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Some Parity Statistics in Integer Partitions

by

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Summary. We study integer partitions with respect to the classical word statistics of levels and descents subject to prescribed parity conditions. For instance, a partition with summands $\lambda_1 \geq \cdots \geq \lambda_k$ may be enumerated according to descents $\lambda_i > \lambda_{i+1}$ while tracking the individual parities of λ_i and λ_{i+1}. There are two types of parity levels, $E = E$ and $O = O$, and four types of parity-descents, $E > E$, $E > O$, $O > E$ and $O > O$, where E and O represent arbitrary even and odd summands. We obtain functional equations and explicit generating functions for the number of partitions of n according to the joint occurrence of the two levels. Then we obtain corresponding results for the joint occurrence of the four types of parity-descents. We also provide enumeration results for the total number of occurrences of each statistic in all partitions of n together with asymptotic estimates for the average number of parity-levels in a random partition.

1. Introduction. A *partition* of n is a representation of n as a sum of positive integers without regard to order. The summands are called parts of the partition. We will write partitions in weakly decreasing order. Thus a partition λ of n into $m > 0$ parts will generally be expressed as $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $\lambda_1 \geq \cdots \geq \lambda_m > 0$ and $|\lambda| = \lambda_1 + \cdots + \lambda_m = n$. The empty partition of 0 will be denoted by $\emptyset$. Further properties of partitions can be found in the standard reference [2].

This work considers the enumeration of partitions with respect to two statistics defined on adjacent parts. The systematic study of partition statistics was initiated by Erdős and Lehner [8] in 1941, who discussed questions about the statistical distributions, over all partitions, of the number of parts, the number of different part sizes, and the size of the largest part. In recent times several mathematicians have examined interesting variations on the theme. For example, Knopfmacher and Robbins [9] considered a variety of

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statistics which count the parts in certain classes of restricted partitions of n that include partitions with distinct parts, partitions into distinct and odd parts, and self-conjugate partitions. In [6] Grabner and Knopfmacher studied gap-free partitions and obtained enumeration functions for partitions with specified gap-sizes as well as a formula for the multiplicity of the largest part.

George Andrews recently rekindled interest in partition statistics which fulfil parity conditions with the publication of a 46-page treatise [1]. The latter investigated a variety of parity questions related to partition identities, and posed some 15 open problems. A more recent work by Grabner, Knopfmacher and Wagner [7] explores several statistics in random partitions. Further useful background on the topic from different perspectives may be found in the research articles [3, 4, 10].

Our study of partitions in this paper is concerned with the classical statistics of *levels* and *descents* whereby the adjacent pair of letters fulfills prescribed parity conditions.

Given the partition λ and integers $s, t \in \{0, 1\}$, we say that $i, \in \{1, \dots, m-1\}$ is an (s, t) *parity-descent* if $\lambda_i \equiv s \pmod 2$ and $\lambda_{i+1} \equiv t \pmod 2$ whenever $\lambda_i > \lambda_{i+1}$, and i is an (s, s) *parity-level* if $\lambda_i = \lambda_{i+1}$ and $\lambda_i \equiv s \pmod 2$. Denote the number of occurrences of an (s, t) parity-descent and of an (s, s) parity-level by D_{st} and L_{ss} respectively.

In this paper we undertake the enumeration of partitions according to the two statistics D_{st} and L_{ss}.

In Section 2 we obtain the generating function for the number of partitions of n with largest part $\lambda_1 = k$ according to the statistics L_{00} and L_{11} (Theorems 2.1 and 2.8), and verify a few known special cases. Subsequently, we compute simpler formulas for the generating functions for the total number of the two types of levels in all partitions of n. Section 3 is devoted to the execution of the same enumeration agenda for the four descent statistics, D_{00}, D_{01}, D_{10} and D_{11}. Section 4 deals with the computation of asymptotic estimates of the average number of both types of parity-levels in a random partition. Lastly, we use Section 5 to highlight two combinatorial identities arising from the algebraic work in previous sections.

2. Enumeration of levels. For $k \geq l \geq 1$ let us define the generating function for the number of partitions with largest part (first two largest parts) equal to k (resp. k, l) subject to the statistics L_0, L_1 as the sum $\sum_\lambda x^{|\lambda|} y_0^{L_0} y_1^{L_1}$, where the sum runs over all partitions λ with $\lambda_1 = k$ (resp. with $\lambda_1 = k$ and $\lambda_2 = l$). This function will be denoted by $F_k(x, y_0, y_1)$ or simply by $F_k(x)$ (resp. by $F_{k,l}(x, y_0, y_1)$ or simply by $F_{k,l}(x)$). Clearly, $F_1(x) = \frac{x}{1-y_{11}x}$. Note that we have simplified notation in this section as follows: $L_{ss} \to L,\ L_{11} \to L_1,\ L_{00} \to L_0$.

By the definitions we have

$$\begin{aligned}
F_{2k}(x) &= x^{2k} + F_{2k,2k}(x) + \sum_{j=1}^{2k-1} F_{2k,j}(x) = x^{2k} + x^{2k} y_0 F_{2k}(x) \\
&\quad + x^{2k} \sum_{j=1}^{2k-1} F_j(x) \\
&= x^{2k}\Big(1 + y_0 F_{2k}(x) + \sum_{j=1}^{k} F_{2j-1}(x) + \sum_{j=1}^{k-1} F_{2j}(x)\Big), \\
F_{2k-1}(x) &= x^{2k-1} + F_{2k-1,2k-1}(x) + \sum_{j=1}^{2k-2} F_{2k-1,j}(x) \\
&= x^{2k-1} + x^{2k-1} y_1 F_{2k-1}(x) + x^{2k-1} \sum_{j=1}^{2k-2} F_j(x) \\
&= x^{2k-1}\Big(1 + y_1 F_{2k-1}(x) + \sum_{j=1}^{k-1} F_{2j-1}(x) + \sum_{j=1}^{k-1} F_{2j}(x)\Big).
\end{aligned}$$

Note that $F_2(x) = x^2(1 + F_1(x) + y_0 F_2(x))$, which implies that

$$F_2(x) = \frac{x^2}{1 - y_0 x^2}(1 + \frac{x}{1 - y_1 x}) = \frac{x^2(1 + (1 - y_1)x)}{(1 - y_0 x^2)(1 - y_1 x)}.$$

Therefore,

(1)
$$\begin{aligned}
F_{2k+2}(x) - x^2 F_{2k}(x) &= x^{2k+2}\big(F_{2k+1}(x) + y_0 F_{2k+2}(x) + (1 - y_0) F_{2k}(x)\big), \\
F_{2k+1}(x) - x^2 F_{2k-1}(x) &= x^{2k+1}\big(y_1 F_{2k+1}(x) + (1 - y_1) F_{2k-1}(x) + F_{2k}(x)\big),
\end{aligned}$$

for all $k \geq 1$.

Define

$$\begin{aligned}
E(u) &= E(u \,|\, y_0, y_1) = \sum_{k\geq 1} F_{2k}(x) u^{2k}, \\
O(u) &= O(u \,|\, y_0, y_1) = \sum_{k\geq 1} F_{2k-1}(x) u^{2k-1}, \\
F(u) &= F(u \,|\, y_0, y_1) = \sum_{k\geq 1} F_k(x) u^{k}.
\end{aligned}$$

Hence, multiplying (1) by u^{2k+2} and u^{2k+1}, respectively, then summing over $k \geq 1$, we obtain

$$
\begin{aligned}
&E(u) - F_2(x)u^2 - x^2u^2E(u)\\
&\quad = xu(O(xu) - F_1(x)xu) + y_0(E(xu) - F_2(x)x^2u^2) + (1-y_0)x^2u^2E(xu),\\
&O(u) - F_1(x)u - x^2u^2O(u)\\
&\qquad\qquad\qquad = y_1(O(xu) - F_1(x)xu) + xuE(xu) + (1-y_1)x^2u^2O(xu),
\end{aligned}
$$

which implies the following result.

THEOREM 2.1. *We have $F(u) = E(u) + O(u)$, where*

$$
\begin{aligned}
E(u) &= \left(\frac{x^2u^2}{1-x^2u^2} + y_0\right)E(xu) + \frac{xu}{1-x^2u^2}O(xu) + \frac{x^2u^2}{1-x^2u^2},\\
O(u) &= \frac{xu}{1-x^2u^2}E(xu) + \left(\frac{x^2u^2}{1-x^2u^2} + y_1\right)O(xu) + \frac{xu}{1-x^2u^2}.
\end{aligned}
$$

EXAMPLE 2.2 (All partitions). By setting $y_0 = y_1 = 1$ in Theorem 2.1 and using the relation $F(u\,|\,1,1) = E(u\,|\,1,1) + O(u\,|\,1,1)$ we obtain

$$F(u\,|\,1,1) = \frac{xu}{1-xu} + \frac{1}{1-xu}F(xu\,|\,1,1).$$

Iterating this recurrence infinitely (with $|x|, |u| < 1$) yields

$$1 + F(u\,|\,1,1) = 1 + \sum_{i\geq 1}\frac{x^iu}{(1-ux)\cdots(1-ux^i)} = \prod_{i\geq 1}\frac{1}{1-ux^i},$$

which is the well known bivariate generating function for the number of partitions of n with largest part k.

EXAMPLE 2.3 (Partitions with no levels). By Theorem 2.1 with $y_0 = y_1 = 0$, we analogously obtain

$$F(u\,|\,0,0) = \frac{xu}{1-xu} + \frac{xu}{1-xu}F(xu|0,0),$$

which may be iterated m times to give

$$F(u\,|\,0,0) = \sum_{m\geq 1}\frac{x^{\binom{m+1}{2}}u^m}{\prod_{i=1}^{m}(1-ux^i)}.$$

The coefficient of u^kx^n gives the number of partitions of n into m distinct parts with largest part k. The case $u = 1$ yields the classical generating function for the number of strict partitions of n into m parts.

2.1. Explicit formulas. In order to find explicit formulas for $E(u)$ and $O(u)$, we need the following definitions and lemma.

DEFINITION 2.4. For any integer a and a finite sequence of integers $W = (w_1, \ldots, w_s)$ the sequence $(a, w_1, \ldots, w_s)$ will be denoted by aw. For a set $\mathcal{A}$ of finite sequences of integers let $a\mathcal{A}$ denote the set $\{aw : w \in \mathcal{A}\}$ and

$a\emptyset = \{(a)\}$. Define $\mathcal{A}_0 = \{0\}$, $\mathcal{B}_0 = \{\emptyset\}$, and for all $m \geq 1$,

$$\begin{aligned}\mathcal{A}_m &= m\mathcal{A}_{m-1}\\ &\quad \cup \{w' = (-(m-1))w \mid w = w_1 \cdots w_s \in \mathcal{A}_{m-1} \text{ with } w_1 \neq m-1\}\\ &\quad \cup \{w' = w_2 \cdots w_s \mid (m-1)w_2 \cdots w_s \in \mathcal{A}_{m-1}\},\\ \mathcal{B}_m &= (-m)\mathcal{B}_{m-1}\\ &\quad \cup \{w' = (m-1)w \mid w = w_1 \cdots w_s \in \mathcal{B}_{m-1} \text{ with } w_1 \neq -(m-1)\}\\ &\quad \cup \{w' = w_2 \cdots w_s \mid (-(m-1))w_2 \cdots w_s \in \mathcal{B}_{m-1}\}.\end{aligned}$$

Let

$$\begin{aligned}K_m(a(z), b(z)) &= \sum_{w_1 \cdots w_s \in \mathcal{A}_m} \prod_{w_j \geq 0} a(x^{w_j} z) \prod_{w_j < 0} b(x^{-w_j} z),\\ L_m(a(z), b(z)) &= \sum_{w_1 \cdots w_s \in \mathcal{B}_m} \prod_{w_j \geq 0} a(x^{w_j} z) \prod_{w_j < 0} b(x^{-w_j} z),\end{aligned} \tag{2}$$

where $K_{-1}(a(z), b(z)) = 1$, $L_{-1}(a(z), b(z)) = 0$ and a, b are arbitrary functions.

EXAMPLE 2.5. Assume that $a(u) = b(u) = 1/(xu)$. Then

$$\begin{aligned}K_m\left(\frac{1}{xu}, \frac{1}{xu}\right) &= \sum_{w_1 \cdots w_s \in \mathcal{A}_m} \frac{1}{x^{s+\sum_{j=1}^s |w_j|} u^s},\\ L_m\left(\frac{1}{xu}, \frac{1}{xu}\right) &= \sum_{w_1 \cdots w_s \in \mathcal{B}_m} \frac{1}{x^{s+\sum_{j=1}^s |w_j|} u^s}.\end{aligned}$$

By Definition 2.4 and induction on m, we obtain

$$\mathcal{A}_{m+1} = (m+1)\mathcal{A}_m \cup \mathcal{B}_m, \qquad \mathcal{B}_{m+1} = (-m-1)\mathcal{B}_m \cup \mathcal{A}_m. \tag{3}$$

So if

$$t_m = K_m\left(\frac{1}{xu}, \frac{1}{xu}\right) + L_m\left(\frac{1}{xu}, \frac{1}{xu}\right)$$

we get

$$t_m = x^{-\binom{m+2}{2}} u^{-m-1} \prod_{j=1}^{m+1} (1 + x^j u).$$

Define

$$M(z) = \begin{pmatrix} a(z) & 1 \\ 1 & b(z) \end{pmatrix}.$$

Then using (2) and induction (on m), we obtain the following lemma.

LEMMA 2.6. *For all $m \geq 0$,*

$$\prod_{j=0}^{m} M(x^j z) = \begin{pmatrix} K_m(a(z), b(z)) & L_m(a(z), b(z)) \\ L_m(b(z), a(z)) & K_m(b(z), a(z)) \end{pmatrix}.$$

Theorem 2.7. *We have*

$$E(u) = \sum_{m\geq 0} \frac{x^{\binom{m+2}{2}} u^{m+2}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)} \big(x^{m+1}uK_{m-1}(a(u),b(u)) + L_{m-1}(a(u),b(u))\big),$$

$$O(u) = \sum_{m\geq 0} \frac{x^{\binom{m+2}{2}} u^{m+2}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)} \big(x^{m+1}uL_{m-1}(b(u),a(u)) + K_{m-1}(b(u),a(u))\big)$$

where

$$a(u) = xu + \frac{y_0(1-x^2u^2)}{xu} \quad \text{and} \quad b(u) = xu + \frac{y_1(1-x^2u^2)}{xu}.$$

Proof. By Theorem 2.1, we have

$$\begin{pmatrix} E(u) \\ O(u) \end{pmatrix} = \frac{xu}{1-x^2u^2} M(u) \begin{pmatrix} E(xu) \\ O(xu) \end{pmatrix} + \rho(u),$$

where $\rho(u) = \frac{xu}{1-x^2u^2}\binom{xu}{1}$. By iterating the above recurrence, we obtain

$$\begin{pmatrix} E(u) \\ O(u) \end{pmatrix} = \sum_{m\geq 0} \frac{x^{\binom{m+2}{2}} u^{m+1}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)} M(u)M(xu)\cdots M(x^{m-1}u) \begin{pmatrix} x^{m+1}u \\ 1 \end{pmatrix}.$$

The result follows using Lemma 2.6. ∎

Theorem 2.7 together with the fact that $F(u) = O(u) + E(u)$ leads to the following result.

Theorem 2.8. *We have*

$$\begin{aligned} F(u) &= F(u \,|\, y_0, y_1) \\ &= \sum_{m\geq 0} \frac{x^{\binom{m+2}{2}} u^{m+1}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)} \big(x^{m+1}uT_{m-1}(a(u),b(u)) + T_{m-1}(b(u),a(u))\big) \end{aligned}$$

where $T_m(a,b) = T_m(a,b \,|\, x,u,y_0,y_1) = K_m(a,b) + L_m(b,a)$, *and* $a(u)$ *and* $b(u)$ *are as in Theorem* 2.7.

Example 2.9. Using Example 2.5 we obtain

$$\begin{aligned} T_m\left(\frac{1}{xu}, \frac{1}{xu} \,\middle|\, x,u,1,1\right) &= K_m\left(\frac{1}{xu}, \frac{1}{xu}\right) + L_m\left(\frac{1}{xu}, \frac{1}{xu}\right) \\ &= x^{-\binom{m+2}{2}} u^{-m-1} \prod_{j=1}^{m+1} (1+x^ju), \end{aligned}$$

so Theorem 2.8 implies

$$F(1) = F(1 \,|\, 1,1) = \sum_{m\geq 0} \frac{x^{m+1}}{\prod_{j=1}^{m+1}(1-x^j)} = \frac{1}{\prod_{j\geq 1}(1-x^j)} - 1,$$

which is the generating function for $p(n)$, the number of unrestricted partitions of $n > 0$.

We can obtain an explicit formula for the generating function $F(u \mid y_0, y_1)$ by noting that each partition can be expressed as a word $a_1 \cdots a_1 \cdots a_k \cdots a_k$ with $a_1 > \cdots > a_k \geq 1$. Thus we obtain the following formula.

THEOREM 2.10. *We have*

$$F(1 \mid y_0, y_1) = \prod_{n=1}^{\infty} \frac{1 + (1 - y_0)x^{2n}}{1 - y_0 x^{2n}} \prod_{n=1}^{\infty} \frac{1 + (1 - y_1)x^{2n-1}}{1 - y_1 x^{2n-1}}.$$

By comparing Theorems 2.8 and 2.10, we get the following corollary.

COROLLARY 2.11. *Define*

$$T_m(a, b) = T_m(a, b \mid x, 1, y_0, y_1) = K_m(a, b) + L_m(b, a)$$

for all $m \geq 0$. *Then*

$$\sum_{m \geq 0} \frac{x^{\binom{m+2}{2}}}{\prod_{j=1}^{m+1}(1 - x^{2j})} \big(x^{m+1} T_{m-1}(a(1), b(1)) + T_{m-1}(b(1), a(1))\big)$$
$$= \prod_{n=1}^{\infty} \frac{1 + (1 - y_0)x^{2n}}{1 - y_0 x^{2n}} \prod_{n=1}^{\infty} \frac{1 + (1 - y_1)x^{2n-1}}{1 - y_1 x^{2n-1}},$$

where $a(1) = x + \frac{y_0(1-x^2)}{x}$ *and* $b(1) = x + \frac{y_1(1-x^2)}{x}$.

2.2. The total number of $(0,0)$ parity-levels. In this section we find the total number of $(0,0)$ parity-levels. We first establish two lemmas which may be proved by straightforward induction arguments.

LEMMA 2.12. *Let* $k_m = K_m\big(\frac{1}{xu}, \frac{1}{xu}\big)$ *and* $\ell_m = L_m\big(\frac{1}{xu}, \frac{1}{xu}\big)$. *Then for all* $m \geq -1$,

$$k_m = \frac{1}{2}\Bigg(\prod_{j=1}^{m+1}\left(1 + \frac{1}{x^j u}\right) + \prod_{j=1}^{m+1}\left(\frac{1}{x^j u} - 1\right)\Bigg),$$
$$\ell_m = \frac{1}{2}\Bigg(\prod_{j=1}^{m+1}\left(1 + \frac{1}{x^j u}\right) - \prod_{j=1}^{m+1}\left(\frac{1}{x^j u} - 1\right)\Bigg).$$

LEMMA 2.13. *Let*

$$k'_m = \frac{d}{dy_0} K_m(a(u), b(u))\bigg|_{y_0 = y_1 = 1}, \qquad \ell'_m = \frac{d}{dy_0} L_m(a(u), b(u))\bigg|_{y_0 = y_1 = 1},$$
$$k''_m = \frac{d}{dy_0} K_m(b(u), a(u))\bigg|_{y_0 = y_1 = 1}, \qquad \ell''_m = \frac{d}{dy_0} L_m(b(u), a(u))\bigg|_{y_0 = y_1 = 1},$$

where $a(u)$ and $b(u)$ are as in Theorem 2.7. Then, for all $m \geq -1$,

$$k'_m = \frac{1}{4}\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)+\prod_{i=j+1}^{m+1}(1+x^iu)\prod_{i=1}^{j+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}$$
$$+\frac{1}{4}\sum_{j=0}^{m} \frac{\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)+(1+x^{j+1}u)\prod_{i=1}^{m+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}},$$

$$\ell'_m = \frac{1}{4}\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)+\prod_{i=j+1}^{m+1}(1+x^iu)\prod_{i=1}^{j+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}$$
$$-\frac{1}{4}\sum_{j=0}^{m} \frac{\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)+(1+x^{j+1}u)\prod_{i=1}^{m+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}},$$

$$\ell''_m = \frac{1}{4}\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)-\prod_{i=j+1}^{m+1}(1+x^iu)\prod_{i=1}^{j+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}$$
$$+\frac{1}{4}\sum_{j=0}^{m} \frac{\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)-(1+x^{j+1}u)\prod_{i=1}^{m+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}},$$

$$k''_m = \frac{1}{4}\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)-\prod_{i=j+1}^{m+1}(1+x^iu)\prod_{i=1}^{j+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}$$
$$-\frac{1}{4}\sum_{j=0}^{m} \frac{\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)-(1+x^{j+1}u)\prod_{i=1}^{m+1}(1-x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}.$$

THEOREM 2.14. *We have*

$$\frac{d}{dy_0}F(u\,|\,y_0,y_1)\bigg|_{y_0=y_1=1} = \frac{u}{2}\sum_{m\geq 1}\frac{x^{m+1}}{\prod_{i=1}^{m+1}(1-x^iu)}\sum_{j=1}^{m}(1-x^ju)\left(1-\frac{\prod_{i=j+1}^{m+1}(1-x^iu)}{\prod_{i=j+1}^{m+1}(1+x^iu)}\right).$$

Proof. By Theorem 2.8, we obtain

$$A = \frac{d}{dy_0}F(u\,|\,y_0,y_1)\bigg|_{y_0=y_1=1} = \sum_{m\geq 0}\frac{x^{\binom{m+2}{2}}u^{m+1}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)}\big(x^{m+1}u(k'_{m-1}+\ell''_{m-1})+k''_{m-1}+\ell'_{m-1}\big),$$

where k'_{m-1}, k''_{m-1}, ℓ'_{m-1} and ℓ''_{m-1} are given in Lemma 2.13. By that lemma,

$$(4)\qquad \begin{cases} k'_m + \ell''_m &= \dfrac{1}{2}\displaystyle\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)+\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}, \\ k''_m + \ell'_m &= \dfrac{1}{2}\displaystyle\sum_{j=0}^{m} \frac{(1-x^{j+1}u)\prod_{i=1}^{m+1}(1+x^iu)-\prod_{i=j+1}^{m+1}(1-x^iu)\prod_{i=1}^{j+1}(1+x^iu)}{x^{\binom{m+2}{2}}u^{m+1}}. \end{cases}$$

Therefore,

$$A = \frac{u}{2}\sum_{m\geq 1}\frac{x^{m+1}}{\prod_{i=1}^{m+1}(1-x^iu)}\sum_{j=1}^{m}(1-x^ju)\left(1-\frac{\prod_{i=j+1}^{m+1}(1-x^iu)}{\prod_{i=j+1}^{m+1}(1+x^iu)}\right),$$

as claimed. ∎

2.3. The total number of $(1,1)$ parity-levels. In this section we find the total number of $(1,1)$ parity-levels.

The next lemma follows immediately from Theorem 2.8 and the definitions.

LEMMA 2.15. *For all $m \geq 0$,*

$$\begin{aligned} \frac{d}{dy_1}K_m(a(u),b(u))\bigg|_{y_0=y_1=1} &= \frac{d}{dy_0}K_m(b(u),a(u))\bigg|_{y_0=y_1=1}, \\ \frac{d}{dy_1}K_m(b(u),a(u))\bigg|_{y_0=y_1=1} &= \frac{d}{dy_0}K_m(a(u),b(u))\bigg|_{y_0=y_1=1}, \\ \frac{d}{dy_1}L_m(a(u),b(u))\bigg|_{y_0=y_1=1} &= \frac{d}{dy_0}L_m(b(u),a(u))\bigg|_{y_0=y_1=1}, \\ \frac{d}{dy_1}L_m(b(u),a(u))\bigg|_{y_0=y_1=1} &= \frac{d}{dy_0}L_m(a(u),b(u))\bigg|_{y_0=y_1=1}. \end{aligned}$$

where $a(u) = xu + \frac{y_0(1-x^2u^2)}{xu}$ *and* $b(u) = xu + \frac{y_1(1-x^2u^2)}{xu}$.

THEOREM 2.16. *We have*

$$\begin{aligned} &\frac{d}{dy_1}F(u\,|\,y_0,y_1)\bigg|_{y_0=y_1=1} \\ &\qquad = \frac{u}{2}\sum_{m\geq 0}\frac{x^{m+1}}{\prod_{j=1}^{m+1}(1-x^ju)}\sum_{j=1}^{m}(1-x^ju)\left(1+\frac{\prod_{i=j+1}^{m+1}(1-x^iu)}{\prod_{i=j+1}^{m+1}(1+x^iu)}\right). \end{aligned}$$

Proof. By Theorem 2.8, Lemma 2.13 and Lemma 2.15, we obtain

$$\begin{aligned} B &= \frac{d}{dy_1}F(u\,|\,y_0,y_1)\bigg|_{y_0=y_1=1} \\ &= \sum_{m\geq 0}\frac{x^{\binom{m+2}{2}}u^{m+1}}{\prod_{j=1}^{m+1}(1-x^{2j}u^2)}\big(x^{m+1}u(k''_{m-1}+\ell'_{m-1})+k'_{m-1}+\ell''_{m-1}\big), \end{aligned}$$

where k'_{m-1}, k''_{m-1}, ℓ'_{m-1} and ℓ''_{m-1} are given in Lemma 2.13. By (4) we have

$$B=\frac{u}{2}\sum_{m\ge 0}\frac{x^{m+1}}{\prod_{j=1}^{m+1}(1-x^ju)}\sum_{j=1}^{m}(1-x^ju)\biggl(1+\frac{\prod_{i=j+1}^{m+1}(1-x^iu)}{\prod_{i=j+1}^{m+1}(1+x^iu)}\biggr),$$

as claimed. ■

3. Enumeration of descents. Let

$$G_k(x)=G_k(x\,|\,z_{00},z_{10},z_{01},z_{11})=\sum_{\lambda}x^{|\lambda|}z_{00}^{D_{00}(\lambda)}z_{10}^{D_{10}(\lambda)}z_{01}^{D_{01}(\lambda)}z_{11}^{D_{11}(\lambda)}$$

be the generating function for the number of partitions of n with largest part k according to the statistics D_{00}, D_{01}, D_{10} and D_{11}. Clearly, $G_1(x)=\frac{x}{1-x}$. Let $k\ge 1$; by the definitions, we have

$$G_{2k}(x)=x^{2k}\Bigl(1+G_{2k}(x)+z_{01}\sum_{j=1}^{k}G_{2j-1}(x)+z_{00}\sum_{j=1}^{k-1}G_{2j}(x)\Bigr),$$

$$G_{2k-1}(x)=x^{2k-1}\Bigl(1+G_{2k-1}(x)+z_{11}\sum_{j=1}^{k-1}G_{2j-1}(x)+z_{10}\sum_{j=1}^{k-1}G_{2j}(x)\Bigr).$$

Note that $G_2(x)=x^2(1+G_2(x)+z_{01}G_1(x))$, which implies that

$$G_2(x)=\frac{x^2(1+(z_{01}-1)x)}{(1-x)(1-x^2)}.$$

Therefore,

(5)
$$G_{2k+2}(x)-x^2G_{2k}(x)=x^{2k+2}\bigl(G_{2k+2}(x)+z_{01}G_{2k+1}(x)+(z_{00}-1)G_{2k}(x)\bigr),$$
$$G_{2k+1}(x)-x^2G_{2k-1}(x)=x^{2k+1}\bigl(G_{2k+1}(x)+(z_{11}-1)G_{2k-1}(x)+z_{10}G_{2k}(x)\bigr),$$

for all $k\ge 1$.

Define

$$GE(u)=GE(u\,|\,z_{00},z_{10},z_{01},z_{11})=\sum_{k\ge 1}G_{2k}(x)u^{2k},$$
$$GO(u)=GO(u\,|\,z_{00},z_{10},z_{01},z_{11})=\sum_{k\ge 1}G_{2k-1}(x)u^{2k-1},$$
$$G(u)=G(u\,|\,z_{00},z_{10},z_{01},z_{11})=\sum_{k\ge 1}G_k(x)u^k.$$

Hence, multiplying (5) by u^{2k+2} and u^{2k+1}, respectively, then summing over

$k \geq 1$, we obtain

$$GE(u) = \frac{1+(z_{00}-1)x^2u^2}{1-x^2u^2}GE(xu) + \frac{z_{01}xu}{1-x^2u^2}GO(xu) + \frac{(1-x^2)u^2G_2(x) - z_{01}x^2u^2G_1(x)}{1-x^2u^2},$$

$$GO(u) = \frac{z_{10}xu}{1-x^2u^2}GE(xu) + \frac{1+(z_{11}-1)x^2u^2}{1-x^2u^2}GO(xu) + \frac{(1-x)uG_1(x)}{1-x^2u^2},$$

which implies the following result.

Theorem 3.1. *Let*

$$Q(u) = \begin{pmatrix} \frac{1+(z_{00}-1)x^2u^2}{xu} & z_{01} \\ z_{10} & \frac{1+(z_{11}-1)x^2u^2}{xu} \end{pmatrix}.$$

Then $G(u) = GE(u) + GO(u)$, *where*

$$\begin{pmatrix} GE(u) \\ GO(u) \end{pmatrix} = \frac{xu}{1-x^2u^2}Q(u)\begin{pmatrix} GE(xu) \\ GO(xu) \end{pmatrix} + \frac{xu}{1-x^2u^2}\begin{pmatrix} xu \\ 1 \end{pmatrix}.$$

In order to find an explicit formula for $G(u)$, we need the following notation. It is easy to see that the matrix product

$$\begin{pmatrix} a(u) & z_{01} \\ z_{10} & b(u) \end{pmatrix}\begin{pmatrix} a(xu) & z_{01} \\ z_{10} & b(xu) \end{pmatrix}\cdots\begin{pmatrix} a(x^mu) & z_{01} \\ z_{10} & b(x^mu) \end{pmatrix}$$

can be represented by the matrix

$$\begin{pmatrix} U_m(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) & V_m(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) \\ V_m(b(u),a(u)\,|\,z_{00},z_{10},z_{01},z_{11}) & U_m(b(u),a(u)\,|\,z_{00},z_{10},z_{01},z_{11}) \end{pmatrix}.$$

Moreover, we obtain the following lemma.

Lemma 3.2. *For all* $m \geq 1$,

$$\begin{aligned} U_m(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) &= a(x^mu)U_{m-1}(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) \\ &\quad + z_{10}V_{m-1}(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}), \\ V_m(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) &= z_{01}U_{m-1}(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) \\ &\quad + b(x^mu)V_{m-1}(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) \end{aligned}$$

with $U_0(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) = a(u)$ *and* $V_0(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) = z_{01}$.

Theorem 3.3. *The generating function* $G(u)$ *is given by*

$$\sum_{m\geq 0} \frac{x^{\binom{m+2}{2}}u^{m+1}\left(x^{m+1}uW_{m-1}(a(u),b(u)\,|\,z_{00},z_{01},z_{10},z_{11}) + W_{m-1}(b(u),a(u)\,|\,z_{00},z_{10},z_{01},z_{11})\right)}{\prod_{i=1}^{m+1}(1-x^{2i}u^2)},$$

where

$$W_m(a(u), b(u) \mid z_{00}, z_{01}, z_{10}, z_{11}) \\ = U_m(a(u), b(u) \mid z_{00}, z_{01}, z_{10}, z_{11}) + V_m(a(u), b(u) \mid z_{00}, z_{10}, z_{01}, z_{11}),$$

and

$$a(u) = \frac{1 + (z_{00} - 1)x^2u^2}{xu} \quad \textit{and} \quad b(u) = \frac{1 + (z_{11} - 1)x^2u^2}{xu}.$$

Proof. By Theorem 3.1, we have

$$\begin{pmatrix} GE(u) \\ GO(u) \end{pmatrix} = \sum_{m \geq 0} \frac{x^{\binom{m+2}{2}} u^{m+1}}{\prod_{i=1}^{m+1}(1 - x^{2i}u^2)} Q(u)Q(xu)\cdots Q(x^{m-1}u) \begin{pmatrix} x^{m+1}u \\ 1 \end{pmatrix}.$$

An equivalent expression for $\binom{GE(u)}{GO(u)}$ is

$$\sum_{m \geq 0} \frac{x^{\binom{m+2}{2}} u^{m+1}}{\prod_{i=1}^{m+1}(1 - x^{2i}u^2)} \\ \cdot \begin{pmatrix} x^{m+1}uU_{m-1}(a(u), b(u) \mid z_{00}, z_{01}, z_{10}, z_{11}) + V_{m-1}(a(u), b(u) \mid z_{00}, z_{01}, z_{10}, z_{11}) \\ x^{m+1}uV_{m-1}(b(u), a(u) \mid z_{00}, z_{10}, z_{01}, z_{11}) + U_{m-1}(b(u), a(u) \mid z_{00}, z_{10}, z_{01}, z_{11}) \end{pmatrix}.$$

The result follows because $G(u) = GE(u) + GO(u)$, and this completes the proof. ∎

Example 3.4. Assume that $z_{00} = z_{01} = z_{10} = z_{11} = 1$. Then $a(u) = b(u) = 1/(xu)$, and

$$u_m = U_m(a(u), b(u) \mid 1,1,1,1) = \frac{1}{x^{m+1}u} u_{m-1} + v_{m-1} \quad \text{with} \quad u_0 = \frac{1}{xu},$$
$$v_m = V_m(a(u), b(u) \mid 1,1,1,1) = u_{m-1} + \frac{1}{x^{m+1}u} v_{m-1} \quad \text{with} \quad v_0 = 1.$$

Therefore, as in Example 2.5, we have

$$w_m = u_m + v_m = \prod_{j=1}^{m+1} \left(\frac{1}{x^j u} + 1 \right) \quad \text{and} \quad u_m - v_m = \prod_{j=1}^{m+1} \left(\frac{1}{x^j u} - 1 \right).$$

Hence,

$$x^{m+1}uw_{m-1} + w_{m-1} = x^{-\binom{m+1}{2}} u^{-m} \prod_{j=1}^{m+1} (1 + x^j u),$$

which, by Theorem 3.3, implies that

$$G(1) = \sum_{m \geq 0} \frac{x^{\binom{m+2}{2}} x^{-\binom{m+1}{2}} \prod_{j=1}^{m+1}(1 + x^j)}{\prod_{i=1}^{m+1}(1 - x^{2i})} = \sum_{m \geq 0} \frac{x^{m+1}}{\prod_{i=1}^{m+1}(1 - x^i)} \\ = \frac{1}{\prod_{i \geq 1}(1 - x^i)} - 1,$$

which is the generating function for the number of all nonempty partitions of n.

3.1. The total number of $(0,0)$ parity-descents. In this section we find the total number of $(0,0)$ parity-descents. Our results rely on Theorem 3.3.

LEMMA 3.5. *Define*

$$u_m^{00} = \frac{d}{dz_{00}} U_m(a(u), b(u) \mid z_{00}, 1, 1, 1)\Big|_{z_{00}=1},$$
$$v_m^{00} = \frac{d}{dz_{00}} V_m(a(u), b(u) \mid z_{00}, 1, 1, 1)\Big|_{z_{00}=1},$$
$$u_m'^{00} = \frac{d}{dz_{00}} U_m(b(u), a(u) \mid z_{00}, 1, 1, 1)\Big|_{z_{00}=1},$$
$$v_m'^{00} = \frac{d}{dz_{00}} V_m(b(u), a(u) \mid z_{00}, 1, 1, 1)\Big|_{z_{00}=1}.$$

Then

$$u_m^{00} + v_m^{00} = \frac{1}{2}\sum_{j=0}^{m} \frac{x^{2j+2}u^2}{1+x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}+1\right) + \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}+1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}-1\right)\right),$$
$$u_m^{00} - v_m^{00} = \frac{1}{2}\sum_{j=0}^{m} \frac{x^{2j+2}u^2}{1-x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}-1\right) + \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}-1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}+1\right)\right),$$
$$u_m'^{00} + v_m'^{00} = \frac{1}{2}\sum_{j=0}^{m} \frac{x^{2j+2}u^2}{1+x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}+1\right) - \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}+1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}-1\right)\right),$$
$$u_m'^{00} - v_m'^{00} = \frac{1}{2}\sum_{j=0}^{m} \frac{x^{2j+2}u^2}{1-x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}-1\right) - \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}-1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}+1\right)\right).$$

Proof. From Example 3.4, we know that

$$u_m = \frac{1}{2}\left(\prod_{j=1}^{m+1}\left(\frac{1}{x^j u}+1\right) + \prod_{j=1}^{m+1}\left(\frac{1}{x^j u}-1\right)\right).$$

By Lemma 3.2 with

$$a(u) = \frac{1+(z_{00}-1)x^2u^2}{xu} \quad \text{and} \quad b(u) = \frac{1+(z_{11}-1)x^2u^2}{xu},$$

we have

$$u_m^{00} = x^{m+1}u \cdot u_{m-1} + \frac{1}{x^{m+1}u}u_{m-1}^{00} + v_{m-1}^{00}, \qquad v_m^{00} = u_{m-1}^{00} + \frac{1}{x^{m+1}u}v_{m-1}^{00}$$

with $u_m^{00} = xu$ and $v_0^{00} = 0$. Thus,

$$u_m^{00} + v_m^{00} = \left(\frac{1}{x^{m+1}u} + 1\right)(u_{m-1}^{00} + v_{m-1}^{00}) + x^{m+1}u \cdot u_{m-1},$$
$$u_m^{00} - v_m^{00} = \left(\frac{1}{x^{m+1}u} - 1\right)(u_{m-1}^{00} - v_{m-1}^{00}) + x^{m+1}u \cdot u_{m-1}.$$

Hence, by induction on m, we obtain the stated formulas for $u_m^{00} + v_m^{00}$ and $u_m^{00} - v_m^{00}$.

Also by Example 3.4, we have

$$v_m = \frac{1}{2}\bigg(\prod_{j=1}^{m+1}\left(\frac{1}{x^j u} + 1\right) - \prod_{j=1}^{m+1}\left(\frac{1}{x^j u} - 1\right)\bigg).$$

Again, by Lemma 3.2 with the same $a(u)$ and $b(u)$, we have

$$u_m'^{00} = \frac{1}{x^{m+1}u}u_{m-1}'^{00} + v_{m-1}'^{00}, \qquad v_m'^{00} = u_{m-1}'^{00} + \frac{1}{x^{m+1}u}v_{m-1}'^{00} + x^{m+1}u \cdot v_{m-1},$$

with $u_m^{00} = xu$ and $v_0^{00} = 0$. Thus,

$$u_m'^{00} + u_m'^{00} = \left(\frac{1}{x^{m+1}u} + 1\right)(u_{m-1}'^{00} + v_{m-1}'^{00}) + x^{m+1}u \cdot v_{m-1},$$
$$u_m'^{00} - v_m'^{00} = \left(\frac{1}{x^{m+1}u} - 1\right)(u_{m-1}'^{00} - v_{m-1}'^{00}) + x^{m+1}u \cdot v_{m-1}.$$

Hence, by induction on m, we obtain the required formulas for $u_m'^{00} + v_m'^{00}$ and $u_m'^{00} - v_m'^{00}$. ∎

THEOREM 3.6. *The generating function* $\frac{d}{dz_{00}}G(u)\big|_{z_{00}=z_{01}=z_{10}=z_{11}=1}$ *is given by*

$$\frac{1}{2}\sum_{m\geq 2}\frac{x^m u}{\prod_{i=1}^m(1-x^i u)}\sum_{j=1}^{m-1}\frac{x^{2j}u^2}{1+x^j u}\bigg(1-\prod_{i=j+1}^{m}\frac{1-x^i u}{1+x^i u}\bigg).$$

Proof. By Theorem 3.3 and using the notation of Lemma 3.5, we find that the generating function $A_{00} = \frac{d}{dz_{00}}G(u)\big|_{z_{00}=z_{01}=z_{10}=z_{11}=1}$ is given by

$$\begin{aligned}A_{00} &= \sum_{m\geq 0}\frac{x^{\binom{m+2}{2}}u^{m+1}\big(x^{m+1}u(u_{m-1}^{00} + v_{m-1}'^{00}) + u_{m-1}'^{00} + v_{m-1}^{00}\big)}{\prod_{i=1}^{m+1}(1-x^{2i}u^2)}\\ &= \sum_{m\geq 2}\frac{x^m u}{2\prod_{i=1}^m(1-x^i u)}\sum_{j=1}^{m-1}\frac{x^{2j}u^2}{1+x^j u}\bigg(1-\prod_{i=j+1}^{m}\frac{1-x^i u}{1+x^i u}\bigg),\end{aligned}$$

as claimed. ∎

3.2. The total number of $(0,1)$ parity-descents. In this section we find the total number of $(0,1)$ parity-descents. By using similar arguments to those in the proof of Lemma 3.5 and Theorem 3.6, we obtain the following results.

LEMMA 3.7. *Define*

$$\begin{aligned}
u_m^{01} &= \frac{d}{dz_{01}} U_m(a(u), b(u) \mid 1, z_{01}, 1, 1)\bigg|_{z_{01}=1},\\
v_m^{01} &= \frac{d}{dz_{01}} V_m(a(u), b(u) \mid 1, z_{01}, 1, 1)\bigg|_{z_{01}=1},\\
u_m'^{01} &= \frac{d}{dz_{01}} U_m(b(u), a(u) \mid 1, z_{01}, 1, 1)\bigg|_{z_{01}=1},\\
v_m'^{01} &= \frac{d}{dz_{01}} V_m(b(u), a(u) \mid 1, z_{01}, 1, 1)\bigg|_{z_{01}=1}.
\end{aligned}$$

Then

$$\begin{aligned}
v_m^{01} + u_m^{01} &= \frac{1}{2}\sum_{j=0}^{m} \frac{x^{j+1}u}{1+x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}+1\right) + \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}+1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}-1\right)\right),\\
v_m^{01} - u_m^{01} &= \frac{1}{2}\sum_{j=0}^{m} \frac{x^{j+1}u}{1-x^{j+1}u}\left(\prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}-1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}+1\right) + \prod_{i=1}^{m+1}\left(\frac{1}{x^i u}-1\right)\right),\\
u_m'^{01} + v_m'^{01} &= \frac{1}{2}\sum_{j=0}^{m} \frac{x^{j+1}u}{1+x^{j+1}u}\left(\prod_{i=1}^{m+1}\left(\frac{1}{x^i u}+1\right) - \prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}+1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}-1\right)\right),\\
u_m'^{01} - v_m'^{01} &= \frac{1}{2}\sum_{j=0}^{m} \frac{x^{j+1}u}{1-x^{j+1}u}\left(\prod_{i=j+1}^{m+1}\left(\frac{1}{x^i u}-1\right)\prod_{i=1}^{j}\left(\frac{1}{x^i u}+1\right) - \prod_{i=1}^{m+1}\left(\frac{1}{x^i u}-1\right)\right).
\end{aligned}$$

THEOREM 3.8. *The generating function* $\frac{d}{dz_{01}}G(u)\big|_{z_{00}=z_{01}=z_{10}=z_{11}=1}$ *is given by*

$$\frac{1}{2}\sum_{m\geq 2} \frac{x^m u}{\prod_{i=1}^{m}(1-x^i u)} \sum_{j=1}^{m-1} \frac{x^j u}{1+x^j u}\left(1 + \prod_{i=j+1}^{m} \frac{1-x^i u}{1+x^i u}\right).$$

The generating function for the total numbers of $(1,0)$ and $(1,1)$ parity-descents may be obtained in a manner analogous to the foregoing cases. The corresponding results are stated below.

THEOREM 3.9. *We have*

$$\begin{aligned}
&\frac{d}{dz_{10}}G(u)\bigg|_{z_{00}=z_{01}=z_{10}=z_{11}=1}\\
&\qquad = \frac{1}{2}\sum_{m\geq 2} \frac{x^m u}{\prod_{i=1}^{m}(1-x^i u)} \sum_{j=1}^{m-1} \frac{x^j u}{1+x^j u}\left(1 - \prod_{i=j+1}^{m} \frac{1-x^i u}{1+x^i u}\right),
\end{aligned}$$

$$\frac{d}{dz_{11}}G(u)\bigg|_{z_{00}=z_{01}=z_{10}=z_{11}=1} = \frac{1}{2}\sum_{m\geq 2}\frac{x^m u}{\prod_{i=1}^{m}(1-x^i u)}\sum_{j=1}^{m-1}\frac{x^{2j}u^2}{1+x^j u}\bigg(1+\prod_{i=j+1}^{m}\frac{1-x^i u}{1+x^i u}\bigg).$$

4. Asymptotics. In this section we consider the behavior of the average number of occurrences of each statistic for large values of n.

THEOREM 4.1. (i) *The average number of* $(0,0)$ *parity-levels in a random partition of* n *is given by*

$$\frac{\sqrt{6n}}{4\pi}\bigg(\log\Big(\frac{3n}{2\pi^2}\Big)+2\gamma-2\bigg)+\frac{3}{4\pi^2}\bigg(\pi^2+1+\log\Big(\frac{6n}{\pi^2}\Big)\bigg)+O\Big(\frac{\log n}{n}\Big).$$

(ii) *The average number of* $(1,1)$ *parity-levels in a random partition of* n *is given by*

$$\frac{\sqrt{6n}}{2\pi}\bigg(\log\Big(\frac{12n}{\pi^2}\Big)+\gamma-1\bigg)+\frac{3}{4\pi^2}\bigg(1+\log\Big(\frac{6n}{\pi^2}\Big)\bigg)+O\Big(\frac{\log n}{n}\Big).$$

Proof. (i) By Theorem 2.10 we have

$$\frac{d}{dy_{00}}F(1\,|\,y_{00},y_{11})\bigg|_{y_{00}=y_{11}=1}=\prod_{n=1}^{\infty}\frac{1}{1-x^n}\sum_{n=1}^{\infty}\frac{x^{4n}}{1-x^{2n}}. \tag{6}$$

Thus we can set $F(x)=\sum_{j\geq 1}\frac{x^{4j}}{1-x^{2j}}$. So

$$F(e^{-t})=\sum_{j\geq 1}\Big(\frac{1}{e^{2jt}-1}-\frac{1}{e^{2jt}}\Big)=\frac{1}{1-e^{2t}}+\sum_{j\geq 1}\frac{1}{e^{2jt}-1}.$$

Note that the Mellin transform of this function is easily found to be $\frac{1}{2^s}\zeta^2(s)\Gamma(s)$, where $\zeta(s)$ is the zeta function and $\Gamma(s)$ is the gamma function (see [5]). So with the help of the Mellin inversion formula (see [5]), the asymptotic expansion around $t=0$ is given by

$$\sum_{j\geq 1}\frac{1}{e^{2jt}-1}=\frac{\log(1/(2t))+\gamma}{2t}+\frac{1}{4}-\frac{t}{72}+O(t^3),$$

where γ is the Euler constant. Note that $\frac{1}{1-e^{2t}}=\frac{-1}{2t}+\frac{1}{2}-\frac{t}{6}+O(t^3)$. Thus, the asymptotic expansion of $F(e^{-t})$ around $t=0$ is

$$F(e^{-t})=\frac{\log(1/t)}{2t}+\frac{\gamma-1-\log 2}{2t}+\frac{3}{4}-\frac{13t}{72}+O(t^3).$$

Hence, by [7, Theorem 2.3] with $a=1/2$ and $b=-1$, we complete the proof.

(ii) To prove the second part we observe that Theorem 2.10 implies

$$\frac{d}{dy_{11}}F(1\,|\,y_{00},y_{11})\bigg|_{y_{00}=y_{11}=1}=\prod_{n=1}^{\infty}\frac{1}{1-x^n}\sum_{n=1}^{\infty}\frac{x^{4n-2}}{1-x^{2n-1}}. \tag{7}$$

Thus we may set $F(x) = \sum_{j\ge 1} \frac{x^{4j-2}}{1-x^{2j-1}}$ and obtain (ii) using a similar argument to the proof of (i). ■

We remark that we are presently unable to use our techniques to obtain the average number of (s,t) parity-descents in a random partition of n for all $(a,b) \in \{0,1\}^2$.

5. Series-product identities. We give two combinatorial identities arising naturally from previous sections without further work. It might be interesting to discover direct proofs of these identities.

The first identity below follows from (6) and Theorem 2.14 or (7), while the second identity is a consequence of Theorem 2.16.

Theorem 5.1. *We have*

$$\frac{1}{2}\sum_{m\ge 1} \frac{x^{m+1}}{\prod_{i=1}^{m+1}(1-x^i)} \sum_{j=1}^{m} (1-x^j)\left(1 - \frac{\prod_{i=j+1}^{m+1}(1-x^i)}{\prod_{i=j+1}^{m+1}(1+x^i)}\right) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \sum_{n=1}^{\infty} \frac{x^{4n}}{1-x^{2n}},$$

$$\frac{1}{2}\sum_{m\ge 1} \frac{x^{m+1}}{\prod_{j=1}^{m+1}(1-x^j)} \sum_{j=1}^{m} (1-x^j)\left(1 + \frac{\prod_{i=j+1}^{m+1}(1-x^i)}{\prod_{i=j+1}^{m+1}(1+x^i)}\right) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \sum_{n=1}^{\infty} \frac{x^{4n-2}}{1-x^{2n-1}}.$$

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