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A MAP MAINTAINING THE ORBITS OF A GIVEN $\mathbb{Z}^d$-ACTION

BY

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Abstract. Giordano et al. (2010) showed that every minimal free $\mathbb{Z}^d$-action of a Cantor space X is orbit equivalent to some $\mathbb{Z}$-action. Trying to avoid the K-theory used there and modifying Forrest's (2000) construction of a Bratteli diagram, we show how to define a (one-dimensional) continuous and injective map F on $X \setminus \{\text{one point}\}$ such that for a residual subset of X the orbits of F are the same as the orbits of a given minimal free $\mathbb{Z}^d$-action.

1. Introduction. In two papers published in 1959 and 1963, Dye [3] contributed to the theory of classification of dynamical systems by proving that any two measure preserving ergodic transformations of a Lebesgue space are orbit equivalent. In 1980 Ornstein and Weiss [9] generalized this result by showing that any two free, ergodic, measure preserving actions of amenable groups are orbit equivalent. In particular, every such action is orbit equivalent to an ergodic action of $\mathbb{Z}$. Further generalizations (in the language of relations) can be found in [1].

The topological theory of orbit equivalence seems to be more complex and more delicate. In particular, it turns out that not all minimal $\mathbb{Z}$-actions fall into the same class of orbit equivalence (see [8] for a deep study of the subject). In the last decade, Giordano, Matui, Putnam and Skau showed that, similarly to the measure case, every minimal $\mathbb{Z}^d$-action of a Cantor space is orbit equivalent to some $\mathbb{Z}$-action (see [6] for the case $d = 2$ and [7] for the general result). They employed quite complicated tools of K-theory, used already in [8], and the main idea was to study certain invariants of orbit equivalence, namely the dimension groups. Though undoubtedly brilliant, the papers are not of the kind with which one should start one's adventure with Cantor dynamics. On the other hand, knowing that a multidimensional system is orbit equivalent to a one-dimensional system, one would like to see the appropriate orbit preserving map. Roughly speaking, the question is

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how to cleverly linearize an orbit which is a d-dimensional lattice, so that the procedure used does not depend on the position of the origin. We try to carry out this beginner's attempt to establish orbit equivalence between $\mathbb{Z}^d$- and $\mathbb{Z}$-actions. Inspired by Forrest's results [4], we construct a Bratteli diagram of a given $\mathbb{Z}^d$-action. The main tool is a kind of marker lemma, but instead of cutting the orbits into pieces that are Voronoi regions, we apply the maximolexicographic order, which was also used in [5]. Though not fully successful, we keep our faith that using this technique one may get a better outcome, perhaps imposing some additional requirements on the structure of markers. At this point we also want to express our gratitude to the referee, whose remarks allowed us to improve the paper.

We consider a compact (perfect) zero-dimensional metrizable space X (i.e. a Cantor space) and a collection $T = \{T_1, \dots, T_d\}$ of commuting homeomorphisms of X. For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ we write $T^{\mathbf{n}}$ for the superposition $T_1^{n_1} \dots T_d^{n_d}$. We say that a system (X, T) is *aperiodic* or that T acts *freely* on X if $T^{\mathbf{n}}(x) \neq x$ for all $x \in X$ and all $\mathbf{n} \neq (0, \dots, 0)$. It is *minimal* if X contains no proper nonempty closed subset which is invariant (a set F is invariant if $T_i F = F$ for all $i = 1, \dots, d$). Equivalently, (X, T) is minimal if and only if the orbit $\{T^{\mathbf{n}}x \colon \mathbf{n} \in \mathbb{Z}^d\}$ of every $x \in X$ is dense.

To perform our construction we will encode a given system (X, T) into a d-dimensional symbolic system over a compact alphabet Λ. As usual, on a compact space $\Lambda^{\mathbb{Z}^d}$ we define shift maps σ_i by setting $(\sigma_i(y))_{\mathbf{n}} = y_{\mathbf{n}+\mathbf{e}_i}$ for all $y \in \Lambda^{\mathbb{Z}^d}$, $\mathbf{n} \in \mathbb{Z}^d$ and $i = 1, \dots, d$, where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^d$ with the only 1 at the ith place. A *d-dimensional symbolic system* is a nonempty closed subset Y of $\Lambda^{\mathbb{Z}^d}$ which is invariant under all σ_i.

We use the following conventions. For a set Λ a function $M : \mathbb{Z}^d \to \Lambda$, i.e. an element of $\Lambda^{\mathbb{Z}^d}$, is called an *array*. For a finite set $A \subset \mathbb{Z}^d$ and an array M we define a *configuration* M_A to be M restricted to A. To comply with the standard notation, for $\mathbf{n} \in \mathbb{Z}^d$ we denote by $M_{\mathbf{n}}$ a single symbol $M_{\{\mathbf{n}\}}$. If $\widetilde{A} = A + \mathbf{m}$ for some $\mathbf{m} \in \mathbb{Z}^d$, and $\widetilde{M}_{\mathbf{n}+\mathbf{m}} = M_{\mathbf{n}}$ for every $\mathbf{n} \in A$, then we say that M_A and $\widetilde{M}_{\widetilde{A}}$ have the same *pattern*. In this case both A and $\widetilde{A}$ represent the *shape* of the pattern. More formally, shapes and patterns are cosets of the equivalence relation based on translation of the domain. Thus one can define inclusion for shapes S, S' as follows: $S' \subset S$ if $A' \subset A$ for some $A' \subset \mathbb{Z}^d$ representing S' and $A \subset \mathbb{Z}^d$ representing S. A shape S is *bounded* if the sets representing S are bounded. A *cube* with *maximal vertex* $\mathbf{v} = (v_1, \dots, v_d)$ and edge length p is the set $\mathcal{K}_p^{\mathbf{v}} = \{\mathbf{n} \in \mathbb{Z}^d \colon v_i - p < n_i \leq v_i\}$. For $p \in \mathbb{N}_0$ and $\mathbf{v} = (p, \dots, p)$ we also write $\mathcal{K}_p = \mathcal{K}_{p+1}^{\mathbf{v}}$, the cube fixed at the origin. We will also use the name 'cube' when referring to shapes based on cubes in $\mathbb{Z}^d$. In a symbolic system (Y, σ), by *blocks* we will mean patterns having bounded shapes. For $p \in \mathbb{N}_0$ we denote the centered cube with edge length $2p+1$ by

$$\bar{\mathcal{K}}_p = \{\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d \colon \max\{|n_1|, \dots, |n_d|\} \le p\}.$$

DEFINITION. A set $F \subset X$ is a *marker of order* $p \in \mathbb{N}_0$ or simply a *p-marker* if:

(i) the elements of $\{T^{\mathbf{n}}F \colon \mathbf{n} \in \bar{\mathcal{K}}_p\}$ are pairwise disjoint,
(ii) $\{T^{\mathbf{n}}F \colon \mathbf{n} \in \bar{\mathcal{K}}_N\}$ is a cover of X for some minimal $N \in \mathbb{N}_0$; the number $2N+1$ is then called the *covering constant* of F.

The following lemma is the base of our construction—it allows us to produce for each x a certain sequence of partitions of $\mathbb{Z}^d$ into finite subsets on which we perform linearization of the orbit. The lemma is quite well known, but we add three demands and so present a proof to ensure that these are easy to obtain just by being more specific in carrying out the standard argument. In particular, the third condition guarantees that the set of those points for which our construction fails is meager (Proposition 10), and that even then we keep partial control over the linearization process by assembling the one-dimensional orbit from finitely many pieces (Lemma 3).

LEMMA 1. *For any aperiodic minimal $\mathbb{Z}^d$-action (X,T) on a compact metrizable zero-dimensional Hausdorff space and any $x_0 \in X$ there is an increasing sequence (p_t) of integers and a descending sequence (F_t) of clopen p_t-markers satisfying:*

(1) $\bigcap_{t=1}^{\infty} F_t = \{x_0\}$,
(2) $p_{t+1} \ge t p_t$,
(3) *the sequence $(p_t/q_t)_{t=1}^{\infty}$, where q_t is a covering constant of F_t, is bounded away from zero.*

Proof. We define F_t recursively. Let $p_1 = 1$. To construct the first marker choose a clopen neighborhood F_1 of x_0 such that $\{T^{\mathbf{n}}F_1 \colon \mathbf{n} \in \bar{\mathcal{K}}_2\}$ consists of pairwise disjoint sets. Since the action is minimal, there is an integer q_1 such that the collection $T^{\mathbf{n}}F_1$, where $\mathbf{n} \in \bar{\mathcal{K}}_{q_1}$, is a cover of X.

Let $B(x_0,r)$ and $\bar{B}(x_0,r)$ denote the open and closed balls with center at x_0 and radius r, respectively. Next choose $\delta_1 < 1$ so that $B(x_0,\delta_1) \subset F_1$. Again by minimality, there is an integer N_1 such that the collection $T^{\mathbf{n}}B(x_0, \frac{1}{2}\delta_1)$, where $\mathbf{n} \in \bar{\mathcal{K}}_{N_1}$, covers X. Suppose that we have already constructed p_t-markers $F_1, \dots, F_n$ and chosen radii $\delta_1, \dots, \delta_n$ with $\delta_t < 1/t$ satisfying $B(x_0,\delta_t) \subset F_t \subset B(x_0,\delta_{t-1})$ (note that we can take $\delta_0 > \operatorname{diam}(X)$) with

$$p_{t+1} = q_t \cdot N_t \quad \text{for } t = 1, \dots, n-1,$$

where N_t denotes the size of the orbit of $B(x_0, \frac{1}{2}\delta_t)$ (i.e. the edge length of an appropriate cube) needed to cover the whole X. To obtain condition (2) of the hypothesis we demand that $N_t \ge t$ (increasing N_t at each step if necessary). To construct a p_{n+1}-marker F_{n+1} with $p_{n+1} = q_n \cdot N_n$, for every $x \in \bar{B}(x_0, \frac{1}{2}\delta_n)$ we choose a clopen neighborhood E_x of x, contained in $B(x_0, \delta_n)$, such that $\{T^{\mathbf{n}}E_x : \mathbf{n} \in \bar{\mathcal{K}}_{2p_{n+1}}\}$ consists of pairwise disjoint sets. From the cover $\{E_x\}$ of the closed ball $\bar{B}(x_0, \frac{1}{2}\delta_n)$ we choose a finite subcover $\mathcal{V} = \{V_l : l = 1, \dots, L\}$. Assume that $V_1 \ni x_0$. Now we set

$$G_1 = V_1, \qquad G_{l+1} = G_l \cup \Big(V_{l+1} \setminus \bigcup_{\mathbf{m} \in \bar{\mathcal{K}}_{2p_{n+1}}} T^{\mathbf{m}} G_l\Big).$$

Finally, $F_{n+1} = G_L$. Obviously, F_{n+1} is clopen and $x_0 \in F_{n+1} \subset B(x_0, \delta_n)$.

We omit the elementary proof that the elements of $\{T^{\mathbf{n}}F_{n+1} : \mathbf{n} \in \bar{\mathcal{K}}_{p_{n+1}}\}$ are pairwise disjoint; we only check that $T^{\mathbf{n}}F_{n+1}$ for $\mathbf{n} \in \bar{\mathcal{K}}_{2p_{n+1}+N_n}$ cover the whole X. Every $x \in \bar{B}(x_0, \frac{1}{2}\delta_n)$ belongs to one of V_l's. Either it was added to $G_l \subset F_{n+1}$ at the lth step of the construction, or it had already been contained in $T^{\mathbf{m}}G_{l-1} \subset T^{\mathbf{m}}F_{n+1}$ for some $\mathbf{m} \in \bar{\mathcal{K}}_{2p_{n+1}}$. Thus $\bar{B}(x_0, \frac{1}{2}\delta_n) \subset \bigcup_{\mathbf{n} \in \bar{\mathcal{K}}_{2p_{n+1}}} T^{\mathbf{n}}F_{n+1}$, and $X \subset \bigcup_{\mathbf{n} \in \bar{\mathcal{K}}_{2p_{n+1}+N_n}} T^{\mathbf{n}}F_{n+1}$ by definition of N_n.

Clearly, $\bigcap_{t=1}^{\infty} F_t \subset \bigcap_{t=1}^{\infty} B(x_0, \delta_t) = \{x_0\}$. To verify (3), we calculate

$$\frac{p_t}{q_t} \geq \frac{p_t}{4p_t + 2N_{t-1} + 1} = \frac{1}{4 + 2/q_{t-1} + 1/p_t} \xrightarrow{t \to \infty} \frac{1}{4}. \ \blacksquare$$

2. t-blocks. Let us fix a summable sequence, $\varepsilon_t = 1/2^{t+3}$, $t \in \mathbb{N}_0$. Let $d_{\overline{\mathbb{N}}_0}$ be a metric on $\overline{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty\}$ given by $d_{\overline{\mathbb{N}}_0}(k, l) = \sum_{t=k+1}^{l} \varepsilon_t$ for $k \leq l$. Let d_X denote a metric on X. We define a metric d_Λ on $\Lambda = \overline{\mathbb{N}}_0 \times X$ by

$$d_\Lambda((m, x), (n, y)) = d_{\overline{\mathbb{N}}_0}(m, n) + d_X(x, y).$$

Note that (Λ, d_Λ) is a compact metric space. Let $\kappa = 1 + \operatorname{diam}(\Lambda)$. By Tikhonov's theorem the product $\Lambda^{\mathbb{Z}^d}$ is also compact when equipped with the pointwise convergence topology. For $\mathbf{n} \in \mathbb{Z}^d$ set

$$|\mathbf{n}| = \max\{n_i : i = 1, \dots, d\}$$

and define a metric compatible with the Tikhonov topology by

$$\rho(M, M') = \sum_{n=1}^{\infty} \frac{1}{\kappa^n} \max_{|\mathbf{n}| = n} d_\Lambda(M_{\mathbf{n}}, M'_{\mathbf{n}}) \quad \text{ for } M, M' \in \Lambda^{\mathbb{Z}^d}.$$

Let (p_t), (q_t), (F_t) be as in Lemma 1. Additionally, let $p_0 = q_0 = 1$, $F_0 = X$, $Q_{-1} = 0$, and $Q_t = \sum_{i=0}^{t} q_i$ for $t \geq 0$. In the construction of (X^*, σ) we replace each $x \in X$ by an array $[x] : \mathbb{Z}^d \to \Lambda$ such that $[x]_{\mathbf{n}} = (t, T^{\mathbf{n}}x)$ if $T^{\mathbf{n}}x \in F_t$ and $T^{\mathbf{n}}x \notin F_{t+1}$, and $[x]_{\mathbf{n}} = (\infty, T^{\mathbf{n}}x)$ if $T^{\mathbf{n}}x$ belongs to all markers. We say that $[x]$ has marker t at position $\mathbf{n} \in \mathbb{Z}^d$ if the first coordinate of $[x]_{\mathbf{n}}$ is equal to t; a configuration $[x]_A$ *contains the marker* t if $[x]$ has marker t on some position $\mathbf{n} \in A$. The space $X^* = \{[x] : x \in X\}$ is homeomorphic to X and in an obvious manner the collection σ of the shifts

σ_i is topologically conjugate to T. According to the definition of a marker, every $[x] \in X^*$ has the following properties:

(1) every configuration in $[x]$ based on a cube with edge length p_t has (at some position) at most one marker $\geq t$,
(2) every configuration in $[x]$ based on a cube with edge length q_t has at least one marker $\geq t$.

Below we describe an inductive algorithm of partitioning every $[x] \in X^*$ into disjoint configurations. The resulting sequence of partitions will allow us to create a local (independent of the shift map) rule of linearizing d-dimensional orbits. On every cone $\mathbf{n} + \mathbb{N}_0^d = \{\mathbf{m} \in \mathbb{Z}^d \colon \mathbf{m} \geq \mathbf{n}\}$, where $\mathbf{n} \in \mathbb{Z}^d$, we define a maximolexicographic order '$<^*$' as follows. For $\mathbf{m} \in \mathbb{N}^d$ let $\mathrm{sort}(\mathbf{m})$ denote the element of $\mathbb{Z}^d$ whose coordinates are equal to those of $\mathbf{m}$, but arranged in a nonincreasing order, and let '$\prec$' be the usual lexicographic order. To define the order on $\mathbb{N}_0^d$ we write $\mathbf{m} <^* \mathbf{m'}$ if either

- $\mathrm{sort}(\mathbf{m}) \prec \mathrm{sort}(\mathbf{m'})$, or
- $\mathrm{sort}(\mathbf{m}) = \mathrm{sort}(\mathbf{m'})$ and $\mathbf{m} \prec \mathbf{m'}$.

The order on $\mathbf{n} + \mathbb{N}_0^d$ is a translation of the order from $\mathbb{N}_0^d$. The relation '$<^*$' is a well-order. The operation of taking minimum with respect to this order will be denoted by '$\min^*$'. In this notation we do not indicate the dependence on the vertex of the cone, because the choice of the order is always clear from the context.

Table 1. The scheme of the maximolexicographic order for $d = 2$. The number 0 is the vertex of a cone; consecutive integers are placed according to the maxlex order on this cone.

⋮				
9	11	13	15	
4	6	8	14	
1	3	7	12	
0	2	5	10	…

First, we define 0-configurations to be single symbols $[x]_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. To proceed inductively, we assume that we have defined t-configurations in such a way that every t-configuration contains exactly one marker u not smaller than t. Denote the position of this marker in a t-configuration $[x]_A$ by $\mathbf{n}(t, A)$. We define a $(t+1)$-configuration as a concatenation of t-configurations as follows. Every $(t+1)$-configuration $[x]_C$ consists of exactly one t-configuration $[x]_A$ with a marker u not less than $t+1$ and all t-configurations $[x]_{A'}$ such that

$$\mathbf{n}(t, A) = \min{}^*\{\mathbf{m} \geq^* \mathbf{n}(t, A') \colon [x] \text{ has marker at least } t+1 \text{ at } \mathbf{m}\},$$

where the ordering '$\geq^*$' is inverse to '$<^*$' defined for the cone $\mathbf{n}(t, A') + \mathbb{N}_0^d$. We obtain $\mathbf{n}(t+1, C) = \mathbf{n}(t, A)$. Roughly speaking, the t-marker of A'

searches for the nearest (in '$<^*$') $(t+1)$-marker of some A, and then the t-configuration A' is glued to A at the $(t+1)$st step of the construction. Thus for each $x \in X$ and $t \in \mathbb{N}$ distinct t-configurations are supported by disjoint subsets of $\mathbb{Z}^d$, and every t-configuration is a part of a $(t+1)$-configuration.

Example. Let $(Z_i)_{i\in\mathbb{N}}$ be a decreasing sequence of subgroups of $\mathbb{Z}^2$ isomorphic to $\mathbb{Z}^2$ and let $G \subset \prod_{i=0}^{\infty} \mathbb{Z}^d/Z_i$ be a two-dimensional odometer (for the definition see e.g. [2]). Denoting by 0_i the neutral element in $\mathbb{Z}^d/Z_i$ and setting

$$F_t = \{(g_n) \in G \colon g_i = 0_i \text{ for } i < t\}$$

we obtain a convenient decreasing sequence of clopen markers. Every point visits the marker F_t regularly, at times belonging to the, perhaps shifted, lattice Z_i. In the process of forming 1-configurations into an array $[x]$, each coordinate (m,n) searches for the nearest (in '$<^*$') 1-marker which is located at some point from $g_1 + Z_1$, where $(g_n) = x$. Then every coordinate marked with a 1-marker searches for the nearest 2-marker situated in $g_2 + Z_2$ to glue its 1-configuration to it, and so on. The regularity of the distribution of t-markers causes that for each t all t-configurations will share just one shape. In particular, if G is a product of two one-dimensional odometers with scales (p_t) and (q_t), these t-configurations will have the shape of a rectangle with edges p_t and q_t.

The patterns of t-configurations will be called *t-blocks*. The collection of all t-blocks which occur in the system X^* will be denoted by $\mathcal{B}_t$. As in [5], we summarize the main properties of t-blocks (see also Fig. 1).

Lemma 2 ([5]). *Let B be a t-block. Then:*

1. *B is a finite concatenation of $(t-1)$-blocks $(t > 0)$.*
2. *B contains exactly one marker u not less than t.*
3. *The marker u not less than t is situated at the maximal vertex of B, i.e. at the maximal vertex of the smallest cube containing the domain of B.*
4. *The shape of B contains a cube with edge length $p_t - Q_{t-1}$.*
5. *The shape of B is contained in a cube with edge length Q_t.*

For every $[x] \in X^*$ we define an equivalence relation $\overset{x}{\sim}$ on $\mathbb{Z}^d$ as follows: $\mathbf{n} \overset{x}{\sim} \mathbf{m}$ if for some t they belong to the domain of the same t-configuration induced by $[x]$. In this case they clearly belong to the same s-configuration for every $s \geq t$.

Lemma 3. *For every $[x] \in X^*$ the lattice $\mathbb{Z}^d$ consists of finitely many equivalence classes* (*of the relation $\overset{x}{\sim}$*).

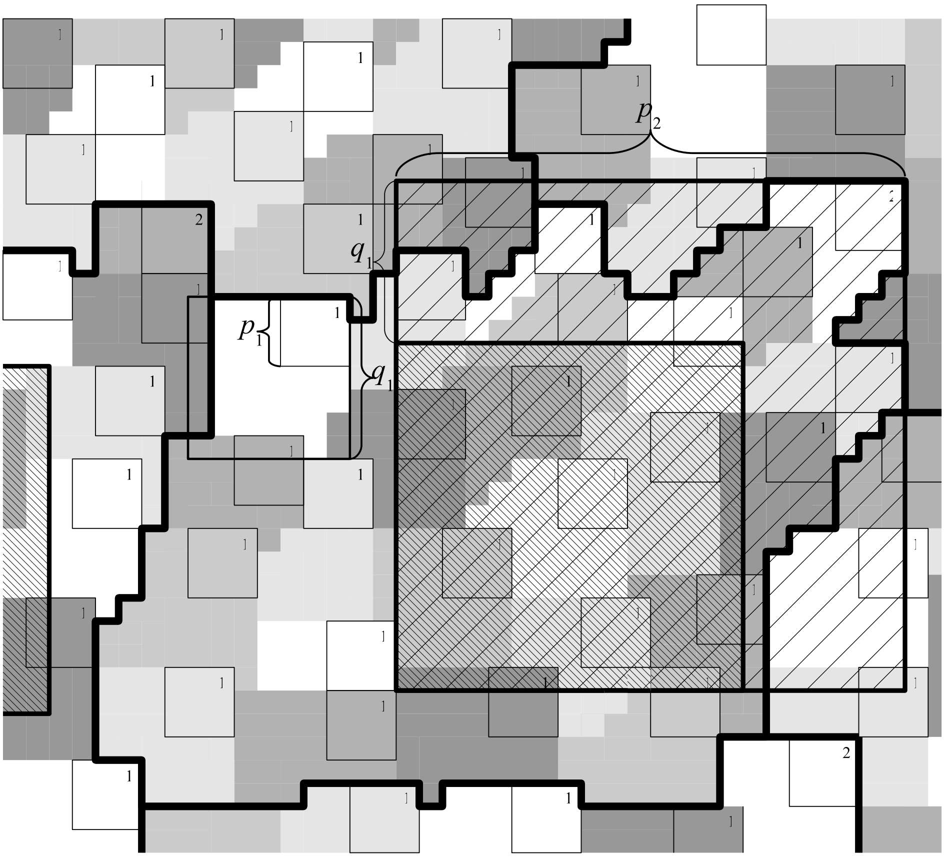

Fig. 1. The construction of 1- and 2-blocks in two dimensions for $p_1 = 3$, $q_1 = 7$, $p_2 = 22$. 1-blocks are distinguished by shades of grey. The bold line separates 2-blocks. Each of the marked squares with edge length p_1 has a unique 1-marker in the upper right corner. The big hatched square is an area with a unique 2-marker.

Proof. Since p_t/q_t is bounded away from zero by some $\delta = 1/N$, and $p_{t+1} \geq tp_t$ for each t, we have

$$\frac{Q_t}{p_t} = \frac{\sum_{i\leq t} q_i}{p_t} \leq \frac{N\sum_{i\leq t} p_i}{p_t} \leq \frac{2Np_t}{p_t} = 2N.$$

Now assume that there are at least m equivalence classes for some $[x] \in X^*$. Choose one element from every class. All these elements lie in a cube $\bar{\mathcal{K}}_R$ for some $R > 0$. According to Lemma 2, for each of the fixed elements (and any t) the t-configuration covering it contains a cube with edge length

$$p_t - Q_{t-1} \geq p_t - 2Np_{t-1} \overset{\text{Lem. 1(2)}}{\geq} p_t\left(1 - \frac{2N}{t-1}\right),$$

which is greater than $\frac{1}{2}p_t$ for sufficiently large t. On the other hand, it is contained in a cube with edge length Q_t. Therefore, all these covering t-configurations are contained in a cube $\bar{\mathcal{K}}_{R+Q_t}$. Thus we have

$$m\left(\frac{p_t}{2}\right)^d \le (2R+1+2Q_t)^d \le (2R+1+4Np_t)^d,$$

hence

$$m \le \left(\frac{4R+2}{p_t}+8N\right)^d.$$

Since $p_t \to \infty$ we conclude that $m \le (8N)^d$. ∎

3. Towers and diagrams. We follow [4] in making the following definitions.

DEFINITION.

- A *tower* is a collection $\{T^{\mathbf{n}}F\colon \mathbf{n} \in A\}$ of disjoint clopen sets, where A is a finite subset of $\mathbb{Z}^d$ containing 0, and F is some clopen set.
- The *levels* of the tower based on $A \in \mathbb{Z}^d$ are the sets $T^{\mathbf{n}}F$, where $\mathbf{n} \in A$. A *traverse* of the tower $\{T^{\mathbf{n}}F\colon \mathbf{n} \in A\}$ is a set of the form $\{T^{\mathbf{n}}x\colon \mathbf{n} \in A\}$, where $x \in F$.
- The *boundary* of a set $A \in \mathbb{Z}^d$ is defined by

  $$\partial A = \{\mathbf{n} \in A\colon \mathbf{n} \pm \mathbf{e}_j \notin A \text{ for some } 1 \le j \le d\},$$

  where the $\mathbf{e}_j$ are the elements of the standard unit basis of $\mathbb{Z}^d$. The *boundary* of the tower is the union of $T^{\mathbf{n}}F$, where $\mathbf{n} \in \partial A$.
- A *KR decomposition* $\mathcal{Q}$ is a collection of towers covering X, whose levels are mutually disjoint. We write $|\mathcal{Q}|$ for the underlying partition of the space.
- A KR decomposition $\mathcal{Q}$ *refines* a KR decomposition Q' if $|\mathcal{Q}|$ refines $|\mathcal{Q}'|$ in the usual sense, and if each traverse of a tower in $\mathcal{Q}$ is a union of traverses of towers in $\mathcal{Q}'$.
- A *Vershik model* for $(X, \mathbb{Z}^d)$ is a sequence of refining KR decompositions. We say that the model is *faithful* if it refines to points.

Any KR decomposition $\mathcal{Q}$ gives rise to a certain collection of partitions of the lattice. Indeed, for each $x \in X$ we define a partition $\mathcal{T}(x)$ of $\mathbb{Z}^d$ into sets A such that for each A the set $\{T^{\mathbf{n}}x\colon \mathbf{n} \in A\}$ is a traverse of a tower of $\mathcal{Q}$. The function $\mathcal{T}(x)$ has the following properties:

P1. If $x \in X$ and $\mathbf{n} \in \mathbb{Z}^d$ then $\mathcal{T}(T^{\mathbf{n}}x)$ is the partition $\mathcal{T}(x)$ shifted by $-\mathbf{n}$.

P2. The atoms of the partitions $\mathcal{T}(x)$ are finite sets of uniformly bounded diameter; the bound is uniform also in $x \in X$.

P3. The map $x \mapsto \mathcal{T}(x)$ is 'uniformly continuous': for every $r > 0$ there is $\delta > 0$ such that if $d_X(x, y) < \delta$ then the partitions $\mathcal{T}(x)$ and $\mathcal{T}(y)$ are the same when restricted to $\bar{\mathcal{K}}_r$.

Denote by $A(x)$ the atom of $\mathcal{T}(x)$ which contains 0.

LEMMA 4 ([4]). *A map $\mathcal{T}$ with properties* P1–P3 *canonically defines a KR decomposition. The boundary of this decomposition is the set* $\{x \in X : 0 \in \partial A(x)\}$.

Given a Vershik model we automatically have a sequence of partition-valued maps $\mathcal{T}_n$ with the property

P4. For each $x \in X$ and $n \geq 1$, $\mathcal{T}_n(x)$ refines $\mathcal{T}_{n+1}(x)$.

If F is a subset of X and ξ is a partition of X then we denote by $F \cap \xi$ the partition of F given by the formula

$$F \cap \xi = \{F \cap E : E \in \xi\}.$$

LEMMA 5. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two KR decompositions, which determine partition-valued functions $\mathcal{T} : X \to \mathbb{Z}^d$ and $\mathcal{T}' : X \to \mathbb{Z}^d$, respectively. If $\mathcal{T}(x)$ refines $\mathcal{T}'(x)$ for each $x \in X$ then there exists a KR decomposition $\mathcal{Q}''$ such that:*

(1) *the partition-valued function associated to $\mathcal{Q}''$ is $\mathcal{T}'(x)$,*
(2) *$|\mathcal{Q}''|$ refines $|\mathcal{Q}|$ and $|\mathcal{Q}'|$.*

In particular, the KR decomposition $\mathcal{Q}''$ refines $\mathcal{Q}$ and $\mathcal{Q}'$.

Proof. Let $\mathcal{Q} = \{W_1, \ldots, W_r\}$ and $\mathcal{Q}' = \{W'_1, \ldots, W'_s\}$, where W_i and W'_j are towers of the form $W_i = \{T^{\mathbf{n}} F_i : \mathbf{n} \in A_i\}$ and $W'_j = \{T^{\mathbf{n}} F'_j : \mathbf{n} \in A'_j\}$. We fix the clopen subsets F_i, $i = 1, \ldots, r$, and F'_j, $j = 1, \ldots, s$, of X, and the associated regions A_i, $A'_j \subset \mathbb{Z}^d$ (we demand that each of these regions contains $\mathbf{0}$). For $j = 1, \ldots, s$ let I_j be the set of all i such that there is $\mathbf{n} \in A'_j$ for which $T^{\mathbf{n}} F'_j \cap F_i$ is nonempty. Denote by $N_{j,i}$ the set of all such $\mathbf{n} \in A'_j$. Let $\xi_i = \{F_i, F_i^c\}$. We introduce a partition of F'_j by $\zeta_j = F'_j \cap \bigvee_{i \in I_j} \bigvee_{\mathbf{n} \in N_{j,i}} T^{-\mathbf{n}} \xi_i$. For $E \in \zeta_j$ we let

$$V_{j,E} = \{T^{\mathbf{n}} E : \mathbf{n} \in A'_j\},$$

and then we define

$$\mathcal{Q}'' = \{V_{j,E} : j = 1, \ldots, r,\ E \in \zeta_j\}.$$

Clearly, each element of $\mathcal{Q}''$ is a tower whose levels are contained in levels of W'_j and whose traverses are the same as in W'_j. Since $\bigcup_{E \in \zeta_j} \bigcup_{\mathbf{n} \in A'_j} T^{\mathbf{n}} E$ is exactly the union of all levels of W'_j, the elements of $\mathcal{Q}''$ cover the whole X. Moreover, if $j \neq k$ then levels of $V_{j,E}$ are disjoint from levels of $V_{k,D}$, where $E \in \zeta_j$, $D \in \zeta_k$. For $E_1, E_2 \in \zeta_j$ we have $T^{\mathbf{m}} E_1 \cap T^{\mathbf{n}} E_2 \subset T^{\mathbf{m}} F_j \cap T^{\mathbf{n}} F_j = \emptyset$ if $\mathbf{m} \neq \mathbf{n}$ and $\mathbf{m}, \mathbf{n} \in A'_j$. If $\mathbf{m} = \mathbf{n}$ then $T^{\mathbf{m}} E_1 \cap T^{\mathbf{n}} E_2 = T^{\mathbf{n}}(E_1 \cap E_2) = \emptyset$, because E_1 and E_2 are disjoint elements of the partition ζ_j. Hence, $\mathcal{Q}''$ is a KR partition that yields the partition-valued function $\mathcal{T}'$. It is obvious that $|\mathcal{Q}''|$ refines $|\mathcal{Q}'|$. But also for each F_i we have

$$F_i = \bigcup_{j=1}^{s} \bigcup_{\mathbf{n}\in N_{j,i}} T^{\mathbf{n}} F'_j \cap F_i.$$

The set $F'_j \cap T^{-\mathbf{n}} F_i$ is a union of elements of ζ_j, so $\bigcup_{\mathbf{n}\in N_{j,i}} T^{\mathbf{n}} F'_j \cap F_i$ is a union of some elements of $\{T^{\mathbf{n}} E \colon E \in \zeta_j,\ \mathbf{n} \in A'_j\}$, and consequently F_i is a union of elements of $|\mathcal{Q}''|$. For $\mathbf{m} \in A_i$ we have

$$T^{\mathbf{m}} F_i = \bigcup_{j=1}^{s} \bigcup_{\mathbf{n}\in N_{j,i}} T^{\mathbf{n}+\mathbf{m}} F'_j \cap T^{\mathbf{m}} F_i.$$

Take any $x \in T^{\mathbf{n}} F'_j \cap F_i$. Since $\mathcal{T}(x)$ refines $\mathcal{T}'(x)$, we have $\mathbf{n} + \mathbf{m} \in A'_j$, so elements of $T^{\mathbf{n}+\mathbf{m}}\zeta_j$ are levels of some towers $V_{j,E} \in \mathcal{Q}''$. Because $T^{\mathbf{n}+\mathbf{m}}(F'_j \cap T^{-\mathbf{n}} F_i)$ is a union of some elements of $T^{\mathbf{n}+\mathbf{m}}\zeta_j$, we conclude again that $T^{\mathbf{m}} F_i$ is a union of elements of $|\mathcal{Q}''|$. Finally, $|\mathcal{Q}''|$ refines $|\mathcal{Q}|$. ∎

PROPOSITION 6. *Suppose that the sequence* $(\mathcal{T}_n)_{n\in\mathbb{N}}$ *obeys* P4 *and each of* $\mathcal{T}_n$ *satisfies* P1, P2, *and* P3. *Then there is a Vershik model* $(\mathcal{Q}_n)$ *for* $(X, \mathbb{Z}^d)$ *such that each* $\mathcal{Q}_n$ *has* $\mathcal{T}_n$ *as its partition-valued function.*

Proof. By Lemma 4 the function $\mathcal{T}_1$ induces a KR decomposition $\mathcal{Q}_1$. Having defined decompositions $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ so that $\mathcal{Q}_{i+1}$ refines $\mathcal{Q}_i$ for each $i = 1, \ldots, n-1$, we again use Lemma 4 to produce $\mathcal{Q}'_{n+1}$ associated to $\mathcal{T}_{n+1}$, and then by Lemma 5 we obtain $\mathcal{Q}_{n+1}$ inducing the same $\mathcal{T}_{n+1}$, but refining $\mathcal{Q}_n$ and $\mathcal{Q}'_{n+1}$. ∎

Denote by $\mathcal{P}_t(x)$ the partition of $\mathbb{Z}^d$ into t-blocks induced by $x \in X$ (as defined in the previous section).

LEMMA 7. *Every partition* $\mathcal{P}_t(x)$, $x \in X$, $t \geq 1$, *has properties* P1–P3. *The sequence* $(\mathcal{P}_t)_{t\geq 1}$ *of partition-valued maps satisfies* P4.

Proof. Condition P1 is obvious, since the partition depends only on the distribution of markers which is appropriately shifted according to the action of $T^{\mathbf{n}}$. To verify P2 note that every atom of the partition $\mathcal{P}_t(x)$ is contained in a cube having edge length Q_t, regardless of x and of the position in $\mathbb{Z}^d$. If x and y are sufficiently close then $T^{\mathbf{n}}(x)$ and $T^{\mathbf{n}}(y)$ fall into the same markers for any fixed finite set of $\mathbf{n}$, which yields P3. P4 is exactly Lemma 2(1). ∎

The path from the sequence of partition-valued maps to a Bratteli diagram is completed by the following theorem:

THEOREM 8 ([4]). *Suppose that* $(X, \mathbb{Z}^d)$ *is a Cantor system with a faithful Vershik model* $(\mathcal{Q}_n)$. *Let* Y_n *be the boundary of the decomposition* $\mathcal{Q}_n$, *i.e. the* *union of the boundaries of its towers. Denote by $Y^{\#}$ the intersection of the (decreasing) sequence (Y_n) and let $Y = \bigcup_{\mathbf{n}\in\mathbb{Z}^d} T^{\mathbf{n}}Y^{\#}$. Then there is a Bratteli diagram $\mathcal{B}$ and a homeomorphism $\phi\colon P(\mathcal{B}) \to X$ for which:*

(1) *if $p, q \in P(\mathcal{B})$ are cofinal then $\phi(p)$ and $\phi(q)$ are coorbital,*
(2) *$x \in X \setminus Y$ implies $\phi^{-1}(x)$ is cofinal with $\phi^{-1}(T^{\mathbf{n}}x)$ for all $\mathbf{n} \in \mathbb{Z}^d$,*
(3) *for a free action, if $\phi^{-1}(x)$ is cofinal with $\phi^{-1}(T^{\mathbf{n}}x)$ for all $\mathbf{n} \in \mathbb{Z}^d$ then $x \in X \setminus Y$.*

The proof reveals the following relationship. In the Bratteli diagram every vertex at the level t represents a unique tower in $\mathcal{Q}_t$. If $(\mathcal{Q}_t)$ comes from the sequence $(\mathcal{P}_t)$ of partition-valued maps, which partition into t-blocks, to every such tower there corresponds a set of t-blocks which differ only in the second coordinate (meaning that the arrangement of markers is identical). For the sake of convenience we will keep using the same symbol v_t for a vertex of the Bratteli diagram and a tower in $\mathcal{Q}_t$, while we will denote by $v_t^{\mathbf{0}}(x)$ a corresponding t-configuration whose maximal vertex (in the sense of Lemma 2(3)) lies at the origin and bears the symbol (t, x). A t-configuration having the same pattern, but with maximal vertex at $-\mathbf{n}$, will be written $v_t^{\mathbf{n}}(x)$. If there is an edge originating at a vertex $v_t \in V_t$ and terminating at $v_{t+1} \in V_{t+1}$ then some t-configuration $v_t^{\mathbf{n}}(T^{-\mathbf{n}}x)$ is an element of the concatenation building up $v_{t+1}^{\mathbf{0}}(x)$. Note that $v_t^{\mathbf{n}}(T^{-\mathbf{n}}x)$ may belong to $v_{t+1}^{\mathbf{0}}(x)$ only if $\mathbf{0} \le^* \mathbf{n} \le^* \mathbf{Q}_t$, where $\mathbf{Q}_t = (Q_t, \dots, Q_t)$. The number of edges connecting v_t to v_{t+1} is equal to the number of occurrences of the towers v_t in v_{t+1}.

We are interested in defining a dynamics on the set of paths of the diagram. Before we do it we need to learn more about the boundaries of the partitions $\mathcal{P}_t$.

4. Boundaries of the partitions $\mathcal{P}_t$. Denote by $B_t^{\mathbf{n},j}$ the set of all elements $x \in X$ such that $\mathbf{n}$ and $\mathbf{n} + \mathbf{e}_j$ belong to distinct atoms of $\mathcal{P}_t(x)$. Clearly, the sequence $(B_t^{\mathbf{n},j})_{t\in\mathbb{N}}$ of sets is descending.

LEMMA 9. *Each $B_t^{\mathbf{n},j}$ is a clopen subset of X.*

Proof. Note that $B_t^{\mathbf{n},j} = T^{-\mathbf{n}}B_t^{\mathbf{0},j}$. Since $T^{\mathbf{n}}$ is a homeomorphism, it is enough to consider $B_t^{\mathbf{0},j}$. By uniform continuity of $x \mapsto \mathcal{P}_t(x)$ (see Lemma 7) both $B_t^{\mathbf{0},j}$ and its complement are open, because for sufficiently close x and y the partitions $\mathcal{P}_t(x)$ and $\mathcal{P}_t(y)$ agree on $\bar{\mathcal{K}}_1$. ∎

DEFINITION. A sequence $(\mathcal{P}_t(x))_{t\in\mathbb{N}}$ of partitions of $\mathbb{Z}^d$ has an *eternal boundary* between $\mathbf{n}$ and $\mathbf{n} + \mathbf{e}_j$ if $\mathbf{n}$ and $\mathbf{n} + \mathbf{e}_j$ belong to distinct atoms of $\mathcal{P}_t(x)$ for all $t \in \mathbb{N}$. In this case we will also say that x has an eternal boundary.

Note that either $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_j$ belong to distinct atoms of $\mathcal{P}_t(x)$ for all $t \in \mathbb{N}$, or there is $t_0 \in \mathbb{N}$ such that $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_j$ are separated only for $t \le t_0$.

PROPOSITION 10. *If X is minimal then the set of all points $x \in X$ with no eternal boundaries is residual.*

Proof. Note first that this set is equal to

$$\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \bigcap_{j=1}^{d} \bigcup_{t\in\mathbb{N}} (B_t^{\mathbf{n},j})^c,$$

thus it is a G_δ.

The property of having no eternal boundaries is $T^{\mathbf{n}}$-invariant for every $\mathbf{n}$, because $[T^{\mathbf{n}}x] = \sigma^{\mathbf{n}}[x]$. Moreover, there exists at least one point $x \in X$ which has no eternal boundaries; indeed, the set of such points has full measure by the arguments of [5] (compare with Lemma 5(2) there; note that we replace the first estimate there by the statement of the current Lemma 1(3)). Thus the orbit of such a point is contained in the set of points with no eternal boundaries and, by minimality of T, it is dense. ∎

REMARK 11. The set of all points which admit eternal boundaries is exactly the set $Y = \bigcup_{\mathbf{n}\in\mathbb{Z}^d} T^{\mathbf{n}} Y^{\#}$ mentioned in Theorem 8, where $Y^{\#} = \bigcap_{t\in\mathbb{N}} Y_t$ and each Y_t is the boundary of the decomposition $\mathcal{Q}_t$. Indeed, $T^{\mathbf{n}}x$ belongs to the boundary of a tower in $\mathcal{Q}_t$ if and only if $T^{\mathbf{n}}x$ and $T^{\mathbf{n}\pm\mathbf{e_j}}x$ belong to distinct towers of $\mathcal{Q}_t$ for certain $1 \le j \le d$, that is, $Y_t = \bigcup_{j=1}^{d} B_t^{\mathbf{0},j}$. Since $\mathcal{Q}_t$ refines $\mathcal{Q}_{t-1}$, for this j the points $T^{\mathbf{n}}x$ and $T^{\mathbf{n}\pm\mathbf{e_j}}x$ belong to distinct towers of $\mathcal{Q}_s$ for all $s \le t$. If $T^{\mathbf{n}}x \in Y^{\#}$ then some j is used infinitely many times, thus we can find j such that $T^{\mathbf{n}}x$ and $T^{\mathbf{n}\pm\mathbf{e_j}}x$ belong to distinct towers for all partitions $\mathcal{Q}_t$, and consequently

$$Y^{\#} = \bigcap_{t} \bigcup_{j=1}^{d} B_t^{\mathbf{0},j} = \bigcup_{j=1}^{d} \bigcap_{t} B_t^{\mathbf{0},j}.$$

5. Construction of the mapping. It was shown in [4] that any Vershik model of a Cantor system (X, Z^d) can be made into a faithful model by refining with a sequence of clopen partitions which refine to points. This procedure preserves both the boundary of the model and the shapes of towers. Thus, in view of the earlier statements the sequence of partitions of $\mathbb{Z}^d$ into t-blocks gives rise to a faithful Vershik model and a Bratteli diagram $\mathcal{B}$. The set $P(\mathcal{B})$ of all infinite paths of the diagram is homeomorphic to X. The vertices that belong to the tth level V_t may be represented by t-blocks occurring in $\mathcal{P}_t$. Denote $\mathcal{O}_T(x) = \{T^{\mathbf{n}}x \colon \mathbf{n} \in \mathbb{Z}^d\}$.

THEOREM 12. *For every free minimal $\mathbb{Z}^d$-action T on a Cantor set X and every $x_0 \in X$ there is a continuous injection $F : X \setminus \{x_0\} \to X$ such* *that*

$$\mathcal{O}_T(x) = \bigcup_{n\in\mathbb{Z}} F^n\{x\}$$

for every x from a residual subset X', namely the set of points having no eternal boundaries. (We assume that the image $F^n\{x\}$, $n < 0$, is empty if F^n is not defined at x.) Moreover, if x is not in X' then

$$\mathcal{O}_T(x) = \bigcup_{j=1}^{J} \bigcup_{n\in\mathbb{Z}} F^n\{x_j\}$$

for a finite set $\{x_1, \dots, x_J\}$.

Proof. Fix x_0 and a sequence of markers descending to x_0 as in Lemma 1. Let $\mathcal{B} = (V, E)$, $V = V_0 \cup V_1 \cup \cdots$, be a Bratteli diagram induced by the sequence of partitions into t-blocks. We introduce an order on the set E of edges of $\mathcal{B}$ in the following way. Consider the set $E(v_{t+1})$ of edges with range $v_{t+1} \in V_{t+1}$. The source of each of these edges represents a tower such that for some $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$ belonging to this tower the t-configuration $v_t^{\mathbf{n}}(T^{-\mathbf{n}}x)$ is contained in the $(t+1)$-configuration $v_{t+1}^{\mathbf{0}}(x)$. Each edge $e \in E(v_{t+1})$ corresponds to one occurrence of the tower v_t in v_{t+1}, so each edge may be given a label $\mathbf{n}$ coming from the appropriate $v_t^{\mathbf{n}}(T^{-\mathbf{n}}x)$. Since these labels are well-ordered by the maximolexicographic order with $\mathbf{0}$ as the vertex of the cone, we obtain a linear order on the set $E(v_{t+1})$.

We define a map on $\mathcal{B}$ mimicking the Vershik map. We cannot use the Vershik map itself, as the order may have more than one maximal path. Let $e(x) = (e_1(x), e_2(x), \dots)$ be an infinite path in $P(\mathcal{B})$ representing $x \in X$. For an edge $e_n = e_n(x)$ let $s(e_n)$ denote the source of the edge, and let $r(e_n)$ be the range of e_n. We define $G(e(x))$ as the path $f = (f_1, f_2, \dots)$ satisfying the following condition: if N is the smallest number such that $e_N(x)$ is not maximal, then let f_N be the edge succeeding $e_N(x)$ in the order on the edges with range $r(e_N(x))$, let $(f_1, \dots, f_{N-1})$ be the minimal path connecting the root to $s(f_N)$, and let $f_k = e_k$ for $k > N$. The map G remains undefined if $e(x)$ is a maximal path. Simultaneously we define $F(e(x))$ as $f = (f_1, f_2, \dots)$ such that: if N is the smallest number such that $e_N(x)$ is not minimal, then let f_N be the edge preceding $e_N(x)$ in the order on the edges with range $r(e_N(x))$, let $(f_1, \dots, f_{N-1})$ be the maximal path connecting the root to $s(f_N)$, and let $f_k = e_k$ for $k > N$. Again, F is undefined if $e(x)$ is a minimal path. However, if $e(x)$ is minimal then all neighbors $\mathbf{e}_i$ of $\mathbf{0}$ belong to t-configurations different from $v_t^{\mathbf{0}}(x)$, so x has an eternal boundary. Moreover, x lies in the intersection of all markers, so $x = x_0$. Clearly, $F\{e(x)\colon x \in X \setminus \{x_0\}\}$ is contained in the domain of G and $G(F(e(x))) = e(x)$. Similarly, $F(G(e(x))) = e(x)$ whenever G is defined at $e(x)$, thus F is injective.

We will now prove continuity of F. Let $[f_1,\dots,f_n]$ denote a cylinder set with fixed n starting edges. If there is $k \le n$ such that f_k is not maximal, while $f_1,\dots,f_{k-1}$ are maximal, then

$$F^{-1}[f_1,\dots,f_n] = [e_1,\dots,e_{k-1},e_k,f_{k+1},\dots,f_n],$$

where $e_1,\dots,e_{k-1},e_k,f_{k+1},\dots,f_n$ is a path such that $e_1,\dots,e_{k-1}$ are minimal and e_k is a successor of f_k. If all f_i's are maximal then

$$F^{-1}[f_1,\dots,f_n] = \bigcup_{e_1\in s^{-1}(r(f_n))} F^{-1}[f_1,\dots,f_n,e_1]$$

with every $F^{-1}[f_1,\dots,f_n,e_1]$ being a cylinder if e_1 is not maximal. If however e_1 is a maximal edge with source $r(f_n)$ then

$$F^{-1}[f_1,\dots,f_n,e_1] = \bigcup_{e_2\in s^{-1}(r(e_1))} F^{-1}[f_1,\dots,f_n,e_1,e_2],$$

where again the union consists of cylinders for e_2 not maximal and some other counterimages $F^{-1}[f_1,\dots,f_n,e_1,e_2]$. Continuing in this way and denoting by $E_{\max}$ the set of all maximal edges we obtain

$$\begin{aligned} F^{-1}[f_1,\dots,f_n] &= \text{a sum of cylinders}\\ &\qquad \cup \underbrace{F^{-1}\{(f_1,\dots,f_n,e_1,e_2,\dots)\colon e_1,e_2,\dots\in E_{\max}\}}_{=\emptyset},\end{aligned}$$

which is an open set.

Since F was defined on the set of all infinite paths of the diagram $\mathcal{B}$ excluding the unique minimal path, and $\mathcal{B}$ is homeomorphic to X, the map F naturally gives a continuous injective map on $X\setminus\{x_0\}$, where x_0 was represented by the minimal path of $\mathcal{B}$. This map will be denoted by the same letter F.

The compatibility of the orbits follows from the fact that for every $x\in X$ with no eternal boundaries and $\mathbf{n}\in\mathbb{Z}^d$, the points x and $T^{\mathbf{n}}x$ are cofinal in the Bratteli diagram, so they belong to the same F-orbit. If x has an eternal boundary then by Lemma 3 the array $[x]$ splits into finitely many (unbounded) pieces. Each of these pieces is tiled by a nested structure of t-blocks, which implies that the elements of one piece are cofinal in $\mathcal{B}$. ∎

Note that the map F may not be surjective. Even for points with no eternal boundaries it may happen that the orbit contains a point represented by the maximal path, thus having empty preimage. However, in the case of $\mathbb{Z}^2$ diagonal odometers (products of one-dimensional odometers), t-blocks are rectangles. The only points with eternal boundaries are those with marker ∞ and those with one horizontal line or one vertical line as eternal boundary. Any other orbit can be linearized according to our construction, giving a doubly infinite sequence. If S and T denote the horizontal and vertical shift on $\mathbb{Z}^2$ then one can additionally define the image of $[x]$, corresponding to the minimal path, to be $ST[x]$, which is represented by a maximal path. In this way one obtains a continuous invertible one-dimensional map defined *on the whole* X, maintaining multidimensional orbits on a residual subset of X. Moreover, one can show that in the case of one-dimensional systems it is possible to distribute markers quite regularly, so that the return time to the tth marker is either p_t or $p_t + 1$. This leaves hope that if one could obtain a specific distribution of markers for an arbitrary system (possibly not so regular in order to obtain a proper representation also in the presence of eternal boundaries) one would be able to prove orbit equivalence of d-dimensional and one-dimensional systems directly.

References

[1] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a simple transformation*, Ergodic Theory Dynam. Systems 1 (1981), 431–450.

[2] M. I. Cortez, $\mathbb{Z}^d$ *Toeplitz arrays*, Discrete Contin. Dynam. Systems 15 (2006), 859–881.

[3] H. Dye, *On groups of measure preserving transformations I, II*, Amer. J. Math. 81 (1959), 119–159; 85 (1963), 551–576.

[4] A. Forrest, *A Bratteli diagram for commuting homeomorphisms of the Cantor set*, Int. J. Math. 11 (2000), 177–200.

[5] B. Frej and A. Kwaśnicka, *Minimal models for* $\mathbb{Z}^d$*-actions*, Colloq. Math. 110 (2008), 461–476.

[6] T. Giordano, H. Matui, I. F. Putnam and C. F. Skau, *Orbit equivalence for Cantor minimal* $\mathbb{Z}^2$*-systems*, J. Amer. Math. Soc. 21 (2008), 863–892.

[7] T. Giordano, H. Matui, I. F. Putnam and C. F. Skau, *Orbit equivalence for Cantor minimal* $\mathbb{Z}^d$*-systems*, Invent. Math. 179 (2010), 119–158.

[8] T. Giordano, I. F. Putnam and C. F. Skau, *Topological orbit equivalence and* C^**-crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.

[9] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: The Rohlin lemma*, Bull. Amer. Math. Soc. 2 (1980), 161–164.

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