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Three-dimensional locally symmetric almost Kenmotsu manifolds

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Abstract. We prove that a three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

1. Introduction. The studies of classification problems on locally symmetric contact metric manifolds started from M. Okumura's work [13] in 1962, where he proved that a locally symmetric Sasakian manifold is of constant sectional curvature 1. Later, the above result was generalized to K-contact metric manifolds by S. Tanno [16] in 1967. In addition, locally symmetric contact metric manifolds of dimension three and five were studied by Blair and Sharma [2] and Blair and Sierra [3], respectively. It is also worth mentioning that Pastore [15] improved the corresponding results of Blair and Sierra [3]. Extending the results of Okumura, Ghosh and Sharma [9] proved that a locally symmetric contact strongly pseudo-convex integrable CR manifold of dimension $2n+1$ (which is supposed to be greater than 3 and not equal to 7) is locally isometric to either the unit sphere $S^{2n+1}(1)$ or the Riemannian product $S^n(4)\times\mathbb{R}^{n+1}$. Finally, in 2006, Boeckx and Cho [4] completed the classification theorem in this framework, proving that a locally symmetric contact metric manifold is locally isometric to either a Sasakian manifold of constant sectional curvature 1 or the unit tangent sphere bundle of a Euclidean space with its standard contact metric structure.

Since almost Kenmotsu manifolds and contact metric manifolds are both special types of almost contact metric manifolds, it is interesting to consider the classification problems on locally symmetric almost Kenmotsu mani-

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folds. In 1972, Kenmotsu [11] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature -1. Recently, Dileo and Pastore [6] classified locally symmetric almost Kenmotsu manifolds under a certain reasonable geometric condition, namely $R(X,Y)\xi = 0$ for any vector fields X and Y orthogonal to the characteristic vector field ξ. As an analogue of the classification results for locally symmetric contact strongly pseudo-convex integrable CR manifolds, the present author and Liu [17] proved that a locally symmetric CR-integrable almost Kenmotsu manifold of dimension $2n+1$, $n > 1$, is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Moreover, a complete classification theorem concerning Riemannian semisymmetric $(k,\mu)'$-almost Kenmotsu manifolds was obtained by Wang and Liu [18].

In this paper, we shall show that a three-dimensional locally symmetric almost Kenmotsu manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$. Some preliminaries required in the proof of our main result are presented in the second section. In the last section, we give a detailed proof of our main results and some immediate corollaries.

2. Preliminaries. According to Blair [1], an *almost contact structure* on a smooth differentiable manifold M^{2n+1} of dimension $2n+1$ is a (ϕ,ξ,η)-structure such that

$$\phi^2 = -\mathrm{id} + \eta\otimes\xi \quad \text{and} \quad \eta(\xi) = 1,$$

where ϕ is a $(1,1)$-type tensor field, the vector field ξ is called the *characteristic* or *Reeb vector field* and η is a 1-form. Let M^{2n+1} be a manifold endowed with a (ϕ,ξ,η)-structure; if there exists a Riemannian metric g on M^{2n+1} such that $g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector fields X, Y, then M^{2n+1} is said to be an *almost contact metric manifold* and the metric g is said to be *compatible* with the almost contact structure.

According to Janssens and Vanhecke [10], an *almost Kenmotsu manifold* is defined as an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ such that η is closed and $d\Phi = 2\eta\wedge\Phi$, where the *fundamental 2-form* Φ of the almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields X, Y on M^{2n+1}. An almost contact metric manifold such that $d\eta = \Phi$ is called a *contact metric manifold* (see Blair [1]).

Given an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$, one can define on $M^{2n+1}\times\mathbb{R}$ an almost complex structure J by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$

and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ (see Blair [1]); if $[\phi, \phi] = -2d\eta \otimes \xi$ then the almost contact metric structure is said to be *normal*. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (see [10, 11]). It is well-known that an almost Kenmotsu manifold is a Kenmotsu manifold if and only if $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ for any vector fields X and Y. A contact metric manifold with the characteristic vector field being Killing is called a *K-contact metric manifold*. A normal contact metric manifold is said to be a *Sasakian manifold*.

Now let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold. We consider three tensor fields $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. By Kim and Pak [12], the three $(1,1)$-type tensor fields l, h' and h are symmetric and satisfy $h'\xi = 0$, $h\xi = 0$, $l\xi = 0$, $\mathrm{tr}(h) = 0$, $\mathrm{tr}(h') = 0$ and $h\phi + \phi h = 0$. The following formulas were proved by Dileo and Pastore [6, 7]:

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.1}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.2}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.3}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr}(h^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{2.5}$$

for any $X, Y \in \mathfrak{X}(M)$, where S, Q, ∇ and $\mathfrak{X}(M)$ denote the Ricci curvature tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the Lie algebra of all differentiable vector fields on M^{2n+1}, respectively. Throughout this paper, we denote by $\mathcal{D}$ the distribution $\mathcal{D} = \ker \eta$ which is of dimension $2n$.

3. Locally symmetric 3-dimensional almost Kenmotsu manifolds. In this section we shall prove our main results. First, we need the following three lemmas.

LEMMA 3.1 ([6, Proposition 6]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. Then $\nabla_\xi h = 0$.*

LEMMA 3.2 ([8, Proposition 2.3]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold. Then the distribution $\mathcal{D}$ has Kählerian leaves if and only if*

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{3.1}$$

for any $X, Y \in \mathfrak{X}(M)$.

Lemma 3.3 ([17]). *Let $(M^3, \phi, \xi, \eta, g)$ be a three-dimensional locally symmetric almost Kenmotsu manifold. Then the Ricci operator is given by*

$$Q = -4h' + \eta \otimes Q\xi - \phi Q \phi + (\eta \circ Q) \otimes \xi - \mathrm{tr}(l)\eta \otimes \xi. \tag{3.2}$$

Proof. As M^3 is of dimension three, the distribution $\mathcal{D}$ of M^3 has Kählerian leaves and hence M^3 is CR-integrable. Proceeding as in the proof of [17, Lemma 4.3] we deduce from [17, eq. (4.3)] that (3.2) is true. ∎

Theorem 3.4. *A three-dimensional locally symmetric almost Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. Let $(M^3, \phi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. Suppose that $h = 0$ identically on M^3. Then from Dileo and Pastore [6, Theorem 3], M^3 is a Kenmotsu manifold of constant sectional curvature -1. In what follows, we shall consider the case $h \neq 0$.

Firstly, by Lemma 3.1 we obtain $\nabla_\xi h = 0$. Making use of this in (2.3) we obtain $\phi l \phi = (h')^2 - 2h' - \phi^2$, and comparing this equation with (2.2) yields

$$R(X, \xi)\xi = -X + \eta(X)\xi - 2h'X - (h')^2 X \tag{3.3}$$

for any $X \in \mathfrak{X}(M)$. Since M^3 is locally symmetric, it is also Ricci semisymmetric, i.e., $R(X, Y) \cdot Q = 0$ for any $X, Y \in \mathfrak{X}(M)$. Thus,

$$g(R(X, Y)Z, QW) + g(R(X, Y)W, QZ) = 0 \tag{3.4}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Replacing $Y = Z = W$ by ξ in (3.4) we get $g(R(X, \xi)\xi, Q\xi) = 0$. On the other hand, taking the inner product of (3.3) with $Q\xi$ gives

$$g(R(X, \xi)\xi, Q\xi) = g(X, -Q\xi + S(\xi, \xi)\xi - 2h'Q\xi - (h')^2 Q\xi).$$

Then it follows that

$$Q\xi - S(\xi, \xi)\xi + 2h'Q\xi + (h')^2 Q\xi = 0. \tag{3.5}$$

Next, we suppose that $h \neq 0$ on a non-empty open subset $\mathcal{U} \subseteq M^3$. Then there exists a unit eigenvector field E of h' with non-zero eigenvalue λ. It is easy to check that ϕE is a unit eigenvector field of h' with eigenvalue $-\lambda \neq 0$. Then $\{\xi, E, \phi E\}$ is a local orthonormal basis on $\mathcal{U}$. Hence, the spectrum of h' on $\mathcal{U}$ is $\mathrm{spec}(h') = \{0, \lambda, -\lambda\}$, with the respective eigenvector fields ξ, E and ϕE. From (2.4) we get $S(\xi, \xi) = -2(\lambda^2 + 1)$. We set

$$Q\xi = k_1 \xi + k_2 E + k_3 \phi E, \tag{3.6}$$

where k_1, k_2 and k_3 are smooth functions on $\mathcal{U}$. Taking the inner product of (3.6) with ξ we obtain $k_1 = -2(\lambda^2 + 1) \neq 0$. Inserting (3.6) into (3.5) yields $k_2(\lambda + 1)^2 = k_3(\lambda - 1)^2 = 0$. Assume that $\lambda^2 \neq 1$ on an open subset $\mathcal{O}$ of $\mathcal{U}$. Then $k_2 = k_3 = 0$, and hence $Q\xi = -2(1 + \lambda^2)\xi$.

It is well-known that the curvature tensor R of a three-dimensional Riemannian manifold (M^3, g) is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}\big(g(Y,Z)X - g(X,Z)Y\big) \tag{3.7}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where r denotes the scalar curvature of M^3. Since ξ is an eigenvector field of the Ricci operator, (3.7) implies that $R(X,Y)\xi = 0$ for any vector fields X, Y orthogonal to ξ. However, by [6, Theorem 5], the spectrum of h' is $\{0, 1, -1\}$, a contradiction.

The above analysis shows that $\lambda^2 = 1$ identically on $\mathcal{U}$, and hence $(h')^2 = -\phi^2$. In this case, (3.5) becomes

$$h'Q\xi = -Q\xi - 4\xi. \tag{3.8}$$

We shall denote by $[0]'$, $[1]'$ and $[-1]'$ the eigenspaces of h' with eigenvalues 0, 1 and -1, respectively. Considering a unit vector field $e \in [1]'$, we see that $\phi e \in [-1]'$ is also a unit vector field, i.e., $\{\xi, e, \phi e\}$ is a local orthonormal basis on $\mathcal{U}$. Taking the inner product of (3.8) with e we obtain $g(Q\xi, e) = 0$. Substituting ξ and e for $Y = Z$ and X respectively in (3.7) we obtain $R(e,\xi)\xi = Qe - (4 + r/2)e$. Comparing this with (3.3) and making use of $(h')^2 = -\phi^2$ we obtain $Qe = (r/2)e$. By Lemma 3.3 and (2.4), the Ricci operator is given by

$$Q\xi = -4\xi + \rho\phi e, \qquad Qe = \frac{r}{2}e \quad \text{and} \quad Q\phi e = \left(4 + \frac{r}{2}\right)\phi e + \rho\xi, \tag{3.9}$$

where ρ is a smooth function on $\mathcal{U}$.

A straightforward calculation gives $g(\nabla_e e, e) = 0$ and $g(\nabla_e e, \xi) = -2$. Next, if we put $g(\nabla_e e, \phi e) = \alpha$, we may write $\nabla_e e = -2\xi + \alpha\phi e$. Using this and Lemma 3.2 we have $\nabla_e \phi e = -\alpha e$. Since M^3 is locally symmetric, it is also Ricci symmetric, i.e., $\nabla Q = 0$. Taking the covariant derivative of the first and second terms of (3.9), both in direction e, and using $\nabla_e\xi = 2e$ and (3.9), we obtain

$$-(8 + r + \alpha\rho)e + e(\rho)\phi e = 0$$

and

$$(8 + r + \alpha\rho)\xi - \tfrac{1}{2}e(r)e + 2(2\alpha - \rho)\phi e = 0,$$

respectively. Then it follows that

$$r = -8 - \tfrac{1}{2}\rho^2, \qquad \alpha = \tfrac{1}{2}\rho \quad \text{and} \quad e(r) = e(\rho) = 0. \tag{3.10}$$

In view of $g(\nabla_\xi e, \xi) = g(\nabla_\xi e, e) = 0$, we may assume that $\nabla_\xi e = \beta\phi e$, and hence from $\nabla_\xi\phi = 0$ we get $\nabla_\xi \phi e = -\beta e$, where β is a smooth function. Similarly, taking the covariant derivative of the first and second terms of (3.9), both in direction ξ, and using (3.9), we obtain $\xi(\rho)\phi e - \beta\rho e = 0$

and $\beta\rho\xi - \frac{1}{2}\xi(r)e + 4\beta\phi e = 0$, respectively. This yields

$$\xi(r) = \xi(\rho) = 0 \quad \text{and} \quad \beta = 0. \tag{3.11}$$

Next, using $\nabla_{\phi e}\xi = 0$ and taking into account $g(\nabla_{\phi e}e, \xi) = g(\nabla_{\phi e}e, e) = 0$ we may set $\nabla_{\phi e}e = \theta\phi e$, and hence by (3.1) we have $\nabla_{\phi e}\phi e = -\theta e$, where θ is a smooth function. Taking the covariant derivative of the first and second terms of (3.9), both in direction ϕe, and using (3.9), we get $\phi e(\rho)\phi e - \theta\rho e = 0$ and $\theta\rho\xi - \frac{1}{2}\phi e(r)e + 4\theta\phi e = 0$, respectively. It follows that

$$\phi e(r) = \phi e(\rho) = 0 \quad \text{and} \quad \theta = 0. \tag{3.12}$$

Obviously, it follows from (3.10)–(3.12) that ρ is a constant and $\beta = \theta = 0$. Therefore,

$$R(e, \phi e)\xi = \nabla_e\nabla_{\phi e}\xi - \nabla_{\phi e}\nabla_e\xi - \nabla_{[e,\phi e]}\xi = \rho e. \tag{3.13}$$

By a simple computation we see that the distribution $[-1]' \oplus [0]'$ is integrable and totally geodesic. Thus, in view of $\nabla_e e = -2\xi + \frac{1}{2}\rho\phi e$, ρ a constant, we deduce that M^3 is locally isometric to a warped product $N \times_f C$, where N is a leaf of the distribution $[-1]' \oplus [0]'$ and C is a curve tangent to $[1]'$.

Applying O'Neill [14, Proposition 35] on the warped product $N \times_f C$ and using $\nabla_{\phi e}e = 0$ and $\nabla_e\phi e = -\frac{1}{2}\rho e$ shows that $\rho = 0$. Therefore, (3.13) implies that $R(e, \phi e)\xi = 0$, and hence $R(X, Y)\xi = 0$ for any vector fields X, Y orthogonal to ξ. Finally, the conclusion follows from Dileo and Pastore [6, Theorem 6]. ∎

From (3.7) we see that local symmetry ($\nabla R = 0$) and Ricci symmetry ($\nabla S = 0$) are equivalent on any three-dimensional Riemannian manifold. Thus, the statement of Theorem 3.4 remains valid even if local symmetry is replaced by Ricci symmetry.

Remark 3.5. From Theorem 3.4, we deduce that a three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Remark 3.6. From Theorem 3.4, we see that the answer to the question proposed by Dileo and Pastore [6, Introduction] is positive for dimension three. However, in higher dimensions the problem is still open.

Remark 3.7. Chinea and Gonzalez [5, Examples] constructed a Kenmotsu structure on the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Note that a strictly almost Kenmotsu structure on $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ was given by Dileo and Pastore [6, pp. 352–353].

Using Theorem 3.4 and [17, Theorem 1.2], we immediately obtain the following result.

THEOREM 3.8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a CR-integrable almost Kenmotsu manifold of dimension ≥ 3. Then it is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

By a $(k, \mu)'$-*almost Kenmotsu manifold* $(M^{2n+1}, \phi, \xi, \eta, g)$, we mean that the characteristic vector field ξ belongs to the $(k, \mu)'$-*nullity distribution*, that is, $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y)$ for any vector fields X, Y on M^{2n+1}, where both k and μ are constants. In addition, an almost Kenmotsu manifold is said to be ϕ-*recurrent* if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z \tag{3.14}$$

for any vector fields X, Y, Z, W, where A denotes a 1-form on M^{2n+1}. Obviously, the vanishing of A in (3.14) reduces to ϕ-symmetry (which is weaker than local symmetry). The following result follows directly from Wang and Liu [18, Theorems 1.1, 1.2] and [19, Theorem 1].

REMARK 3.9. Let $(M^3, \phi, \xi, \eta, g)$ be a $(k, \mu)'$-almost Kenmotsu manifold of dimension 3. Then the following statements are equivalent:

(1) M^3 is locally symmetric, i.e., $\nabla R = 0$;
(2) M^3 is ϕ-recurrent, i.e., (3.14) holds;
(3) M^3 is Riemannian semisymmetric, i.e., $R \cdot R = 0$;
(4) M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

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