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On definably proper maps

by

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Abstract. In this paper we work in o-minimal structures with definable Skolem functions, and show that: (i) a Hausdorff definably compact definable space is definably normal; (ii) a continuous definable map between Hausdorff locally definably compact definable spaces is definably proper if and only if it is a proper morphism in the category of definable spaces. We give several other characterizations of definably proper, including one involving the existence of limits of definable types. We also prove the basic properties of definably proper maps and the invariance of definably proper (and definably compact) in elementary extensions and o-minimal expansions.

1. Introduction. Let $\mathbb{M} = (M, <, \ldots)$ be an arbitrary o-minimal structure with definable Skolem functions. In this paper we show that Hausdorff definably compact definable spaces are definably normal (Theorem 2.11). We also show a local almost everywhere curve selection for Hausdorff locally definably compact definable spaces (Theorem 2.18).

Theorem 2.11 was only known in special cases: it was proved by Berarducci and Otero for definable manifolds in o-minimal expansions of real closed fields ([1, Lemma 10.4]—the proof there works as well in o-minimal expansions of ordered groups); it was proved in [9] for definably compact groups in arbitrary o-minimal structures. Theorem 2.18 is an extension of the almost everywhere curve selection for M^n in arbitrary o-minimal structures proved by Peterzil and Steinhorn [17, Theorem 2.3].

In Corollary 4.6 and Proposition 4.8 we show that definably compact is invariant under elementary extensions and o-minimal expansions of $\mathbb{M}$. In Proposition 4.10 we show that if $\mathbb{M}$ is an o-minimal expansion of the ordered set of real numbers, then definably compact corresponds to compact. These

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invariance and comparison results extend similar ones for definably compact subsets of M^n in arbitrary o-minimal structures and answer partially a question from [17].

In the authors' recent work on the formalism of the six Grothendieck operations on o-minimal sheaves [7], [8] we require the basic theory of morphisms proper in the category of o-minimal spectral spaces similar to the theory of proper morphisms in semialgebraic geometry [3, Section 9] (and also in algebraic geometry [12, Chapter II, Section 4] or [11, Chapter II, Section 5.4]). Here, in Section 3, we provide such a theory by giving a category theory characterization of definably proper maps (as separated and universally closed morphisms in the category of definable spaces) and by proving the basic properties of such morphisms.

In Theorems 4.4 and 4.9 we show that definably proper is invariant under elementary extensions and o-minimal expansions of $\mathbb{M}$. In Theorem 4.11 we prove that if $\mathbb{M}$ is an o-minimal expansion of the ordered set of real numbers, then definably proper corresponds to proper. These invariance and comparison results transfer to the notion of proper morphism in the category of o-minimal spectral spaces.

The formalism of the six Grothendieck operations on o-minimal sheaves [7], [8] provides the cohomological ingredients required for the computation of the subgroup of m-torsion points of a definably compact, abelian definable group G, thus extending the main result of [6] which was proved in o-minimal expansions of ordered fields using the o-minimal singular (co)homology. This result is enough to settle Pillay's conjecture for definably compact definable groups [19], [15] in arbitrary o-minimal structures: see [5]. Pillay's conjecture is a nonstandard analogue of Hilbert's 5° problem for locally compact topological groups; roughly it says that after taking the quotient by a "small subgroup" (a smallest type-definable subgroup of bounded index) the quotient when equipped with the the so-called logic topology is a compact real Lie group of the same dimension.

Finally in Section 5 we prove that definable compactness of Hausdorff definable spaces can be characterized by the existence of limits of definable types (Theorem 5.2), extending a remark by Hrushovski and Loeser [14] in the affine case. In Theorem 5.3 we prove a corresponding characterization of definably proper maps between Hausdorff locally definably compact definable spaces which, when transferred to morphisms proper in the category of o-minimal spectral spaces, is the analogue of the valuative criterion for properness in algebraic geometry [12, Chapter II, Theorem 4.7]. As is known, in o-minimal structures with definable Skolem functions, definable types correspond to valuations [16], [18].

2. On definably compact spaces

2.1. Hausdorff definably compact spaces. Here we will show that if $\mathbb{M}$ has definable Skolem functions, then Hausdorff definably compact definable spaces are definably normal.

Below, we will assume the reader is familiar with basic o-minimality notions (see for example [4]). Below, by definable we mean definable in $\mathbb{M}$ possibly with parameters. Recall also that $\mathbb{M}$ *has definable Skolem functions* if for every uniformly definable family $\{F_t\}_{t\in T}$ of definable sets, there is a definable map $f : T \to \bigcup_t F_t$ such that for each $t \in T$ we have $f(t) \in F_t$.

Recall the notion of definable spaces [4]:

Definition 2.1. A *definable space* is a tuple $(X, (X_i, \theta_i)_{i\leq k})$ where:

- $X = \bigcup_{i\leq k} X_i$;
- each $\theta_i : X_i \to M^{n_i}$ is an injection such that $\theta_i(X_i)$ is a definable subset of M^{n_i} with the induced topology;
- for all i, j, $\theta_i(X_i \cap X_j)$ is an open definable subset of $\theta_i(X_i)$ and the transition maps $\theta_{ij} : \theta_i(X_i \cap X_j) \to \theta_j(X_i \cap X_j) : x \mapsto \theta_j(\theta_i^{-1}(x))$ are definable homeomorphisms.

We call the (X_i, θ_i)'s the *definable charts of* X and set

$$\dim X = \max\{\dim \theta_i(X_i) : i = 1, \ldots, k\}.$$

If all the $\theta_i(X_i)$'s are open definable subsets of some M^n, we say that X is a *definable manifold of dimension* n.

A definable space X has a topology such that each X_i is open and the θ_i's are homeomorphisms: a subset U of X is open in this topology if and only if for each i, $\theta_i(U \cap X_i)$ is an open definable subset of $\theta_i(X_i)$.

A map $f : X \to Y$ between definable spaces with definable charts $(X_i, \theta_i)_{i\leq k}$ and $(Y_j, \delta_j)_{j\leq l}$ respectively is a *definable map* if

- for all i and j with $f(X_i) \cap Y_j \neq \emptyset$, $\delta_j \circ f \circ \theta_i^{-1} : \theta_i(X_i) \to \delta_j(Y_j)$ is a definable map between definable sets.

We say that a definable space is *affine* if it is definably homeomorphic to a definable set with the induced topology.

The construction above defines the *category of definable spaces with definable continuous maps* which we denote by Def. All topological notions on definable spaces are relative to the topology above. Note however that often we will have to replace topological notions on definable spaces by their definable analogues.

We say that a subset A of a definable space X is *definable* if for each i, $\theta_i(A \cap X_i)$ is a definable subset of $\theta_i(X_i)$. A definable subset A of a definable space X is naturally a definable space and its topology is the induced topology, thus we also call them *definable subspaces*.

In nonstandard o-minimal structures closed and bounded definable sets are not compact. Thus we have to replace the notion of compactness by a suitable definable analogue.

Let X be a definable space and $C \subseteq X$ a definable subset. By a *definable curve in* C we mean a continuous definable map $\alpha : (a,b) \to C \subseteq X$, where $a < b$ are in $M \cup \{-\infty, \infty\}$. We say that a definable curve $\alpha : (a,b) \to C \subseteq X$ in C is *completable in* C if both limits $\lim_{t\to a^+} \alpha(t)$ and $\lim_{t\to b^-} \alpha(t)$ exist in C, equivalently if there exists a continuous definable map $\overline{\alpha} : [a,b] \to C \subseteq X$ such that the following diagram is commutative:

$$\begin{tikzcd} (a,b) \arrow[r, "\alpha"] \arrow[d, hook] & C \subseteq X \\ {[a,b]} \arrow[ur, dashed, "\overline{\alpha}"'] & \end{tikzcd}$$

DEFINITION 2.2. Let X be a definable space and $C \subseteq X$ a definable subset. We say that C is *definably compact* if every definable curve in C is completable in C (see [17]).

The following is easy:

FACT 2.3. *Suppose* $\mathbb{M}$ *has definable Skolem functions. Let* $f : X \to Y$ *be a continuous definable map between definable spaces. If* $K \subseteq X$ *is a definably compact definable subset, then so is* $f(K) \subseteq Y$.

For definable subsets $X \subseteq M^n$ with their induced topology (i.e., affine definable spaces) the notion of definably compact is very well behaved. Indeed, we have [17, Theorem 2.1]:

FACT 2.4. *A definable subset* $X \subseteq M^n$ *is definably compact if and only if it is closed and bounded in* M^n.

However, in general, definably compact definable subsets of a definable space are not Hausdorff and are not even necessarily closed subsets:

EXAMPLE 2.5 (Non-Hausdorff and nonclosed definably compact subsets). Let $a,b,c,d \in M$ be such that $c < a < b < d$. Let X be the definable space with definable charts $(X_i, \theta_i)_{i=1,2}$ given by $X_1 = (\{\langle x,y\rangle \in [c,d]\times[c,d] : x = y\} \setminus \{\langle b,b\rangle\}) \cup \{\langle b,a\rangle\} \subseteq M^2$, $X_2 = \{\langle x,y\rangle \in [c,d]\times[c,d] : x = y\} \subseteq M^2$ and $\theta_i = \pi_{|X_i}$ where $\pi : M^2 \to M$ is the projection onto the first coordinate. Then any open definable neighborhood in X of $\langle b,a\rangle$ intersects any open definable neighborhood in X of $\langle b,b\rangle$. Clearly X is definably compact but not Hausdorff, and X_2 is a definably compact subset which is not closed (in X).

It is desirable to work in a situation where definably compact subsets are closed. We will show that this is the case in Hausdorff definable spaces when $\mathbb{M}$ has definable Skolem functions. First, we need to introduce some notation.

Let X be a definable space and let $(X_i, \theta_i)_{i\leq k}$ be the definable charts of X with $\theta_i(X_i) \subseteq M^{n_i}$. Let $N = n_1 + \cdots + n_k$ and fix $* \in M$. For each $i \leq k$, let $\pi_i : M^N = M^{n_1} \times \cdots \times M^{n_k} \to M^{n_i}$ be the natural projection and let $\rho_i : M^{n_i} \to M^N$ be the inclusion with $\rho_i(M^{n_i}) = \{*\} \times \cdots \times \underbrace{M^{n_i}}_{\text{position } i} \times \cdots \times \{*\} \subseteq M^N$. Identify each M^{n_i} with $\rho_i(M^{n_i}) \subseteq M^N$. Identify also each $\theta_i(X_i)$ with $\rho_i(\theta_i(X_i)) \subset M^N$ and each θ_i with $\rho_i \circ \theta_i$.

For $a \in X$, let $I_a = \{i \leq k : a \in X_i\}$ and set

$$D(a) = \Big\{ \langle \langle d_1^-, d_1^+ \rangle, \ldots, \langle d_N^-, d_N^+ \rangle \rangle \in M^{2N} : \\ \theta_j(a) \in \pi_j\Big(\prod_{l=1}^{N} (d_l^-, d_l^+) \Big) \text{ for all } j \in I_a \Big\}.$$

Consider the finite set $I_X = \{I \subseteq \{1, \ldots, k\} : I = I_a \text{ for some } a \in X\}$. Then each $X_I = \{x \in X : I_x = I\}$ with $I \in I_X$ is a definable subset and $X = \bigsqcup_{I \in I_X} X_I$. Therefore, $\{D(a)\}_{a\in X}$ is a uniformly definable family of definable sets, since it is defined by the first-order formula

$$\bigvee_{I\in I_X} \Big[(a \in X_I) \wedge \bigwedge_{j\in I} \bigwedge_{l=N_{j-1}+1}^{N_j} (d_l^- < \theta_j(a)_l < d_l^+) \Big]$$

where for each $i \leq k$ we set $N_i = n_1 + \cdots + n_i$ and where $\theta_j(a)_l$ is the l-coordinate of $\theta_j(a) \in M^N$.

For $d, d' \in D(a)$ we set $d \preceq d'$ if $\prod_{l=1}^N (d_l^-, d_l^+) \subseteq \prod_{l=1}^N (d_l'^-, d_l'^+)$, and write $d \prec d'$ whenever $\prod_{l=1}^N (d_l^-, d_l^+) \subset \prod_{l=1}^N (d_l'^-, d_l'^+)$.

The following are immediate:

- (D0) The relation $\preceq$ on $D(a)$ is a definable downwards directed order on $D(a)$.
- (D1) $D(a) \subseteq M^{2N}$ is an open definable subset.
- (D2) If $d \in D(a)$ then $\{d' \in D(a) : d' \prec d\}$ is an open definable subset of $D(a)$.

For $a \in X$ and $d = \langle \langle d_1^-, d_1^+ \rangle, \ldots, \langle d_N^-, d_N^+ \rangle \rangle \in D(a)$, set

$$U(a, d) = \bigcap_{j\in I_a} \theta_j^{-1}\Big(\theta_j(X_j) \cap \pi_j\Big(\prod_{l=1}^N (d_l^-, d_l^+) \Big) \Big).$$

Then $\{U(a, d)\}_{d\in D(a)}$ is a uniformly definable system of fundamental open definable neighborhoods of a in X.

The following will also be useful:

(D3) If $a, a' \in X$ are such that $I_{a'} \subseteq I_a$, then for every $d \in D(a) \cap D(a')$ we have $U(a,d) \subseteq U(a',d)$.

Finally we will also require:

(D4) If $a \in X$ and W is an open definable neighborhood of a, then $\{d \in D(a) : U(a,d) \subset W\}$ is an open definable subset of $D(a)$.

If $B \subseteq X$ is a definable subset and $\epsilon : B \to M^{2N}$ is a definable map such that $\epsilon(x) \in D(x)$ for all $x \in B$, then

$$U(B,\epsilon) = \bigcup_{x \in B} U(x, \epsilon(x))$$

is an open definable neighborhood of B in X. This implies:

Remark 2.6. The notions of open (resp. closed) in a definable space X are first-order in the sense that if $(A_t)_{t\in T}$ is a uniformly definable family of definable subsets of X, then the set of all $t \in T$ such that A_t is an open (resp. a closed) subset of X is a definable set.

Recall that a topological space X is *regular* if the following equivalent conditions hold:

- for every $a \in X$ and $S \subseteq X$ closed such that $a \notin S$, there are open disjoint subsets U and V of X such that $a \in U$ and $S \subseteq V$;
- for every $a \in X$ and $W \subseteq X$ open such that $a \in W$, there is an open subset V of X such that $a \in V$ and $\overline{V} \subseteq W$.

Proposition 2.7. *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a Hausdorff definable space. Then for any $a \in X$ and any definably compact subset $K \subseteq X$ such that $a \notin K$, there are finitely many definably compact subsets K_i $(i = 1, \ldots, l)$ of K, continuous definable functions $\epsilon_i : K_i \to M^{2N}$ with $\epsilon_i(x) \in D(x)$ for all $x \in K_i$, and a $d \in D(a)$ such that*

$$K \subseteq \bigcup_{i=1}^{l} U(K_i, \epsilon_i), \qquad U(a,d) \cap \bigcup_{i=1}^{l} U(K_i, \epsilon_i) = \emptyset.$$

In particular, if X is a Hausdorff, definably compact definable space, then X is regular.

Proof. We fix $a \in X$ and prove the result by induction on the dimension of definably compact subsets $K \subseteq X$ such that $a \notin K$.

If $\dim K = 0$, then this follows because X is Hausdorff. Assume the result holds for every definably compact subset L of X such that $a \notin L$ and $\dim L < \dim K$.

Since X is Hausdorff, for each $x \in K$ there are $d' \in D(x)$ and $d \in D(a)$ such that $U(a,d) \cap U(x,d') = \emptyset$. By definable Skolem functions there are definable maps $g : K \to M^{2N}$ and $h : K \to M^{2N}$ such that, for all $x \in K$,

$$g(x) \in D(a), \quad h(x) \in D(x), \quad U(a, g(x)) \cap U(x, h(x)) = \emptyset.$$

Since, by Remark 2.6, continuity is first-order, the subset of K where either g or h is not continuous is a definable subset. By working in charts and using [4, Chapter 3, (2.11), and Chapter 4, (1.8)] this definable subset has dimension $< \dim K$, and if L is the closure of this subset, then $\dim L < \dim K$. By induction hypothesis, there are finitely many definably compact subsets L_i ($i = 1, \ldots, k$) of L, continuous definable functions $\epsilon_i : L_i \to M^{2N}$ with $\epsilon_i(x) \in D(x)$ for all $x \in L_i$, and a $d_L \in D(a)$ such that

$$L \subseteq \bigcup_{i=1}^{k} U(L_i, \epsilon_i), \quad U(a, d_L) \cap \bigcup_{i=1}^{k} U(L_i, \epsilon_i) = \emptyset.$$

Let $K' = K \setminus \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$. Then K' is definably compact and both $g_| : K' \to M^{2N}$ and $h_| : K' \to M^{2N}$ are continuous. We show that there is $d_{K'} \in D(a)$ such that $d_{K'} \preceq g_|(x)$ for all $x \in K'$.

Write $g(x) = \langle\langle g^-(x)_1, g^+(x)_1\rangle, \ldots, \langle g^-(x)_N, g^+(x)_N\rangle\rangle \in D(a)$ where, for each $l = 1, \ldots, N$, $g^-(x)_l$ and $g^+(x)_l$ are the two l-components of $g(x)$.

By Fact 2.3, for each $l = 1, \ldots, N$, let $d_l^- = \max\{g^-(x)_l : x \in K'\}$ and $d_l^+ = \min\{g^+(x)_l : x \in K'\}$. Since each d_l^- equals $g^-(z)_l$ for some $z \in K'$, and similarly each d_l^+ equals $g^+(z')_l$ for some $z' \in K'$, we have $d_{K'} := \langle\langle d_1^-, d_1^+\rangle, \ldots, \langle d_N^-, d_N^+\rangle\rangle \in D(a)$. By construction we also have $d_{K'} \preceq g_l(x)$ for all $x \in K'$.

To finish the proof, choose $d \preceq d_L, d_{K'}$ by (D0) and, for each $i = 1, \ldots, k$, set $K_i = L_i$ and also take $K_{k+1} = K'$ and $\epsilon_{k+1} = h_{|K'}$. Then, by construction,

$$K \subseteq \bigcup_{i=1}^{k+1} U(K_i, \epsilon_i), \quad U(a, d) \cap \bigcup_{i=1}^{k+1} U(K_i, \epsilon_i) = \emptyset. \ ■$$

The following is now immediate:

Corollary 2.8. *Suppose that $\mathbb{M}$ has definable Skolem functions, and X is a Hausdorff definable space. If K is a definably compact subset of X, then K is a closed definable subset.*

We will require the following:

Lemma 2.9. *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a Hausdorff, definably connected, definable space and $K \subseteq X$ a definably compact subset. Let $\epsilon : K \to M^{2N}$ be a definable continuous map such that $\epsilon(x) \in D(x)$ for all $x \in K$, and suppose that for each $w \in K$ there is*

$d \in D(w)$ such that $\epsilon(w) \prec d$ and $\overline{U(w,d)}$ is definably compact. Then

$$\bigcup_{x\in K} \overline{U(x,\epsilon(x))}$$

is a closed definably compact definable neighborhood of K. In particular,

$$\overline{U(K,\epsilon)} = \overline{\bigcup_{x\in K} U(x,\epsilon(x))} = \bigcup_{x\in K} \overline{U(x,\epsilon(x))}.$$

Proof. Let $\alpha : (a,b) \to \bigcup_{x\in K} \overline{U(x,\epsilon(x))}$ be a definable curve. We have to show that $\lim_{t\to b^-} \alpha(t)$ exists in $\bigcup_{x\in K} \overline{U(x,\epsilon(x))}$.

By definable Skolem functions there is a definable map $\beta : (a,b) \to K$ such that for each $t \in (a,b)$ we have

$$\alpha(t) \in \overline{U\big(\beta(t), \epsilon(\beta(t))\big)}.$$

By o-minimality, after shrinking (a,b) if necessary, i.e., after replacing a by $a' \in (a,b)$, we may assume that β is a definable curve in K. Since K is definably compact, let $w = \lim_{t\to b^-} \beta(t) \in K$. Let also $\overline{\beta} : (a,b] \to K$ be the continuous definable map such that $\overline{\beta}_{|(a,b)} = \beta_{|(a,b)}$.

Recall that $\epsilon\circ\overline{\beta}(b) = \epsilon(w) \in D(w)$ and $D(w) \subseteq M^{2N}$ is an open definable subset by (D1). Since $\epsilon : K \to M^{2N}$ is continuous, it follows from the continuity of $\epsilon\circ\overline{\beta} : (a,b] \to M^{2N}$ at b that there is $a' \in (a,b)$ such that $\epsilon\circ\overline{\beta}(t) \in D(w)$ for all $t \in [a',b]$.

Since for each $j \in I_w$, X_j is an open definable neighborhood of w, by continuity, after shrinking $(a',b]$ if necessary, we may assume that $\overline{\beta}(t) \in X_j$ for all $t \in [a',b]$ and all $j \in I_w$. Thus $I_w \subseteq I_{\overline{\beta}(t)}$ for all $t \in [a',b]$. Therefore, by (D3), for all $t \in [a',b]$ we have $U(\overline{\beta}(t), \epsilon(\overline{\beta}(t))) \subseteq U(w, \epsilon(\overline{\beta}(t)))$.

In particular, for each $t \in [a',b)$ we have

$$\alpha(t) \in \overline{U(w, \epsilon(\overline{\beta}(t)))}.$$

By hypothesis there is $d \in D(w)$ such that $\epsilon(w) = \epsilon(\overline{\beta}(b)) \prec d$ and $\overline{U(w,d)}$ is definably compact. By (D2) and continuity of $\epsilon\circ\overline{\beta} : [a',b] \to D(w) \subseteq M^{2N}$, after shrinking $(a',b]$ if necessary, we may further assume $\epsilon(\overline{\beta}(t)) \prec d$ for all $t \in [a',b]$. Therefore,

$$\alpha(t) \in \overline{U(w,d)}$$

for all $t \in [a',b)$.

As $\overline{U(w,d)}$ is definably compact, the limit $\lim_{t\to b^-} \alpha(t) \in \overline{U(w,d)}$ exists. Let $v = \lim_{t\to b^-} \alpha(t) \in \overline{U(w,d)}$. We want to show that $v \in \overline{U(w,\epsilon(w))}$. Suppose not and set $L = \overline{U(w,\epsilon(w))}$. Since L is a definably compact subset of $\overline{U(w,d)}$, by Proposition 2.7 there are finitely many definably compact subsets L_i $(i = 1,\dots,k)$ of L, continuous definable functions $\epsilon_i : L_i \to M^{2N}$

with $\epsilon_i(x) \in D(x)$ for all $x \in L_i$, and a $d_L \in D(v)$ such that

$$L \subseteq \bigcup_{i=1}^{k} U(L_i, \epsilon_i), \qquad U(v, d_L) \cap \bigcup_{i=1}^{k} U(L_i, \epsilon_i) = \emptyset.$$

We have $U(w, \epsilon(w)) \subseteq L \subseteq \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$. If $U(w, \epsilon(w)) = \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$ then $U(w, \epsilon(w)) = L = \overline{U(w, \epsilon(w))}$, and so $U(w, \epsilon(w))$ is a closed and open definable subset of X. Since X is definably connected we would have $U(w, \epsilon(w)) = X$ and so $v \in \overline{U(w, \epsilon(w))}$, which is a contradiction.

Since $U(w, \epsilon(w)) \subset \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$ and $\bigcup_{i=1}^{k} U(L_i, \epsilon_i)$ is an open definable neighborhood of w, by (D4) there is $a'' \in [a', b]$ such that $U(w, \epsilon(\overline{\beta}(t))) \subset \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$ for all $t \in [a'', b]$. Therefore, for each $t \in [a'', b]$ we have

$$\alpha(t) \in \overline{\bigcup_{i=1}^{k} U(L_i, \epsilon_i)}.$$

This implies $v \in \overline{\bigcup_{i=1}^{k} U(L_i, \epsilon_i)}$, contradicting $U(v, d_L) \cap \bigcup_{i=1}^{k} U(L_i, \epsilon_i) = \emptyset$.

By Corollary 2.8, $\bigcup_{x \in K} \overline{U(x, \epsilon(x))}$ is closed and hence

$$\overline{U(K, \epsilon)} = \overline{\bigcup_{x \in K} U(x, \epsilon(x))} = \bigcup_{x \in K} \overline{U(x, \epsilon(x))}. \ \blacksquare$$

Recall that a definable space X is *definably normal* if the following equivalent conditions hold:

(1) for any disjoint closed definable subsets Z_1 and Z_2 of X there are disjoint open definable subsets U_1 and U_2 of X such that $Z_i \subseteq U_i$ for $i = 1, 2$.

(2) for every $S \subseteq X$ closed definable and $W \subseteq X$ open definable such that $S \subseteq W$, there is an open definable subsets U of X such that $S \subseteq U$ and $\overline{U} \subseteq W$.

In general, regular does not imply definably normal:

EXAMPLE 2.10 (Regular non-definably normal definable space). Assume that $\mathbb{M} = (M, <)$ is a dense linearly ordered set with no end points. Let $a, b, c, d \in M$ be such that $c < a < b < d$ and let $X = (c, d) \times (c, d) \setminus \{\langle a, b\rangle\}$. Since X is affine, it is regular. Note also that the only open definable subsets of X are the intersections with X of definable subsets of M^2 which are finite unions of nonempty finite intersections $W_1 \cap \cdots \cap W_k$ where each W_i is either an open box in M^2, $\{\langle x, y\rangle \in M^2 : x < y\}$ or $\{\langle x, y\rangle \in M^2 : y < x\}$.

Let $C = \{\langle x, y\rangle \in X : x = a\}$ and $D = \{\langle x, y\rangle \in X : y = b\}$. Then C and D are closed disjoint definable subsets of X. However, by the description of open definable subsets of X, there are no open disjoint definable subsets U and V of X such that $C \subseteq U$ and $D \subseteq V$.

THEOREM 2.11. *Suppose that $\mathbb{M}$ has definable Skolem functions. If X is a Hausdorff, definably compact definable space, then X is definably normal. In fact, for every closed definable subset $K \subseteq X$ and every open definable subset $V \subseteq X$, if $K \subseteq V$ then there are finitely many definably compact subsets K_i $(i = 1, \ldots, l)$ of K and continuous definable functions $\epsilon_i : K_i \to M^{2N}$ with $\epsilon_i(x) \in D(x)$ for all $x \in K_i$ such that*

$$K \subseteq \bigcup_{i=1}^{l} U(K_i, \epsilon_i), \qquad \bigcup_{i=1}^{l} \overline{U(K_i, \epsilon_i)} \subseteq V.$$

Proof. Clearly we may assume that X is definably connected, and we can fix an open definable subset $V \subseteq X$. We prove the result by induction on the dimension of closed definable subsets $K \subseteq X$ such that $K \subset V$.

If $\dim K = 0$ then the result follows since X is regular (Proposition 2.7). So assume that the result holds for every closed definable subset L such that $L \subseteq V$ and $\dim L < \dim K$.

Since X is regular (Proposition 2.7), for each $x \in K$ there is $d \in D(x)$ such that $\overline{U(x, d)} \subseteq V$. Since the property "$d \in D(x)$ and $\overline{U(x, d)} \subseteq V$" is first-order (Remark 2.6), by definable Skolem functions there is a definable map $\delta : K \to M^{2N}$ such that, for all $x \in K$,

$$\delta(x) \in D(x), \qquad \overline{U(x, \delta(x))} \subseteq V.$$

By definable Skolem functions again and by (D2), there is a definable map $\epsilon : K \to M^{2N}$ such that, for all $x \in K$,

$$\epsilon(x) \in D(x), \qquad \epsilon(x) \prec \delta(x), \qquad \overline{U(x, \epsilon(x))} \subseteq V.$$

Since, by Remark 2.6, continuity is first-order, the subset of K where ϵ is not continuous is a definable subset. By working in charts and using [4, Chapter 3, (2.11), and Chapter 4, (1.8)] this definable subset has dimension $< \dim K$, and if L is the closure of this subset, then $\dim L < \dim K$. By induction hypothesis, there are finitely many definably compact subsets L_i $(i = 1, \ldots, k)$ of L and continuous definable functions $\epsilon_i : L_i \to M^{2N}$ with $\epsilon_i(x) \in D(x)$ for all $x \in L_i$ such that

$$L \subseteq \bigcup_{i=1}^{k} U(L_i, \epsilon_i), \qquad \bigcup_{i=1}^{k} \overline{U(L_i, \epsilon_i)} \subseteq V.$$

Let $K' = K \setminus \bigcup_{i=1}^{k} U(L_i, \epsilon_i)$. Then K' is a closed definable subset and $\epsilon' = \epsilon_| : K' \to M^{2N}$ is continuous. Furthermore, for each $w \in K'$ there is $d = \delta(w) \in D(w)$ such that $\epsilon'(w) \prec d$ and $\overline{U(w, d)}$ is definably compact. Therefore, by Lemma 2.9, we have $\overline{U(K', \epsilon')} \subseteq V$.

For each $i = 1, \ldots, k$, set $K_i = L_i$ and also take $K_{k+1} = K'$ and $\epsilon_{k+1} = \epsilon'$. Then, by construction,

$$K \subseteq \bigcup_{i=1}^{k+1} U(K_i, \epsilon_i), \qquad \bigcup_{i=1}^{k+1} \overline{U(K_i, \epsilon_i)} \subseteq V. \ \blacksquare$$

Definable normality gives the shrinking lemma (compare with [4, Chapter 6, (3.6)]):

Corollary 2.12 (The shrinking lemma). *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a Hausdorff definably compact definable space. If $\{U_i : i = 1, \ldots, n\}$ is a covering of X by open definable subsets, then there are definable open subsets V_i and definable closed subsets C_i of X $(1 \le i \le n)$ with $V_i \subseteq C_i \subseteq U_i$ and $X = \bigcup\{V_i : i = 1, \ldots, n\}$.*

2.2. Local almost everywhere curve selection. To prove our results about definably proper maps later we will need a local version of an extension to definable spaces of the almost everywhere curve selection [17, Theorem 2.3]:

Fact 2.13. *If $C \subseteq M^n$ is a definable subset which is not closed, then there is a definable set $E \subseteq \overline{C} \setminus C$ such that $\dim E < \dim(\overline{C} \setminus C)$ and for every $x \in \overline{C} \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point.*

We say that *almost everywhere curve selection holds for a definable space* X if for every definable subset $C \subseteq X$ which is not closed, there is a definable set $E \subseteq \overline{C} \setminus C$ such that $\dim E < \dim(\overline{C} \setminus C)$ and for every $x \in \overline{C} \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point.

For general definable spaces, even affine ones, even if $\mathbb{M}$ has definable Skolem functions, almost everywhere curve selection does not hold:

Example 2.14. (1) In $\mathbb{M} = (\mathbb{Q}, <)$, for the definable set $D = \{\langle x, y\rangle \in \mathbb{Q}^2 : 0 < y < x\}$ there is no definable curve in D with limit $d = \langle 0, 0\rangle$. (This example is from [17].)

(2) Let $\Gamma = (\mathbb{R}, <, 0, -, +, (q)_{q\in\mathbb{Q}})$. Let $\Gamma_0 = \{0\} \times \Gamma$ and $\Gamma_1 = \{1\} \times \Gamma$, and ∞ be a new symbol such that $\langle 0, x\rangle < \infty < \langle 1, y\rangle$ for all $x, y \in \mathbb{R}$. Let $M = \Gamma_0 \cup \{\infty\} \cup \Gamma_1$ be equipped with the natural induced total order from $<$. Let $\mathbb{M}$ be the structure obtained by including on Γ_0 and on Γ_1 the structure from Γ. Then $\mathbb{M}$ has definable Skolem functions (since each copy of Γ has definable Skolem functions by [4, Chapter 6, (1.2)]). However, for the definable set $D = \{\langle\langle 0, x\rangle, \langle 0, y\rangle\rangle \in M^2 : x, y > 0\}$ there is no definable curve in D with limit $d = \langle\langle 0, 0\rangle, \infty\rangle$. Indeed, any definable curve in D will be definable in Γ and so its graph will be a piecewise linear subset of D [4, Chapter 1, (7.8)]. By piecewise linearity there are no definable bijections between bounded and unbounded intervals, and so no definable curve in D

will have $d = \langle\langle 0,0\rangle,\infty\rangle$ as a limit point. (This example is essentially the same as the Γ_∞ from [14, Section 4.1]; the only difference is that we added a new copy of Γ, the Γ_1, so that our M has no endpoints.)

In both cases, if $X = D \cup \{d\}$ then almost everywhere curve selection does not hold for the definable space X since in X we have $\overline{D} \setminus D = \{d\}$.

Almost everywhere curve selection fails for $X \subseteq M^2$ in Example 2.14 because X is not a locally closed subset of M^2:

Lemma 2.15. *Suppose that X is a definable space and that almost everywhere curve selection holds for X. Then it also holds for every locally closed definable subset of X.*

Proof. Let Z be a closed definable subset of X and let $C \subseteq Z$ be a definable subset which is not closed in Z. Since Z is closed, $\overline{C} = \mathrm{cl}_Z(C) \subseteq Z$ (the closure of C in Z), so $\overline{C} \setminus C = \mathrm{cl}_Z(C) \setminus C \neq \emptyset$ and the result follows by the assumption on X.

Let U be an open definable subset of X and let $C \subseteq U$ be a definable subset which is not closed in U. Note that $\overline{C} \cap U = \mathrm{cl}_U(C)$. Let $B = C \cup (\overline{C} \setminus U) = C \cup ((\overline{C} \setminus C) \setminus U)$. Then $\emptyset \neq \mathrm{cl}_U(C) \setminus C = \overline{C} \cap U \setminus C = (\overline{C} \setminus C) \cap U = \overline{B} \setminus B$ and the result follows by applying the assumption on X to B. Note that any definable curve in B with limit a point in $\overline{B} \setminus B \subseteq U$ must enter U, and so gives a definable curve in $C = B \cap U$.

Let $Z \cap U$ be a general locally closed definable subset of X, where Z is a closed definable subset and U is an open definable subset. Let $C \subseteq Z \cap U$ be a definable subset which is not closed in $Z \cap U$. Then $\mathrm{cl}_{Z\cap U}(C) = \overline{C} \cap U = \mathrm{cl}_U(C)$ and $\mathrm{cl}_U(C) \setminus C = \mathrm{cl}_{Z\cap U}(C) \setminus C \neq \emptyset$, and therefore the result follows from the previous case. ∎

Lemma 2.16. *Suppose that X is a definable space and V and W are open definable subsets such that $V \cup W = X$ and almost everywhere curve selection holds for V and W. Then it also holds for X.*

Proof. Let $C \subseteq X = V \cup W$ be a definable subset which is not closed. Let $C_V = C \cap V \subseteq V$ and let $C_W = C \cap W \subseteq W$. Then we have $C = C_V \cup C_W$, $\overline{C} = \overline{C_V} \cup \overline{C_W}$ and $\mathrm{cl}_V(C_V) = \overline{C_V} \cap V = \overline{C} \cap V$, and similarly $\mathrm{cl}_W(C_W) = \overline{C_W} \cap W = \overline{C} \cap W$. So $\overline{C} = (\overline{C} \cap V) \cup (\overline{C} \cap W) = \mathrm{cl}_V(C_V) \cup \mathrm{cl}_W(C_W)$. Hence, $\overline{C} \setminus C = (\mathrm{cl}_V(C_V) \setminus C) \cup (\mathrm{cl}_W(C_W) \setminus C) = (\mathrm{cl}_V(C_V) \setminus C_V) \cup (\mathrm{cl}_W(C_W) \setminus C_W)$.

If C_V is not closed in V, by the hypothesis, there is a definable set $F_V \subseteq \mathrm{cl}_V(C_V) \setminus C_V$ such that $\dim F_V < \dim(\mathrm{cl}_V(C_V) \setminus C_V)$ and for every $x \in \mathrm{cl}_V(C_V) \setminus (C_V \cup F_V)$ there is a definable curve in C_V which has x as a limit point. Similarly, if C_W is not closed in W, there is a definable set $F_W \subseteq \mathrm{cl}_W(C_W) \setminus C_W$ such that $\dim F_W < \dim(\mathrm{cl}_W(C_W) \setminus C_W)$ and for every $x \in \mathrm{cl}_W(C_W) \setminus (C_W \cup F_W)$ there is a definable curve in C_W which has x as a limit point. Let E_V be F_V if the latter exists, and $\emptyset$ otherwise. Similarly, let E_W be F_W if it exists, and $\emptyset$ otherwise. Let $E = E_V \cup E_W$. Since $\overline{C} \setminus C = (\mathrm{cl}_V(C_V) \setminus C_V) \cup (\mathrm{cl}_W(C_W) \setminus C_W)$ we have $E \subseteq \overline{C} \setminus C$. As $C = C_V \cup C_W$, for every $x \in \overline{C} \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point. Since $\dim E = \max\{\dim E_V, \dim E_W\}$ and $\dim \overline{C} \setminus C = \max\{\dim(\mathrm{cl}_V(C_V) \setminus C_V), \dim(\mathrm{cl}_W(C) \setminus C_W)\}$ we also have $\dim E < \dim(\overline{C} \setminus C)$, as required. ∎

By Fact 2.13, Lemma 2.15 and an induction argument using Lemma 2.16 we see that:

COROLLARY 2.17. *Almost everywhere curve selection holds for locally closed definable subsets of definable manifolds.*

Let X be a definable space. We say that X is *locally definably compact* if every $x \in X$ has a definably compact neighborhood.

We now have the following extension of almost everywhere curve selection to the nonaffine case, which will be useful later:

THEOREM 2.18 (Local almost everywhere curve selection). *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a Hausdorff, locally definably compact definable space. If $C \subseteq X$ is a definable subset which is not closed, then for every $z \in \overline{C} \setminus C$ there is a definable open neighborhood V of z in X such that $\overline{V}$ is definably compact, and there is a definable set $E \subseteq (\overline{C} \setminus C) \cap V$ such that* $\dim E < \dim((\overline{C} \cap V) \setminus (C \cap V))$, *and for every $x \in (\overline{C} \cap V) \setminus ((C \cap V) \cup E)$ there is a definable curve in $C \cap V$ which has x as a limit point.*

Proof. By the assumption on X we get V such that $\overline{V}$ is definably compact and so definably normal (Theorem 2.11). The result then follows once we show that almost everywhere curve selection holds for definably normal, definably compact definable spaces Y.

Let Y be such a space. Consider the definable charts $(U_i, \phi_i)_{i=1}^{l}$ of Y. Since Y is definably normal, by the shrinking lemma there are open definable subsets V_i $(1 \le i \le l)$ and closed definable subsets C_i $(1 \le i \le l)$ such that $V_i \subseteq C_i \subseteq U_i$ and $Y = \bigcup\{V_i : i = 1, \ldots, l\}$. Since each C_i is definably compact and each ϕ_i is a definable homeomorphism, we see that each $\phi_i(C_i)$ is a closed (and bounded) definable subset of M^{n_i}, and hence, by Fact 2.13 and Lemma 2.15, each $\phi_i(C_i)$ and so each C_i has almost everywhere curve selection. Therefore, by Lemma 2.15, each V_i has almost everywhere curve selection. Now we conclude by induction on l, using Lemma 2.16, that Y has almost everywhere curve selection. ∎

The second part of the proof of Theorem 2.18 shows:

Corollary 2.19. *Almost everywhere curve selection holds for definably normal, definably compact definable spaces, even without assuming that* $\mathbb{M}$ *has definable Skolem functions.*

3. Proper morphisms in Def

3.1. Preliminaries. Here we recall some preliminary notions for the category Def whose objects are definable spaces and whose morphisms are continuous definable maps between definable spaces.

Let $f : X \to Y$ be a morphism in Def. We say that:

- $f : X \to Y$ is *closed in* Def (i.e., *definably closed*) if for every object A of Def such that A is a closed subset of X, its image $f(A)$ is a closed (definable) subset of Y.
- $f : X \to Y$ is a *closed* (resp. *open*) *immersion* if $f : X \to f(X)$ is a homeomorphism and $f(X)$ is a closed (resp. open) subset of Y.

Proposition 3.1. *In the category* Def *the* cartesian square *of any two morphisms* $f : X \to Z$ *and* $g : Y \to Z$ *in* Def *exists and is given by a commutative diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where the morphisms p_X *and* p_Y *are known as* projections. *The cartesian square has the following universal property: for any other object* Q *of* Def *and morphisms* $q_X : Q \to X$ *and* $q_Y : Q \to Y$ *of* Def *for which the following diagram commutes:*

$$\begin{tikzcd} Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_Y"] \arrow[ddr, bend right, "q_X"'] & & \\ & X \times_Z Y \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

there exists a unique natural morphism $u : Q \to X \times_Z Y$ *(called the* mediating morphism) *making the whole diagram commute. As with all universal constructions, the cartesian square is unique up to a definable homeomorphism.*

Proof. The usual fiber product $X \times_Z Y = \{\langle x, y\rangle \in X \times Y : f(x) = g(y)\}$ (a closed definable subspace of the definable space $X \times Y$) together with the restrictions $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ of the usual projections determine a cartesian square in the category Def. ∎

Given a morphism $f : X \to Y$ in Def, the corresponding *diagonal morphism* is the unique morphism $\Delta : X \to X \times_Y X$ in Def given by the universal property of cartesian squares:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_Y"'] \arrow[d, "p_X"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

We say that $f : X \to Y$ is *separated in* Def if the corresponding diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion.

We say that an object Z in Def is *separated in* Def if the morphism $Z \to \{\mathrm{pt}\}$ to a point is separated.

REMARK 3.2. Since in the above diagram we have $p_X \circ \Delta = p_Y \circ \Delta = \mathrm{id}_X$, it is clear that the following are equivalent:

(1) $f : X \to Y$ (resp. Z) is separated in Def.
(2) The image of the corresponding diagonal morphism $\Delta : X \to X \times_Y X$ is a closed (definable) subset of $X \times_Y X$ (resp. the diagonal Δ_Z of Z is a closed (definable) subset of $Z \times Z$).

Let Z be an object of Def and $s : Z' \to Z$ a morphism in Def.

- By a *morphism over* Z *in* Def we mean a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\
& Z
\end{tikzcd}$$

  of morphisms in Def.
- We call $s : Z' \to Z$ a *base extension in* Def, and the induced commutative diagram

$$\begin{tikzcd}
X \times_Z Z' \arrow[r, "f'"] \arrow[dr] & Y \times_Z Z' \arrow[d] \\
& Z'
\end{tikzcd}$$

  where $f' = f \times \mathrm{id}_{Z'}$ and the down arrows are the natural projections is called the *base extension in* Def *of*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\
& Z
\end{tikzcd}
$$

> Note that since the f' above is completely determined by the corresponding morphism over Z we will often just say that $f' : X \times_Z Z' \to Y \times_Z Z'$ is the corresponding *base extension morphism.*

Let $f : X \to Y$ be a morphism in Def. We say that $f : X \to Y$ is *universally closed in* Def if for any morphism $g : Y' \to Y$ in Def the morphism $f' : X' \to Y'$ in Def obtained from the cartesian square

$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
$$

in Def is closed in Def.

Definition 3.3. We say that a morphism $f : X \to Y$ in Def is *proper in* Def if $f : X \to Y$ is separated and universally closed in Def.

Definition 3.4. We say that an object Z of Def is *complete in* Def if the morphism $Z \to \{\mathrm{pt}\}$ is proper in Def.

Below we will relate the notion of proper in Def and complete in Def with the usual notions of definably proper and definably compact.

3.2. Separated and proper in Def**.** Here we list the main properties of morphisms separated or proper in Def.

From Remark 3.2 and the way cartesian squares are defined in Def we easily obtain the following:

Remark 3.5. Let $f : X \to Y$ be a morphism in Def. Then the following are equivalent:

(1) $f : X \to Y$ is separated in Def.
(2) The fibers $f^{-1}(y)$ of f are Hausdorff (with the induced topology).

Directly from the definitions (as in [10, Chapter I, Propositions 5.5.1 and 5.5.5]) or more easily from Remark 3.5 the following is immediate:

Proposition 3.6. *In the category* Def*:*

(1) *Injective continuous definable maps are separated in* Def.
(2) *A composition of two morphisms separated in* Def *is separated in* Def.

(3) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd}$$

be a morphism over Z *in* Def *and let* $Z' \to Z$ *be a base extension in* Def. *If* $f : X \to Y$ *is separated in* Def, *then the corresponding base extension morphism* $f' : X \times_Z Z' \to Y \times_Z Z'$ *is separated in* Def.

(4) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[dr, "p'"'] & Y' \arrow[d, "q'"] \\ & Z \end{tikzcd}$$

be morphisms over Z *in* Def. *If* $f : X \to Y$ *and* $f' : X' \to Y'$ *are separated in* Def, *then the corresponding product morphism* $f \times f' : X \times_Z X' \to Y \times_Z Y'$ *is separated in* Def.

(5) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms such that* $g \circ f$ *is separated in* Def, *then* f *is separated in* Def.

(6) *A morphism* $f : X \to Y$ *is separated in* Def *if and only if* Y *can be covered by finitely many open definable subsets* V_i *such that* $f_| : f^{-1}(V_i) \to V_i$ *is separated in* Def.

Directly from the definitions (as in [11, Chapter II, Proposition 5.4.2 and Corollary 5.4.3], see also [3, Section 9]) one has the following. For the reader's convenience we include some details:

PROPOSITION 3.7. *In the category* Def:

(1) *Closed immersions are proper in* Def.
(2) *A composition of two morphisms proper in* Def *is proper in* Def.
(3) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd}$$

be a morphism over Z *in* Def *and* $Z' \to Z$ *a base extension in* Def. *If* $f : X \to Y$ *is proper in* Def, *then the corresponding base extension morphism* $f' : X \times_Z Z' \to Y \times_Z Z'$ *is proper in* Def.

(4) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[dr, "p'"'] & Y' \arrow[d, "q'"] \\ & Z \end{tikzcd}$$

be morphisms over Z in Def. *If $f : X \to Y$ and $f' : X' \to Y'$ are proper in* Def*, then the corresponding product morphism $f \times f' : X \times_Z X' \to Y \times_Z Y'$ is proper in* Def.

(5) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is proper in* Def*, then:*

(i) *f is proper in* Def;

(ii) *if g is separated in* Def *and f is surjective, then g is proper in* Def.

(6) *A morphism $f : X \to Y$ is proper in* Def *if and only if Y can be covered by finitely many open definable subsets V_i such that $f_| : f^{-1}(V_i) \to V_i$ is proper in* Def.

Proof. (1) Let $X \to Y$ be a closed immersion and $Y' \to Y$ a morphism in Def. Since $X \times_Y Y' \to Y \times_Y Y' = Y'$ is also a closed immersion, it is closed in Def. So $X \to Y$ is universally closed, and it is separated by Proposition 3.6(1).

(2) Let $X \to Y$ and $Y \to Z$ be morphisms proper in Def and let $Z' \to Z$ be a morphism in Def. Since $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ and $X \times_Z Z' \to Z'$ is $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$, the result follows from the fact that the composition of morphisms closed in Def is closed in Def, and from Proposition 3.6(2).

(3) Since $X \times_Z Z' = X \times_Y (Y \times_Z Z')$, for every morphism $W \to Y \times_Z Z'$ we have

$$(X \times_Z Z') \times_{Y \times_Z Z'} W = (X \times_Y (Y \times_Z Z')) \times_{Y \times_Z Z'} W = X \times_Y W.$$

Hence, since $X \times_Y W \to W$ is closed in Def by hypothesis, the result follows by using also Proposition 3.6(3).

(4) The product morphism $X \times_Z X' \to Y \times_Z Y'$ is the composition of the base extension $X \times_Z X' \to Y \times_Z X'$, the identification $Y \times_Z X' = X' \times_Z Y$ and the base extension $X' \times_Z Y \to Y' \times_Z Y$. So (4) follows from (1) and (3).

(5) Let $X \to Y$ and $Y \to Z$ be morphisms in Def such that the composition $X \to Y \to Z$ is proper in Def.

(i) Let $Y' \to Y$ be a morphism in Def. Then $X \times_Z Y' \to Y'$ obtained with the composition $Y' \to Y \to Z$ is the same as the composition of $X \times_Z Y' \to X \times_Y Y'$, which is surjective, with $X \times_Y Y' \to Y'$. Therefore, since $X \times_Z Y' \to Y'$ is closed in Def, so is $X \times_Y Y' \to Y'$, and the result follows by using also Proposition 3.6(5).

(ii) Let $Z' \to Z$ be a morphism in Def. Then

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f\times \mathrm{id}_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

is a commutative diagram with $f \times \mathrm{id}_{Z'}$ surjective and p closed in Def by hypothesis. It follows that p' is closed in Def as required.

(6) Suppose that $f : X \to Y$ is a morphism in Def and let $\{V_i\}_{i\leq k}$ be a finite cover of Y by open definable subsets. If $g : Y' \to Y$ is a morphism in Def, then $\{f^{-1}(V_i)\}_{i\leq k}$ (resp. $\{g^{-1}(V_i)\}_{i\leq k}$) is a finite cover of X (resp. Y') by open definable subsets, and $\{f^{-1}(V_i) \times_Y g^{-1}(V_i)\}_{i\leq k}$ is a finite cover of $X \times_Y Y'$ by open definable subsets. One the other hand, $f^{-1}(V_i) \times_Y g^{-1}(V_i) = f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$ and

$$\begin{tikzcd} f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \arrow[r, "i"] \arrow[d, "p'_i"] & X \times_Y Y' \arrow[d, "p'"] \\ g^{-1}(V_i) \arrow[r, "j"] & Y' \end{tikzcd}$$

is a commutative diagram with i and j the inclusions, p' the projection and p'_i the restriction of p'. Since p' is closed in Def if and only if each p'_i is closed in Def, the result follows by using also Proposition 3.6(6). ∎

Corollary 3.8. *Let $f : X \to Y$ be a morphism in* Def *and $Z \subseteq X$ an object in* Def *which is complete in* Def. *Then:*

1. *Z is a closed (definable) subset of X.*
2. *$f_{|Z} : Z \to Y$ is proper in* Def.
3. *$f(Z) \subseteq Y$ is (definable) complete in* Def.
4. *If $f : X \to Y$ is proper in* Def *and $C \subseteq Y$ is an object in* Def *which is complete in* Def, *then $f^{-1}(C) \subseteq X$ is (definable) complete in* Def.

From Proposition 3.7 we also obtain in a standard way the following:

Corollary 3.9. *Let* **B** *be a full a subcategory of the category of definable spaces* Def *whose set of objects is:*

- *closed under taking locally closed definable subspaces of objects of* **B***;*
- *closed under taking cartesian products of objects of* **B**.

Then the following are equivalent:

1. *Every object X of* **B** *is* completable in **B**, *i.e., there exists an object X' of* **B** *which is complete in* Def *together with an open immersion*

$i : X \hookrightarrow X'$ in $\mathbf{B}$ *with $i(X)$ dense in X'. Such an $i : X \hookrightarrow X'$ is called a* completion *of X in* $\mathbf{B}$*.*

(2) *Every morphism $f : X \to Y$ in* $\mathbf{B}$ *is* completable in $\mathbf{B}$, *i.e., there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of morphisms in $\mathbf{B}$ *such that i is a completion of X in* $\mathbf{B}$*, and j is a completion of Y in* $\mathbf{B}$*.*

(3) *Every morphism $f : X \to Y$ in* $\mathbf{B}$ *has a* proper extension in $\mathbf{B}$, *i.e., there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

of morphisms in $\mathbf{B}$ *such that ι is a open immersion with $\iota(X)$ dense in P and $\overline{f}$ proper in* Def*.*

Proof. Assume that (1) holds. Let $h : X \to Y$ be a morphism in $\mathbf{B}$. Let $j : Y \to Y'$ be a completion of Y in $\mathbf{B}$. Choose also a completion $g : X \to X''$ of X in $\mathbf{B}$ and note that $g \times j : X \times Y \to X'' \times Y'$ is a completion of $X \times Y$ in $\mathbf{B}$ (since $X'' \times Y'$ is complete in Def by Proposition 3.7(4)). Let X' be the closure of $(g \times j)(\Gamma(h))$ in $X'' \times Y'$. Then $i : X \to X'$ given by $i = (g \times j) \circ (\mathrm{id}_X \times h)$ is a completion of X in $\mathbf{B}$ (by Proposition 3.7(1) & (5)), and the restriction of the projection $X'' \times Y' \to Y'$ to X' is a morphism $h' : X' \to Y'$ completing a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of morphisms in $\mathbf{B}$ as required in (2).

Assume that (2) holds. Let $h : X \to Y$ be a morphism in $\mathbf{B}$. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of morphisms in $\mathbf{B}$ such that i is a completion of X in $\mathbf{B}$, and j is a completion of Y in $\mathbf{B}$. Let $P = h'^{-1}(j(Y))$ (an open definable subspace of X') and

$\overline{h} = j^{-1} \circ h'_{|P} : P \to Y$ where $j^{-1} : j(Y) \to Y$ is the inverse of $j : Y \to j(Y)$ which is a definable homeomorphism. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "h"'] & P \arrow[d, "\overline{h}"] \\ & Y \end{tikzcd}$$

of morphisms in $\mathbf{B}$ such that $\iota = i : X \to P$ is a definable open immersion with $\iota(X)$ dense in P and $\overline{h}$ is proper in Def (since $h' : X' \to Y'$ is proper in Def by Corollary 3.8(2)) as required in (3).

Assume that (3) holds. Let X be an object of $\mathbf{B}$. Take $h : X \to \{\mathrm{pt}\}$ to be the morphism in $\mathbf{B}$ to a point. Applying (3) to this morphism we obtain (1). ∎

3.3. Definably proper maps. Here we recall the definition of a definably proper map between definable spaces and prove its main properties. A special case of this theory appears in [4, Chapter 6, Section 4] in the context of affine definable spaces in o-minimal expansions of ordered groups.

Definition 3.10. A continuous definable map $f : X \to Y$ between definable spaces X and Y is called *definably proper* if for every definably compact definable subset K of Y its inverse image $f^{-1}(K)$ is a definably compact definable subset of X.

From the definitions we deduce:

Remark 3.11. A definable space X is definably compact if and only if the map $X \to \{\mathrm{pt}\}$ is definably proper.

Typical examples of definably proper continuous definable maps are: (i) $f : X \to Y$ where X is a definably compact definable space and Y is any definable space; (ii) the projection $X \times Y \to Y$ where X is a definably compact definable space and Y is any definable space; (iii) closed definable immersions.

The following is proved just as in the affine case in o-minimal expansions of ordered groups treated in [4, Chapter 6, Lemma (4.5)]:

Theorem 3.12. *Let $f : X \to Y$ be a continuous definable map. Suppose that every definably compact subset of Y is a closed subset (e.g. $\mathbb{M}$ has definable Skolem functions and Y is Hausdorff). Then the following are equivalent:*

(1) *f is definably proper.*
(2) *For every definable curve $\alpha : (a, b) \to X$ and every continuous definable map $[a, b] \to Y$ forming a commutative diagram*

$$\begin{tikzcd}
(a,b) \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\
{[a,b]} \arrow[r] \arrow[ur, dashed, "\overline{\alpha}"'] & Y
\end{tikzcd}$$

there is at least one continuous definable map $[a,b] \to X$ making the whole diagram commutative.

Proof. Assume that (1) holds. Let $\alpha : (a,b) \to X$ be a definable curve in X such that $f \circ \alpha$ is completable in Y, say $\lim_{t \to b^-} f \circ \alpha(t) = y \in Y$. Take $c \in (a,b)$ and set $K = \{f(\alpha(t)) : t \in [c,b)\} \cup \{y\} \subseteq Y$. Then K is a definably compact definable subset of Y, and so $f^{-1}(K)$ is a definably compact definable subset of X containing $\alpha((c,b))$. Thus α must be completable in $f^{-1}(K)$, hence in X.

Assume that (2) holds. Suppose that f is not definably proper. Then there is a definably compact definable subset K of Y such that $f^{-1}(K)$ is not a definably compact definable subset of X. Thus there is a definable curve $\alpha : (a,b) \to f^{-1}(K) \subseteq X$ in $f^{-1}(K)$ which is not completable in $f^{-1}(K)$. Since $f^{-1}(K)$ is closed (by assumption on Y, K is closed), α is not completable. But $f \circ \alpha : (a,b) \to K \subseteq Y$ is completable, which contradicts (2). ∎

By Theorem 3.12 we have the following result which summarizes the main properties of definably proper maps.

Corollary 3.13. *Let* $\mathbf{A}$ *be a full subcategory of* Def *such that every definably compact subset of an object of* $\mathbf{A}$ *is a closed subset. Suppose that the set of objects of* $\mathbf{A}$ *is:*

- *closed under taking locally closed definable subsets of objects of* $\mathbf{A}$*;*
- *closed under taking cartesian products of objects of* $\mathbf{A}$*.*

Then in the category $\mathbf{A}$*:*

(1) *Closed immersions are definably proper.*
(2) *A composition of two definably proper morphisms is definably proper.*
(3) *Let*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\
& Z
\end{tikzcd}$$

be a morphism over Z *in* $\mathbf{A}$ *and* $Z' \to Z$ *a base extension in* $\mathbf{A}$*. If* $f : X \to Y$ *is definably proper, then the corresponding base extension morphism* $f' : X \times_Z Z' \to Y \times_Z Z'$ *is definably proper.*

(4) *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[dr, "p'"'] & Y' \arrow[d, "q'"] \\ & Z \end{tikzcd}$$

be morphisms over Z in **A**. *If $f : X \to Y$ and $f' : X' \to Y'$ are definably proper, then the corresponding product morphism $f \times f' : X \times_Z X' \to Y \times_Z Y'$ is definably proper.*

(5) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is definably proper, then:*

(i) *f is definably proper;*

(ii) *if $\mathbb{M}$ has definable Skolem functions, then $g_{|f(X)} : f(X) \to Z$ is definably proper.*

(6) *A morphism $f : X \to Y$ is definably proper if and only if Y can be covered by finitely many open definable subsets V_i such that $f_| : f^{-1}(V_i) \to V_i$ is definably proper.*

Proof. (1) Consider the commutative diagram

$$\begin{tikzcd} (a,b) \arrow[r, "\alpha"] \arrow[dd] & X \arrow[d, "f"] \\ & f(X) \arrow[d] \\ {[a,b]} \arrow[r, "\overline{\alpha}"'] \arrow[uur, dashed, "\gamma'"] \arrow[ur, dashed, "\gamma"'] & Y \end{tikzcd}$$

where $f : X \to Y$ is a definable closed immersion and we assume we have α such that $\overline{\alpha}$ exists. We must show that γ' exists. As the inclusion $f(X) \subseteq Y$ is closed and we have $f \circ \alpha$ such that $\overline{\alpha}$ exists, γ exists. So, since $f : X \to f(X)$ is a definable homeomorphism, we let $\gamma' = f^{-1} \circ \gamma$.

(2) Consider the commutative diagram

$$\begin{tikzcd} (a,b) \arrow[r, "\alpha"] \arrow[dd] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ {[a,b]} \arrow[r, "\overline{\alpha}"'] \arrow[uur, dashed, "\gamma'"] \arrow[ur, dashed, "\gamma"'] & Z \end{tikzcd}$$

where we assume that α is such that $\overline{\alpha}$ exists. We must show that γ' exists.

Since $g : Y \to Z$ is definably proper and $f \circ \alpha$ is such that $\overline{\alpha}$ exists, by Theorem 3.12, γ exists. Since $f : X \to Y$ is definably proper and α is such that γ exists, by Theorem 3.12, γ' exists.

(3) Since the base extension morphism is a special case of the product morphism, the result follows from (4) below.

(4) Consider the commutative diagram

$$\begin{tikzcd}
 & (a,b) \arrow[r] \arrow[dl, "p_X \circ \alpha"'] \arrow[d] & {[a,b]} \arrow[dll, dashed, "\alpha"'] \arrow[dl, dashed, "\gamma'"'] \arrow[ddl, dashed] \arrow[d, "\overline{\alpha}"] & \\
X \arrow[ddr] \arrow[drr, bend right=60, "f"] & X \times_Z X' \arrow[l, "p_X"] \arrow[r, "f \times f'"'] \arrow[d, "p_{X'}"'] & Y \times_Z Y' \arrow[d, "q_Y"] \arrow[dr, "q_{Y'}"] & \\
 & X' \arrow[d] \arrow[rr, bend right=60, "f'"'] & Y \arrow[dl] & Y' \arrow[dll] \\
 & Z & &
\end{tikzcd}$$

where we assume that α is such that $\overline{\alpha}$ exists. We must show that $\gamma' : [a, b] \to X \times_Z X'$ exists. Since $f : X \to Y$ is definably proper and $p_X \circ \alpha$ is such that $q_Y \circ \overline{\alpha}$ exists, by Theorem 3.12, $[a, b] \to X$ exists. Since $f' : X' \to Y'$ is definably proper and $p_{X'} \circ \alpha$ is such that $q_{Y'} \circ \overline{\alpha}$ exists, by Theorem 3.12, $[a, b] \to X'$ exists. So we let γ' be the morphism given by the universal property of cartesian squares.

(5) (i) Consider the commutative diagram

$$\begin{tikzcd}
(a,b) \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\
{[a,b]} \arrow[r, "\overline{\alpha}"] \arrow[ur, dashed, "\gamma'"] \arrow[dr, bend right, "g \circ \overline{\alpha}"'] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

where we assume that α is such that $\overline{\alpha}$ exists. We must show that γ' exists. Since $g \circ f : X \to Y$ is definably proper and α is such that $g \circ \overline{\alpha}$ exists, by Theorem 3.12, γ' exists.

(ii) Consider the commutative diagram

$$\begin{tikzcd}
 & X \arrow[d, "f"] \\
(a,b) \arrow[ur, bend left, "\beta"] \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, "g"] \\
{[a,b]} \arrow[uur, dashed, "\gamma"] \arrow[ur, dashed, "\gamma'"'] \arrow[r, "\overline{\alpha}"'] & Z
\end{tikzcd}$$

where we assume that α is such that $\overline{\alpha}$ exists. We must show that γ' exists. Since f is surjective, by definable Skolem functions let β be such that $\alpha = f \circ \beta$. Since $g \circ f : X \to Y$ is definably proper and β is such that $\overline{\alpha}$ exists, by Theorem 3.12, γ exists. Now take $\gamma' = f \circ \gamma$.

(6) One implication is clear. Suppose that there are open definable subsets $V_1, \ldots, V_l$ of Y such that each restriction $f_| : f^{-1}(V_i) \to V_i$ is definably proper. Let $\alpha : (a,b) \to X$ be a definable curve such that $f \circ \alpha : (a,b) \to Y$ is completable. Without loss of generality it is enough to show that $\lim_{t \to b^-} \alpha(t)$ exists in X. Let $z = \lim_{t \to b^-} f \circ \alpha(t) \in Y$ and let i be such that $z \in V_i$. By continuity, let $c \in (a,b)$ be such that $f \circ \alpha([c,b)) \subseteq V_i$. Then $\alpha_| : (c,b) \to f^{-1}(V_i) \subseteq X$ is a definable curve in $f^{-1}(V_i)$ such that $f_| \circ \alpha_| : (c,b) \to V_i$ is completable. By hypothesis, $\alpha_| : (c,b) \to f^{-1}(V_i) \subseteq X$ is completable in $f^{-1}(V_i)$, and so $\lim_{t \to b^-} \alpha(t)$ exists in X as required. ∎

From Corollary 3.13 we obtain as in Corollary 3.9 the following analogue for definably proper. In the case of o-minimal expansions of real closed fields this can be read off from [4, Chapter 10, (2.6) and (2.7)].

Corollary 3.14. *Let* $\mathbf{B}$ *be a full subcategory of* Def. *Suppose that the set of objects of* $\mathbf{B}$ *is:*

- *closed under taking locally closed definable subspaces of objects of* $\mathbf{B}$*;*
- *closed under taking cartesian products of objects of* $\mathbf{B}$*.*

Then the following are equivalent:

(1) *Every object* X *of* $\mathbf{B}$ *is* definably completable in $\mathbf{B}$, *i.e., there exists a definably compact space* X' *in* $\mathbf{B}$ *together with a definable open immersion* $i : X \hookrightarrow X'$ *in* $\mathbf{B}$ *with* $i(X)$ *dense in* X'. *Such an* $i : X \hookrightarrow X'$ *is called a* definable completion *of* X *in* $\mathbf{B}$.

(2) *Every morphism* $f : X \to Y$ *in* $\mathbf{B}$ *is* definably completable in $\mathbf{B}$, *i.e., there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of morphisms in $\mathbf{B}$ *such that* i *is a definable completion of* X *in* $\mathbf{B}$, *and* j *is a definable completion of* Y *in* $\mathbf{B}$.

(3) *Every morphism* $f : X \to Y$ *in* $\mathbf{B}$ *has a* definably proper extension in $\mathbf{B}$, *i.e., there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

of morphisms in $\mathbf{B}$ *such that* ι *is a definable open immersion with* $\iota(X)$ *dense in* P *and* $\overline{f}$ *definably proper.*

If $\mathbf{B} = \mathrm{Def}$, we do not mention $\mathbf{B}$ and we talk of *definably completable, definable completion* and *definably proper extension.*

3.4. Definably proper and proper in Def**.** Assuming that $\mathbb{M}$ has definable Skolem functions, we will: (i) show that a definably proper map between Hausdorff locally definably compact definable spaces is the same as a morphism proper in Def; (ii) prove the definable analogue of the topological characterization of the notion of proper continuous maps (as closed maps with compact and Hausdorff fibers).

THEOREM 3.15. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *and* Y *be Hausdorff definable spaces with* Y *locally definably compact. Let* $f : X \to Y$ *be a continuous definable map. Then the following are equivalent:*

(1) f *is proper in* Def.
(2) f *is definably proper.*

Proof. Assume that (1) holds. Let $\gamma : (a, b) \to X$ be a definable curve in X and suppose that $f \circ \gamma : (a, b) \to Y$ is completable. By Theorem 3.12, we need to show that $\gamma : (a, b) \to X$ is completable in X. By assumption $f \circ \gamma$ extends to a continuous definable map $g : [a, b] \to Y$. Consider a cartesian square of continuous definable maps

$$\begin{tikzcd}
(a,b) \arrow[dr, dashed, "\gamma'"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right] & & \\
& X \times_Y [a,b] \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& {[a,b]} \arrow[r, "g"] & Y
\end{tikzcd}$$

together with the continuous definable map $\gamma' : (a,b) \to X \times_Y [a,b]$ obtained from the maps $\gamma : (a,b) \to X$ and $(a,b) \hookrightarrow [a,b]$.

Consider $\overline{\gamma'((a,b))} \subseteq X \times_Y [a,b]$. By assumption, $f' : X \times_Y [a,b] \to [a,b]$ is definably closed. So $f'(\overline{\gamma'((a,b))})$ is a closed definable subset of $[a,b]$. But $(a,b) = f'(\gamma'((a,b))) \subseteq f'(\overline{\gamma'((a,b))})$ and so $f'(\overline{\gamma'((a,b))}) = [a,b]$. Hence there are $u, v \in \overline{\gamma'((a,b))}$ such that $f'(u) = a$ and $f'(v) = b$. Since f' is the restriction of the projection $X \times [a,b] \to [a,b]$, f' is definably open. Therefore, $\lim_{t\to a^+} \gamma'(t) = u$ and $\lim_{t \to b^-} \gamma'(t) = v$ and $\gamma' : (a,b) \to X \times_Y [a,b]$ is completable in $X \times_Y [a,b]$. Thus $\gamma = g' \circ \gamma'$ is completable in X as required.

Assume that (2) holds. Since $f : X \to Y$ is separated in Def (Remark 3.5), it is enough to show that f is universally closed in Def. For that it suffices to consider a cartesian square of continuous definable maps

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and show that f' is definably closed (i.e., closed in Def).

Let $A \subseteq X \times_Y Z$ be a closed definable subset and suppose that $f'(A) \subseteq Z$ is not closed. By local almost everywhere curve selection (Theorem 2.18) there is $z \in \overline{f'(A)} \setminus f'(A)$ together with a definable curve $\beta : (a,b) \to f'(A) \subseteq Y$ such that $\lim_{t \to b^-} \beta(t) = z$. By replacing (a,b) with a smaller subinterval we may assume that $\lim_{t\to a^+} \beta(t)$ exists in Z, so β is completable in Z. By definable Skolem functions, after replacing (a,b) with a smaller subinterval, there exists a definable curve $\gamma : (a,b) \to X$ in X such that for every $t \in (a,b)$ we have $\langle \gamma(t), \beta(t)\rangle \in A$. Since $f \circ \gamma = g \circ \beta$ and β is completable in Z, $g \circ \beta$ is completable in Y. Thus by (2) and Theorem 3.12, γ is completable in X and $\lim_{t\to b^-} \gamma(t)$ exists in X; call it x. If $\alpha = \langle \gamma, \beta \rangle : (a,b) \to X \times_Y Z$, then $\lim_{t \to b^-} \alpha(t) = \langle x, z\rangle \in X \times_Y Z$ and so $\langle x, z \rangle \in A$ because A is closed. But then $z = f'(x,z) \in f'(A)$, which is absurd. ∎

Note that the assumption that Y is locally definably compact is needed:

EXAMPLE 3.16. Consider the setting of Example 2.14 and let $X = D$, $Y = D \cup \{d\}$ and $f : X \to Y$ be the inclusion. Then $f : X \to Y$ is definably proper, Y is not locally definably compact and f is not definably closed.

COROLLARY 3.17. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *be a Hausdorff definable space. Then the following are equivalent:*

(1) X *is definably compact.*
(2) X *is complete in* Def.

The following is the definable analogue of the topological characterization of the notion of proper continuous maps (as closed maps with compact and Hausdorff fibers). A similar result appears in the semialgebraic case [3, Theorem 12.5]:

THEOREM 3.18. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *and* Y *be Hausdorff definable spaces with* Y *locally definably compact. Let* $f : X \to Y$ *be a continuous definable map. Then the following are equivalent:*

(1) f *is definably proper.*
(2) f *is definably closed and has definably compact fibers.*

Proof. Assume that (1) holds. Then $f : X \to Y$ has definably compact fibers and, by Theorem 3.15, f is definably closed.

Assume that (2) holds. Let K be a definably compact definable subset of Y. Let $\alpha : (a, b) \to f^{-1}(K)$ be a definable curve in $f^{-1}(K)$. Suppose that $\lim_{t\to b^-} \alpha(t)$ does not exist in $f^{-1}(K)$. Then this limit does not exist in X either, since $f^{-1}(K)$ is a closed definable subset of X (by Corollary 2.8, K is closed). Therefore, if $d \in (a, b)$, then for every $e \in [d, b)$, $\alpha([e, b))$ is a closed definable subset of X contained in $f^{-1}(K)$. Indeed, we can first replace a by $a' \in (a, b)$ if necessary so that α is injective, and so $\alpha((a, b))$ has a definable total order such that α is increasing; then if $\alpha([e, b))$ is not closed, one can use local almost everywhere curve selection (Theorem 2.18) to obtain a definable curve $\delta : (a', b') \to \alpha([e, b))$ with say $\lim_{t\to b'^-} \delta(t) \in \mathrm{cl}_X(\alpha([e, b))) \setminus \alpha([e, b))$; after replacing a' by some $a'' \in (a', b')$ if necessary, δ will be strictly increasing, but then we would have $\lim_{t\to b^-} \alpha(t) = \lim_{t\to b'^-} \delta(t)$.

By assumption, for every $e \in [d, b)$, $f \circ \alpha([e, b))$ is then a closed definable subset of Y contained in K. Since K is definably compact, the limit $\lim_{t\to b^-} f \circ \alpha(t)$ exists in K; call it c. Hence, $c \in f \circ \alpha([e, b))$ for every $e \in [d, b)$. Since the definable subset $\{t \in [d, b) : f \circ \alpha(t) = c\}$ is a finite union of points and intervals, it follows that there is $d' \in [d, b)$ such that $f \circ \alpha(t) = c$ for all $t \in [d', b)$. Thus $\alpha([d', b)) \subseteq f^{-1}(c) \subseteq f^{-1}(K)$. Since $f^{-1}(c)$ is definably compact, $\lim_{t\to b^-} \alpha(t)$ exists in $f^{-1}(K)$, which is absurd. ∎

By Example 3.16 the assumption that Y is locally definably compact is needed.

4. Invariance and comparison results

4.1. Definably proper in elementary extensions. Here $\mathbb{S}$ is an elementary extension of $\mathbb{M}$, and we consider the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ from the category of definable spaces and continuous definable maps to the category of $\mathbb{S}$-definable spaces and continuous $\mathbb{S}$-definable maps. This functor sends a definable space X to the $\mathbb{S}$-definable space $X(\mathbb{S})$ and sends a continuous definable map $f : X \to Y$ to the continuous $\mathbb{S}$-definable map $f^{\mathbb{S}} : X(\mathbb{S}) \to Y(\mathbb{S})$. We show that: (i) f is proper in Def if and only if $f^{\mathbb{S}}$ is proper in $\mathrm{Def}(\mathbb{S})$ (Theorem 4.3); (ii) if $\mathbb{M}$ has definable Skolem functions and Y is Hausdorff, then f is definably proper if and only if $f^{\mathbb{S}}$ is $\mathbb{S}$-definably proper (Theorem 4.4).

The following is easy and well known:

Fact 4.1. *If $\mathbb{M}$ has definable Skolem functions, then so does $\mathbb{S}$.*

Since the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ is a monomorphism from the boolean algebra of definable subsets of a definable space X to the boolean algebra of $\mathbb{S}$-definable subsets of $X(\mathbb{S})$ and it commutes with:

- the interior and closure operations,
- the image and inverse image under (continuous) definable maps,

we have:

Lemma 4.2. *Let $f : X \to Y$ be a morphism in* Def. *Then the following are equivalent:*

(1) *f is closed in* Def *(i.e., definably closed).*
(2) *$f^{\mathbb{S}}$ is closed in* $\mathrm{Def}(\mathbb{S})$ *(i.e., $\mathbb{S}$-definably closed).*

Proof. Assume that (1) holds. Let $A \subseteq X(\mathbb{S})$ be a closed $\mathbb{S}$-definable subset and suppose that $f^{\mathbb{S}}(A)$ is not a closed subset of $Y(\mathbb{S})$. Then there is a uniformly definable family $\{A_t : t \in T\}$ of definable subsets of X such that $A = A_s(\mathbb{S})$ for some $s \in T(\mathbb{S})$. Since the property of t saying that A_t is closed is first-order, after replacing T by a definable subset we may assume that for all $t \in T$, A_t is a closed definable subset of X. We also see that $\{f(A_t) : t \in T\}$ is a uniformly definable family of definable subsets of Y such that $f^{\mathbb{S}}(A) = f^{\mathbb{S}}(A_s(\mathbb{S}))$. Let E be the definable subset of T of all t such that $f(A_t)$ is not closed. Since $s \in E(\mathbb{S})$, we have $E \neq \emptyset$, which is a contradiction since by assumption, for every $t \in T$, $f(A_t)$ is a closed definable subset of Y.

Assume that (2) holds. Let $A \subseteq X$ be a closed definable subset. Then $A(\mathbb{S}) \subseteq X(\mathbb{S})$ is a closed $\mathbb{S}$-definable subset and, by assumption, $f(A)(\mathbb{S}) =$

$f^{\mathbb{S}}(A(\mathbb{S}))$ is a closed $\mathbb{S}$-definable subset of $Y(\mathbb{S})$. So $f(A)$ is a closed definable subset of Y. ∎

Since the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ sends open (resp. closed) definable immersions to open (resp. closed) $\mathbb{S}$-definable immersions and sends cartesian squares in Def to cartesian squares in $\mathrm{Def}(\mathbb{S})$, we have, using Lemma 4.2:

Theorem 4.3. *Let $f : X \to Y$ a morphism in* Def. *Then the following are equivalent:*

(1) *f is proper (resp. separated) in* Def.
(2) *$f^{\mathbb{S}}$ is proper (resp. separated) in* $\mathrm{Def}(\mathbb{S})$.

We also have:

Theorem 4.4. *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X and Y be definable spaces with Y Hausdorff. Let $f : X \to Y$ be a continuous definable map. Then the following are equivalent:*

(1) *f is definably proper.*
(2) *$f^{\mathbb{S}}$ is $\mathbb{S}$-definably proper.*

Proof. First note that $\mathbb{S}$ has definable Skolem functions (Fact 4.1) and $Y(\mathbb{S})$ is a Hausdorff $\mathbb{S}$-definable space (since Hausdorff is a first-order property). Using Corollary 2.8 and Theorem 3.12 in $\mathbb{M}$ and Corollary 2.8 and Theorem 3.12 in $\mathbb{S}$, we deduce the result from:

Claim 4.5. *The following are equivalent:*

(1) *There is a definable curve $\alpha : (a, b) \to X$ such that $f \circ \alpha : (a, b) \to Y$ is completable in Y but α is not completable in X.*
(2) *There is an $\mathbb{S}$-definable curve $\beta : (c, d) \to X(\mathbb{S})$ such that the $\mathbb{S}$-definable curve $f^{\mathbb{S}} \circ \beta : (c, d) \to Y(\mathbb{S})$ is completable in $Y(\mathbb{S})$ but β is not completable in $X(\mathbb{S})$.*

To prove the claim, assume first that (1) holds; then (2) holds with $(c, d) = (a, b)(\mathbb{S})$ and $\beta = \alpha^{\mathbb{S}}$ since "α is continuous", "$f \circ \alpha : (a, b) \to Y$ is completable in Y" and "α is not completable in X" are first-order properties.

Now assume that (2) holds; then (1) holds since "β is continuous", "$f^{\mathbb{S}} \circ \beta : (c, d) \to Y(\mathbb{S})$ is completable in $Y(\mathbb{S})$" and "β is not completable in $X(\mathbb{S})$" are first-order properties in the parameters defining β (together with c and d). ∎

The proof of Claim 4.5 above actually shows:

Corollary 4.6. *Let X be a definable space. Then the following are equivalent:*

(1) *X is definably compact.*
(2) *$X(\mathbb{S})$ is $\mathbb{S}$-definably compact.*

4.2. Definably proper in o-minimal expansions. Here $\mathbb{S}$ is an o-minimal expansion of $\mathbb{M}$ and we again consider the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ as in Section 4.1. This time the functor sends a definable space X to the $\mathbb{S}$-definable space X and sends a continuous definable map $f : X \to Y$ to the continuous $\mathbb{S}$-definable map $f : X \to Y$. We show that if $\mathbb{M}$ has definable Skolem functions, X and Y are Hausdorff and Y is locally definably compact, then f is definably proper if and only if $f^{\mathbb{S}}$ is $\mathbb{S}$-definably proper, and f is proper in Def if and only if $f^{\mathbb{S}}$ is proper in $\mathrm{Def}(\mathbb{S})$ (Theorem 4.9).

FACT 4.7. *If $\mathbb{M}$ has definable Skolem functions, then so does $\mathbb{S}$.*

Proof. By Fact 4.1 we may assume that both $\mathbb{M}$ and $\mathbb{S}$ are ω-saturated. In this case, by the (observations before the) proof of [4, Chapter 6, (1.2)] (see also Comment (1.3) there), $\mathbb{S}$ has definable Skolem functions if and only if every nonempty $\mathbb{S}$-definable subset $X \subseteq M$ defined with parameters in $\overline{a} = a_1, \dots, a_l$ has an element in $\mathrm{dcl}_{\mathbb{S}}(\overline{a})$.

So let X be an $\mathbb{S}$-definable subset of M. By o-minimality, X is a finite union of points $\{c_0, \dots, c_m\} \subseteq M$ and open intervals $I_0, \dots, I_n \subseteq M$ with end points in $M \cup \{-\infty, \infty\}$ with all the c_i's and the end points of the I_k's in $\mathrm{dcl}_{\mathbb{S}}(\overline{a})$. Thus X is definable over $\mathrm{dcl}_{\mathbb{S}}(\overline{a})$ using just equality and the order relation, hence X is $\mathbb{M}$-definable. Since $\mathbb{M}$ has definable Skolem functions, X has a point in $\mathrm{dcl}_{\mathbb{M}}(\mathrm{dcl}_{\mathbb{S}}(\overline{a}))$. Since $\mathbb{S}$ is an expansion of $\mathbb{M}$, we have $\mathrm{dcl}_{\mathbb{M}}(\mathrm{dcl}_{\mathbb{S}}(\overline{a})) \subseteq \mathrm{dcl}_{\mathbb{S}}(\mathrm{dcl}_{\mathbb{S}}(\overline{a})) = \mathrm{dcl}_{\mathbb{S}}(\overline{a})$. ∎

The shrinking lemma (Corollary 2.12) gives the following:

PROPOSITION 4.8. *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a Hausdorff definable space. Then the following are equivalent:*

(1) *X is definably compact.*
(2) *X is $\mathbb{S}$-definably compact.*

Proof. By Theorem 2.11, X is definably normal. Let $(X_i, \phi_i)_{i \le l}$ be the definable charts of X. By the shrinking lemma, there are open definable subsets V_i $(1 \le i \le l)$ and closed definable subsets C_i $(1 \le i \le l)$ such that $V_i \subseteq C_i \subseteq X_i$ and $X = \bigcup\{C_i : i = 1, \dots, l\}$.

Then X is definably compact if and only if each C_i is a definably compact definable subset of X if and only if each $\phi_i(C_i)$ is also a definably compact definable subset of M^{n_i}, and therefore, by [17, Theorem 2.1], if and only if each $\phi_i(C_i)$ is a closed and bounded definable subset of M^{n_i}. Similarly, X is $\mathbb{S}$-definably compact if and only if each C_i is an $\mathbb{S}$-definably compact $\mathbb{S}$-definable subset of X if and only if each $\phi_i(C_i)$ is also an $\mathbb{S}$-definably compact $\mathbb{S}$-definable subset of M^{n_i}, and therefore, by [17, Theorem 2.1] in $\mathbb{S}$, if and only if each $\phi_i(C_i)$ is a closed and bounded $\mathbb{S}$-definable subset of M^{n_i}. Since "closed" and "bounded" are preserved under going to $\mathbb{S}$, the result follows. ∎

Theorem 4.9. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *and* Y *be Hausdorff definable spaces with* Y *locally definably compact. Then the following are equivalent:*

(1) f *is proper in* Def*.*
(2) f *is definably proper.*
(3) f *is* $\mathbb{S}$*-definably proper.*
(4) f *is proper in* $\mathrm{Def}(\mathbb{S})$*.*

Proof. First note that since Y is locally definably compact, $Y(\mathbb{S})$ is locally $\mathbb{S}$-definably compact. By Theorem 3.15 in $\mathbb{M}$ and in $\mathbb{S}$ it is enough to prove that f is definably proper if and only if f is $\mathbb{S}$-definably proper. Using the fact that Y is locally definably compact and Proposition 4.8, one can show this as in [4, Chapter 6, (4.8), Exercise 2] (see p. 170 for the solution). ∎

4.3. Definably proper in topology. Here $\mathbb{M}$ is an o-minimal expansion of the ordered set of real numbers, and we consider the functor Def $\to$ Top from the category of definable spaces and continuous definable maps to the category of topological spaces and continuous maps. We show that if $\mathbb{M}$ has definable Skolem functions, then, for Hausdorff locally definably compact definable spaces, definably proper is the same as proper, and proper in Def is the same as proper in Top.

As before, we have:

Proposition 4.10. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *be a Hausdorff definable space. Then the following are equivalent:*

(1) X *is definably compact.*
(2) X *is compact.*

Proof. Follow the proof of Proposition 4.8 using the Heine–Borel theorem (a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded) instead of [17, Theorem 2.1]. ∎

A similar result holds in the semialgebraic case with a completely different proof [3, Theorem 9.11]:

Theorem 4.11. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *and* Y *be Hausdorff definable spaces with* Y *locally definably compact. Then the following are equivalent:*

(1) f *is proper in* Def*.*
(2) f *is definably proper.*
(3) f *is proper.*
(4) f *is proper in* Top*.*

Proof. First note that Y is locally compact. Next recall that f is proper if $f^{-1}(K) \subseteq X$ is a compact subset for every $K \subseteq Y$ compact subset, and

f is proper in Top if it is separated and universally closed in the category Top of topological spaces. Also, it is well known that f is proper if and only if f is proper in Top if and only if f is closed and has compact fibers (see [2, Chapter 1, §10, Theorem 1]).

By Theorem 3.15 it is enough to prove that f is definably proper if and only if f is proper. Using Theorem 3.18 and Proposition 4.10 one can show this as in [4, Chapter 6, (4.8), Exercise 3] (see p. 170 for the solution). ∎

5. Definably compact, definably proper and definable types. Here we show that definable compactness of Hausdorff definable spaces in o-minimal structures with definable Skolem functions can also be characterized by the existence of limits of definable types—extending a similar result in the affine case [14, Remark 4.2.15]. The corresponding characterization for definably proper maps between Hausdorff, locally definably compact definable spaces is also given (Theorem 5.3).

Let X be a definable space. A *type on* X is an ultrafilter α of definable subsets of X. A type α on X is a *definable type on* X if for every uniformly definable family $\{F_t\}_{t\in T}$ of definable subsets of X with $T \subseteq M^n$ for some n, there is a definable subset $T(\alpha) \subseteq T$ such that $F_t \in \alpha$ if and only if $t \in T(\alpha)$.

If α is a type on X and $x \in X$, we say that *x is a limit of* α if for every open definable subset U of X such that $x \in U$ we have $U \in \alpha$.

For affine definable spaces the existence of limits of definable types gives another criterion for definable compactness (see [14, Remark 4.2.15]). Since the proof is not written down in [14], for convenience, we include the details.

FACT 5.1. *Let $Z \subseteq M^n$ be a definable set. Then the following are equivalent:*

(1) *Z is closed and bounded (i.e., definably compact).*
(2) *Every definable type on Z has a limit in Z.*

Proof. Assume that (1) holds. Let α be a definable type on Z. For each $i = 1, \dots, n$, let $\pi_i : M^n \to M$ be the projection onto the i-coordinate and let $Z_i = \pi_i(Z)$ and $\alpha_i = \widetilde{\pi}_i(\alpha)$ (the definable type on Z_i determined by the collection of definable subsets $\{A \subseteq Z_i : \pi_i^{-1}(A) \in \alpha\}$). By [16, Lemma 2.3], no α_i is a cut, and so, since each Z_i is bounded, there is $a_i \in M$ such that α_i is either determined by $x = a_i$ or $\{b < x < a_i : b \in M,\ b < a_i\}$ or $\{a_i < x < b : b \in M,\ a_i < b\}$. In either case a_i is the limit of α_i in M. Clearly $a = \langle a_1, \dots, a_n\rangle \in M^n$ is the limit of α in M^n, and since Z is closed, $a \in Z$ as required.

Assume that (2) holds. Let $\alpha : (a, b) \to Z$ be a definable curve. Then the collection $\{\alpha([t, b)) : t \in (a, b)\}$ of definable subsets of Z determines a type β on Z such that $S \in \beta$ if and only if $\alpha([t, b)) \subseteq S$ for some $t \in (a, b)$. This type β is definable since for any uniformly definable family $\{F_l\}_{l\in L}$ of definable subsets of Z we have $\{l \in L : F_l \in \beta\} = \{l \in L : \exists t \in (a,b)\ (\alpha([t,b)) \subseteq F_l)\}$. By hypothesis, β has a limit z in Z. It equals $\lim_{t\to b^-} \alpha(t)$ since for every $d \in D(z)$ we have $U(z,d) \in \beta$, so $\alpha([t,b)) \subseteq U(z,d)$ for some $t \in (a,b)$. ∎

We can use the shrinking lemma to extend this result to nonaffine Hausdorff definable spaces:

THEOREM 5.2. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *be a Hausdorff definable space. Then the following are equivalent:*

(1) X *is definably compact.*
(2) *Every definable type on* X *has a limit in* X.

Proof. By Theorem 2.11, X is definably normal. Let $(X_i,\theta_i)_{i\le k}$ be the definable charts of X with $\theta_i(X_i) \subseteq M^{n_i}$. By the shrinking lemma there are definable open subsets V_i and definable closed subsets C_i of X $(1 \le i \le n)$ with $V_i \subseteq C_i \subseteq X_i$ and $X = \bigcup\{C_i : i = 1,\dots,n\}$. We notice that: (i) X is definably compact if and only if each C_i is definably compact; (ii) every definable type on X has a limit in X if and only if for each i, every definable type on C_i has a limit in C_i. Since $\theta_{i|} : C_i \to \theta_i(C_i) \subseteq M^{n_i}$ is a definable homeomorphism, the result now follows by Fact 5.1. ∎

We also have the following definable types criterion for definably proper:

THEOREM 5.3. *Suppose that* $\mathbb{M}$ *has definable Skolem functions. Let* X *and* Y *be Hausdorff definable spaces with* Y *locally definably compact. Let* $f : X \to Y$ *be a continuous definable map. Then the following are equivalent:*

(1) f *is definably proper.*
(2) *For every definable type* α *on* X, *if* $\widetilde{f}(\alpha)$ *has a limit in* Y, *then* α *has a limit in* X.

Proof. Assume that (1) holds. Let α be a definable type on X with $\lim \widetilde{f}(\alpha) = y \in Y$. Since Y is locally definably compact, there is a definable open neighborhood V of y in Y such that $\overline{V}$ is definably compact. So, $f^{-1}(\overline{V})$ is a definably compact definable subset of X, and α is a definable type on $f^{-1}(\overline{V})$. But then, by Theorem 5.2, α has a limit in $f^{-1}(\overline{V})$, hence in X.

Assume that (2) holds. Suppose that f is not definably proper. Then there is a definably compact definable subset K of Y such that $f^{-1}(K)$ is not a definably compact definable subset of X. Thus by Theorem 5.2 there is a definable type α on $f^{-1}(K)$ with no limit in $f^{-1}(K)$. Since $f^{-1}(K)$ is closed (by Corollary 2.8, K is closed), α does not have a limit in X. But $\widetilde{f}(\alpha)$ is a definable type on $K \subseteq Y$ and has a limit by Theorem 5.2, which contradicts (2). ∎

The following was observed in [14, Remark 4.2.15] in the affine case but the same proof works here.

FACT 5.4. *Suppose that $\mathbb{M}$ has definable Skolem functions. Let X be a definable space and $C \subseteq X$ a definable subset which is not closed. If $x \in \overline{C} \setminus C$, then there is a definable type α on C such that x is a limit of α.*

Proof. Consider the definable set $D(x)$ with the relation $\preceq$ (a definable downwards directed order). By [14, Lemma 4.2.18] (or [13, Lemma 2.19]), there is a definable type β on $D(x)$ such that for every $d \in D(x)$ we have $\{d' \in D(x) : d' \preceq d\} \in \beta$.

Since $x \in \overline{C}$, we have $U(x,d) \cap C \neq \emptyset$ for every $d \in D(x)$. By definable Skolem functions, there is a definable map $h : D(x) \to C$ such that $h(d)$ is in $U(x,d) \cap C$ for every $d \in D(x)$. Let $\alpha = \widetilde{h}(\beta)$ be the definable type on C determined by the collection $\{A \subseteq C : h^{-1}(A) \in \beta\}$ of definable subsets. Clearly, x is a limit of α. ∎

By Example 3.16 and Fact 5.4, in Theorem 5.3 the assumption that Y is locally definably compact is needed. Note that, by the same example, this observation applies also if one replaces the Peterzil–Steinhorn definition of definably compact (using definable curves [17]) by the Hrushovski–Loeser definition of definably compact (using definable types [14]).

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