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LEFT GENERAL FRACTIONAL MONOTONE APPROXIMATION THEORY

Abstract. We introduce left general fractional Caputo style derivatives with respect to an absolutely continuous strictly increasing function g. We give various examples of such fractional derivatives for different g. Let f be a p-times continuously differentiable function on $[a, b]$, and let L be a linear left general fractional differential operator such that $L(f)$ is non-negative over a closed subinterval I of $[a, b]$. We find a sequence of polynomials Q_n of degree $\leq n$ such that $L(Q_n)$ is non-negative over I, and furthermore f is approximated uniformly by Q_n over $[a, b]$.

The degree of this constrained approximation is given by an inequality using the first modulus of continuity of $f^{(p)}$. We finish with applications of the main fractional monotone approximation theorem for different g. On the way to proving the main theorem we establish useful related general results.

1. Introduction and preparation. The topic of monotone approximation started in [10] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k.

Furthermore in [1], the author generalized the result of [2] to linear fractional differential operators.

To describe the motivating result here we need

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Definition 1 ([4, p. 50]). Let $\alpha > 0$ with $\lceil \alpha \rceil = m$ ($\lceil \cdot \rceil$ is the ceiling of the number). Consider $f \in C^m([-1,1])$. We define the *left Caputo fractional derivative* of f of order α as follows:

$$(D_{*-1}^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt$$

for any $x \in [-1,1]$, where Γ is the gamma function, $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\, dt$, $\nu > 0$. We set

$$D_{*-1}^{0} f(x) = f(x),$$
$$D_{*-1}^{m} f(x) = f^{(m)}(x), \quad \forall x \in [-1,1].$$

We proved

Theorem 2 ([1]). *Let $0 \le h \le k \le p$ be integers, and let f be a real function such that $f^{(p)}$ continuous on $[-1,1]$ with modulus of continuity $\omega_1(f^{(p)}, \delta)$, $\delta > 0$, there. Let $\alpha_j(x)$, $j = h, h+1, \ldots, k$, be real functions, defined and bounded on $[-1,1]$, and assume that for all $x \in [0,1]$ either $\alpha_h(x) \ge \alpha > 0$ or $\alpha(x) \le \beta < 0$. Let $\alpha_0 = 0 < \alpha_1 \le 1 < \alpha_2 \le 2 < \cdots < \alpha_p \le p$ be real numbers. Let $D_{*-1}^{\alpha_j} f$ stand for the left Caputo fractional derivative of f of order α_j anchored at -1. Consider the linear left fractional differential operator*

$$L := \sum_{j=h}^{k} \alpha_j(x) [D_{*-1}^{\alpha_j}]$$

and suppose that, throughout $[0,1]$,

$$L(f) \ge 0. \tag{1}$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\le n$ such that

$$L(Q_n) \ge 0 \quad \textit{throughout } [0,1], \tag{2}$$

and

$$\max_{-1 \le x \le 1} |f(x) - Q_n(x)| \le C n^{k-p} \omega_1(f^{(p)}, 1/n), \tag{3}$$

where C is independent of n and f.

Notice that the monotonicity property is only true on $[0,1]$: see (1), (2). However the approximation property (3) holds over the whole interval $[-1,1]$.

In this article we extend Theorem 2 to much more general linear left fractional differential operators.

We use the following generalized fractional integral.

DEFINITION 3 (see also [7, p. 99]). The *left generalized fractional integral* of a function f with respect to a given function g is defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, and $\alpha > 0$. Assume that $g \in AC([a,b])$ (absolutely continuous functions) is strictly increasing, and $f \in L_\infty([a,b])$. We set

$$(I^\alpha_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t)\, dt, \qquad x \geq a;$$

clearly $(I^\alpha_{a+;g} f)(a) = 0$. When g is the identity function id, we get $I^\alpha_{a+;\mathrm{id}} = I^\alpha_{a+}$, the ordinary left Riemann–Liouville fractional integral, where

$$(I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt, \qquad x \geq a,$$

with $(I^\alpha_{a+} f)(a) = 0$.

When $g(x) = \ln x$ on $[a,b]$, $0 < a < b < \infty$, we get

DEFINITION 4 ([7, p. 110]). Let $0 < a < b < \infty$ and $\alpha > 0$. The *left Hadamard fractional integral* of order α of $f \in L_\infty([a,b])$ is given by

$$(J^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y}\right)^{\alpha-1} \frac{f(y)}{y}\, dy, \qquad x \geq a.$$

DEFINITION 5. The *left fractional exponential integral* of $f \in L_\infty([a,b])$ is defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, and $\alpha > 0$. We set

$$(I^\alpha_{a+;e^x} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (e^x - e^t)^{\alpha-1} e^t f(t)\, dt, \qquad x \geq a.$$

DEFINITION 6. Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$, and $A > 1$. We introduce the fractional integral

$$(I^\alpha_{a+;A^x} f)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_a^x (A^x - A^t)^{\alpha-1} A^t f(t)\, dt, \qquad x \geq a.$$

DEFINITION 7. Let $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, and $f \in L_\infty([a,b])$. We set

$$(K^\alpha_{a+;x^\sigma} f)(x) = \frac{1}{\Gamma(\alpha)} \int_z^x (x^\sigma - t^\sigma)^{\alpha-1} f(t) \sigma t^{\sigma-1}\, dt, \qquad x \geq a.$$

We introduce the following general fractional derivatives.

DEFINITION 8. Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([a,b])$ (the space of functions f with $f^{(m-1)} \in AC([a,b])$). We define the *left general fractional derivative* of f of order α with respect to g as follows:

$$(D^\alpha_{*a;g} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) f^{(m)}(t)\, dt$$

for any $x \in [a,b]$, where Γ is the gamma function.

We set

$$D^m_{*\alpha;g}f(x)=f^{(m)}(x),$$
$$D^0_{*a;g}f(x)=f(x),\qquad \forall x\in[a,b].$$

If $g=\mathrm{id}$, then $D^\alpha_{*a}f=D^\alpha_{*a;\mathrm{id}}f$ is the left Caputo fractional derivative.

So we have the specific general left fractional derivatives.

Definition 9.

$$D^\alpha_{*a;\ln x}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x\left(\ln\frac{x}{y}\right)^{m-\alpha-1}\frac{f^{(m)}(y)}{y}\,dy,\qquad x\ge a>0,$$

$$D^\alpha_{*a;e^x}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x(e^x-e^t)^{m-\alpha-1}e^tf^{(m)}(t)\,dt,\qquad x\ge a,$$

$$D^\alpha_{*a;A^x}f(x)=\frac{\ln A}{\Gamma(m-\alpha)}\int_a^x(A^x-A^t)^{m-\alpha-1}A^tf^{(m)}(t)\,dt,\qquad x\ge a,$$

$$(D^\alpha_{*a;x^\sigma}f)(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x^\sigma-t^\sigma)^{m-\alpha-1}\sigma t^{\sigma-1}f^{(m)}(t)\,dt,\qquad x\ge a\ge 0.$$

We need a modification of

Theorem 10 (Trigub, [11], [12]). *Let $g\in C^p([-1,1])$, $p\in\mathbb{N}$. Then there exists a real polynomial $q_n(x)$ of degree $\le n$ such that*

$$\max_{-1\le x\le 1}|g^{(j)}(x)-q_n^{(j)}(x)|\le R_pn^{j-p}\omega_1(g^{(p)},1/n),$$

$j=0,1,\dots,p$, where R_p is independent of n and g.

Remark 11. Let $a<b$. Let $\varphi:[-1,1]\to[a,b]$ be defined by

$$x=\varphi(t)=\frac{b-a}{2}t+\frac{b+a}{2}.$$

Clearly φ is a 1-1 and onto map. We get

$$x'=\varphi'(t)=\frac{b-a}{2},$$

and

$$t=\frac{2x-b-a}{b-a}=2\frac{x}{b-a}-\frac{b+a}{b-a}. \tag{4}$$

In fact,

$$\varphi(-1)=a\quad\text{and}\quad\varphi(1)=b.$$

Theorem 12. *Let $f\in C^p([a,b])$, $p\in\mathbb{N}$. Then there exist real polynomials $Q_n^*(x)$ of degree $\le n\in\mathbb{N}$ such that*

$$\max_{a\le x\le b}|f^{(j)}(x)-Q_n^{*(j)}(x)|\le R_p\left(\frac{b-a}{2n}\right)^{p-j}\omega_1\left(f^{(p)},\frac{b-a}{2n}\right), \tag{5}$$

$j=0,1,\dots,p$, where R_p is independent of n and g.

Proof. We use Theorem 10 and Remark 11.

Since $f \in C^p([a,b])$, it is clear that the function

$$g(t) = f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right), \qquad t \in [-1,1],$$

is in $C^p([-1,1])$. We notice that

$$\frac{dg(t)}{dt} = \frac{df\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)}{dt} = f'(x)\frac{b-a}{2},$$

and thus

$$g'(t) = f'(x)\frac{b-a}{2} = f'\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}.$$

Moreover

$$g''(t) = \frac{df'\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)}{dt}\,\frac{b-a}{2}.$$

Since as before

$$\frac{df'\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)}{dt} = f''(x)\frac{b-a}{2},$$

we obtain

$$g''(x) = f''(x)\frac{(b-a)^2}{2^2}.$$

In general,

$$g^{(j)}(t) = f^{(j)}(x)\frac{(b-a)^j}{2^j}$$

for $j = 0, 1, \ldots, p$. Hence by Theorem 10, for any $t \in [-1,1]$, we have

$$|g^{(j)}(t) - q_n^{(j)}(t)| \le R_p n^{j-p} \omega_1(g^{(p)}, 1/n) \tag{6}$$

for $j = 0, 1, \ldots, p$, where R_p is independent of n and g.

Notice that

$$q_n^{(j)}(t) \overset{(4)}{=} q_n^{(j)}\left(\frac{2x-b-a}{b-a}\right), \qquad j = 0, 1, \ldots, p.$$

So, for $t \in [-1,1]$, we have

$$q_n(t) = q_n\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right) =: Q_n^*(x), \qquad x \in [a,b],$$

a polynomial of degree n.

Also

$$Q_n^{*\prime}(x) = \frac{dq_n\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)}{dx} = \frac{dq_n(t)}{dt}\,\frac{dt}{dx} = q_n'(t)\frac{2}{b-a}. \tag{7}$$

That is,

$$q_n'(t) = Q_n^{*\prime}(x)\frac{b-a}{2}.$$

Similarly we get

$$Q_n^{*\prime\prime}(x) = \frac{dQ_n^{*\prime}(x)}{dx} \overset{(7)}{=} \frac{dq_n'\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)}{dx} \frac{2}{b-a}$$
$$= \frac{dq_n'(t)}{dt} \frac{dt}{dx} \frac{2}{b-a} = q_n''(t) \frac{2^2}{(b-a)^2}.$$

Hence

$$q_n''(t) = Q_n^{*\prime\prime}(x) \frac{(b-a)^2}{2^2}.$$

In general,

$$q_n^{(j)}(t) = Q_n^{*(j)}(x) \frac{(b-a)^j}{2^j}, \qquad j = 0, 1, \ldots, p.$$

Thus we have

$$\text{L.H.S.}(6) = \frac{(b-a)^j}{2^j} |f^{(j)}(x) - Q_n^{*(j)}(x)|$$

for $j = 0, 1, \ldots, p$ and $x \in [a, b]$.

Next we observe that

$$(8) \qquad \omega_1(g^{(p)}, 1/n) = \sup_{\substack{|t_1 - t_2| \le 1/n \\ t_1, t_2 \in [-1,1]}} |g^{(p)}(t_1) - g^{(p)}(t_2)|$$
$$\sup_{\substack{|x_1 - x_2| \le \frac{b-a}{2n} \\ x_1, x_2 \in [a,b]}} \frac{(b-a)^p}{2^p} |f^{(p)}(x_1) - f^{(p)}(x_2)| = \frac{(b-a)^p}{2^p} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right),$$

since for any t_1, $t_2 \in [-1, 1]$ with $|t_1 - t_2| \le 1/n$ the corresponding $x_1, x_2 \in [a, b]$ satisfy

$$|x_1 - x_2| \le \frac{b-a}{2n}.$$

Finally, by (6) we can find

$$\frac{(b-a)^j}{2^j} |f^{(j)}(x) - Q_n^{*(j)}(x)| \le R_p n^{j-p} \frac{(b-a)^p}{2^p} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)$$

for $j = 0, 1, \ldots, p$, and so

$$|f^{(j)}(x) - Q_n^{*(j)}(x)| \le R_p \frac{(b-a)^{p-j}}{(2n)^{p-j}} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right),$$

for any $x \in [a, b]$ and $j = 0, 1, \ldots, p$, proving the claim. ∎

Remark 13. Let $g \in AC([a, b])$ be increasing over $[a, b]$, and let $\alpha > 0$. Suppose $g(a) = c$, $g(b) = d$. We want to calculate

$$I = \int_a^b (g(b) - g(t))^{\alpha-1} g'(t)\, dt.$$

Consider the function

$$f(y) = (g(b) - y)^{\alpha-1} = (d - y)^{\alpha-1}, \quad \forall y \in [c, d].$$

We have $f(y) \geq 0$, $f(d) = \infty$ when $0 < \alpha < 1$, but f is measurable on $[c, d]$. By [8, exercise 13d, p. 107],

$$(f \circ g)(t)g'(t) = (g(b) - g(t))^{\alpha-1} g'(t)$$

is measurable on $[a, b]$, and

$$I = \int_c^d (d - y)^{\alpha-1}\, dy = \frac{(d - c)^\alpha}{\alpha}$$

(notice that $(d - y)^{\alpha-1}$ is Riemann integrable). That is,

$$I = \frac{(g(b) - g(a))^\alpha}{\alpha}.$$

Similarly,

$$\int_a^x (g(x) - g(t))^{\alpha-1} g'(t)\, dt = \frac{(g(x) - g(a))^\alpha}{\alpha}, \quad \forall x \in [a, b]. \tag{9}$$

THEOREM 14. *Let $\alpha > 0$, $\mathbb{N} \ni m = \lceil \alpha \rceil$, and $f \in C^m([a, b])$. Then $(D^\alpha_{*a;g} f)(x)$ is continuous in $x \in [a, b]$.*

Proof. By [3, p. 78], we know that g^{-1} exists and is strictly increasing on $[g(a), g(b)]$. Since g is continuous on $[a, b]$, so is g^{-1} on $[g(a), g(b)]$. Hence $f^{(m)} \circ g^{-1}$ is a continuous function on $[g(a), g(b)]$.

If $\alpha = m \in \mathbb{N}$, then the claim is trivial.

We treat the case of $0 < \alpha < m$. The function

$$G(z) = (g(x) - z)^{m-\alpha-1} (f^{(m)} \circ g^{-1})(z)$$

is integrable on $[g(a), g(x)]$, and by assumption $g : [a, b] \to [g(a), g(b)]$ is absolutely continuous.

Since g is strictly increasing, the function

$$(g(x) - g(t))^{m-\alpha-1} g'(t) (f^{(m)} \circ g^{-1})(g(t))$$

is integrable on $[a, x]$ (see [6]). Furthermore (see also [6]),

$$\begin{aligned}
&\frac{1}{\Gamma(m - \alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{m-\alpha-1} (f^{(m)} \circ g^{-1})(z)\, dz \\
&\qquad = \frac{1}{\Gamma(m - \alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) (f^{(m)} \circ g^{-1})(g(t))\, dt \\
&\qquad = (D^\alpha_{*a;g} f)(x), \quad \forall x \in [a, b].
\end{aligned}$$

And we can write

$$(D^{\alpha}_{*a;g}f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{g(a)}^{g(x)} (g(x)-z)^{m-\alpha-1}(f^{(m)} \circ g^{-1})(z)\,dz,$$

$$(D^{\alpha}_{*a;g}f)(y) = \frac{1}{\Gamma(m-\alpha)} \int_{g(a)}^{g(y)} (g(y)-z)^{m-\alpha-1}(f^{(m)} \circ g^{-1})(z)\,dz.$$

Here $a \le x \le y \le b$, and $g(a) \le g(x) \le g(y) \le g(b)$, and $0 \le g(x) - g(a) \le g(y) - g(a)$.

Let $\lambda = g(x) - z$; then $z = g(x) - \lambda$. Thus

$$(D^{\alpha}_{*a;g}f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{g(x)-g(a)} \lambda^{m-\alpha-1}(f^{(m)} \circ g^{-1})(g(x)-\lambda)\,d\lambda.$$

Clearly, if $g(a) \le z \le g(x)$, then $-g(a) \ge -z \ge -g(x)$, and $g(x) - g(a) \ge g(x) - z \ge 0$, i.e. $0 \le \lambda \le g(x) - g(a)$.

Similarly

$$(D^{\alpha}_{*a;g}f)(y) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{g(y)-g(a)} \lambda^{m-\alpha-1}(f^{(m)} \circ g^{-1})(g(y)-\lambda)\,d\lambda. \tag{10}$$

Hence

$$(D^{\alpha}_{*a;g}f)(y) - (D^{\alpha}_{*a;g}f)(x) = \frac{1}{\Gamma(m-\alpha)} \cdot \Big[\int_{0}^{g(x)-g(y)} \lambda^{m-\alpha-1}\big((f^{(m)} \circ g^{-1})(g(y)-\lambda) - (f^{(m)} \circ g^{-1})(g(x)-\lambda)\big)\,d\lambda + \int_{g(x)-g(a)}^{g(y)-g(a)} \lambda^{m-\alpha-1}(f^{(m)} \circ g^{-1})(g(y)-\lambda)d\lambda\Big].$$

Thus we obtain

$$|(D^{\alpha}_{*a;g}f)(y) - (D^{\alpha}_{*a;g}f)(x)| \le \frac{1}{\Gamma(m-\alpha)} \cdot \Big[\frac{(g(x)-g(a))^{m-\alpha}}{m-\alpha}\omega_1(f^{(m)} \circ g^{-1}, |g(y)-g(x)|) + \frac{\|f^{(m)} \circ g^{-1}\|_{\infty,[g(a),g(b)]}}{m-\alpha}((g(y)-g(a))^{m-\alpha} - (g(x)-g(a))^{m-\alpha})\Big] =: (\xi).$$

As $y \to x$, we have $g(y) \to g(x)$ (since $g \in AC([a,b])$). So $(\xi) \to 0$. As a result

$$(D^{\alpha}_{*a;g}f)(y) \to (D^{\alpha}_{*a;g}f)(x),$$

proving that $(D^{\alpha}_{*a;g}f)(x)$ is continuous in $x \in [a,b]$. ∎

2. Main result. We will prove

THEOREM 15. *Assume that $g \in AC([a,b])$ is strictly increasing with $g(b) - g(a) > 1$. Let $0 \le h \le k \le p$ be integers, and let $f \in C^p([a,b])$, $a < b$, with modulus of continuity $\omega_1(f^{(p)}, \delta)$, $0 < \delta \le b - a$. Let $\alpha_j(x)$, $j = h, h+1, \dots, k$, be real functions, defined and bounded on $[a,b]$ and assume that, for all $x \in [g^{-1}(1+g(a)), b]$, either $\alpha_h(x) \ge \alpha^* > 0$, or $\alpha_n(x) \le \beta^* < 0$. Let $\alpha_0 = 0 < \alpha_1 \le 1 < \alpha_2 \le 2 < \cdots < \alpha_p \le p$ be real numbers. Consider the linear left general fractional differential operator*

$$L = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*a;g}],$$

and suppose that throughout $[g^{-1}(1+g(a)), b]$,

$$L(f) \ge 0.$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\le n$ such that

$$L(Q_n) \ge 0 \quad \text{throughout } [g^{-1}(1+g(a)), b],$$

and

$$\max_{x\in[a,b]} |f(x) - Q_n(x)| \le C n^{k-p} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \tag{11}$$

where C is independent of n and f.

Proof. Let $Q_n^*(x)$ be as in Theorem 12.

We have

$$(D^{\alpha_j}_{*a;g} f)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t) f^{(j)}(t)\,dt,$$

$$(D^{\alpha_j}_{*a;g} Q_n^*)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t) Q_n^{*(j)}(t)\,dt,$$

for $j = 1, \dots, p$.

Also

$$(D^{j}_{*a;g} f)(x) = f^{(j)}(x), \quad (D^{j}_{*a;g} Q_n^*)(x) = Q_n^{*(j)}(x), \quad j = 1, \dots, p.$$

By [9], g' exists a.e., and g' is measurable and non-negative.

We notice that

$$|(D^{\alpha_j}_{*a;g} f)(x) - D^{\alpha_j}_{*a;g} Q_n^*(x)|$$
$$= \frac{1}{\Gamma(j-\alpha_j)} \left| \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t)(f^{(j)}(t) - Q_n^{*(j)}(t))\,dt \right|$$

$$\le \frac{1}{\Gamma(j-\alpha_j)} \int_a^x (g(x)-g(t))^{j-\alpha_j-1} g'(t) |f^{(j)}(t) - Q_n^{*(j)}(t)|\, dt$$

$$\overset{(5)}{\le} \frac{1}{\Gamma(j-\alpha_j)} \Big(\int_a^x (g(x)-g(t))^{j-\alpha_j-1} g'(t)\, dt \Big) R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big)$$

$$\overset{(9)}{=} \frac{(g(x)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big)$$

$$\le \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big).$$

Hence for all $x \in [a,b]$,

$$|(D_{*a;g}^{\alpha_j} f)(x) - D_{*a;g}^{\alpha_j} Q_n^*(x)| \le \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big)$$

and

$$\max_{x\in[a,b]} |D_{*a;g}^{\alpha_j} f(x) - D_{*a;g}^{\alpha_j} Q_n^*(x)| \le \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big), \tag{12}$$

for $j = 0, 1, \ldots, p$.

Above we set $D_{*a;g}^{0} f(x) = f(x)$ and $D_{*a;g}^{0} Q_n^*(x) = Q_n^*(x)$, for all $x \in [a,b]$, and $\alpha_0 = 0$, i.e. $\lceil \alpha_0 \rceil = 0$.

Define

$$s_j = \sup_{a\le x\le b} |\alpha_h^{-1}(x)\alpha_j(x)|, \qquad j = h, \ldots, k,$$

and

$$\eta_n = R_p \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big) \sum_{j=h}^{k} s_j \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \Big(\frac{b-a}{2n}\Big)^{p-j}. \tag{13}$$

I. Suppose $\alpha_h(x) \ge \alpha^* > 0$ throughout $[g^{-1}(1+g(a)), b]$. Let $Q_n(x)$ be the real polynomial of degree $\le n$ that corresponds to $f(x) + \eta_n (h!)^{-1} x^h$, $x \in [a,b]$, so by Theorem 12 and (12) we get

$$\max_{x\in[a,b]} |D_{*a;g}^{\alpha_j} (f(x) + \eta_n (h!)^{-1} x^h) - (D_{*a;g}^{\alpha_j} Q_n)(x)| \le \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \Big(\frac{b-a}{2n}\Big)^{p-j} \omega_1\Big(f^{(p)}, \frac{b-a}{2n}\Big) \tag{14}$$

for $j = 0, 1, \ldots, p$.

In particular (for $j=0$)

(15) $$\max_{x\in[a,b]}|(f(x)+\eta_n(h!)^{-1}x^h)-Q_n(x)|\le R_p\left(\frac{b-a}{2n}\right)^p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right),$$

and

$$\begin{aligned}
&\max_{x\in[a,b]}|f(x)-Q_n(x)|\\
&\le \eta_n(h!)^{-1}(\max(|a|,|b|))^h+R_p\left(\frac{b-a}{2n}\right)^p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)\\
&=\eta_n(h!)^{-1}\max(|a|^h,|b|^h)+R_p\left(\frac{b-a}{2n}\right)^p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)\\
&=R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)\sum_{j=h}^k s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}\\
&\qquad\times(h!)^{-1}\max(|a|^h,|b|^h)+R_p\left(\frac{b-a}{2n}\right)^p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)\\
&\le R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)n^{k-p}\\
&\quad\times\left[\sum_{j=h}^k s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2}\right)^{p-j}(h!)^{-1}\max(|a|^h,|b|^h)+\left(\frac{b-a}{2}\right)^p\right].
\end{aligned}$$

proving (11).

Notice that for $j=h+1,\dots,k$,

(16) $$(D^{\alpha_j}_{*a;g}x^h)=\frac{1}{\Gamma(j-\alpha_j)}\int_a^x(g(x)-g(t))^{j-\alpha_j-1}g'(t)(t^h)^{(j)}\,dt=0.$$

Here

$$L=\sum_{j=h}^k\alpha_j(x)[D^{\alpha_j}_{*a;g}],$$

and suppose $Lf\ge 0$ throughout $[g^{-1}(1+g(a)),b]$. So over $g^{-1}(1+g(a))\le x\le b$, we get

$$\begin{aligned}
\alpha_h^{-1}(x)L(Q_n(x))&\overset{(16)}{=}\alpha_h^{-1}(x)L(f(x))+\frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h))\\
&\qquad+\sum_{j=h}^k\alpha_h^{-1}(x)\alpha_j(x)\left[D^{\alpha_j}_{*a;g}Q_n(x)-D^{\alpha_j}_{*a;g}f(x)-\frac{\eta_n}{h!}D^{\alpha_j}_{*a;g}x^h\right]\\
&\overset{(14)}{\ge}\frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h))-\sum_{j=h}^k s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)
\end{aligned}$$

$$\overset{(13)}{=} \frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h)) - \eta_n = \eta_n\left(\frac{D^{\alpha_h}_{*a;g}(x^h)}{h!} - 1\right)$$
$$= \eta_n\left(\frac{1}{\Gamma(h-\alpha_h)h!}\int_a^x (g(x)-g(t))^{h-\alpha_h-1}g'(t)(t^h)^{(h)}\,dt - 1\right)$$
$$= \eta_n\left(\frac{h!}{h!\Gamma(h-\alpha_h)}\int_a^x (g(x)-g(t))^{h-\alpha_h-1}g'(t)\,dt - 1\right)$$
$$\overset{(9)}{=} \eta_n\left(\frac{(g(x)-g(a))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1\right)$$
$$= \eta_n\left(\frac{(g(x)-g(a))^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right)$$
$$\geq \eta_n\left(\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right) \geq 0.$$

Clearly here $g(x)-g(a) \geq 1$.

Moreover, $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0 \leq h-\alpha_h < 1$ and $1 \leq h-\alpha_h+1 < 2$. Thus

$$\Gamma(h-\alpha_h+1) \leq 1 \quad \text{and} \quad 1-\Gamma(h-\alpha_h+1) \geq 0. \tag{17}$$

Hence

$$L(Q_n(x)) \geq 0 \qquad \text{for } x \in [g^{-1}(1+g(a)), b].$$

II. Suppose $\alpha_h(x) \leq \beta^* < 0$ throughout $[g^{-1}(1+g(a)), b]$.

Let $Q_n(x)$, $x \in [a,b]$, be a real polynomial of degree $\leq n$, according to Theorem 12 and (12), so that

$$\max_{x\in[a,b]} |D^{\alpha_j}_{*a;g}(f(x) - \eta_n(h!)^{-1}x^h) - (D^{\alpha_j}_{*a;g}Q_n)(x)| \leq \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p\left(\frac{b-a}{2n}\right)^{p-j}\omega_1\left(f^{(p)}, \frac{b-a}{2n}\right) \tag{18}$$

for $j = 0,1,\ldots,p$.

In particular (for $j=0$)

$$\max_{x\in[a,b]} |(f(x) - \eta_n(h!)^{-1}x^h) - Q_n(x)| \leq R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right),$$

and

$$\max_{x\in[a,b]} |f(x)-Q_n(x)| \leq \eta_n(h!)^{-1}(\max(|a|,|b|))^h + R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)$$
$$= \eta_n(h!)^{-1}\max(|a|^h,|b|^h) + R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right),$$

etc. We find again that

$$\max_{x\in[a,b]}|f(x)-Q_n(x)|\le R_p\Bigg[\left(\frac{b-a}{2}\right)^p+(h!)^{-1}\max(|a|^h,|b|^h)$$
$$\cdot\left(\sum_{j=h}^{k}s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2}\right)^{p-j}\right)\Bigg]n^{k-p}\omega_1\left(f^{(p)},\frac{b-a}{2n}\right),$$

reproving (11).

Here again

$$L=\sum_{j=h}^{k}\alpha_j(x)[D^{\alpha_j}_{*a;g}],$$

and suppose $Lf\ge 0$ throughout $[g^{-1}(1+g(a)),b]$. So over $g^{-1}(1+g(a))\le x\le b$, we get

$$\alpha_h^{-1}(x)L(Q_n(x))\overset{(16)}{=}\alpha_h^{-1}(x)L(f(x))-\frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h))$$
$$+\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[D^{\alpha_j}_{*a;g}Q_n(x)-D^{\alpha_j}_{*a;g}f(x)+\frac{\eta_n}{h!}D^{\alpha_j}_{*a;g}x^h\right]$$
$$\overset{(18)}{\le}-\frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h))$$
$$+\left(\sum_{j=h}^{k}s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}\right)R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)$$
$$\overset{(13)}{=}-\frac{\eta_n}{h!}(D^{\alpha_h}_{*a;g}(x^h))+\eta_n=\eta_n\left(1-\frac{D^{\alpha_h}_{*a;g}(x^h)}{h!}\right)$$
$$=\eta_n\left(1-\frac{1}{\Gamma(h-\alpha_h)h!}\int_a^x(g(x)-g(t))^{h-\alpha_h-1}g'(t)(t^h)^{(h)}\,dt\right)$$
$$=\eta_n\left(1-\frac{h!}{h!\Gamma(h-\alpha_h)}\int_a^x(g(x)-g(t))^{h-\alpha_h-1}g'(t)\,dt\right)$$
$$\overset{(9)}{=}\eta_n\left(1-\frac{(g(x)-g(a))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)$$
$$=\eta_n\left(\frac{\Gamma(h-\alpha_h+1)-(g(x)-g(a))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)$$
$$\overset{(17)}{\le}\eta_n\left(\frac{1-(g(x)-g(a))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\le 0.$$

Hence again

$$L(Q_n(x))\ge 0,\qquad \forall x\in[g^{-1}(1+g(a)),b].$$

The case of $\alpha_h=h$ is trivially deduced from the above. The proof of the theorem is now complete. ∎

REMARK 16. By Theorem 14, $D_{*a;g}^{\alpha_j} f$ are continuous functions, $j = 0, 1, \ldots, p$. Suppose that $\alpha_h(x), \ldots, \alpha_k(x)$ are continuous functions on $[a, b]$, and $L(f) \geq 0$ on $[g^{-1}(1+g(a)), b]$ is replaced by $L(f) > 0$ on $[g^{-1}(1+g(a)), b]$. Disregard the assumption made in the main theorem on $\alpha_h(x)$. For $n \in \mathbb{N}$, let $Q_n(x)$ be the $Q_n^*(x)$ of Theorem 12, and f as in Theorem 12 (same as in Theorem 15). Then $Q_n(x)$ converges to $f(x)$ at the Jackson rate $1/n^{p+1}$ ([5, p. 18, Theorem VIII]) and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[a, b]$, $L(Q_n) > 0$ on $[g^{-1}(1 + g(a)), b]$ for all n sufficiently large.

3. Applications (of Theorem 15)

1) When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Here we assume that $b > ae$, $\alpha_h(x)$ restriction true on $[ae, b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D_{*a;\ln x}^{\alpha_j} f] \geq 0$$

throughout $[ae, b]$. Then $L(Q_n) \geq 0$ on $[ae, b]$.

2) When $g(x) = e^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \ln(1+e^a)$, $\alpha_h(x)$ restriction true on $[\ln(1+e^a), b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D_{*a;e^x}^{\alpha_j} f] \geq 0$$

throughout $[\ln(1 + e^a), b]$.

Then $L(Q_n) \geq 0$ on $[\ln(1 + e^a), b]$.

3) When $A > 1$, $g(x) = A^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \log_A(1+A^a)$, $\alpha_h(x)$ restriction true on $[\log_A(1+A^a), b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D_{*a;A^x}^{\alpha_j} f] \geq 0$$

throughout $[\log_A(1 + A^a), b]$. Then $L(Q_n) \geq 0$ on $[\log_A(1 + A^a), b]$.

4) When $\sigma > 0$, $g(x) = x^\sigma$, $0 \leq a < b < \infty$.

Here we assume that $b > (1 + a^\sigma)^{1/\sigma}$, $\alpha_h(x)$ restriction true on $[(1 + a^\sigma)^{1/\sigma}, b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D_{*a;x^\sigma}^{\alpha_j} f] \geq 0$$

throughout $[(1 + a^\sigma)^{1/\sigma}, b]$. Then $L(Q_n) \geq 0$ on $[(1 + a^\sigma)^{1/\sigma}, b]$.

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