambda m\rangle_{B\rtimes_{\sigma,v}H}=\langle x\rtimes_\lambda l,(y\rtimes_\lambda m)(a\rtimes_{\rho,u}h)^*\rangle_{B\rtimes_{\sigma,v}H},
$$

we have only to show that for any $a\in A$, $x,y\in X$ and $h\in H$,

$$
\langle(a\rtimes_{\rho,u}h)(x\rtimes_\lambda 1),y\rtimes_\lambda 1\rangle_{B\rtimes_{\sigma,v}H}=\langle x\rtimes_\lambda 1,(a\rtimes_{\rho,u}h)^*(y\rtimes_\lambda 1)\rangle_{B\rtimes_{\sigma,v}H}.
$$

For any $a \in A$ and $x, y \in X$, we have

$$\begin{aligned}\langle (a \rtimes_{\rho,u} 1)(x \rtimes_\lambda 1), y \rtimes_\lambda 1\rangle_{B\rtimes_{\sigma,v}H} &= \langle ax \rtimes_\lambda 1, y \rtimes_\lambda 1\rangle_{B\rtimes_{\sigma,v}H} \\ &= \langle ax, y\rangle_B = \langle x \rtimes_\lambda 1, (a \rtimes_{\rho,u} 1)^*(y \rtimes_\lambda 1)\rangle_{B\rtimes_{\sigma,v}H}.\end{aligned}$$

Also, for any $x, y \in X$ and $h \in H$,

$$\begin{aligned}&\langle (1 \rtimes_{\rho,u} h)(x \rtimes_\lambda 1), y \rtimes_\lambda 1\rangle_{B\rtimes_{\sigma,v}H} \\ &\quad= \big\langle [S(h_{(4)}) \cdot_\lambda [h_{(1)} \cdot_\lambda x]]\widehat{v}(S(h_{(3)}), h_{(2)}), [h^*_{(5)} \cdot_\lambda y]\big\rangle_B \rtimes_{\sigma,v} h^*_{(6)} \\ &\quad= \langle \widehat{u}(S(h_{(2)}), h_{(1)})x, [h^*_{(3)} \cdot_\lambda y]\rangle_B \rtimes_{\sigma,v} h^*_{(4)} \\ &\quad= \big\langle x \rtimes_\lambda 1, \widehat{u}(S(h_{(2)}), h_{(1)})^*[h^*_{(3)} \cdot_\lambda y] \rtimes_\lambda h^*_{(4)}\big\rangle_{B\rtimes_{\sigma,v}H} \\ &\quad= \langle x \rtimes_\lambda 1,\ (1 \rtimes_{\rho,u} h)^*(y \rtimes_\lambda 1)\rangle_{B\rtimes_{\sigma,v}H}.\end{aligned}$$

Thus condition (8) in [9, Lemma 1.3] is satisfied. Moreover, for any $a \in A$, $b \in B$, $x \in X$ and $h, l, m \in H$,

$$\begin{aligned}&(a \rtimes_{\rho,u} h)[(x \rtimes_\lambda l)(b \rtimes_{\sigma,v} m)] \\ &\quad= a[h_{(1)} \cdot_\lambda x]\big[h_{(2)} \cdot_{\sigma,v} [l_{(1)} \cdot_{\sigma,v} b]\big]\widehat{v}(h_{(3)}, l_{(2)})\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\ &\quad= a[h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)})[h_{(3)}l_{(2)} \cdot_{\sigma,v} b]\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\ &\quad= [(a \rtimes_{\rho,u} h)(x \rtimes_\lambda l)](b \rtimes_{\sigma,v} m).\end{aligned}$$

Thus condition (7) in [9, Lemma 1.3] is satisfied. Since X is of finite type, there are finite subsets $\{w_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ in X such that

$$x = \sum_{i=1}^n w_i\langle w_i, x\rangle_B = \sum_{j=1}^m {}_A\langle x, z_j\rangle z_j$$

for any $x \in X$. Then we have the following lemma:

Lemma 4.1. *With the above notation, if $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system, then for any $x \in X$ and $h \in H$,*

$$\begin{aligned}x \rtimes_\lambda h &= \sum_{i=1}^n (w_i \rtimes_\lambda 1)\langle w_i \rtimes_\lambda 1, x \rtimes_\lambda h\rangle_{B\rtimes_\sigma H} \\ &= \sum_{j=1}^m {}_{A\rtimes_\rho H}\langle x \rtimes_\lambda h, z_j \rtimes_\lambda 1\rangle(z_j \rtimes_\lambda 1).\end{aligned}$$

Proof. For any $x \in X$ and $h \in H$,

$$\sum_{i=1}^n (w_i \rtimes_\lambda 1)\langle w_i \rtimes_\lambda 1, x \rtimes_\lambda h\rangle_{B\rtimes_\sigma H} = \sum_{i=1}^n w_i\langle w_i, x\rangle_B \rtimes_\lambda h = x \rtimes_\lambda h.$$

Also,

$$\begin{aligned}\sum_{j=1}^{m} {}_{A\rtimes_\rho H}\langle x\rtimes_\lambda h, z_j\rtimes_\lambda 1\rangle(z_j\rtimes_\lambda 1)\\
&= \sum_{j=1}^{m} {}_A\langle [h_{(2)}S(h_{(1)})\cdot_\lambda x], [S(h_{(3)})^*\cdot_\lambda z_j]\rangle [h_{(4)}\cdot_\lambda z_j]\rtimes_\lambda h_{(5)}\\
&= \sum_{j=1}^{m} \big[h_{(2)}\cdot_\lambda {}_A\langle [S(h_{(1)})\cdot_\lambda x], z_j\rangle z_j\big]\rtimes_\lambda h_{(3)}\\
&= \big[h_{(2)}\cdot_\lambda [S(h_{(1)})\cdot_\lambda x]\big]\rtimes_\lambda h_{(3)} = x\rtimes_\lambda h.\end{aligned}$$

Therefore, we obtain the conclusion. ∎

For any Hilbert C^*-bimodule Y, l-Ind$[Y]$ and r-Ind$[Y]$ denote its left and right indices, respectively.

Corollary 4.2. *With the above notation and assumptions,*

$$\text{l-Ind}[X\rtimes_\lambda H] = \text{l-Ind}[X]\rtimes_\sigma 1, \qquad \text{r-Ind}[X\rtimes_\lambda H] = \text{r-Ind}[X]\rtimes_\rho 1.$$

Proof. By the definitions of the left and right indices of a Hilbert C^*-bimodule,

$$\text{l-Ind}[X\rtimes_\lambda H] = \sum_{j=1}^{m}\langle z_j, z_j\rangle_B\rtimes_\sigma 1 = \text{l-Ind}[X]\rtimes_\sigma 1,$$

$$\text{r-Ind}[X\rtimes_\lambda H] = \sum_{i=1}^{n} {}_A\langle w_i, w_i\rangle\rtimes_\rho 1 = \text{r-Ind}[X]\rtimes_\rho 1. \ ∎$$

Proposition 4.3. *With the above notation and assumptions, $X\rtimes_\lambda H$ is a Hilbert $A\rtimes_\rho H$ - $B\rtimes_\sigma H$-bimodule of finite type with*

$$\text{l-Ind}[X\rtimes_\lambda H] = \text{l-Ind}[X]\rtimes_\sigma 1, \qquad \text{r-Ind}[X\rtimes_\lambda H] = \text{r-Ind}[X]\rtimes_\rho 1.$$

Proof. This is immediate by Lemma 4.1, Corollary 4.2 and [9, Lemma 1.3]. ∎

Lemma 4.4. *With the above notation, if $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system and X is an A-B-equivalence bimodule, then for any $x\in X$ and $h\in H$,*

$$\begin{aligned}x\rtimes_\lambda h &= \sum_{i=1}^{n}(w_i\rtimes_\lambda 1)\langle w_i\rtimes_\lambda 1, x\rtimes_\lambda h\rangle_{B\rtimes_{\sigma,z}H}\\
&= \sum_{j=1}^{m} {}_{A\rtimes_{\rho,w}H}\langle x\rtimes_\lambda h, z_j\rtimes_\lambda 1\rangle(z_j\rtimes_\lambda 1).\end{aligned}$$

Proof. For any $x \in X$ and $h \in H$,

$$\sum_{i=1}^{n}(w_i \rtimes_\lambda 1)\langle w_i \rtimes_\lambda 1, x \rtimes_\lambda h\rangle_{B\rtimes_{\sigma,v}H} = \sum_{i=1}^{n} w_i\langle w_i, x\rangle_B \rtimes_\lambda h = x \rtimes_\lambda h.$$

Also,

$$\begin{aligned}\sum_{j=1}^{m} {}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda h, z_j \rtimes_\lambda 1\rangle(z_j \rtimes_\lambda 1)\\ &= \sum_{j=1}^{m} {}_A\langle x, [S(h_{(1)})^* \cdot_\lambda z_j]\rangle[h_{(2)} \cdot_\lambda z_j] \rtimes_\lambda h_{(3)}\\ &= \sum_{j=1}^{m} x\langle[S(h_{(1)})^* \cdot_\lambda z_j], [h_{(2)} \cdot_\lambda z_j]\rangle_B \rtimes_\lambda h_{(3)}\\ &= \sum_{j=1}^{m} x[h_{(1)} \cdot_\sigma \langle z_j, z_j\rangle_B] \rtimes_\lambda h_{(2)} = x \rtimes_\lambda h.\end{aligned}$$

Therefore, we obtain the conclusion. ∎

LEMMA 4.5. *With the above notation and assumptions, if X is an A-B-equivalence bimodule, then the Hilbert $A \rtimes_{\rho,u} H$ - $B \rtimes_{\sigma,v} H$-bimodule is full with both-sided inner products.*

Proof. For any $x, y \in X$, ${}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda 1, y \rtimes_\lambda 1\rangle = {}_A\langle x, y\rangle \rtimes_{\rho,u} 1$. Since ${}_{A\rtimes_{\rho,u}H}\langle X \rtimes_\lambda H, X \rtimes_\lambda H\rangle$ is a closed ideal of $A \rtimes_{\rho,u} H$, for any $x, y \in X$ and $h \in H$ we have

$$({}_A\langle x, y\rangle \rtimes_{\rho,u} 1)(1 \rtimes_{\rho,u} h) = {}_A\langle x, y\rangle \rtimes_{\rho,u} h \in {}_{A\rtimes_{\rho,u}H}\langle X \rtimes_\lambda H, X \rtimes_\lambda H\rangle.$$

Since ${}_A\langle X, X\rangle = A$, we obtain

$${}_{A\rtimes_{\rho,u}H}\langle X \rtimes_\lambda H, X \rtimes_\lambda H\rangle = A \rtimes_{\rho,u} H.$$

Also, for any $x, y \in X$ and $h \in H$,

$$\langle x \rtimes_\lambda 1, y \rtimes_\lambda h\rangle_{B\rtimes_{\sigma,v}H} = \langle x, y\rangle_B \rtimes_{\sigma,v} h \in \langle X \rtimes_\lambda H, X \rtimes_\lambda H\rangle_{B\rtimes_{\sigma,v}H}.$$

Since $\langle X, X\rangle_B = B$, we conclude that

$$\langle X \rtimes_\lambda H, X \rtimes_\lambda H\rangle_{B\rtimes_{\sigma,v}H} = B \rtimes_{\sigma,v} H. \blacksquare$$

COROLLARY 4.6. *With the above notation and assumptions, suppose that X is an A-B-equivalence bimodule. Then $X \rtimes_\lambda H$ is an $A \rtimes_{\rho,u} H$ - $B \rtimes_{\sigma,v} H$-equivalence bimodule.*

Proof. By Lemma 4.5, it suffices to show that

$${}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle(z \rtimes_\lambda m) = (x \rtimes_\lambda h)\langle y \rtimes_\lambda l, z \rtimes_\lambda m\rangle_{B\rtimes_{\sigma,v}H}$$

for any $x, y, z \in X$ and $h, l, m \in H$. Since X is an A-B-equivalence bimodule,

$$
\begin{aligned}
&(x \rtimes_\lambda h)\langle y \rtimes_\lambda l, z \rtimes_\lambda m\rangle_{B\rtimes_{\sigma,v}H} \\
&\quad = x\big[h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l^*_{(2)}, S(l_{(1)})^*)[l^*_{(3)} \cdot_{\sigma,v} \langle y, z\rangle_B]\widehat{v}(l^*_{(4)}, m_{(1)})\big]\widehat{v}(h_{(2)}, l^*_{(5)}m_{(2)}) \\
&\qquad\quad \rtimes_\lambda h_{(3)}l^*_{(6)}m_{(3)} \\
&\quad = x[h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l^*_{(2)}, S(l_{(1)})^*)]\big[h_{(2)} \cdot_{\sigma,v} [l^*_{(3)} \cdot_{\sigma,v} \langle y, z\rangle_B]\big] \\
&\qquad\quad \times [h_{(3)} \cdot_{\sigma,v} \widehat{v}(l^*_{(4)}, m_{(1)})]\widehat{v}(h_{(4)}, l^*_{(5)}m_{(2)}) \rtimes_\lambda h_{(5)}l^*_{(6)}m_{(3)} \\
&\quad = x[h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l^*_{(2)}, S(l^*_{(1)}))]\widehat{v}(h_{(2)}, l^*_{(3)})[h_{(3)}l^*_{(4)} \cdot_{\sigma,v} \langle y, z\rangle_B] \\
&\qquad\quad \times \widehat{v}(h_{(4)}l^*_{(5)}, m_{(1)}) \rtimes_\lambda h_{(5)}l^*_{(6)}m_{(2)} \\
&\quad = x\widehat{v^*}(h_{(1)}l^*_{(2)}, S(l^*_{(1)}))[h_{(2)}l^*_{(3)} \cdot_{\sigma,v} \langle y, z\rangle_B]\widehat{v}(h_{(3)}l^*_{(4)}, m_{(1)}) \rtimes_\lambda h_{(4)}l^*_{(5)}m_{(2)}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda h, y \rtimes_\lambda l\rangle(z \rtimes_\lambda m) \\
&\quad = {}_A\big\langle x, [S(h_{(2)}l^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(1)}l^*_{(2)})^*, l_{(1)})\big\rangle[h_{(3)}l^*_{(4)} \cdot_\lambda z]\widehat{v}(h_{(4)}l^*_{(5)}, m_{(1)}) \\
&\qquad\quad \rtimes_\lambda h_{(5)}l^*_{(6)}m_{(2)} \\
&\quad = x\big\langle [S(h_{(2)}l^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(1)}l^*_{(2)})^*, l_{(1)}), [h_{(3)}l^*_{(4)} \cdot_\lambda z]\big\rangle_B\widehat{v}(h_{(4)}l^*_{(5)}, m_{(1)}) \\
&\qquad\quad \rtimes_\lambda h_{(5)}l^*_{(6)}m_{(2)} \\
&\quad = x\widehat{v^*}(h_{(1)}l^*_{(2)}, S(l^*_{(1)}))[h_{(2)}l^*_{(3)} \cdot_{\sigma,v} \langle y, z\rangle_B]\widehat{v}(h_{(3)}l^*_{(4)}, m_{(1)}) \rtimes_\lambda h_{(4)}l^*_{(5)}m_{(2)}.
\end{aligned}
$$

This yields the conclusion. ∎

By the above discussions, we obtain the following:

Corollary 4.7.

(1) *Let $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ be a twisted covariant system. Suppose that X is an A-B-equivalent bimodule. Then $X \rtimes_\lambda H$ is an $A \rtimes_{\rho,u} H$-$B \rtimes_{\sigma,v} H$-equivalence bimodule.*
(2) *Let $(A, B, X, \rho, \sigma, \lambda, H^0)$ be a covariant system. Then $X \rtimes_\lambda H$ is a Hilbert $A \rtimes_\rho H$-$B \rtimes_\sigma H$-bimodule of finite type.*

In the situation of Corollary 4.7(1), let $X \rtimes_\lambda H$ be the crossed product associated to a twisted covariant system $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$, where X is an A-B-equivalence bimodule. Then we define the dual covariant system to $X \rtimes_\lambda H$ as follows: Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the dual coactions of H on $A \rtimes_{\rho,u} H$ and $B \rtimes_{\sigma,v} H$ of (ρ, u) and (σ, v), respectively. Let $\widehat{\lambda}$ be the dual coaction of H on $X \rtimes_\lambda H$ defined by

$$\widehat{\lambda}(x \rtimes_\lambda h) = (x \rtimes_\lambda h_{(1)}) \otimes h_{(2)}$$

for any $x \in X$ and $h \in H$. Then by easy computations, we can see that

$$(A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, X \rtimes_\lambda H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H)$$

is a covariant system. Hence we obtain the following:

COROLLARY 4.8. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively. Then the following conditions are equivalent:*

(1) *(ρ, u) is strongly Morita equivalent to (σ, v).*
(2) *The dual coaction $\widehat{\rho}$ of (ρ, u) is strongly Morita equivalent to the dual coaction $\widehat{\sigma}$ of (σ, v).*

Proof. By the above discussion, it is clear that (1) implies (2). Conversely, suppose (2) holds. Then we can see that $\widehat{\widehat{\rho}}$ is strongly Morita equivalent to $\widehat{\widehat{\sigma}}$, where $\widehat{\widehat{\rho}}$ and $\widehat{\widehat{\sigma}}$ are the dual coactions of $\widehat{\rho}$ and $\widehat{\sigma}$, respectively. By [11, Theorem 3.3], there is an isomorphism Ψ of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ such that $\widehat{\widehat{\rho}}$ is exterior equivalent to the twisted coaction

$$\big((\Psi \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}) \circ \Psi^{-1}, (\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N)\big).$$

Hence by Lemma 3.12, $\widehat{\widehat{\rho}}$ is strongly Morita equivalent to $(\rho \otimes \mathrm{id}, u \otimes I_N)$. Thus, by Lemma 3.16 and Corollary 3.8, $\widehat{\widehat{\rho}}$ is strongly Morita equivalent to (ρ, u). Similarly $\widehat{\widehat{\sigma}}$ is strongly Morita equivalent to (σ, v). Therefore, by Corollary 3.8, (ρ, u) is strongly Morita equivalent to (σ, v). ∎

Also, in the situation of Corollary 4.7(2), we can see that

$$(A \rtimes_\rho H,\ B \rtimes_\sigma H,\ X \rtimes_\lambda H,\ \widehat{\rho},\ \widehat{\sigma},\ \widehat{\lambda}, H)$$

is a covariant system in the same way as above.

5. Duality. In this section, we present a duality theorem for a crossed product of a Hilbert C^*-bimodule of finite type by a (twisted) coaction of a finite-dimensional C^*-Hopf algebra, in the same way as in [11]. As mentioned in Section 1, Guo and Zhang [5] have already obtained a duality result using the language of multiplicative unitary elements and Kac systems. We give our duality result because our approach to coactions of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra is a useful addition to the main result in Section 6.

First, suppose condition (1) or (2) in Corollary 4.7 holds. In both cases, we can consider the dual covariant systems

$$(A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, X \rtimes_\lambda H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H),$$
$$(A \rtimes_\rho H, B \rtimes_\sigma H, X \rtimes_\lambda H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H).$$

Let Λ be the set of all triplets (i, j, k) where $i, j = 1, \dots, d_k$ and $k = 1, \dots, K$ and $\sum_{k=1}^K d_k^2 = N$. For each $I = (i, j, k) \in \Lambda$, let W_I^ρ, V_I^ρ be elements in $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$W_I^\rho = \sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k, \qquad V_I^\rho = (1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau)(W_I^\rho \rtimes_{\widehat{\rho}} 1^0).$$

Similarly for each $I=(i,j,k)\in\Lambda$, we define elements

$$W_I^\sigma=\sqrt{d_k}\rtimes_{\sigma,v}w_{ij}^k,\qquad V_I^\sigma=(1\rtimes_{\sigma,v}1\rtimes_{\widehat{\sigma}}\tau)(W_I^\sigma\rtimes_{\widehat{\sigma}}1^0)$$

in $B\rtimes_{\sigma,v}H\rtimes_{\widehat{\sigma}}H^0$. We regard $M_N(\mathbf{C})$ as a Hilbert $M_N(\mathbf{C})$-$M_N(\mathbf{C})$-bimodule in the usual way. Let $X\otimes M_N(\mathbf{C})$ be the exterior tensor product of X and $M_N(\mathbf{C})$, which is a Hilbert $A\otimes M_N(\mathbf{C})$ - $B\otimes M_N(\mathbf{C})$-bimodule. In the same way as in Lemma 2.1, we can see that $X\otimes M_N(\mathbf{C})$ is of finite type. Let $\{f_{IJ}\}_{I,J\in\Lambda}$ be the system of matrix units of $M_N(\mathbf{C})$. Let Ψ_X be a linear map from $X\otimes M_N(\mathbf{C})$ to $X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0$ defined by

$$\Psi_X\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big)=\sum_{I,J}V_I^{\rho*}(x_{IJ}\rtimes_\lambda 1\rtimes_{\widehat{\lambda}}1^0)V_J^\sigma$$

for any $x_{IJ}\in X$. Let Ψ_A and Ψ_B be isomorphisms of $A\otimes M_N(\mathbf{C})$ and $B\otimes M_N(\mathbf{C})$ onto $A\rtimes_{\rho,u}H\rtimes_{\widehat{\rho}}H^0$ and $B\rtimes_{\sigma,v}H\rtimes_{\widehat{\sigma}}H^0$ defined by

$$\Psi_A\Big(\sum_{I,J}a_{IJ}\otimes f_{IJ}\Big)=\sum_{I,J}V_I^{\rho*}(a_{IJ}\rtimes_{\rho,u}1\times_{\widehat{\rho}}1^0)V_J^\rho,$$

$$\Psi_B\Big(\sum_{I,J}b_{IJ}\otimes f_{IJ}\Big)=\sum_{I,J}V_I^{\sigma*}(a_{IJ}\rtimes_{\sigma,v}1\times_{\widehat{\sigma}}1^0)V_J^\sigma,$$

for any $a_{IJ}\in A$ and $b_{IJ}\in B$ (see [11]).

Lemma 5.1. *With the above notation,*

(1) $$\Psi_X\Big(\Big(\sum_{I,J}a_{IJ}\otimes f_{IJ}\Big)\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big)\Big)=\Psi_A\Big(\sum_{I,J}a_{IJ}\otimes f_{IJ}\Big)\Psi_X\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big),$$

(2) $$\Psi_X\Big(\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big)\Big(\sum_{I,J}b_{IJ}\otimes f_{IJ}\Big)\Big)=\Psi_X\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big)\Psi_B\Big(\sum_{I,J}b_{IJ}\otimes f_{IJ}\Big),$$

(3) $${}_{A\rtimes_{\rho,u}H\rtimes_{\widehat{\rho}}H^0}\Big\langle\Psi_X\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big),\Psi_X\Big(\sum_{I,J}y_{IJ}\otimes f_{IJ}\Big)\Big\rangle=\Psi_A\Big({}_{A\otimes M_N(\mathbf{C})}\Big\langle\sum_{I,J}x_{IJ}\otimes f_{IJ},\sum_{I,J}y_{IJ}\otimes f_{IJ}\Big\rangle\Big),$$

(4) $$\Big\langle\Psi_X\Big(\sum_{I,J}x_{IJ}\otimes f_{IJ}\Big),\Psi_X\Big(\sum_{I,J}y_{IJ}\otimes f_{IJ}\Big)\Big\rangle_{B\rtimes_{\sigma,v}H\rtimes_{\widehat{\sigma}}H^0}=\Psi_B\Big(\Big\langle\sum_{I,J}x_{IJ}\otimes f_{IJ},\sum_{I,J}y_{IJ}\otimes f_{IJ}\Big\rangle_{B\otimes M_N(\mathbf{C})}\Big)$$

for any $a_{IJ}\in A$, $b_{IJ}\in B$, $x_{IJ},y_{IJ}\in X$ and $I,J\in\Lambda$.

Proof. This is immediate by routine computations. Indeed,

$$\Psi_X\Big(\Big(\sum_{I,J} a_{IJ}\otimes f_{IJ}\Big)\Big(\sum_{I,J} x_{IJ}\otimes f_{IJ}\Big)\Big)=\sum_{I,J,L} V_I^{\rho *}(a_{IL}x_{LJ}\rtimes_{\lambda} 1\rtimes_{\widehat{\lambda}} 1^0)V_J^{\sigma}.$$

On the other hand, by [11, Lemma 3.1],

$$\begin{aligned}
&\Psi_A\Big(\sum_{I,J} a_{IJ}\otimes f_{IJ}\Big)\Psi_X\Big(\sum_{L,M} x_{LM}\otimes f_{LM}\Big)\\
&\quad=\sum_{I,J,M} V_I^{\rho *}(a_{IJ}\rtimes_{\rho,u} 1\rtimes_{\widehat{\rho}} 1^0)(1\rtimes_{\rho,u} 1\rtimes_{\widehat{\rho}}\tau)(x_{JM}\rtimes_{\lambda} 1\rtimes_{\widehat{\lambda}} 1^0)V_M^{\sigma}\\
&\quad=\sum_{I,J,M} V_I^{\rho *}(a_{IJ}x_{JM}\rtimes_{\lambda} 1\rtimes_{\widehat{\lambda}} 1^0)V_M^{\sigma}.
\end{aligned}$$

Thus we obtain (1). Similarly we can obtain (2). Also, by [11, Lemma 3.1],

$$\begin{aligned}
&{}_{A\rtimes_{\rho,u}H\rtimes_{\widehat{\rho}}H^0}\Big\langle \Psi_X\Big(\sum_{I,J} x_{IJ}\otimes f_{IJ}\Big),\Psi_X\Big(\sum_{I,J} y_{IJ}\otimes f_{IJ}\Big)\Big\rangle\\
&\quad=\sum_{I,J,I_1,J_1} {}_{A\rtimes_{\rho,u}H\rtimes_{\widehat{\rho}}H^0}\langle V_I^{\rho *}(x_{IJ}\rtimes_{\lambda} 1\rtimes_{\widehat{\lambda}} 1^0)V_J^{\sigma}, V_{I_1}^{\rho *}(y_{I_1J_1}\rtimes_{\lambda} 1\rtimes_{\widehat{\lambda}} 1^0)V_{J_1}^{\sigma}\rangle\\
&\quad=\sum_{I,J,I_1} V_I^{\rho *}({}_{A\rtimes_{\rho,u}H}\langle x_{IJ}\rtimes_{\lambda} 1, y_{I_1J}\rtimes_{\lambda} 1\rangle\rtimes_{\widehat{\rho}}\tau)V_{I_1}^{\rho}\\
&\quad=\sum_{I,J,I_1} V_I^{\rho *}({}_A\langle x_{IJ}, y_{I_1J}\rangle\rtimes_{\rho,u}\rtimes_{\widehat{\rho}} 1^0)V_{I_1}^{\rho}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Psi_A\Big({}_{A\otimes M_N(\mathbb{C})}\Big\langle\sum_{I,J} x_{IJ}\otimes f_{IJ},\sum_{I_1,J_1} y_{I_1,J_1}\otimes f_{I_1,J_1}\Big\rangle\Big)\\
&\quad=\sum_{I,J,I_1}\Psi_A({}_A\langle x_{IJ}, y_{I_1J}\rangle\otimes f_{II_1})=\sum_{I,J,I_1} V_I^{\rho *}({}_A\langle x_{IJ}, y_{I_1J}\rangle\rtimes_{\rho,u} 1\rtimes_{\widehat{\rho}} 1^0)V_{I_1}^{\rho}.
\end{aligned}$$

Thus we obtain (3). Furthermore,

$$\begin{aligned}
&\Big\langle\Psi_X\Big(\sum_{I,J} x_{IJ}\otimes f_{IJ}\Big),\Psi_X\Big(\sum_{I_1,J_1} y_{I_1J_1}\otimes f_{I_1J_1}\Big)\Big\rangle_{B\rtimes_{\sigma,v}H\rtimes_{\widehat{\sigma}}H^0}\\
&\qquad\qquad=\sum_{I,J,J_1} V_J^{\sigma *}(\langle x_{IJ}, y_{IJ_1}\rangle_B\rtimes_{\sigma,v} 1\rtimes_{\widehat{\sigma}} 1^0)V_{J_1}^{\sigma}\\
&\qquad\qquad=\Psi_B\Big(\Big\langle\sum_{I,J} x_{IJ}\otimes f_{IJ},\sum_{I_1,J_1} y_{I_1J_1}\otimes f_{I_iJ_i}\Big\rangle_{B\otimes M_N(\mathbb{C})}\Big).
\end{aligned}$$

Thus we obtain (4). ■

From the above lemma, we can see that Ψ_X is injective. Next, we show that Ψ_X is surjective.

Lemma 5.2. *With the above notation,*

$$(X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} 1^0) = X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0.$$

Proof. Let $x \in X$, $h \in H$ and $\phi \in H^0$. Since

$$\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0)^*(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) = 1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} 1^0$$

by [10, Proposition 3.18], we have

$$\begin{aligned}
x \rtimes_\lambda h \rtimes_{\widehat{\lambda}} \phi
&= \sum_{i,j,k}(x \rtimes_\lambda h \rtimes_{\widehat{\lambda}} \phi)(\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0)^*(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) \\
&\quad \times (\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) \\
&= \sum_{i,j,k,j_1,j_2} d_k\big((x \rtimes_\lambda h)\big[\phi \cdot_{\widehat{\sigma}} \big(\widehat{v}(S(w_{j_1j_2}^k), w_{ij_1}^k)^* \rtimes_{\sigma,v} w_{j_2j}^{k*}\big)\big] \rtimes_{\widehat{\lambda}} \tau\big) \\
&\quad \times (1 \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) \\
&= \sum_{i,j,k,j_1,j_2,j_3} d_k\phi(w_{j_3j}^{k*})\big((x \rtimes_\lambda h)(\widehat{v}(S(w_{j_1j_2}^k), w_{ij_1}^k)^* \rtimes_{\sigma,v} w_{j_2j_3}^{k*}) \rtimes_{\widehat{\lambda}} 1^0\big) \\
&\quad \times (1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(1 \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0).
\end{aligned}$$

Hence we obtain the conclusion. ∎

Let E_1^σ be the canonical conditional expectation from $B \rtimes_{\sigma,v} H$ to B defined by $E_1^\sigma(b \rtimes_{\sigma,v} h) = \tau(h)b$ for any $b \in B$ and $h \in H$. Let E_1^λ be the linear map from $X \rtimes_\lambda H$ onto X defined by

$$E_1^\lambda(x \rtimes_\lambda h) = \tau(h)x$$

for any $x \in X$ and $h \in H$.

Lemma 5.3. *With the above notation, for any $x \in X$ and $h \in H$,*

$$\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* E_1^\lambda\big((\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)(x \rtimes_\lambda h)\big) = x \rtimes_\lambda h.$$

Proof. This is also immediate by routine computations. Indeed, for any $x \in X$ and $h \in H$, by [17, Theorem 2.2],

$$\begin{aligned}
&\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* E_1^\lambda\big((\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)(x \rtimes_\lambda h)\big) \\
&= \sum_{i,j,k,j_1,j_2,s,s_1,s_2,s_3} d_k\widehat{u^*}(w_{ss_1}^{k*}, w_{si}^k)\big[w_{s_1s_2}^{k*} \cdot_\lambda [w_{ij_1}^k \cdot_\lambda x]\big][w_{s_2s_3}^{k*} \cdot_{\sigma,v} \widehat{v}(w_{j_1j_2}^k, h_{(1)})] \\
&\qquad \rtimes_\lambda \tau(w_{j_2j}^k h_{(2)})w_{s_3j}^{k*}
\end{aligned}$$

$$= \sum_{i,j,k,j_1,j_2,s_2,s_3} d_k x\widehat{v}^*(w^{k*}_{is_2}, w^k_{ij_1})[w^{k*}_{s_2s_3} \cdot_{\sigma,v} \widehat{v}(w^k_{j_1j_2}, h_{(1)})] \rtimes_\lambda \tau(w^k_{j_2j}h_{(2)})w^{k*}_{s_3j}$$
$$= \sum_{i,j,k,j_1,j_2,s_2,s_3} d_k x\widehat{v}(w^{k*}_{is_2}w^k_{ij_1}, h_{(1)})\widehat{v}^*(w^{k*}_{s_2s_3}, w^k_{j_1j_2}h_{(2)})\tau(w^k_{j_2j}h_{(3)}) \rtimes_\lambda w^{k*}_{s_3j}$$
$$= \sum_{j,k,s_2} d_k x\tau(w^k_{s_2j}h) \rtimes_\lambda S(w^k_{js_2})$$
$$= \sum_{i,k,s_2} d_k x\tau(w^k_{s_2j}h_{(1)}) \rtimes_\lambda S(w^k_{js_2}h_{(2)}S(h_{(3)}))$$
$$= x \rtimes_\lambda S(\tau(Neh_{(1)})S(h_{(2)})) = x \rtimes_\lambda h. \ \blacksquare$$

Lemma 5.4. *With the above notation,*

$$(1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \phi)(x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0) = x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} \phi = (x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \phi)$$

for any $x \in X$ and $\phi \in H^0$.

Proof. For any $x \in X$ and $\phi \in H^0$,

$$\begin{aligned}(1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \phi)(x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0) &= [\phi_{(1)} \cdot_{\widehat{\lambda}} (x \rtimes_\lambda 1)] \rtimes_{\widehat{\lambda}} \phi_{(2)} = x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} \phi \\ &= (x \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \phi). \ \blacksquare\end{aligned}$$

Lemma 5.5. *With the above notation, Ψ_X is surjective.*

Proof. By Lemma 5.2, it suffices to show that for any $b \in B$, $x \in X$ and $h, l \in H$, there is $y \in X \otimes M_N(\mathbb{C})$ such that

$$\Psi_X(y) = (x \rtimes_\lambda h \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\widehat{\sigma}} 1^0).$$

By Lemma 5.3 and [10, Proposition 3.18],

$$x \rtimes_\lambda h = \sum_I W^{\rho*}_I \big(E^\lambda_1(W^\rho_I(x \rtimes_\lambda h)) \rtimes_\lambda 1\big),$$
$$b \rtimes_{\sigma,v} l = \sum_I \big(E^\sigma_1((b \rtimes_{\sigma,v} l)W^{\sigma*}_I) \rtimes_{\sigma,v} 1\big)W^\sigma_I.$$

Thus

$$\begin{aligned}&(x \rtimes_\lambda h \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\widehat{\sigma}} 1^0) \\ &\quad = \sum_{I,J} (W^{\rho*}_I \rtimes_{\widehat{\rho}} 1^0)\big(E^\lambda_1(W^\rho_I(x \rtimes_\lambda h)) \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} \tau\big) \\ &\qquad\quad \times \big(E^\sigma_1((b \rtimes_{\sigma,v} l)W^{\sigma*}_J) \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau\big)(W^\sigma_J \rtimes_{\widehat{\sigma}} 1^0).\end{aligned}$$

Since

$$E^\lambda_1(W^\rho_I(x \rtimes_\lambda h)) \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} \tau = (1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau)\big(E^\lambda_1(W^\rho_I(x \rtimes_\lambda h)) \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0\big)$$

by Lemma 5.4, we have

$$
\begin{aligned}
&(x \rtimes_\lambda h \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\widehat{\sigma}} 1^0)\\
&\qquad = \sum_{I,J} V_I^{\rho*}\big[E_1^\lambda(W_I^\rho(x \rtimes_\lambda h))E_1^\sigma((b \rtimes_{\sigma,v} l)W_J^{\sigma*}) \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0\big]V_J^\sigma .
\end{aligned}
$$

Since $E_1^\lambda(W_I^\rho(x \rtimes_\lambda h))E_1^\sigma((b \rtimes_{\sigma,v} l)W_J^{\sigma*}) \in X$, we obtain the conclusion. ∎

Let $\widehat{V^\rho}$ be the linear map from H to $A \rtimes_{\rho,u} H$ defined by $\widehat{V^\rho}(h) = 1 \rtimes_{\rho,u} h$ for any $h \in H$. By [10], $\widehat{V^\rho}$ is a unitary element in $\mathrm{Hom}(H, A \rtimes_{\rho,u} H)$. Let V^ρ be the unitary element in $(A \rtimes_{\rho,u} H) \otimes H^0$ induced by $\widehat{V^\rho}$. Similarly, we also define the unitary elements $\widehat{V^\sigma} \in \mathrm{Hom}(H, B \rtimes_{\sigma,v} H)$ and $V^\sigma \in (B \rtimes_{\sigma,v} H) \otimes H^0$.

Lemma 5.6. *With the above notation, for any $x \in X$ and $h \in H$,*

$$[h \cdot_\lambda x] \rtimes_\lambda 1 = \widehat{V^\rho}(h_{(1)})(x \rtimes_\lambda 1)\widehat{V^{\sigma*}}(h_{(2)}).$$

Proof. This is also immediate by routine computations. Indeed, for any $x \in X$ and $h \in H$,

$$
\begin{aligned}
&\widehat{V^\rho}(h_{(1)})(x \rtimes_\lambda 1)\widehat{V^{\sigma*}}(h_{(2)})\\
&\quad = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_{\sigma,v} \widehat{v}^*(S(h_{(7)}), h_{(8)})]\widehat{v}(h_{(3)}, S(h_{(6)})) \rtimes_\lambda h_{(4)}S(h_{(5)})\\
&\quad = [h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, S(h_{(5)})h_{(6)})\widehat{v^*}(h_{(3)}S(h_{(4)}), h_{(7)}) \rtimes_\lambda 1 = [h \cdot_\lambda x] \rtimes_\lambda 1. \ ∎
\end{aligned}
$$

Theorem 5.7 (cf. Guo and Zhang [5, Theorem 2.7]). *Let A, B be unital C^*-algebras and H a finite-dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Then:*

(1) *If X is an A-B-equivalence bimodule and $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system, then there is a linear isomorphism Ψ_X from $X \otimes M_N(\mathbb{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ which satisfies conditions* (1)–(4) *in Lemma* 5.1, *where $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ is an $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ - $B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0$ -equivalence bimodule and $X \otimes M_N(\mathbb{C})$ is an exterior tensor product of an A-B-equivalence bimodule X and an $M_N(\mathbb{C})$-$M_N(\mathbb{C})$-equivalence bimodule $M_N(\mathbb{C})$. Furthermore, there are unitary elements $U \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ and $V \in (B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0) \otimes H^0$ such that*

$$U\widehat{\widehat{\lambda}}(x)V^* = \big((\Psi_X \otimes \mathrm{id}) \circ (\lambda \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi_X^{-1}\big)(x)$$

for any $x \in X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$.

(2) *If X is a Hilbert A-B-bimodule of finite type and $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system, then there is a linear isomorphism Ψ_X from $X \otimes M_N(\mathbb{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ which satisfies conditions* (1)–(4) *in Lemma* 5.1, *where $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ is a Hilbert $A \rtimes_\rho H \rtimes_{\widehat{\rho}} H^0$ - $B \rtimes_\sigma H \rtimes_{\widehat{\sigma}} H^0$-bimodule of finite type and $X \otimes M_N(\mathbb{C})$ is an exterior tensor product of a Hilbert A-B-bimodule X of finite type and the $M_N(\mathbb{C})$-$M_N(\mathbb{C})$-equivalence bimodule*

$M_N(\mathbb{C})$. *Furthermore, there are unitary elements* $U \in (A \rtimes_\rho H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ *and* $V \in (B \rtimes_\sigma H \rtimes_{\widehat{\sigma}} H^0) \otimes H^0$ *such that*

$$U\widehat{\widehat{\lambda}}(x)V^* = \big((\Psi_X \otimes \mathrm{id}) \circ (\lambda \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi_X^{-1}\big)(x)$$

for any $x \in X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$.

Proof. (1) Let Ψ_X be as in Lemma 5.1. By Lemmas 5.1 and 5.5, we see that Ψ_X is a linear isomorphism from $X \otimes M_N(\mathbb{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$. By [11, Theorem 3.3], there are unitary elements U and V in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ and $(B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0) \otimes H^0$, respectively such that

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} = (\Psi_A \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi_A^{-1},$$
$$\mathrm{Ad}(V) \circ \widehat{\widehat{\sigma}} = (\Psi_B \otimes \mathrm{id}) \circ (\sigma \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi_B^{-1}.$$

Let V^ρ and V^σ be as above. For any $\sum_{I,J} x_{IJ} \otimes f_{IJ} \in X \otimes M_N(\mathbb{C})$,

$$\begin{aligned} &U\widehat{\widehat{\lambda}}\Big(\Psi_X\Big(\sum_{I,J} x_{IJ} \otimes f_{IJ}\Big)\Big)V^* \\ &= \sum_{I,J}(V_I^{\rho*} \otimes 1^0)V^\rho\ \widehat{\widehat{\lambda}}\big((1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau)(x_{IJ} \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times V^{\sigma*}(V_J^\sigma \otimes 1^0) \end{aligned}$$

by [11, Lemma 3.1], since

$$U = \sum_I (V_I^{\rho*} \otimes 1^0)V^\rho\widehat{\widehat{\rho}}(V_I^\rho), \qquad V = \sum_I (V_I^{\sigma*} \otimes 1^0)V^\sigma\widehat{\widehat{\sigma}}(V_I^\sigma).$$

Since

$$\widehat{\widehat{\rho}}(1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau) = V^{\rho*}((1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0)V^\rho,$$
$$\widehat{\widehat{\sigma}}(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) = V^{\sigma*}((1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) \otimes 1^0)V^\sigma$$

by [10, proof of Proposition 3.19], we have

$$\begin{aligned} &U\widehat{\widehat{\lambda}}\Big(\Psi_X\Big(\sum_{I,J} x_{IJ} \otimes f_{IJ}\Big)\Big)V^* \\ &\qquad = \sum_{I,J}(V_I^{\rho*} \otimes 1^0)V^\rho((x_{IJ} \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0) \otimes 1^0)V^{\sigma*}(V_J^\sigma \otimes 1^0) \\ &\qquad = \sum_{I,J}(V_I^{\rho*} \otimes 1^0)\lambda(x_{IJ} \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0)(V_J^\sigma \otimes 1^0) \end{aligned}$$

by Lemma 5.6, where we identify X with $X \rtimes_\lambda 1$ and $X \rtimes_\lambda 1 \rtimes_{\widehat{\lambda}} 1^0$. On the other hand,

$$((\Psi_X \otimes \mathrm{id}) \circ (\lambda \otimes \mathrm{id}))(x_{IJ} \otimes f_{IJ}) = (\Psi_X \otimes \mathrm{id})(\lambda(x_{IJ}) \otimes f_{IJ}).$$

We write $\lambda(x_{IJ}) = \sum_i y_{IJi} \otimes \phi_i$, where $\phi_i \in H^0$ and $y_{IJi} \in X$ for any I, J, i. Then

$$
\begin{aligned}
((\Psi_X\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id}))(x_{IJ}\otimes f_{IJ}) &= \sum_{I,J,i} V_I^{\rho *}(y_{IJi}\rtimes_\lambda 1\rtimes_{\widehat{\lambda}} 1^0)V_J^\sigma\otimes\phi_i \\
&= \sum_{I,J}(V_I^{\rho *}\otimes 1^0)\lambda(x_{IJ}\rtimes_\lambda\rtimes_{\widehat{\lambda}}1^0)(V_J^\sigma\otimes 1^0).
\end{aligned}
$$

This yields the conclusion.

(2) can be proved in the same way. ∎

6. The strong Morita equivalence for coactions and the Rokhlin property. For a unital C^*-algebra A, we set

$$
\begin{aligned}
c_0(A) &= \Big\{(a_n)\in l^\infty(\mathbb{N},A)\;\Big|\;\lim_{n\to\infty}\|a_n\|=0\Big\},\\
A^\infty &= l^\infty(\mathbb{N},A)/c_0(A).
\end{aligned}
$$

We denote by $[a_n]$ the element in A^∞ corresponding to $(a_n)\in l^\infty(\mathbb{N},A)$. We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences and set

$$A_\infty = A^\infty\cap A'.$$

Let X be a Hilbert A-B-bimodule of finite type, where B is a unital C^*-algebra. We define X^∞ in the same way as above. We set

$$
\begin{aligned}
c_0(X) &= \Big\{(x_n)\in l^\infty(\mathbb{N},X)\;\Big|\;\lim_{n\to\infty}\|x_n\|=0\Big\},\\
X^\infty &= l^\infty(\mathbb{N},X)/c_0(X).
\end{aligned}
$$

We denote by $[x_n]$ the element in X^∞ determined by $(x_n)\in l^\infty(\mathbb{N},X)$. We regard X^∞ as an A^∞-B^∞-bimodule as follows: for any $[a_n]\in A^\infty$, $[b_n]\in B^\infty$ and $[x_n]\in X^\infty$,

$$[a_n][x_n]=[a_nx_n],\qquad [x_n][b_n]=[x_nb_n].$$

Also, we define the left A^∞-valued and right B^∞-valued inner product as follows: for any $[x_n],[y_n]\in X^\infty$,

$${}_{A^\infty}\langle[x_n],[y_n]\rangle=[{}_A\langle x_n,y_n\rangle],\qquad \langle[x_n],[y_n]\rangle_{B^\infty}=[\langle x_n,y_n\rangle_B].$$

By [15, Lemma 2.5] and easy computations, the above definitions are independent of any choices made. We identify X with the Hilbert A^∞-B^∞-subbimodule of X^∞ consisting of the equivalence classes of constant sequences. Also, we can see that X^∞ is a complex vector space satisfying conditions (1)–(8) in [9, Lemma 1.3]. Since X is of finite type, there are finite subsets $\{u_i\}_{i=1}^n,\{v_j\}_{j=1}^m\subset X$ such that for any $x\in X$,

$$\sum_{i=1}^n u_i\langle u_i,x\rangle_B = x = \sum_{j=1}^m {}_A\langle x,v_j\rangle v_j.$$

Then we can regard $u_i, v_j \in X$ as elements in X^∞ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Thus X^∞ is a Hilbert A^∞-B^∞-bimodule of finite type by [9, Lemma 1.3]. Furthermore, if X is an A-B-equivalence bimodule, then X^∞ is an A^∞-B^∞-equivalence bimodule.

LEMMA 6.1. *With the above notation, suppose that X is an A-B-equivalence bimodule. Let $b \in B^\infty$. If $xb = 0$ for any $x \in X$, then $b = 0$, where we regard X as a Hilbert A^∞-B^∞-subbimodule of X^∞.*

Proof. Since $b \in B^\infty$, we write $b = [b_m]$, where $b_m \in B$ for any $m \in \mathbb{N}$. Since $xb = 0$, we have $\|xb_m\| \to 0$ $(m \to \infty)$. For any $y \in X$,

$$\|\langle y, x\rangle_B\, b_m\| = \|\langle y, xb_m\rangle_B\| \le \|y\|\,\|xb_m\| \to 0 \quad (m \to \infty)$$

by [15, Lemma 2.5]. On the other hand, there are $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\sum_{i=1}^n \langle y_i, x_i\rangle_B = 1$ since X is full with the right B-valued inner product. Hence

$$\|b_m\| = \Big\|\sum_{i=1}^n \langle y_i, x_i\rangle_B b_m\Big\| \le \sum_{i=1}^n \|\langle y_i, x_i\rangle_B\, b_m\| \to 0.$$

Therefore $b = 0$. ∎

We are in a position to present the main result in this paper. Before doing so, we give the definitions of approximate representability and the Rokhlin property for a coaction of a finite-dimensional C^*-Hopf algebra on a unital C^*-algebra, and make a remark on the definitions.

DEFINITION 6.2 (cf. [11, Definitions 4.3 and 5.1]). Let (ρ, u) be a twisted coaction of a finite-dimensional C^*-Hopf algebra H^0 on a unital C^*-algebra A. We say that (ρ, u) is *approximately representable* if there is a unitary element $w \in A^\infty \otimes H^0$ satisfying the following conditions:

(1) $\rho(a) = (\mathrm{Ad}(w) \circ \rho^A_{H^0})(a)$ for any $a \in A$,
(2) $u = (w \otimes 1^0)(\rho^{A^\infty}_{H^0} \otimes \mathrm{id})(w)(\mathrm{id} \otimes \Delta^0)(w^*)$,
(3) $u = (\rho^\infty \otimes \mathrm{id})(w)(w \otimes 1^0)(\mathrm{id} \otimes \Delta^0)(w^*)$.

Also, we say that (ρ, u) has the *Rokhlin property* if the dual coaction $\widehat{\rho}$ of H on $A \rtimes_\rho H$ is approximately representable.

By [11, Corollary 6.4], a coaction ρ of H^0 on A has the Rokhlin property if and only if there is a projection $p \in A_\infty$ such that $e \cdot_{\rho^\infty} p = 1/N$, where $N = \dim(H)$.

THEOREM 6.3. *Let H be a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. Let ρ and σ be coactions of H^0 on unital C^*-algebras A and B, respectively. Suppose that ρ is strongly Morita equivalent to σ. Then ρ has the Rokhlin property if and only if σ has the Rokhlin property.*

Proof. Since ρ and σ are strongly Morita equivalent, there are an A-B-equivalence bimodule X and a coaction λ of H^0 on X with respect to (A, B, ρ, σ). From Rieffel [16, proof of Proposition 2.1], we obtain the following: Since X is full with the right B-valued inner product, there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ such that $\sum_{i=1}^n \langle x_i, y_i \rangle_B = 1$. Let $E = A \otimes M_n(\mathbb{C})$ and consider X^n as an E-B-equivalence bimodule in the usual way. Let $x = (x_i)_{i=1}^n, y = (y_i)_{i=i}^n \in X^n$. Let $z = {}_E\langle y, y\rangle^{1/2} x$ and let $q = {}_E\langle z, z\rangle \in E$. Then q is a projection in E. Let π be the map from B to E defined by $\pi(b) = {}_E\langle zb, z\rangle$ for any $b \in B$. Then π is an isomorphism of B onto qEq.

Suppose that ρ has the Rokhlin property. Then by [11, Corollary 6.4] there is a projection $p \in A_\infty$ such that $e \cdot_{\rho^\infty} p = 1/N$. We regard $(X^\infty)^n$ as an E^∞-B^∞-equivalence bimodule in the usual way. Since $p \otimes I_n \in E^\infty$, there are

$$u_1, \ldots, u_m, v_1, \ldots, v_m \in (X^\infty)^n$$

such that $p \otimes I_n = \sum_{k=1}^m {}_{E^\infty}\langle u_k, v_k\rangle$. We write

$$u_k = (u_{k1}, \ldots, u_{kn}), \qquad v_k = (v_{k1}, \ldots, v_{kn}),$$

where $u_{ki}, v_{ki} \in X^\infty$ for $k = 1, \ldots, m$ and $i = 1, \ldots, n$. Thus

$$p \otimes I_n = \sum_{k=1}^m [{}_{A^\infty}\langle u_{ki}, v_{kj}\rangle]_{i,j=1}^n.$$

Hence

$$(***) \qquad \sum_{k=1}^m {}_{A^\infty}\langle u_{ki}, v_{kj}\rangle = \begin{cases} p, & i = j, \\ 0, & i \neq j. \end{cases}$$

We note that since $p \in A_\infty$, we have $q(p \otimes I_n)q = q(p \otimes I_n) \in (qM_n(A)q)^\infty \cap (qM_n(A)q)'$. Let π^∞ be the isomorphism of B^∞ onto $(qM_n(A)q)^\infty$ induced by π. Let $p_1 = (\pi^\infty)^{-1}(q(p \otimes I_n)q)$. Then p_1 is a projection in B_∞ since $\pi(B) = qM_n(A)q$. We show that $e \cdot_{\sigma^\infty} p_1 = 1/N$. Since $q = {}_E\langle z, z\rangle$,

$$\begin{aligned} q(p \otimes I_n)q &= \sum_{k=1}^m {}_{E^\infty}\big\langle {}_E\langle z, z\rangle u_k, {}_E\langle z, z\rangle v_k \big\rangle \\ &= \sum_{k=1}^m {}_{E^\infty}\big\langle z\langle z, u_k\rangle_{B^\infty} \langle v_k, z\rangle_{B^\infty}, z \big\rangle \\ &= \pi^\infty\Big(\sum_{k=1}^m \langle z, u_k\rangle_{B^\infty} \langle v_k, z\rangle_{B^\infty} \Big). \end{aligned}$$

Thus

$$p_1 = \sum_{k=1}^m \langle z, u_k\rangle_{B^\infty} \langle v_k, z\rangle_{B^\infty} = \sum_{k=1}^m \big\langle z, {}_{E^\infty}\langle u_k, v_k\rangle z \big\rangle_{B^\infty}.$$

Since $z \in X^n$, we write $z = (z_i)_{i=1}^n$, where $z_i \in X$ for $i = 1, \ldots, n$. Hence by $(***)$,

$$p_1 = \left\langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \sum_{k=1}^{m} [\,{}_{A^\infty}\langle u_{ki}, v_{kj}\rangle]_{i,j=1}^{n} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right\rangle_{B^\infty}$$
$$= \sum_{i,j=1}^{n} \left\langle z_i, \sum_{k=1}^{m} {}_{A^\infty}\langle u_{ki}, v_{kj}\rangle z_j \right\rangle_{B^\infty} = \sum_{i=1}^{n} \langle z_i, pz_i\rangle_{B^\infty}.$$

For any $w \in X$,

$$w[e \cdot_{\sigma^\infty} p_1] = \sum_{i=1}^{n} w\langle [S(e_{(1)}^*) \cdot_\lambda z_i], [e_{(2)} \cdot_{\lambda^\infty} pz_i]\rangle_{B^\infty}$$
$$= \sum_{i=1}^{n} {}_A\langle w, [S(e_{(1)}^*) \cdot_\lambda z_i]\rangle [e_{(2)} \cdot_{\lambda^\infty} pz_i]$$
$$= \sum_{i=1}^{n} {}_A\langle [e_{(2)} S(e_{(1)}) \cdot_\lambda w], [S(e_{(3)}^*) \cdot_\lambda z_i]\rangle\, [e_{(4)} \cdot_{\lambda^\infty} pz_i]$$
$$= \sum_{i=1}^{n} \big[e_{(2)} \cdot_\rho {}_A\langle [S(e_{(1)}) \cdot_\lambda w], z_i\rangle\big] [e_{(3)} \cdot_{\lambda^\infty} pz_i]$$
$$= \sum_{i=1}^{n} \big[e_{(2)} \cdot_{\lambda^\infty} p[S(e_{(1)}) \cdot_\lambda w]\langle z_i, z_i\rangle_B\big]$$
$$= [e_{(2)} \cdot_{\rho^\infty} p][e_{(3)} S(e_{(1)}) \cdot_\lambda w].$$

Since $e = \sum_{i,k} \frac{d_k}{N} w_{ii}^k$, we get

$$w[e \cdot_{\sigma^\infty} p_1] = \sum_{i,j,k,j_1} \frac{d_k}{N} [w_{jj_1}^k \cdot_{\rho^\infty} p][w_{j_1 i}^k S(w_{ij}^k) \cdot_\lambda w]$$
$$= \sum_{j,k} \frac{d_k}{N} [w_{jj}^k \cdot_{\rho^\infty} p] w = [e \cdot_{\rho^\infty} p] w = \frac{1}{N} w.$$

Thus $e \cdot_{\sigma^\infty} p_1 = 1/N$ by Lemma 6.1. This gives the conclusion by [11, Corollary 6.4]. ∎

COROLLARY 6.4. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively. Suppose that they are strongly Morita equivalent. Then:*

(1) *(ρ, u) has the Rokhlin property if and only if so does (σ, v).*
(2) *(ρ, u) is approximately representable if and only if so is (σ, v).*

Proof. (1) Suppose that (ρ, u) has the Rokhlin property. Then so does $\widehat{\widehat{\rho}}$ by [11, Proposition 5.5]. Also, since (ρ, u) and (σ, v) are strongly Morita equivalent, so are $\widehat{\widehat{\rho}}$ and $\widehat{\widehat{\sigma}}$ by Corollary 4.8. Thus (σ, v) has the Rokhlin property by Theorem 6.3 and [11, Proposition 5.5].

(2) Suppose that (ρ, u) is approximately representable. Then $\widehat{\rho}$ has the Rokhlin property by the definition and [11, Proposition 4.6]. Since (ρ, u) and (σ, v) are strongly Morita equivalent, so are $\widehat{\rho}$ and $\widehat{\sigma}$ by Corollary 4.8. Thus by Theorem 6.3, $\widehat{\sigma}$ has the Rokhlin property. Hence by the definition and [11, Proposition 4.6], (σ, v) is approximately representable. ∎

7. Application. Let A and B be unital C^*-algebras and H a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. Suppose that A is strongly Morita equivalent to B. Let ρ be a coaction of H^0 on A. By [16, Proposition 2.1], there are $n \in \mathbb{N}$ and a full projection $q \in M_n(A)$ such that B is isomorphic to $qM_n(A)q$. We identify B with $qM_n(A)q$. Suppose that $(\rho \otimes \mathrm{id})(q) \sim q \otimes 1^0$ in $M_n(A) \otimes H^0$. Then there is a partial isometry $w \in M_n(A) \otimes H^0$ such that $w^*w = (\rho \otimes \mathrm{id})(q)$, $ww^* = q \otimes 1^0$.

Lemma 7.1. *With the above notation, there is a partial isometry $z \in M_n(A) \otimes H^0$ such that $z^*z = (\rho \otimes \mathrm{id})(q)$, $zz^* = q \otimes 1^0$ and $\widehat{z}(1) = q$.*

Proof. We note that $\widehat{w^*}(1) = \widehat{w}(1)^*$. Since $w^*w = (\rho \otimes \mathrm{id})(q)$ and $ww^* = q \otimes 1^0$, we obtain

$$\widehat{w^*}(1)\widehat{w}(1) = (\mathrm{id} \otimes \epsilon^0)((\rho \otimes \mathrm{id})(q)) = q, \quad \widehat{w}(1)\widehat{w^*}(1) = q.$$

Let $z = (\widehat{w^*}(1) \otimes 1^0)w$. Then $\widehat{z}(1) = \widehat{w^*}(1)\widehat{w}(1) = q$. Also,

$$z^*z = w^*(\widehat{w}(1) \otimes 1^0)(\widehat{w^*}(1) \otimes 1^0)w = (\rho \otimes \mathrm{id})(q),$$
$$zz^* = (\widehat{w^*}(1) \otimes 1^0)ww^*(\widehat{w}(1) \otimes 1^0) = q \otimes 1^0. \ ∎$$

Let

$$\sigma = \mathrm{Ad}(z) \circ (\rho \otimes \mathrm{id}_{M_n(\mathbb{C})}),$$
$$u = (z \otimes 1^0)(\rho \otimes \mathrm{id}_{M_n(\mathbb{C})} \otimes \mathrm{id}_{H^0})(z)(\mathrm{id}_{M_n(A)} \otimes \Delta^0)(z^*).$$

We note that $u \in B \otimes H^0 \otimes H^0$. We shall show that (σ, u) is a twisted coaction of H^0 on B, which is strongly Morita equivalent to ρ. We sometimes identify $A \otimes H^0 \otimes M_n(\mathbb{C})$ with $A \otimes M_n(\mathbb{C}) \otimes H^0$.

Lemma 7.2. *With the above notation, σ is a weak coaction of H^0 on B.*

Proof. For any $x \in M_n(A)$,

$$\sigma(qxq) = z(\rho \otimes \mathrm{id})(qxq)z^* = (q \otimes 1^0)z(\rho \otimes \mathrm{id})(x)z^*(q \otimes 1^0).$$

Hence σ is a map from B to $B \otimes H^0$. Also, by routine computations, we can see that σ is a homomorphism of B to $B \otimes H^0$ with $\sigma(q) = q \otimes 1^0$. Furthermore, since $\widehat{z}(1) = q$, for any $x \in M_n(A)$ we have

$$\begin{aligned}(\mathrm{id} \otimes \epsilon^0)(\sigma(qxq)) &= (\mathrm{id} \otimes \epsilon^0)\big((q \otimes 1^0)z(\rho \otimes \mathrm{id})(x)z^*(q \otimes 1^0)\big) \\ &= q\widehat{z}(1)(\mathrm{id} \otimes \epsilon^0)((\rho \otimes \mathrm{id})(x))\widehat{z^*}(1)q = qxq.\end{aligned}$$

Thus σ is a weak coaction of H^0 on B. ∎

LEMMA 7.3. *With the above notation, (σ, u) is a twisted coaction of H^0 on B.*

Proof. By routine computations, we can see that $uu^* = u^*u = q\otimes 1^0\otimes 1^0$. Thus u is a unitary element in $B \otimes H^0 \otimes H^0$. For any $x \in M_n(A)$, we have

$$\begin{aligned}((\sigma \otimes \mathrm{id}_{H^0}) \circ \sigma)(qxq) \\ &= (z \otimes 1^0)(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z)\big((\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id})\big)(qxq) \\ &\quad \times (\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z^*)(z^* \otimes 1^0).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta^0) \circ \sigma)(qxq) \\ &= (z \otimes 1^0)(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z)\big((\mathrm{id} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id})\big)(qxq) \\ &\quad \times (\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z^*)(z^* \otimes 1^0).\end{aligned}$$

Since $(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}) = (\mathrm{id} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id})$, we obtain

$$(\sigma \otimes \mathrm{id}_{H^0}) \circ \sigma = \mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta^0) \circ \sigma.$$

Also,

$$\begin{aligned}(u \otimes 1^0)(\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0})(u) \\ &= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(z \otimes 1^0) \\ &\quad \times (\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0})\big((\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z)(\mathrm{id} \otimes \Delta^0)(z^*)\big).\end{aligned}$$

On the other hand, since $(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}) = (\mathrm{id} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id})$,

$$\begin{aligned}&(\sigma \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u)(\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0)(u) \\ &= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(z \otimes 1^0) \\ &\quad \times (\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0})((\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z)) \\ &\quad \times (\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})((\mathrm{id} \otimes \Delta^0)(z^*)) \\ &\quad \times (\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0)((\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})(z))(\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0})((\mathrm{id} \otimes \Delta^0)(z^*)).\end{aligned}$$

We can see that

$$(\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0}) \circ (\mathrm{id} \otimes \Delta^0) = (\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})$$

by easy computations. Furthermore, we note that

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0) \circ (\mathrm{id} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0}) \circ (\mathrm{id} \otimes \Delta^0) \circ (\rho \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}).\end{aligned}$$

Thus since

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0)\big((\rho \otimes \mathrm{id} \otimes \mathrm{id}_{H^0})((\rho \otimes \mathrm{id})(q))\big) \\ &= (\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes \Delta^0)\big((\mathrm{id} \otimes \Delta^0)((\rho \otimes \mathrm{id})(q))\big),\end{aligned}$$

it follows that

$$\begin{aligned}
&(\sigma\otimes\mathrm{id}_{H^0}\otimes\mathrm{id}_{H^0})(u)(\mathrm{id}\otimes\mathrm{id}_{H^0}\otimes\Delta^0)(u)\\
&\qquad=(z\otimes 1^0\otimes 1^0)(\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0}\otimes\mathrm{id}_{H^0})(z\otimes 1^0)\\
&\qquad\quad\times(\mathrm{id}\otimes\Delta^0\otimes\mathrm{id}_{H^0})((\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0})(z))\\
&\qquad\quad\times(\mathrm{id}\otimes\mathrm{id}_{H^0}\otimes\Delta^0)\big((\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0})((\rho\otimes\mathrm{id})(q))\big)\\
&\qquad\quad\times(\mathrm{id}\otimes\Delta^0\otimes\mathrm{id}_{H^0})((\mathrm{id}\otimes\Delta^0)(z^*))\\
&\qquad=(z\otimes 1^0\otimes 1^0)(\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0}\otimes\mathrm{id}_{H^0})(z\otimes 1^0)\\
&\qquad\quad\times(\mathrm{id}\otimes\Delta^0\otimes\mathrm{id}_{H^0})\big((\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0})(z)(\mathrm{id}\otimes\Delta^0)(z^*)\big).
\end{aligned}$$

Hence we obtain

$$(u\otimes 1^0)(\mathrm{id}\otimes\Delta^0\otimes\mathrm{id}_{H^0})(u)=(\sigma\otimes\mathrm{id}_{H^0}\otimes\mathrm{id}_{H^0})(u)(\mathrm{id}\otimes\mathrm{id}_{H^0}\otimes\Delta^0)(u).$$

Furthermore, since $\widehat{z}(1)=q$, for any $h\in H$ we have

$$\begin{aligned}
&(\mathrm{id}\otimes h\otimes\epsilon^0)(u)=\widehat{z}(h_{(1)})[h_{(2)}\cdot_{\rho\otimes\mathrm{id}}q]\widehat{z^*}(h_{(3)})=(\mathrm{id}\otimes h)(\sigma(q))=\epsilon(h)q,\\
&(\mathrm{id}\otimes\epsilon^0\otimes h)(u)=\widehat{z}(1)[1\cdot_{\rho\otimes\mathrm{id}}\widehat{z}(h_{(1)})]\widehat{z^*}(h_{(2)})=\widehat{z}(1)\epsilon(h)=\epsilon(h)q.
\end{aligned}$$

Therefore, (σ,u) is a twisted coaction of H^0 on B. ∎

Let f be a minimal projection in $M_n(\mathbb{C})$ and let p be a full projection in $M_n(A)$ defined by $p=1_A\otimes f$. Let $X=pM_n(A)q$. We regard X as an A-B-equivalence bimodule in the usual way, where we identify A and B with $pM_n(A)p$ and $qM_n(A)q$, respectively. Then we can regard X as a set $\{[a_1,\dots,a_n]q\mid a_i\in A,\ i=1,\dots,n\}$. Let λ be the linear map from X to $X\otimes H^0$ defined by

$$\begin{aligned}
\lambda([a_1,\dots,a_n]q)&=[\rho(a_1),\dots,\rho(a_n)](\rho\otimes\mathrm{id})(q)z^*\\
&=[\rho(a_1),\dots,\rho(a_n)]z^*(q\otimes 1^0)
\end{aligned}$$

for any $[a_1,\dots,a_n]q\in X$.

Lemma 7.4. *With the above notation, λ is a twisted coaction of H^0 on X with respect to (A,B,ρ,σ,u).*

Proof. By routine computations, we can see that λ is a weak coaction of H^0 on X with respect to (A,B,ρ,σ,u). For any $[a_1,\dots,a_n]q\in X$,

$$\begin{aligned}
&((\lambda\otimes\mathrm{id}_{H^0})\circ\lambda)([a_1,\dots,a_n]q)\\
&\qquad=[((\rho\otimes\mathrm{id}_{H^0})\circ\rho)(a_1),\dots,((\rho\otimes\mathrm{id}_{H^0})\circ\rho)(a_n)]\\
&\qquad\quad\times(\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0})(z^*)(z^*\otimes 1^0).
\end{aligned}$$

On the other hand, since $(\rho\otimes\mathrm{id}_{H^0})\circ\rho=(\mathrm{id}\otimes\Delta^0)\circ\rho$, we obtain

$$((\mathrm{id}\otimes\Delta^0)\circ\lambda)([a_1,\dots,a_n]q)u^*$$
$$=[((\mathrm{id}\otimes\Delta^0)\circ\rho)(a_1),\dots,((\mathrm{id}\otimes\Delta^0)\circ\rho)(a_n)]$$
$$\times(\rho\otimes\mathrm{id}\otimes\mathrm{id}_{H^0})(z^*)(z^*\otimes 1^0).$$

Hence for any $[a_1,\dots,a_n]q\in X$,

$$((\lambda\otimes\mathrm{id}_{H^0})\circ\lambda)([a_1,\dots,a_n]q)=((\mathrm{id}\otimes\Delta^0)\circ\lambda)([a_1,\dots,a_n]q)u^*.$$

Thus λ is a twisted coaction of H^0 on X with respect to (A,B,ρ,σ,u). ■

THEOREM 7.5. *Let A be a unital C^*-algebra and H a finite-dimensional C^*-Hopf algebra with dual C^*-Hopf algebra H^0. Let ρ be a coaction of H^0 on A with the Rokhlin property. Let q be a full projection in a C^*-algebra $M_n(A)$ such that*

$$(\rho\otimes\mathrm{id}_{M_n(\mathbb{C})})(q)\sim q\otimes 1^0$$

in $M_n(A)\otimes H^0$. Let $B=qM_n(A)q$. Then there is a coaction of H^0 on B with the Rokhlin property.

Proof. By Lemmas 7.3 and 7.4, there is a twisted coaction (σ,u) such that (σ,u) is strongly Morita equivalent to ρ. By Corollary 6.4, (σ,u) has the Rokhlin property. Furthermore, by [11, Theorem 9.6], there is a unitary element $y\in B\otimes H^0$ such that

$$(y\otimes 1^0)(\sigma\otimes\mathrm{id}_{H^0})u(\mathrm{id}\otimes\Delta^0)(y^*)=1_B\otimes 1^0\otimes 1^0.$$

Let $\sigma_1=\mathrm{Ad}(y)\circ\sigma$. Then σ_1 is a coaction of H^0 on B with the Rokhlin property by easy computations since σ_1 is exterior equivalent to (σ,u). ■

Let A be a UHF-algebra of type N^∞, where N is the dimension of a finite-dimensional C^*-Hopf algebra H. In [11], we showed that there is a coaction ρ of H^0 on A with the Rokhlin property.

COROLLARY 7.6. *With the above notation, for any unital C^*-algebra B that is strongly Morita equivalent to A, there is a coaction σ of H^0 on B with the Rokhlin property.*

Proof. By [16, Proposition 2.1] there are $n\in\mathbb{N}$ and a full projection $q\in M_n(A)$ such that B is isomorphic to $qM_n(A)q$. We identify B with $qM_n(A)q$. Let ρ be a coaction of H^0 on A with the Rokhlin property. Then by [11, Lemma 10.10], $(\rho\otimes\mathrm{id}_{M_n(\mathbb{C})})(q)\sim q\otimes 1^0$ in $M_n(A)\otimes H^0$ since A has cancellation. Therefore, by Theorem 7.5 we obtain the conclusion. ■

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