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On the ideal convergence of sequences of quasi-continuous functions

by

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Abstract. For any Borel ideal $\mathcal{I}$ we describe the $\mathcal{I}$-Baire system generated by the family of quasi-continuous real-valued functions. We characterize the Borel ideals $\mathcal{I}$ for which the ideal and ordinary Baire systems coincide.

1. Introduction. Laczkovich and Recław [LR09] and (independently) Debs and Saint Raymond [DSR09] characterized the Borel ideals $\mathcal{I}$ for which the first $\mathcal{I}$-Baire class (the family of pointwise ideal limits of sequences of continuous functions) is equal to the classical first Baire class. They formulated four equivalent conditions, two in the language of game theory, and two via some combinatorial properties of the ideal $\mathcal{I}$. Moreover, they showed that for every non-pathological ideal $\mathcal{I}$, if the class of functions considered is closed under sums and scaling, then the first $\mathcal{I}$-Baire class for such functions is equal to the family of all classical limits of such functions.

Our goal is to describe the $\mathcal{I}$-Baire system generated by the family of quasi-continuous real-valued functions. The notion of quasi-continuity has been introduced by Kempisty [Kem32]. It plays an important role in the theory of real functions, particularly in the study of separately continuous functions (see e.g. [BT10]). It is worth noticing that unlike the continuous functions, the class of quasi-continuous functions is not closed under arithmetic sums. This suggests that conditions on $\mathcal{I}$ under which the first $\mathcal{I}$-Baire class for quasi-continuous functions is equal to the ordinary first Baire class for such functions can be different from the conditions given in [LR09] and [DSR09]. We prove that this is indeed true (see Theorem 4.4).

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In Section 3 we consider two combinatorial properties of ideals: being weakly Ramsey and ω-+-diagonalizability. For Borel ideals those properties are equivalent, which has been proven in terms of some game $G(\mathcal{I})$ by Laflamme [Laf96]. We will characterize those properties by properties of ideal limits of sequences of quasi-continuous functions. Next, in Section 4 we apply those results to characterize the $\mathcal{I}$-Baire system generated by the family of quasi-continuous real-valued functions defined on a Baire space X. We finish with a characterization of weakly Ramsey filters in several fashions, similarly to [LR09]; to do so, we use a characterization of weakly Ramsey filters given recently by Kwela [Kwe15].

2. Preliminaries. We use standard set-theoretic and topological notation. In particular, the set of natural numbers is identified with the ordinal ω. We denote by $[\omega]^\omega$ the family of all infinite subsets of ω. The cardinality of a set A is denoted by $\mathrm{card}(A)$.

Let X be a topological space. For $A \subset X$ we denote by $\operatorname{int} A$ and $\operatorname{cl} A$ the interior and closure of A, respectively. We say that a set A is *nowhere meager* in an open set $U \subset X$ if $A \cap V$ is non-meager for each non-empty open set $V \subset U$.

2.1. Quasi-continuous and pointwise discontinuous functions. We denote by $\mathrm{C}(X)$ the class of all continuous real-valued functions defined on X. The class of all functions $f\colon X \to \mathbb{R}$ with the Baire property is denoted by $\mathrm{Baire}(X)$. We will write C (Baire, respectively) instead of $\mathrm{C}(X)$ ($\mathrm{Baire}(X)$, respectively) if X is fixed.

A function $f\colon X \to \mathbb{R}$ is *quasi-continuous at a point* $x_0 \in X$ if for each open set $U \ni x_0$ and each open set $V \ni f(x_0)$ there is a non-empty open set $W \subset U$ with $f(W) \subset V$ [Kem32]; and f is *quasi-continuous* if it is quasi-continuous at each $x \in X$. The class of all quasi-continuous real-valued functions defined on a space X is denoted by $\mathrm{QC}(X)$ (or QC if X is fixed). We say that a set $U \subset X$ is *semi-open* in X if $U \subset \operatorname{cl}\operatorname{int} U$. It is known that $f\colon X \to \mathbb{R}$ is quasi-continuous iff $f^{-1}(U)$ is semi-open for each open set $U \subset \mathbb{R}$.

A function $f\colon X \to \mathbb{R}$ is *pointwise discontinuous* if the set $\mathcal{C}(f)$ of continuity points of f is dense in X (see e.g. [Kur58, Chapter I.13.III]). The class of all pointwise discontinuous functions defined on a space X is denoted by $\mathrm{PWD}(X)$ (or PWD if X is fixed).

Let X be a Baire space. Then for any quasi-continuous function $f\colon X \to \mathbb{R}$ the set $\mathcal{C}(f)$ is dense (hence residual) in X, thus $\mathrm{QC}(X) \subset \mathrm{PWD}(X) \subset \mathrm{Baire}(X)$.

Note that $\mathrm{PWD}(X)$ is closed with respect to addition, while $\mathrm{QC}(X)$ may not be closed, even for $X = \mathbb{R}$. Recall also that the sum of a continuous function and quasi-continuous one is quasi-continuous. More properties of quasi-continuous functions can be found e.g. in [Bor96].

We will use the following well-known facts.

LEMMA 2.1. *Assume that X is a Baire space and $f\colon X \to \mathbb{R}$.*

(1) *If $f \notin \mathrm{PWD}(X)$ then there are reals $\alpha < \beta$ such that the sets $A = f^{-1}[(-\infty, \alpha)]$ and $B = f^{-1}[(\beta, \infty)]$ are both dense in some non-empty open set $U \subset X$.*
(2) *If $f \notin \mathrm{Baire}(X)$ then there are reals $\alpha < \beta$ such that the sets $A = f^{-1}[(-\infty, \alpha)]$ and $B = f^{-1}[(\beta, \infty)]$ are both nowhere meager in some non-empty open set $U \subset X$.*

LEMMA 2.2 (Borsik [Bor96, Lemma 1]). *Let X be a metric space, $F \subset X$ a closed and nowhere dense set, $U \subset X$ non-empty semi-open and $F \subset \mathrm{cl}\, U$. Then there is a sequence $(U_n)_{n<\omega}$ of non-empty semi-open sets such that:*

(1) *$F \subset \mathrm{cl}\, U_n$ for each $n < \omega$;*
(2) *$\bigcup_{n<\omega} U_n = U \setminus F$.*

LEMMA 2.3. *Suppose that X is a metric space and $U \subset X$ is a semi-open set. Then there exists a quasi-continuous function $g\colon U \to \mathbb{R}$ such that for each $x \in \mathrm{cl}\, U \setminus U$, $y \in \mathbb{R}$ and $\varepsilon > 0$ and for every open neighbourhood $V \subset X$ of x, there exists a non-empty open set $V' \subset U \cap V$ with*

$$\forall x' \in V' \quad |g(x') - y| < \varepsilon,$$

i.e. any extension of g is quasi-continuous on $\mathrm{cl}\, U$.

Proof. Let $(U_n)_n$ be a sequence of pairwise disjoint non-empty semi-open sets which satisfy the assertions of Lemma 2.2 for $F = \mathrm{cl}\, U \setminus \mathrm{int}\, U$. Let $(q_n)_n$ be a sequence of all rationals. Define

$$g(x) = \begin{cases} q_n & \text{for } x \in U_n,\ n \in \omega, \\ 0 & \text{otherwise.} \end{cases} \blacksquare$$

2.2. Ideal convergence. An *ideal* on ω is a non-empty family of subsets of ω closed under taking finite unions and subsets. If not explicitly mentioned otherwise, we assume that an ideal is proper ($\neq \mathcal{P}(\omega)$) and contains all finite sets. We denote by FIN the ideal of all finite subsets of ω. We can talk about ideals on any other countable set by identifying this set with ω via a fixed bijection.

For an ideal $\mathcal{I}$ we define the *dual filter* to $\mathcal{I}$ as $\mathcal{I}^\star = \{A\colon \omega \setminus A \in \mathcal{I}\}$. We denote by $\mathcal{I}^+$ the set of all subsets of ω which do not belong to $\mathcal{I}$.

By identifying subsets of ω with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the product topology of $\{0,1\}^\omega$. It is known that $\mathcal{P}(\omega)$ with this topology is a metrizable compact Polish space without isolated points (it is homeomorphic to the Cantor set). An ideal $\mathcal{I}$ is an F_σ *ideal* (*analytic ideal*, respectively) if $\mathcal{I}$ is an F_σ subset of $\mathcal{P}(\omega)$ (a continuous image of a G_δ subset of $\mathcal{P}(\omega)$, respectively).

Let $\mathcal{I}$ be an ideal on ω. Let $x_n \in \mathbb{R}$ ($n \in \omega$) and $x \in \mathbb{R}$. We say that the sequence (x_n) is *$\mathcal{I}$-convergent* to x if

$$\{n \in \omega : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$$

for every $\varepsilon > 0$. We then write $\mathcal{I}\text{-}\lim x_n = x$. If $\mathcal{I} = \mathrm{FIN}$, then the $\mathcal{I}$-convergence is equivalent to the classical convergence (see [FNS13]).

2.3. Combinatorial properties of ideals and the game $G(\mathcal{I})$. Let $\mathcal{I}$ be an ideal. We will consider the following properties of the dual filter $\mathcal{I}^\star$.

Definition 2.4 ([Laf96]).

(1) We call a tree $\mathcal{T} \subset \omega^{<\omega}$ an *$\mathcal{X}$-tree* for some $\mathcal{X} \subset [\omega]^\omega$ if for each $s \in \mathcal{T}$ there is an $\mathcal{X}_s \in \mathcal{X}$ such that $s \frown n \in \mathcal{T}$ for all $n \in \mathcal{X}_s$.
(2) $\mathcal{I}^\star$ is *weakly Ramsey* if any $\mathcal{I}^\star$-tree has a branch in $\mathcal{I}^+$.
(3) $\mathcal{I}^\star$ is *ω-+-diagonalizable* if there are sets $\{X_n \in \mathcal{I}^+ : n \in \omega\}$ such that for each $F \in \mathcal{I}^\star$ there is an $n \in \omega$ with $X_n \subset F$.

For an ideal $\mathcal{I}$ define an infinite game $G(\mathcal{I})$ as follows: player I in the nth move plays an element C_n of the ideal, and then player II plays any element $a_n \notin C_n$. Player I wins when $\{a_n : n \in \omega\} \in \mathcal{I}$. Otherwise player II wins. This game was investigated by Laflamme [Laf96], who denoted it by $\mathcal{G}(\mathcal{F}, \omega, \mathcal{F}^+)$ for $\mathcal{F} = \mathcal{I}^\star$.

Theorem 2.5 ([Laf96]). *For any ideal $\mathcal{I}$ and the game $G(\mathcal{I})$:*

(1) *I has a winning strategy if and only if $\mathcal{I}^\star$ is not weakly Ramsey, i.e. there exists an $\mathcal{I}^\star$-tree $\mathcal{T}$ with every branch in $\mathcal{I}$;*
(2) *II has a winning strategy if and only if $\mathcal{I}^\star$ is ω-+-diagonalizable.*

Note that theorem above shows that ω-+-diagonalizability implies the weakly Ramsey property. It seems that for many ideals which occur in the literature it is easier to construct winning strategies for player I or II than to verify directly that $\mathcal{I}^\star$ has the relevant combinatorial properties. In Proposition 2.7 we consider some classes of ideals which are—in our opinion—simple and important.

The ideal $\mathcal{I}$ is *dense* if for every infinite A there exists an infinite $B \subset A$ with $B \in \mathcal{I}$. Note that $\mathcal{I}$ is not dense iff $\mathcal{I} = \mathrm{FIN}$ or $\mathcal{I} = \mathrm{FIN} \oplus \mathcal{J}$ for some ideal $\mathcal{J}$.

Lemma 2.6. *If an ideal $\mathcal{I}$ is not dense then player II has a winning strategy in the game $G(\mathcal{I})$.*

Proof. Since $\mathcal{I}$ is not dense, there is an infinite set $A \subset \omega$ such that no infinite subset of A belongs to $\mathcal{I}$. Then player II wins if in the nth move he chooses a number $a_n \in A \setminus (C_n \cup \{a_i : i < n\})$. ∎

PROPOSITION 2.7. *For the game $G(\mathcal{I})$:*

(1) *if $\mathcal{I}$ is an analytic P-ideal then I has a winning strategy if $\mathcal{I}$ is dense, otherwise II has a winning strategy (hence $G(\mathcal{I})$ is determined);*
(2) *if $\mathcal{I} = \mathrm{NWD} = \{A \subset \mathbb{Q}\colon A$ is nowhere dense in $\mathbb{Q}\}$ then II has a winning strategy;*
(3) *if $\mathcal{I} = \mathrm{CONV} = \{A \subset \mathbb{Q}\colon A$ has finitely many cluster points$\}$ then II has a winning strategy.*

Proof. (1) *The case of an analytic P-ideal.* By [Sol99], $\mathcal{I} = \{A \subset \omega\colon \lim_{n\to\infty} \phi(A \setminus \{0, 1, \ldots, n\}) = 0\}$ for some lsc submeasure ϕ. For each $\varepsilon > 0$ let $A_\varepsilon = \{k \in \omega\colon \phi(\{k\}) \geq \varepsilon\}$. We have two possibilities: either

(i) there exists $\varepsilon > 0$ such that A_ε is infinite, or
(ii) for each $\varepsilon > 0$ there is $N_\varepsilon \in \omega$ with $\phi(\{k\}) < \varepsilon$ for each $k > N_\varepsilon$.

Observe that the first condition characterizes analytic P-ideals which are not dense. Then, by Lemma 2.6, II has a winning strategy. In the second case in the nth move I can play $C_n = \{k \in \omega\colon k \leq N_{1/n^2}\}$. Then, regardless of the a_n played by II, $\phi(\{a_i\}_{i>n}) \leq \sum_{k=n}^{\infty} 1/k^2 \to 0$. Thus, $\{a_i\colon i \in \omega\} \in \mathcal{I}$, and I wins.

(2) *Case* "$\mathcal{I} = \mathrm{NWD}$". Let $\{B_n\}_{n\in\omega}$ be a basis of $\mathbb{Q}$. If in the nth move II plays $a_n \in B_n \setminus C_n \neq \emptyset$ then $\{a_n\}_{n\in\omega}$ is dense in $\mathbb{Q}$.

(3) *Case* "$\mathcal{I} = \mathrm{CONV}$". A strategy for II can be constructed using the definition of $\mathcal{I}$ (we leave this as an exercise). On the other hand, since $\mathrm{CONV} \subset \mathrm{NWD}$, if I has a winning strategy in $G(\mathrm{CONV})$ then the same strategy works in $G(\mathrm{NWD})$. From Borel determinacy of $G(\mathrm{CONV})$ and $G(\mathrm{NWD})$ (see Proposition 2.8 and use transposition) it follows that the existence of a winning strategy for II in $G(\mathrm{NWD})$ implies the existence of a winning strategy for II in $G(\mathrm{CONV})$. ∎

In particular, Proposition 2.7(1) implies that I has a winning strategy in $G(\mathcal{I})$ for

$$\mathcal{I} = \mathcal{I}_{1/n} = \left\{A \subset \omega\colon \sum_{n\in A} \frac{1}{n} < \infty\right\}$$

and for

$$\mathcal{I} = \mathcal{I}_d = \left\{A \subset \omega\colon \limsup_{n\to\infty} \frac{\mathrm{card}(A \cap \{0, 1, \ldots, n-1\})}{n} = 0\right\}.$$

Moreover, the example of $\mathcal{I}_{1/n}$ shows that the family of F_σ-ideals for which player II has a winning strategy forms a proper subset of the class of all F_σ-ideals.

PROPOSITION 2.8 (folklore, cf. also [LR09, Prop. 3]). *$G(\mathcal{I})$ is determined for every Borel ideal $\mathcal{I}$.*

Proof. Since $\mathcal{I}$ is a Borel ideal, the game $G(\mathcal{I})$ is Borel with respect to the discrete topology on $\mathcal{P}(\omega)$ (we can consider ω as a subset of $\mathcal{P}(\omega)$). Hence the assertion follows directly from Martin's Theorem on Borel determinacy [Kec95, Th. 20.5]. ∎

COROLLARY 2.9. *If $\mathcal{I}$ is a Borel ideal then $\mathcal{I}^\star$ is ω-+-diagonalizable iff it is weakly Ramsey.*

By Proposition 2.8, for all ideals considered in Proposition 2.7 the game $G(\mathcal{I})$ is determined, thus the negations of the conditions which characterize the existence of a winning strategy for II also characterize the existence of a winning strategy for I.

3. $\mathcal{I}$-limits of sequences of QC functions

PROPOSITION 3.1. *Suppose X is a Baire space, $f_n\colon X \to \mathbb{R}$, $n \in \omega$, are quasi-continuous and $f = \mathcal{I}\text{-}\lim f_n$. If $\mathcal{I}^\star$ is ω-+-diagonalizable then f is pointwise discontinuous.*

Proof. Suppose f is not pointwise discontinuous. By Lemma 2.1(1) there are a non-empty open set $U \subset X$ and reals $\alpha < \beta$ such that $A = f^{-1}[(-\infty, \alpha)]$ and $B = f^{-1}[(\beta, \infty)]$ are dense in U. Moreover, we may assume that A is nowhere meager in U, i.e. $A \cap W$ is non-meager for every non-empty open set $W \subset U$. Since $\mathcal{I}^\star$ is ω-+-diagonalizable, there exists a family $\{X_n : n < \omega\} \subset \mathcal{I}^+$ such that for each $F \in \mathcal{I}^\star$ there is n with $X_n \subset F$. For each $x \in U \cap A$ we have $F_x = \{n : f_n(x) < \alpha\} \in \mathcal{I}^\star$, thus there is $n_x < \omega$ with $X_{n_x} \subset F_x$. Since X is a Baire space, there is n for which the set $\{x \in U \cap A : n_x = n\}$ is dense in some non-empty open set $W \subset U$. For each $k \in X_n$, since f_k is quasi-continuous, we have $f_k(x) \le \alpha$ for each $x \in W$. Now, let $x_0 \in W \cap B$. Then $F = \{k : f_k(x_0) > \beta\} \in \mathcal{I}^\star$, so $F \cap X_n \neq \emptyset$, a contradiction. ∎

PROPOSITION 3.2. *Suppose $\mathcal{I}^\star$ is not weakly Ramsey. Let X be a metric space. For every meager $M \subset X$ and $f\colon X \to \mathbb{R}$ with $\{x : f(x) \neq 0\} \subset M$ there exists a sequence of quasi-continuous functions $f_n\colon X \to \mathbb{R}$ such that*

$$\forall x \in X \quad \{n \in \omega : f_n(x) \neq f(x)\} \in \mathcal{I}.$$

Proof. Let $\mathcal{T}$ be an $\mathcal{I}^\star$-tree with every branch in $\mathcal{I}$. Let $M = \bigcup_{n\in\omega} M_n$, where each M_n is nowhere dense. We may also assume that for each $s \in \mathcal{T}$:

(i) $\mathcal{X}_s > s$ (i.e. $\min \mathcal{X}_s > \max s$);
(ii) $\mathcal{X}_{s\frown n} \subset \mathcal{X}_s$ for each $s \in \mathcal{T}$ and $n \in \omega$ such that $s \frown n \in \mathcal{T}$.

Let $\{U_t : t \in \omega^{<\omega}\}$ be a family of semi-open subsets of X fulfilling the following conditions (such a family exists by Lemma 2.2):

(iii) $U_\emptyset = X$;
(iv) for each $t \in \omega^{<\omega}$ and $n \in \omega$, $U_{t\frown n} \subset U_t$;

(v) for each $t \in \omega^{<\omega}$ and $n \neq m$, $n, m \in \omega$, $U_{t\frown n} \cap U_{t\frown m} = \emptyset$;
(vi) for each $t \in \omega^{<\omega}$, $M_t := M_{\text{length}(t)} \cap U_t = U_t \setminus \bigcup_{n\in\omega} U_{t\frown n}$;
(vii) for each $t \in \omega^{<\omega}$ and $n \in \omega$, $M_t \subset \operatorname{cl} U_{t\frown n}$.

Note that from the above construction it follows that $M = \bigcup_{t\in\omega^{<\omega}} M_t$.

For each $t \in \omega^{<\omega}$ and $n \in \omega$ let

$$\operatorname{last}(t \frown n) := n, \qquad \sigma(t \frown n) := \sigma(t) \frown \sigma_t(n),$$

where $\sigma_t(n)$ is the $(n+1)$th element of $\mathcal{X}_{\sigma(t)}$, i.e. $\sigma_t(n) = p$ iff $p \in \mathcal{X}_{\sigma(t)}$ and $\operatorname{card}(\mathcal{X}_{\sigma(t)} \cap \{0, 1, \ldots, p-1\}) = n$. Then $\sigma\colon \omega^{<\omega} \to \mathcal{T}$ is a bijection (see Fig. 1).

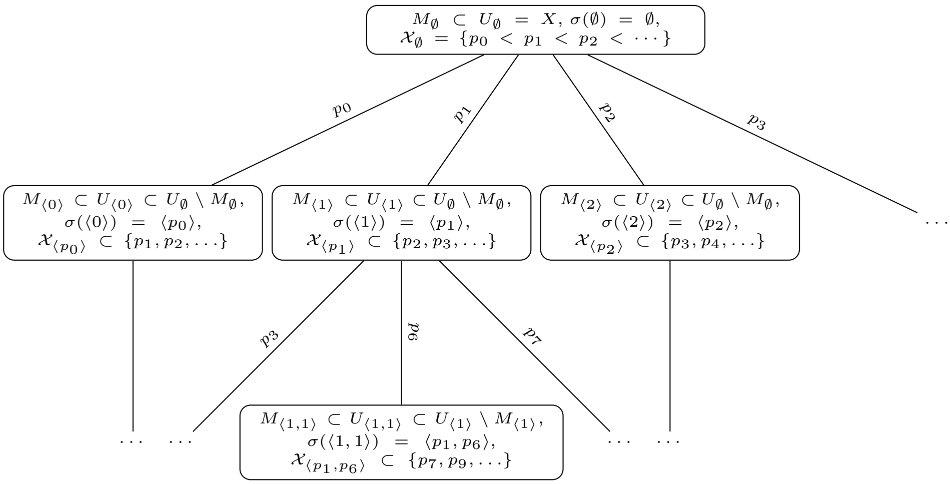

Fig. 1. The tree $\mathcal{T}$. To draw the last level of this diagram we assume that $\mathcal{X}_{\langle p_1\rangle} = \{p_3, p_6, p_7, p_9, \ldots\}$.

Additionally, for each $\bar{t} \in \omega^\omega$ define

$$\sigma(\bar{t}) := \bigcup_{l\in\omega} \sigma(\bar{t}{\restriction}l) = \langle \operatorname{last}(\sigma(\bar{t}{\restriction}1)), \operatorname{last}(\sigma(\bar{t}{\restriction}2)), \ldots\rangle.$$

Recall that by (i) and (ii), $\sigma(\bar{t})$ is a strictly increasing sequence. Moreover, for each $l \in \omega$,

$$\operatorname{last}(\sigma(\bar{t}){\restriction}l) = \operatorname{last}(\sigma(\bar{t}{\restriction}l)).$$

Using our notation, from the fact that $\mathcal{T}$ is an $\mathcal{I}^\star$-tree it follows that for each $\bar{t} \in \omega^\omega$ and $l \in \omega$, we have

$$\mathcal{X}_{\sigma(\bar{t}{\restriction}l)} \in \mathcal{I}^\star \quad \text{and} \quad \operatorname{range}(\sigma(\bar{t})) \in \mathcal{I}.$$

For each $t \in \omega^{<\omega}$ let $g_t\colon U_t \to \mathbb{R}$ be a quasi-continuous function such that any extension of g_t is quasi-continuous at each $x \in \operatorname{cl} U_t$ (such a function exists by Lemma 2.3). Let

$$f_n(x)=\begin{cases} f(x) & \text{if there exists a } t\in\omega^{<\omega} \text{ with } x\in M_t \text{ and } n\in\mathcal{X}_{\sigma(t)},\\ g_t(x) & \text{if there exists a } t\in\omega^{<\omega} \text{ with } x\in U_t \text{ and } n=\operatorname{last}(\sigma(t)),\\ 0 & \text{in any other case.}\end{cases}$$

For each $\bar t\in\omega^\omega$ let $N_{\bar t}:=\bigcap_{n\in\omega}U_{\bar t\restriction n}$. Then $\bigcup_{\bar t\in\omega^\omega}N_{\bar t}=X\setminus\bigcup_{t\in\omega^{<\omega}}M_t=X\setminus M$. We claim that:

(1) $\{n\in\omega\colon f_n(x)\neq f(x)\}\in\mathcal{I}$ for each $x\in M_t$ ($t\in\omega^{<\omega}$);
(2) $\{n\in\omega\colon f_n(x)\neq 0\}\in\mathcal{I}$ for each $x\in N_{\bar t}$ ($\bar t\in\omega^\omega$);
(3) f_n is quasi-continuous for each n.

The first claim follows immediately from the first case of the definition of f_n and the fact that $\mathcal{X}_{\sigma(t)}\in\mathcal{I}^\star$ for each $t\in\omega^{<\omega}$.

To see the second claim, observe that if $x\in N_{\bar t}$ then

$$\{n\colon f_n(x)\neq 0\}\subset\{\operatorname{last}(\sigma(\bar t\restriction l))\colon l\in\omega\}=\operatorname{range}(\sigma(\bar t))\in\mathcal{I}.$$

Finally, we have to show that for each $n\in\omega$ and $x\in X$, f_n is quasi-continuous at x. Fix $n\in\omega$. First, assume that $x\in N_{\bar t}$ for some $\bar t\in\omega^\omega$. We will show that f_n is quasi-continuous on some $U_t\ni x$. We have two possibilities:

(1) $n=\operatorname{last}(\sigma(\bar t\restriction l))$ for some $l\in\omega$ (i.e. $n\in\operatorname{range}(\sigma(\bar t))$), or
(2) $n\neq\operatorname{last}(\sigma(\bar t\restriction l))$ for each $l\in\omega$ (i.e. $n\notin\operatorname{range}(\sigma(\bar t))$).

In the first case, $f_n\restriction U_{t\restriction l}=g_{t\restriction l}$ (by the second case of the definition of f_n). For (2) observe that by the monotonicity of $\sigma(\bar t)$ there exists l such that

$$\operatorname{last}(\sigma(\bar t\restriction l))=\operatorname{last}(\sigma(\bar t)\restriction l)<n<\operatorname{last}(\sigma(\bar t)\restriction(l+1))=\operatorname{last}(\sigma(\bar t\restriction(l+1))).$$

Then it follows from (i) that

$$\operatorname{range}(\sigma(\bar t\restriction l))<n<\mathcal{X}_{\sigma(\bar t\restriction l+1)},$$

so n and $\sigma(\bar t\restriction(l+1))$ do not fulfill the conditions from the first or second case of the definition of f_n. Thus $f_n=0$ on $U_{\bar t\restriction(l+1)}$ (thus quasi-continuous).

Next, assume that $x\in M_t$ for some $t\in\omega^{<\omega}$. We have three possibilities:

(1) $n\in\{\sigma(t\restriction 1),\ldots,\sigma(t\restriction\operatorname{length}(t))\}$ (i.e. $n\in\operatorname{range}(\sigma(t))$);
(2) $n\in\mathcal{X}_{\sigma(t)}$;
(3) $n\notin\operatorname{range}(\sigma(t))\cup\mathcal{X}_{\sigma(t)}$.

In the first case, $n=\sigma(t\restriction l)$ for some $l\in\omega$, and so $x\in M_t\subset U_{t\restriction l}$. Then the situation is exactly as in the paragraph above: $f_n\restriction U_{t\restriction l}=g_{t\restriction l}$ is quasi-continuous.

Now consider the second case. Let $t'=\sigma^{-1}[\sigma(t)\frown n]$. By (v), $x\in M_t\subset\operatorname{cl}U_{t'}$ and, by the second case of the definition of f_n, $f_n\restriction U_{t'}=g_{t'}$. Thus, regardless of its value at x, f_n is quasi-continuous at x.

In the last case, $f_n(x) = 0$. Let $t' = t \frown m$ be such that $\operatorname{last}(\sigma(t')) > n$. Then $x \in M_t \subset \operatorname{cl} U_{t'}$, and n and $\sigma(t')$ do not fulfill the conditions from the first or second case of the definition of f_n. Thus $f_n{\restriction}U_{t'}$ is constant (and equal to 0). ∎

4. $\mathcal{I}$-Baire system generated by the family of QC functions. For any family $\mathcal{F}$ of real-valued functions defined on X there is a smallest family $\mathcal{B}(\mathcal{F})$ of all real-valued functions defined on X which contains $\mathcal{F}$ and which is closed under taking limits of sequences. This family is called the *Baire system* generated by $\mathcal{F}$. One method of generating $\mathcal{B}(\mathcal{F})$ from $\mathcal{F}$ consists in iteration of limits:

- $\mathcal{B}_0(\mathcal{F}) = \mathcal{F}$;
- $\mathcal{B}_\alpha(\mathcal{F}) = \mathrm{LIM}(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(\mathcal{F}))$ for $\alpha > 0$,

where $\mathrm{LIM}(\mathcal{G})$ denotes the family of all limits of sequences from $\mathcal{G}$. Then $\mathcal{B}(\mathcal{F}) = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(\mathcal{F})$. This system was described in 1899 by Baire in the case when $\mathcal{F}$ is the family of all continuous functions defined on a topological space X.

In a similar way one can define the $\mathcal{I}$-Baire system generated by the family $\mathcal{F}$, replacing $\mathrm{LIM}(\mathcal{F})$ by the family $\mathcal{I}$-$\mathrm{LIM}(\mathcal{F})$ of all $\mathcal{I}$-limits of sequences of functions from $\mathcal{F}$ (cf. [FNS13]). Recall that the $\mathcal{I}$-Baire system generated by the family of continuous functions defined on a Polish space X was partly characterized in [LR09], [DSR09] and [FS12] for analytic ideals $\mathcal{I}$. In this section we study the $\mathcal{I}$-Baire system generated by the family of quasi-continuous functions defined on a Baire space X.

The Baire system generated by the family $\mathrm{QC}(X)$ has been described by Grande [Gra88] for $X = \mathbb{R}$ and by Richter [Ric02] for any metric Baire space X. If X is a metric Baire space then:

- $\mathcal{B}_1(\mathrm{QC}) = \mathrm{LIM}(\mathrm{QC}) = \mathrm{PWD}$;
- $\mathcal{B}_\alpha(\mathrm{QC}) = \mathrm{LIM}(\mathrm{PWD}) = \mathrm{Baire}$ for $\alpha \geq 2$.

THEOREM 4.1. *Suppose $\mathcal{I}^\star$ is ω-$+$-diagonalizable and X is a metric Baire space. Then:*

(1) $\mathcal{I}$-$\mathcal{B}_1(\mathrm{QC}) = \mathcal{I}$-$\mathrm{LIM}(\mathrm{QC}) = \mathrm{PWD}$*;*
(2) $\mathcal{I}$-$\mathcal{B}_\alpha(\mathrm{QC}) = \mathcal{I}$-$\mathrm{LIM}(\mathrm{PWD}) = \mathrm{Baire}$ *for* $\alpha \geq 2$.

Proof. The inclusion $\mathcal{I}$-$\mathrm{LIM}(\mathrm{QC}) \subset \mathrm{PWD}$ follows from Proposition 3.1, and the opposite one from $\mathrm{LIM}(\mathrm{QC}) = \mathrm{PWD}$ and the assumption $\mathrm{FIN} \subset \mathcal{I}$. Similarly, to prove $\mathcal{I}$-$\mathrm{LIM}(\mathrm{PWD}) = \mathcal{I}$-$\mathrm{LIM}(\mathrm{Baire}) = \mathrm{Baire}$ we have to show that $\mathcal{I}$-$\mathrm{LIM}(\mathrm{Baire}) \subset \mathrm{Baire}$. Fix a sequence $(f_n)_n$ of functions with the Baire property and suppose $f = \mathcal{I}$-$\lim_n(f_n)$ does not have the Baire property. Then, by Lemma 2.1(2), there are a non-empty open set $U \subset X$ and reals $\alpha < \beta$ such that $A = f^{-1}[(-\infty, \alpha)]$ and $B = f^{-1}[(\beta, \infty)]$ are nowhere meager in U. Since $\mathcal{I}^*$ is ω-$+$-diagonalizable, there exists a family $\{X_n : n < \omega\} \subset \mathcal{I}^+$ such that for each $F \in \mathcal{I}^*$ there is n with $X_n \subset F$. For each $x \in U \cap A$ we have $F_x = \{n : f_n(x) < \alpha\} \in \mathcal{I}^*$, thus there is $n_x < \omega$ with $X_{n_x} \subset F_x$. Since X is a Baire space, there is n for which the set $\{x \in U \cap A : n_x = n\}$ is nowhere meager in some non-empty open set $W \subset U$. Fix $k \in X_n$. Since f_k has the Baire property, this easily implies that the set $A_0 = \{x \in W : \forall k \in X_n\ \ f_k(x) \le \alpha\}$ is a residual subset of W. Similarly, we can find a non-empty open set $V \subset W$ and $m < \omega$ such that $B_0 = \{x \in V : \forall k \in X_m\ \ f_k(x) \ge \beta\}$ is a residual subset of V. Fix any point $x \in A_0 \cap B_0$. Then $\{k < \omega : f_k(x) \le \alpha\} \in \mathcal{I}^+$ and $\{k < \omega : f_k(x) \ge \beta\} \in \mathcal{I}^+$, contrary to the $\mathcal{I}$-convergence of $(f_n(x))_n$. ■

THEOREM 4.2. *Suppose that $\mathcal{I}^\star$ is not weakly Ramsey and X is a metric Baire space. Then for every $f : X \to \mathbb{R}$ with the Baire property there exists a sequence of quasi-continuous functions $f_n : X \to \mathbb{R}$ such that $f = \mathcal{I}\text{-}\lim f_n$. Thus* Baire $\subset \mathcal{I}\text{-}\mathcal{B}_1(\mathrm{QC})$ *in X.*

Proof. Suppose $f : X \to \mathbb{R}$ has the Baire property. Then there exists a residual G_δ set G such that $f{\restriction}G$ is continuous. Let

$$A = \Big\{x \in X \setminus G : \varlimsup_{t \to x,\, t \in G} |f(t)| = \infty\Big\}.$$

Note that A is closed. Denote by X_0 the set of all isolated points of X. Clearly, $X_0 \subset G$. Let $g : X \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \varlimsup\limits_{t \to x,\, t \in G} f(t) & \text{for } x \in X \setminus (X_0 \cup A), \\ f(x) & \text{for } x \in X_0, \\ 0 & \text{for } x \in A. \end{cases}$$

Then g is of the first Baire class (cf. [Kur58, par. 31, VI]), hence it is the pointwise limit of a sequence $(g_n)_n$ of continuous functions. Moreover, $g(x) = f(x)$ for $x \in G$. Let $h = f - g$; then $\{x \in X : h(x) \neq 0\}$ is meager. By Proposition 3.2 there exists a sequence $(h_n)_n \in \mathrm{QC}$ which $\mathcal{I}$-converges to h. Then $f_n = g_n + h_n$ is quasi-continuous and $\mathcal{I}\text{-}\lim f_n = g + h = f$. ■

Recall that if $\mathcal{I}$ is an analytic ideal then the $\mathcal{I}$-limit of any sequence of Baire measurable real-valued functions has the Baire property (cf. [Kat72]). Thus we obtain the following corollary.

COROLLARY 4.3. *Suppose that $\mathcal{I}$ is an analytic ideal, $\mathcal{I}^\star$ is not weakly Ramsey and X is a metric Baire space. Then $\mathcal{I}\text{-}\mathcal{B}_\alpha(\mathrm{QC}) =$* Baire *for each $\alpha \ge 1$. (Hence in this case the $\mathcal{I}$-Baire system generated by the family* QC *has order ≤ 1.)*

There are ideals for which the assertion of Corollary 4.3 does not hold. In fact, it does not hold for any maximal ideal (see e.g. [FNS13, Example 7.27]).

4.1. Borel ideals. Conditions characterizing the Borel ideals for which the first $\mathcal{I}$-Baire class equals the first Baire class were obtained by Laczkovich and Recław [LR09], and by Debs and Saint Raymond [DSR09]. They proved the equivalence of four conditions, two of which give a combinatorial characterizations of $\mathcal{I}$, and two are formulated in the language of "Katětov order" and "containing an isomorphic copy".

Proposition 2.8 and Theorems 4.1, 4.2 yield similar characterizations of the Borel ideals $\mathcal{I}$ for which $\mathcal{I}\text{-}\mathcal{B}_1(\mathrm{QC}) = \mathcal{B}_1(\mathrm{QC})$.

THEOREM 4.4. *Suppose that $\mathcal{I}$ is a Borel ideal. Then the following conditions are equivalent:*

(1) $\mathcal{I}\text{-}\mathcal{B}_1(\mathrm{QC}) = \mathcal{B}_1(\mathrm{QC})$ *for every metric Baire space X;*
(2) $\mathcal{I}\text{-}\mathcal{B}_1(\mathrm{QC}) = \mathcal{B}_1(\mathrm{QC})$ *for $X = \mathbb{R}$;*
(3) *$\mathcal{I}^\star$ is ω-+-diagonalizable;*
(4) *$\mathcal{I}^\star$ is weakly Ramsey.*

Recently Kwela [Kwe15] proved that it is possible to characterize weakly Ramsey ideals in the language of "containing an isomorphic copy" of some ideal. He denoted by $\overline{WR}$ the ideal generated on $\omega \times \omega$ by two kinds of sets:

- all sets of the form $\{n\} \times \omega$ (vertical lines);
- all sets $A \subset \omega \times \omega$ with the property: if $(i, j), (m, n) \in A$ and $(i, j) \neq (m, n)$ then either $m > i + j$ or $i > m + n$.

THEOREM 4.5 ([Kwe15]). *For any ideal $\mathcal{I}$ the following conditions are equivalent:*

(4) *$\mathcal{I}^\star$ is weakly Ramsey;*
(5) $\overline{WR} \not\leq_K \mathcal{I}$;
(6) $\overline{WR} \not\sqsubseteq \mathcal{I}$.

COROLLARY 4.6. *For Borel ideals $\mathcal{I}$ all conditions* (1)–(6) *are equivalent.*

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