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LEFT GENERAL FRACTIONAL MONOTONE APPROXIMATION THEORY

Abstract. We introduce left general fractional Caputo style derivatives with respect to an absolutely continuous strictly increasing function $g$. We give various examples of such fractional derivatives for different $g$. Let $f$ be a $p$-times continuously differentiable function on $[a,b]$, and let $L$ be a linear left general fractional differential operator such that $L(f)$ is non-negative over a closed subinterval $I$ of $[a,b]$. We find a sequence of polynomials $Q_n$ of degree $\leq n$ such that $L(Q_n)$ is non-negative over $I$, and furthermore $f$ is approximated uniformly by $Q_n$ over $[a,b]$.

The degree of this constrained approximation is given by an inequality using the first modulus of continuity of $f^{(p)}$. We finish with applications of the main fractional monotone approximation theorem for different $g$. On the way to proving the main theorem we establish useful related general results.

1. Introduction and preparation. The topic of monotone approximation started in [10] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer $k$, approximate a given function whose $k$th derivative is $\geq 0$ by polynomials having this property.

In [2] the authors replaced the $k$th derivative with a linear differential operator of order $k$.

Furthermore in [1], the author generalized the result of [2] to linear fractional differential operators.

To describe the motivating result here we need

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Definition 1 ([4, p. 50]). Let \( \alpha > 0 \) with \( \lceil \alpha \rceil = m \) (\( \lceil \cdot \rceil \) is the ceiling of the number). Consider \( f \in C^m([-1, 1]) \). We define the \textit{left Caputo fractional derivative} of \( f \) of order \( \alpha \) as follows:

\[
(D^\alpha \ast_{-1} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt
\]

for any \( x \in [-1, 1] \), where \( \Gamma \) is the gamma function, \( \Gamma(\nu) = \int_{0}^{\infty} e^{-t} t^{\nu-1} \, dt \), \( \nu > 0 \). We set

\[
D^0 \ast_{-1} f(x) = f(x),
\]

\[
D^m \ast_{-1} f(x) = f^{(m)}(x), \quad \forall x \in [-1, 1].
\]

We proved Theorem 2 ([1]). Let \( 0 \leq h \leq k \leq p \) be integers, and let \( f \) be a real function such that \( f^{(p)} \) continuous on \([-1, 1]\) with modulus of continuity \( \omega_1(f^{(p)}, \delta), \delta > 0 \), there. Let \( \alpha_j(x), j = h, h+1, \ldots, k, \) be real functions, defined and bounded on \([-1, 1]\), and assume that for all \( x \in [0, 1] \) either \( \alpha_h(x) \geq \alpha > 0 \) or \( \alpha(x) \leq \beta < 0 \). Let \( \alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \cdots < \alpha_p \leq p \) be real numbers. Let \( D^\alpha \ast_{-1} f \) stand for the left Caputo fractional derivative of \( f \) of order \( \alpha_j \) anchored at \(-1\). Consider the linear left fractional differential operator

\[
L := \sum_{j=h}^{k} \alpha_j(x)[D^\alpha \ast_{-1}]
\]

and suppose that, throughout \([0, 1]\),

\[
(1) \quad L(f) \geq 0.
\]

Then, for any \( n \in \mathbb{N} \), there exists a real polynomial \( Q_n(x) \) of degree \( \leq n \) such that

\[
(2) \quad L(Q_n) \geq 0 \quad \text{throughout} \quad [0, 1],
\]

and

\[
(3) \quad \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1(f^{(p)}, 1/n),
\]

where \( C \) is independent of \( n \) and \( f \).

Notice that the monotonicity property is only true on \([0, 1]\): see (1), (2). However the approximation property (3) holds over the whole interval \([-1, 1]\).

In this article we extend Theorem 2 to much more general linear left fractional differential operators.

We use the following generalized fractional integral.
DEFINITION 3 (see also [7] p. 99). The left generalized fractional integral of a function \( f \) with respect to a given function \( g \) is defined as follows:

Let \( a, b \in \mathbb{R}, a < b, \) and \( \alpha > 0. \) Assume that \( g \in AC([a,b]) \) (absolutely continuous functions) is strictly increasing, and \( f \in L_\infty([a,b]) \). We set

\[
(I^\alpha_{a+:g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1}g'(t)f(t) \, dt, \quad x \geq a;
\]

clearly \((I^\alpha_{a+:g}f)(a) = 0.\) When \( g \) is the identity function \( id \), we get \( I^\alpha_{a+:id} = I^\alpha_{a+} \), the ordinary left Riemann–Liouville fractional integral, where

\[
(I^\alpha_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}f(t) \, dt, \quad x \geq a,
\]

with \((I^\alpha_{a+}f)(a) = 0.\)

When \( g(x) = \ln x \) on \([a,b], 0 < a < b < \infty, \) we get

DEFINITION 4 ([7] p. 110). Let \( 0 < a < b < \infty \) and \( \alpha > 0. \) The left Hadamard fractional integral of order \( \alpha \) of \( f \in L_\infty([a,b]) \) is given by

\[
(J^\alpha_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{y} \right)^{\alpha-1}\frac{f(y)}{y} \, dy, \quad x \geq a.
\]

DEFINITION 5. The left fractional exponential integral of \( f \in L_\infty([a,b]) \) is defined as follows: Let \( a, b \in \mathbb{R}, a < b, \) and \( \alpha > 0. \) We set

\[
(I^\alpha_{a+:e^t}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (e^x - e^t)^{\alpha-1}e^t f(t) \, dt, \quad x \geq a.
\]

DEFINITION 6. Let \( a, b \in \mathbb{R}, a < b, \alpha > 0, f \in L_\infty([a,b]), \) and \( A > 1. \) We introduce the fractional integral

\[
(I^\alpha_{a+:A^t}f)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_a^x (A^x - A^t)^{\alpha-1}A^t f(t) \, dt, \quad x \geq a.
\]

DEFINITION 7. Let \( \alpha, \sigma > 0, 0 \leq a < b < \infty, \) and \( f \in L_\infty([a,b]). \) We set

\[
(K^\alpha_{a+:x^\sigma}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^z (x^\sigma - t^\sigma)^{\alpha-1}f(t) \sigma t^{\sigma-1} \, dt, \quad x \geq a.
\]

We introduce the following general fractional derivatives.

DEFINITION 8. Let \( \alpha > 0 \) and \([\alpha] = m.\) Consider \( f \in AC^m([a,b]) \) (the space of functions \( f \) with \( f^{(m-1)} \in AC([a,b]) \)). We define the left general fractional derivative of \( f \) of order \( \alpha \) with respect to \( g \) as follows:

\[
(D^\alpha_{*a;g}f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1}g'(t)f^{(m)}(t) \, dt
\]

for any \( x \in [a,b], \) where \( \Gamma \) is the gamma function.
We set
\[ D_{\alpha}^m g(x) = f^{(m)}(x), \]
\[ D_{\alpha}^0 g(x) = f(x), \quad \forall x \in [a,b]. \]

If \( g = \text{id} \), then \( D_{\alpha}^\alpha f = D_{\alpha;\text{id}}^\alpha f \) is the left Caputo fractional derivative.

So we have the specific general left fractional derivatives.

**Definition 9.**
\[ D_{\alpha;\ln}^x f(x) = \frac{1}{\Gamma(m - \alpha)} \int_a^x \left( \ln \frac{y}{x} \right)^{m-\alpha-1} \frac{f^{(m)}(y)}{y} \, dy, \quad x \geq a > 0, \]
\[ D_{\alpha;e}^x f(x) = \frac{1}{\Gamma(m - \alpha)} \int_a^x (e^x - e^t)^{m-\alpha-1} e^t f^{(m)}(t) \, dt, \quad x \geq a, \]
\[ D_{\alpha:A}^x f(x) = \frac{\ln A}{\Gamma(m - \alpha)} \int_a^x (A^x - A^t)^{m-\alpha-1} A^t f^{(m)}(t) \, dt, \quad x \geq a, \]
\[ (D_{\alpha;x^\sigma}^\alpha f)(x) = \frac{1}{\Gamma(m - \alpha)} \int_a^x (x^\sigma - t^\sigma)^{m-\alpha-1} \sigma t^{\sigma-1} f^{(m)}(t) \, dt, \quad x \geq a \geq 0. \]

We need a modification of

**Theorem 10 (Trigub, [11], [12]).** Let \( g \in C^p([-1,1]), p \in \mathbb{N}. \) Then there exists a real polynomial \( q_n(x) \) of degree \( \leq n \) such that
\[ \max_{-1 \leq x \leq 1} |g^{(j)}(x) - q_n^{(j)}(x)| \leq R_p n^{j-p} \omega_1(g^{(p)}, 1/n), \]
\( j = 0, 1, \ldots, p, \) where \( R_p \) is independent of \( n \) and \( g. \)

**Remark 11.** Let \( a < b. \) Let \( \varphi : [-1,1] \to [a,b] \) be defined by
\[ x = \varphi(t) = \frac{b - a}{2} t + \frac{b + a}{2}. \]

Clearly \( \varphi \) is a 1-1 and onto map. We get
\[ x' = \varphi'(t) = \frac{b - a}{2}, \]
and
\[ t = \frac{2x - b - a}{b - a} = 2 \frac{x}{b - a} - \frac{b + a}{b - a}. \]

In fact,
\[ \varphi(-1) = a \quad \text{and} \quad \varphi(1) = b. \]

**Theorem 12.** Let \( f \in C^p([a,b]), p \in \mathbb{N}. \) Then there exist real polynomials \( Q_n^*(x) \) of degree \( \leq n \in \mathbb{N} \) such that
\[ \max_{a \leq x \leq b} |f^{(j)}(x) - Q_n^{(j)}(x)| \leq R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( f^{(p)}, \frac{b - a}{2n} \right), \]
\( j = 0, 1, \ldots, p, \) where \( R_p \) is independent of \( n \) and \( g. \)
Proof. We use Theorem 10 and Remark 11.

Since \( f \in C^p([a, b]) \), it is clear that the function
\[
g(t) = f\left(\frac{b-a}{2} t + \frac{b+a}{2}\right), \quad t \in [-1, 1],
\]
is in \( C^p([-1, 1]) \). We notice that
\[
\frac{dg(t)}{dt} = \frac{df\left(\frac{b-a}{2} t + \frac{b+a}{2}\right)}{dt} = f'(x) \frac{b-a}{2},
\]
and thus
\[
g'(t) = f'(x) \frac{b-a}{2} = f'\left(\frac{b-a}{2} t + \frac{b+a}{2}\right) \frac{b-a}{2}.
\]
Moreover
\[
g''(t) = \frac{df'\left(\frac{b-a}{2} t + \frac{b+a}{2}\right)}{dt} \frac{b-a}{2}.
\]
Since as before
\[
\frac{df'\left(\frac{b-a}{2} t + \frac{b+a}{2}\right)}{dt} = f''(x) \frac{b-a}{2},
\]
we obtain
\[
g''(x) = f''(x) \frac{(b-a)^2}{2^2}.
\]
In general,
\[
g^{(j)}(t) = f^{(j)}(x) \frac{(b-a)^j}{2^j}
\]
for \( j = 0, 1, \ldots, p \). Hence by Theorem 10 for any \( t \in [-1, 1] \), we have
\[
|g^{(j)}(t) - q_n^{(j)}(t)| \leq R_p n^{-p} \omega_1(g^{(p)}, 1/n)
\]
for \( j = 0, 1, \ldots, p \), where \( R_p \) is independent of \( n \) and \( g \).

Notice that
\[
q_n^{(j)}(t) \overset{[x]}{=} q_n\left(\frac{2x - b - a}{b - a}\right), \quad j = 0, 1, \ldots, p.
\]
So, for \( t \in [-1, 1] \), we have
\[
q_n(t) = q_n\left(\frac{2}{b - a} x - \frac{b + a}{b - a}\right) =: Q^*_n(x), \quad x \in [a, b],
\]
a polynomial of degree \( n \).

Also
\[
Q^*_n'(x) = \frac{d}{dx}\left(\frac{2}{b - a} x - \frac{b + a}{b - a}\right) = \frac{d}{dt}\left(\frac{dt}{dx}\right) = \frac{q'_n(t)}{2} \frac{2}{b - a}.
\]
That is,
\[
q'_n(t) = Q^*_n(x) \frac{b - a}{2}.
\]
Similarly we get
\[ Q_n''(x) = \frac{dQ_n''(x)}{dx} \left( \frac{2}{b-a} x - \frac{b+a}{b-a} \right) \frac{2}{b-a} = \frac{dq_n'(t)}{dt} \frac{2}{dx} = q_n''(t) \frac{2}{(b-a)^2}. \]

Hence
\[ q_n''(t) = Q_n''(x) \frac{(b-a)^2}{2^2}. \]

In general,
\[ q_n^{(j)}(t) = Q_n^{(j)}(x) \frac{(b-a)^j}{2^j}, \quad j = 0, 1, \ldots, p. \]

Thus we have
\[ \text{L.H.S.}(6) = \frac{(b-a)^j}{2^j} |f^{(j)}(x) - Q_n^{(j)}(x)| \]

for \( j = 0, 1, \ldots, p \) and \( x \in [a, b] \).

Next we observe that
\[ \omega_1(g^{(p)}, 1/n) = \sup_{|t_1 - t_2| \leq 1/n} \sup_{t_1, t_2 \in [-1, 1]} |g^{(p)}(t_1) - g^{(p)}(t_2)| \]
\[ \sup_{|x_1 - x_2| \leq \frac{b-a}{2n}} \frac{(b-a)^p}{2^p} |f^{(p)}(x_1) - f^{(p)}(x_2)| = \frac{(b-a)^p}{2^p} \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right), \]

since for any \( t_1, t_2 \in [-1, 1] \) with \( |t_1 - t_2| \leq 1/n \) the corresponding \( x_1, x_2 \in [a, b] \) satisfy
\[ |x_1 - x_2| \leq \frac{b-a}{2n}. \]

Finally, by (6) we can find
\[ \frac{(b-a)^j}{2^j} |f^{(j)}(x) - Q_n^{(j)}(x)| \leq R_{p^n/j} \frac{(b-a)^p}{2^p} \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right) \]

for \( j = 0, 1, \ldots, p \), and so
\[ |f^{(j)}(x) - Q_n^{(j)}(x)| \leq R_{p^n/j} \frac{(b-a)^p}{(2n)^{p-j}} \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right), \]

for any \( x \in [a, b] \) and \( j = 0, 1, \ldots, p \), proving the claim.

Remark 13. Let \( g \in AC([a, b]) \) be increasing over \([a, b]\), and let \( \alpha > 0 \). Suppose \( g(a) = c, g(b) = d \). We want to calculate
\[ I = \int_a^b (g(b) - g(t))^{\alpha-1} g'(t) \, dt. \]
Consider the function
\[ f(y) = (g(b) - y)^{\alpha - 1} = (d - y)^{\alpha - 1}, \quad \forall y \in [c, d]. \]
We have \( f(y) \geq 0, f(d) = \infty \) when \( 0 < \alpha < 1 \), but \( f \) is measurable on \([c, d]\).
By \([8\text{, exercise 13d, p. 107}]\),
\[ (f \circ g)(t)g'(t) = (g(b) - g(t))^{\alpha - 1}g'(t) \]
is measurable on \([a, b]\), and
\[ I = \int_c^d (d - y)^{\alpha - 1} dy = \frac{(d - c)^\alpha}{\alpha} \]
(notice that \((d - y)^{\alpha - 1}\) is Riemann integrable). That is,
\[ I = \frac{(g(b) - g(a))^{\alpha}}{\alpha}. \]
Similarly,
\[ \int_a^x (g(x) - g(t))^{\alpha - 1}g'(t) dt = \frac{(g(x) - g(a))^{\alpha}}{\alpha}, \quad \forall x \in [a, b]. \]

**Theorem 14.** Let \( \alpha > 0, \mathbb{N} \ni m = [\alpha], \) and \( f \in C^m([a, b]) \). Then \((D^\alpha_{a}g f)(x)\) is continuous in \( x \in [a, b] \).

**Proof.** By \([3\text{, p. 78}]\), we know that \( g^{-1} \) exists and is strictly increasing on \([g(a), g(b)]\). Since \( g \) is continuous on \([a, b]\), so is \( g^{-1} \) on \([g(a), g(b)]\). Hence \( f^{(m)} \circ g^{-1} \) is a continuous function on \([g(a), g(b)]\).

If \( \alpha = m \in \mathbb{N} \), then the claim is trivial.

We treat the case of \( 0 < \alpha < m \). The function
\[ G(z) = (g(x) - z)^{m-\alpha-1}(f^{(m)} \circ g^{-1})(z) \]
is integrable on \([g(a), g(x)]\), and by assumption \( g : [a, b] \to [g(a), g(b)] \) is absolutely continuous.

Since \( g \) is strictly increasing, the function
\[ (g(x) - g(t))^{m-\alpha-1}g'(t)(f^{(m)} \circ g^{-1})(g(t)) \]
is integrable on \([a, x]\) (see \([6]\)). Furthermore (see also \([6]\)),
\[
\frac{1}{\Gamma(m - \alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{m-\alpha-1}(f^{(m)} \circ g^{-1})(z) dz
\]
\[ = \frac{1}{\Gamma(m - \alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1}g'(t)(f^{(m)} \circ g^{-1})(g(t)) dt
\]
\[ = (D^\alpha_{a}g f)(x), \quad \forall x \in [a, b]. \]
And we can write

\[
(D_{\alpha:a:g}^\alpha f)(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^{g(x)} g(x) - z^{m-\alpha-1} (f^{(m)} \circ g^{-1})(z) \, dz,
\]

\[
(D_{\alpha:a:g}^\alpha f)(y) = \frac{1}{\Gamma(m - \alpha)} \int_0^{g(y)} g(y) - z^{m-\alpha-1} (f^{(m)} \circ g^{-1})(z) \, dz.
\]

Here \( a \leq x \leq y \leq b \), and \( g(a) \leq g(x) \leq g(y) \leq g(b) \), and \( 0 \leq g(x) - g(a) \leq g(y) - g(a) \).

Let \( \lambda = g(x) - z \); then \( z = g(x) - \lambda \). Thus

\[
(D_{\alpha:a:g}^\alpha f)(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^{g(x) - g(a)} \lambda^{m-\alpha-1} (f^{(m)} \circ g^{-1})(g(x) - \lambda) \, d\lambda.
\]

Clearly, if \( g(a) \leq z \leq g(x) \), then \(-g(a) \geq -z \geq -g(x)\), and \( g(x) - g(a) \geq g(x) - z \geq 0 \), i.e. \( 0 \leq \lambda \leq g(x) - g(a) \).

Similarly

\[
(D_{\alpha:a:g}^\alpha f)(y) = \frac{1}{\Gamma(m - \alpha)} \int_0^{g(y) - g(a)} \lambda^{m-\alpha-1} (f^{(m)} \circ g^{-1})(g(y) - \lambda) \, d\lambda.
\]

Hence

\[
(D_{\alpha:a:g}^\alpha f)(y) - (D_{\alpha:a:g}^\alpha f)(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^{g(x) - g(y)} \lambda^{m-\alpha-1} \left( (f^{(m)} \circ g^{-1})(g(y) - \lambda) - (f^{(m)} \circ g^{-1})(g(x) - \lambda) \right) \, d\lambda
\]

\[+ \int_0^{g(x) - g(a)} \lambda^{m-\alpha-1} (f^{(m)} \circ g^{-1})(g(y) - \lambda) d\lambda.\]

Thus we obtain

\[
|(D_{\alpha:a:g}^\alpha f)(y) - (D_{\alpha:a:g}^\alpha f)(x)| \leq \frac{1}{\Gamma(m - \alpha)} \int_0^{g(x) - g(y)} \lambda^{m-\alpha-1} \left( (f^{(m)} \circ g^{-1})(g(y) - \lambda) - (f^{(m)} \circ g^{-1})(g(x) - \lambda) \right) \, d\lambda
\]

\[+ \int_0^{g(x) - g(a)} \lambda^{m-\alpha-1} (f^{(m)} \circ g^{-1})(g(y) - \lambda) d\lambda.
\]

As \( y \to x \), we have \( g(y) \to g(x) \) (since \( g \in AC([a, b]) \)). So \( (\xi) \to 0 \). As a result

\[
(D_{\alpha:a:g}^\alpha f)(y) \to (D_{\alpha:a:g}^\alpha f)(x),
\]

proving that \( (D_{\alpha:a:g}^\alpha f)(x) \) is continuous in \( x \in [a, b] \).
2. Main result. We will prove

Theorem 15. Assume that \( g \in AC([a, b]) \) is strictly increasing with \( g(b) - g(a) > 1 \). Let \( 0 \leq h < k \leq p \) be integers, and let \( f \in C^p([a, b]) \), \( a < b \), with modulus of continuity \( \omega_1(f^{(p)}, \delta), 0 < \delta \leq b - a \). Let \( \alpha_j(x), j = h, h+1, \ldots, k \), be real functions, defined and bounded on \([a, b]\) and assume that, for all \( x \in [g^{-1}(1+g(a)), b] \), either \( \alpha_h(x) \geq \alpha^* > 0 \), or \( \alpha_n(x) \leq \beta^* < 0 \). Let \( \alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \cdots < \alpha_p \leq p \) be real numbers. Consider the linear left general fractional differential operator

\[
L = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*a;g}],
\]

and suppose that throughout \([g^{-1}(1+g(a)), b]\),

\[
L(f) \geq 0.
\]

Then, for any \( n \in \mathbb{N} \), there exists a real polynomial \( Q_n(x) \) of degree \( \leq n \) such that

\[
L(Q_n) \geq 0 \quad \text{throughout} \quad [g^{-1}(1+g(a)), b],
\]

and

\[
(11) \quad \max_{x \in [a, b]} |f(x) - Q_n(x)| \leq Cn^{k-p} \omega_1\left((f^{(p)}) \left(\frac{b-a}{2n}\right)\right),
\]

where \( C \) is independent of \( n \) and \( f \).

Proof. Let \( Q_n^*(x) \) be as in Theorem [12]

We have

\[
(D^{\alpha_j}_{*a;g}f)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t) f^{(j)}(t) \, dt,
\]

\[
(D^{\alpha_j}_{*a;g}Q_n^*)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t) Q_n^{(j)}(t) \, dt,
\]

for \( j = 1, \ldots, p \).

Also

\[
(D^j_{*a;g}f)(x) = f^{(j)}(x), \quad (D^j_{*a;g}Q_n^*)(x) = Q_n^{(j)}(x), \quad j = 1, \ldots, p.
\]

By [9], \( g' \) exists a.e., and \( g' \) is measurable and non-negative.

We notice that

\[
|(D^{\alpha_j}_{*a;g}f)(x) - D^{\alpha_j}_{*a;g}Q_n^*(x)|
\]

\[
= \frac{1}{\Gamma(j - \alpha_j)} \left| \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t) (f^{(j)}(t) - Q_n^{(j)}(t)) \, dt \right|
\]
Hence for all \( x \) for \( j \) for \( n \) : 

\[
\max_{x \in [a, b]} |D_{*a;g}^{\alpha_j} f(x) - D_{*a;g}^{\alpha_j} Q_n^*(x)| 
\leq \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( f(p), \frac{b - a}{2n} \right),
\]

for \( j = 0, 1, \ldots, p \).

Above we set \( D_{*a;g}^0 f(x) = f(x) \) and \( D_{*a;g}^0 Q_n^*(x) = Q_n^*(x) \), for all \( x \in [a, b] \), and \( \alpha_0 = 0 \), i.e. \( [\alpha_0] = 0 \).

Define 

\[
s_j = \sup_{a \leq x \leq b} |a^{-1}(x)\alpha_j(x)|, \quad j = h, \ldots, k,
\]

and 

\[
\eta_n = R_p \omega_1 \left( f(p), \frac{b - a}{2n} \right) \sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b - a}{2n} \right)^{p-j}.
\]

I. Suppose \( \alpha_h(x) \geq \alpha^* > 0 \) throughout \( [g^{-1}(1 + g(a)), b] \). Let \( Q_n(x) \) be the real polynomial of degree \( \leq n \) that corresponds to \( f(x) + \eta_n(h!)^{-1} x^h \), \( x \in [a, b] \), so by Theorem 12 and (12) we get 

\[
\max_{x \in [a, b]} |D_{*a;g}^{\alpha_j} (f(x) + \eta_n(h!)^{-1} x^h) - (D_{*a;g}^{\alpha_j} Q_n)(x)| 
\leq \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( f(p), \frac{b - a}{2n} \right)
\]

for \( j = 0, 1, \ldots, p \).
In particular (for \( j = 0 \))

\[
(15) \quad \max_{x \in [a, b]} |(f(x) + \eta_n(h!)^{-1}x^h) - Q_n(x)| \leq R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( f(p), \frac{b-a}{2n} \right),
\]

and

\[
\max_{x \in [a, b]} |f(x) - Q_n(x)|
\]

\[
\leq \eta_n(h!)^{-1} \max(|a|, |b|)^h + R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( f(p), \frac{b-a}{2n} \right)
\]

\[
= \eta_n(h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( f(p), \frac{b-a}{2n} \right)
\]

\[
= R_p \omega_1 \left( f(p), \frac{b-a}{2n} \right) \sum_{j=h}^{k} s_j (g(b) - g(a))^{j-\alpha_j} \frac{(b-a)^{p-j}}{\Gamma(j - \alpha_j + 1)} \times (h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( f(p), \frac{b-a}{2n} \right)
\]

\[
\leq R_p \omega_1 \left( f(p), \frac{b-a}{2n} \right) \sum_{j=h}^{k} s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2} \right)^{p-j} (h!)^{-1} \max(|a|^h, |b|^h) + \left( \frac{b-a}{2} \right)^p
\]

proving \((11)\).

Notice that for \( j = h + 1, \ldots, k \),

\[
(16) \quad (D_{s\alpha;g}^\alpha x^h) = \frac{1}{\Gamma(j - \alpha_j)} \int_a^x (g(x) - g(t))^{j-\alpha_j-1} g'(t)(t^h)^j \, dt = 0.
\]

Here

\[
L = \sum_{j=h}^{k} \alpha_j(x)[D_{s\alpha;g}^{\alpha_j}],
\]

and suppose \( Lf \geq 0 \) throughout \([g^{-1}(1 + g(a)), b]\). So over \( g^{-1}(1 + g(a)) \leq x \leq b \), we get

\[
\alpha_h^{-1}(x)L(Q_n(x)) \geq \alpha_h^{-1}(x)L(f(x)) + \frac{\eta_n}{h!} (D_{s\alpha;g}^{\alpha_h} x^h)
\]

\[
+ \sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[ D_{s\alpha;g}^{\alpha_j} Q_n(x) - D_{s\alpha;g}^{\alpha_j} f(x) - \frac{\eta_n}{h!} D_{s\alpha;g}^{\alpha_j} x^h \right]
\]

\[
\geq \frac{\eta_n}{h!} (D_{s\alpha;g}^{\alpha_h} x^h) - \sum_{j=h}^{k} s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2} \right)^{p-j} R_p \omega_1 \left( f(p), \frac{b-a}{2n} \right)
\]
\[ \frac{\eta_n}{h!} (D_{a,g}^{\alpha_h}(x^h)) = \eta_n \left( \frac{D_{a,g}^{\alpha_h}(x^h)}{h!} - 1 \right) \]

\[ = \eta_n \left( \frac{1}{\Gamma(h - \alpha_h)h!} \int_a^x (g(x) - g(t))^{h - \alpha_h} g'(t)(t^h(h) dt - 1) \right) \]

\[ = \eta_n \left( \frac{h!}{h! \Gamma(h - \alpha_h)} \int_a^x (g(x) - g(t))^{h - \alpha_h} g'(t) dt - 1 \right) \]

\[ = \eta_n \left( \frac{(g(x) - g(a))^{h - \alpha_h}}{\Gamma(h - \alpha_h + 1)} - 1 \right) \]

\[ = \eta_n \left( \frac{(g(x) - g(a))^{h - \alpha_h} - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \]

\[ \geq \eta_n \left( \frac{1 - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \geq 0. \]

Clearly here \( g(x) - g(a) \geq 1. \)

Moreover, \( \Gamma(1) = 1, \Gamma(2) = 1, \) and \( \Gamma \) is convex and positive on \( (0, \infty). \)

Here \( 0 \leq h - \alpha_h < 1 \) and \( 1 \leq h - \alpha_h + 1 < 2. \) Thus

\[ \Gamma(h - \alpha_h + 1) \leq 1 \quad \text{and} \quad 1 - \Gamma(h - \alpha_h + 1) \geq 0. \]

Hence

\[ L(Q_n(x)) \geq 0 \quad \text{for} \quad x \in [g^{-1}(1 + g(a)), b]. \]

II. Suppose \( \alpha_h(x) \leq \beta^* < 0 \) throughout \( [g^{-1}(1 + g(a)), b]. \)

Let \( Q_n(x), \ x \in [a, b], \) be a real polynomial of degree \( \leq n, \) according to Theorem 12 and (12), so that

\[ \max_{x \in [a, b]} |D_{a,g}^{\alpha_j}(f(x) - \eta_n(h!)^{-1}x^h) - (D_{a,g}^{\alpha_j}Q_n)(x)| \]

\[ \leq \frac{(g(b) - g(a))^{j - \alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( f(p), \frac{b - a}{2n} \right) \]

for \( j = 0, 1, \ldots, p. \)

In particular (for \( j = 0 \))

\[ \max_{x \in [a, b]} |(f(x) - \eta_n(h!)^{-1}x^h) - Q_n(x)| \leq R_p \left( \frac{b - a}{2n} \right)^{p} \omega_1 \left( f(p), \frac{b - a}{2n} \right), \]

and

\[ \max_{x \in [a, b]} |f(x) - Q_n(x)| \leq \eta_n(h!)^{-1} \max(|a|, |b|)^h + R_p \left( \frac{b - a}{2n} \right)^{p} \omega_1 \left( f(p), \frac{b - a}{2n} \right)\]

\[ = \eta_n(h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b - a}{2n} \right)^{p} \omega_1 \left( f(p), \frac{b - a}{2n} \right), \]
etc. We find again that
\[
\max_{x \in [a,b]} |f(x) - Q_n(x)| \leq R_p \left[ \left( \frac{b-a}{2} \right)^p + (h!)^{-1} \max(|a|^h, |b|^h) \right] \left( \sum_{j=h}^{k} s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2n} \right)^{p-j} \right) n^{k-p} \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right),
\]

Here again
\[
L = \sum_{j=h}^{k} \alpha_j(x)[D^\alpha_{a;g}],
\]
and suppose \(Lf \geq 0\) throughout \([g^{-1}(1+g(a)), b]\). So over \(g^{-1}(1+g(a)) \leq x \leq b\), we get
\[
\alpha_h^{-1}(x)L(Q_n(x)) \leq \alpha_h^{-1}(x)L(f(x)) - \frac{\eta_n}{h!}(D^\alpha_{a;g}(x^h)) \sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[ D^\alpha_{a;g} Q_n(x) - D^\alpha_{a;g} f(x) + \frac{\eta_n}{h!} D^\alpha_{a;g} x^h \right] \leq -\frac{\eta_n}{h!}(D^\alpha_{a;g}(x^h)) \sum_{j=h}^{k} s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2n} \right)^{p-j} R_p \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right)
\]
and hence
\[
\eta_n \left( 1 - \frac{1}{\Gamma(h - \alpha_h)h^!} \int_a^x (g(x) - g(t))^{h-\alpha_h-1} g'(t)(t^h)^{(h)} dt \right) \leq \eta_n \left( 1 - \frac{(g(x) - g(a))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \leq \eta_n \left( 1 - \frac{(g(x) - g(a))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \leq 0.
\]
Hence again
\[
L(Q_n(x)) \geq 0, \quad \forall x \in [g^{-1}(1+g(a)), b].
\]
The case of \(\alpha_h = h\) is trivially deduced from the above. The proof of the theorem is now complete.
Remark 16. By Theorem 14, $D^{\alpha_j}_{*;g}f$ are continuous functions, $j = 0, 1, \ldots, p$. Suppose that $\alpha_h(x), \ldots, \alpha_k(x)$ are continuous functions on $[a, b]$, and $L(f) \geq 0$ on $[g^{-1}(1+g(a)), b]$ is replaced by $L(f) > 0$ on $[g^{-1}(1+g(a)), b]$. Disregard the assumption made in the main theorem on $\alpha_h(x)$. For $n \in \mathbb{N}$, let $Q_n(x)$ be the $Q^{*}_{n}(x)$ of Theorem 12, and $f$ as in Theorem 12 (same as in Theorem 15). Then $Q_n(x)$ converges to $f(x)$ at the Jackson rate $1/n^{p+1}$ ([5, p. 18, Theorem VIII]) and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[a, b]$, $L(Q_n) > 0$ on $[g^{-1}(1+g(a)), b]$ for all $n$ sufficiently large.

3. Applications (of Theorem 15)

1) When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Here we assume that $b > ae, \alpha_h(x)$ restriction true on $[ae, b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*;\ln x}f] \geq 0$$

throughout $[ae, b]$. Then $L(Q_n) \geq 0$ on $[ae, b]$.

2) When $g(x) = e^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \ln(1+e^a), \alpha_h(x)$ restriction true on $[\ln(1+e^a), b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*;e^x}f] \geq 0$$

throughout $[\ln(1+e^a), b]$.

Then $L(Q_n) \geq 0$ on $[\ln(1+e^a), b]$.

3) When $A > 1, g(x) = A^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \log_A(1+A^a), \alpha_h(x)$ restriction true on $[\log_A(1+A^a), b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*;A^x}f] \geq 0$$

throughout $[\log_A(1+A^a), b]$. Then $L(Q_n) \geq 0$ on $[\log_A(1+A^a), b]$.

4) When $\sigma > 0, g(x) = x^\sigma, 0 \leq a < b < \infty$.

Here we assume that $b > (1+a^\sigma)^{1/\sigma}, \alpha_h(x)$ restriction true on $[(1+a^\sigma)^{1/\sigma}, b]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x)[D^{\alpha_j}_{*;x^\sigma}f] \geq 0$$

throughout $[(1+a^\sigma)^{1/\sigma}, b]$. Then $L(Q_n) \geq 0$ on $[(1+a^\sigma)^{1/\sigma}, b]$. 
References


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