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Existence of positive radial solutions for the elliptic equations on an exterior domain

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Abstract. We discuss the existence of positive radial solutions of the semilinear elliptic equation

$$\begin{cases} -\Delta u = K(|x|)f(u), & x \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $N \geq 3$, $K : [r_0, \infty) \to \mathbb{R}^+$ is continuous and $0 < \int_{r_0}^{\infty} rK(r)\,dr < \infty$, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $f(0) = 0$. Under the conditions related to the asymptotic behaviour of $f(u)/u$ at 0 and infinity, the existence of positive radial solutions is obtained. Our conditions are more precise and weaker than the superlinear or sublinear growth conditions. Our discussion is based on the fixed point index theory in cones.

1. Introduction. In this paper we discuss the existence of positive radial solutions for the semilinear elliptic boundary value problem (BVP)

$$\begin{cases} -\Delta u = K(|x|)f(u), & x \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.1}$$

in the exterior domain $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, where $N \geq 3$, $r_0 > 0$, $\partial u/\partial n$ is the outward normal derivative on $\partial\Omega$, $K : [r_0, \infty) \to \mathbb{R}^+$ is a coefficient function, $f : \mathbb{R}^+ \to \mathbb{R}$ is a nonlinear function, and α, β are constant. Throughout this paper, we assume that the following conditions hold:

(A1) $K \in C([r_0, \infty), \mathbb{R}^+)$ and $0 < \int_{r_0}^{\infty} rK(r)\,dr < \infty$;
(A2) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f(0) = 0$;

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(A3) $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

Since $f(0) = 0$, it is clear that 0 is a trivial radial solution of (1.1), and we shall be interested in obtaining a positive radial solution.

For the special case of (1.1) with Dirichlet boundary condition, the existence of positive radial solutions has been discussed by several authors [4–9]. Using the method of upper and lower solutions, a priori estimates or fixed point index theory, these authors obtained some existence results. The purpose of this paper is to obtain more precise existence results for positive radial solutions for (1.1). Our main result is related to the principal eigenvalue λ_1 of the radially symmetric elliptic eigenvalue problem (EVP)

$$(1.2)\qquad \begin{cases} -\Delta u = \lambda K(|x|)u, & x \in \Omega, \\ \alpha u + \beta \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ u = u(|x|), & \lim\limits_{|x|\to\infty} u(|x|) = 0. \end{cases}$$

Indeed, in Section 2 we will prove that (1.2) has a minimal positive real eigenvalue λ_1 (Lemmas 2.2 and 2.3).

We introduce the notation

$$(1.3)\qquad \begin{aligned} f_0 &= \liminf_{u\to 0} \frac{f(u)}{u}, & f^0 &= \limsup_{u\to 0} \frac{f(u)}{u}, \\ f_\infty &= \liminf_{u\to \infty} \frac{f(u)}{u}, & f^\infty &= \limsup_{u\to \infty} \frac{f(u)}{u}. \end{aligned}$$

Our main result can be stated as follows:

THEOREM 1.1. *Suppose that assumptions* (A1)–(A3) *hold. If f satisfies one of the following conditions:*

(H1) $f^0 < \lambda_1$, $f_\infty > \lambda_1$;
(H2) $f_0 > \lambda_1$, $f^\infty < \lambda_1$,

then BVP (1.1) *has a classical positive radial solution.*

Noting that λ_1 is an eigenvalue of (1.2), conditions (H1) and (H2) cannot be improved. If the strict inequalities in (H1) or (H2) of Theorem 1.1 are weakened to nonstrict inequalities, the existence of radial solution of (1.1) cannot be guaranteed. For example, for $f(u) = \lambda_1 u + 1$, we have $f_0 > \lambda_1$ and $f^\infty = \lambda_1$, but by the Fredholm alternative, (1.1) has no radial solution. Therefore, conditions (H1) and (H2) in Theorem 1.1 are sharp.

In Theorem 1.1, condition (H1) allows that $f(u)$ is of superlinear growth in u, and (H2) allows that $f(u)$ is of sublinear growth in u at the origin and at infinity. If $f \in C^1(\mathbb{R}^+)$, then by (1.3),

$$f_0 = f^0 = f'(0), \quad f_\infty = \liminf_{u\to\infty} f'(u), \quad f^\infty = \limsup_{u\to\infty} f'(u).$$

Hence, from Theorem 1.1 we obtain

COROLLARY 1.2. *Suppose that assumptions* (A1)–(A3) *hold. If f is in $C^1(\mathbb{R}^+)$ and one of the following conditions holds:*

(F1) $f'(0) < \lambda_1$, $\liminf_{u\to\infty} f'(u) > \lambda_1$;
(F2) $f'(0) > \lambda_1$, $\limsup_{u\to\infty} f'(u) < \lambda_1$,

then BVP (1.1) *has a classical positive radial solution.*

To show the applicability of our main result, we consider the Gelfand problem on the exterior domain Ω,

$$\begin{cases} -\Delta u = K(|x|)e^u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.4}$$

In Lemma 2.2, we will prove that the principal eigenvalue λ_1 of (1.2) satisfies

$$\lambda_1 > (N-1) \Big/ \int_{r_0}^{\infty} rK(r)\,dr. \tag{1.5}$$

If

$$0 < \int_{r_0}^{\infty} rK(r)\,dr \le N-1, \tag{1.6}$$

then the nonlinearity $f(u) = e^u$ of (1.4) satisfies

$$f'(0) = 1 \le (N-1) \Big/ \int_{r_0}^{\infty} rK(r)\,dr < \lambda_1, \qquad \liminf_{u\to\infty} f'(u) = \infty > \lambda_1.$$

Hence condition (F1) holds. By Corollary 1.1, we have

COROLLARY 1.3. *Let $K \in C([r_0,\infty),\mathbb{R}^+)$ and it satisfy* (1.6). *Then the Gelfand problem* (1.4) *has a positive radial solution.*

In the recent paper [1], the authors discuss the existence of positive radial solutions of (1.1) with the more general nonlinear boundary condition

$$\frac{\partial u}{\partial n} + c(u)u = 0, \qquad x \in \partial\Omega. \tag{1.7}$$

Under the assumption that $f : [0,\infty) \to \mathbb{R}$ is a C^1 function sublinear at ∞ (i.e. $f^\infty = 0$), they obtain existence and multiplicity results by the method of upper and lower solutions. But their results and methods are not applicable to the case where f satisfies (H1) or (H2). In particular, the results in [1] cannot be applied to the Gelfand problem (1.4).

The proof of Theorem 1.1 is based on the fixed point index theory in cones, which will be given in Section 3. Some preliminaries are introduced in Section 2.

2. Preliminaries. Let $I = [0,1]$ and $\mathbb{R}^+ = [0,\infty)$. Let $C(I)$ denote the Banach space of all continuous functions $v(t)$ on I with the norm $\|v\|_C = \max_{t\in I} |v(t)|$, and let $C^+(I)$ be the cone of all nonnegative functions in $C(I)$.

For radially symmetric solutions, upon writing $r = |x|$, BVP (1.1) becomes an ordinary differential equation boundary value problem on the infinite interval $[r_0,\infty)$:

$$\begin{cases} -u''(r) - \dfrac{N-1}{r}u'(r) = K(r)f(u(r)), \quad r \in [r_0,\infty), \\ \alpha u(r_0) - \beta u'(r_0) = 0, \quad u(\infty) = 0, \end{cases} \tag{2.1}$$

where $u(\infty) = \lim_{r\to\infty} u(r)$. Making the variable transformations

$$t = {r_0}^{N-2}/r^{N-2}, \quad r = r_0 t^{-1/(N-2)}, \quad v(t) = u(r(t)), \tag{2.2}$$

we convert (2.1) to an ordinary differential equation boundary value problem on $(0,1]$:

$$\begin{cases} -v''(t) = a(t)f(v(t)), \quad t \in (0,1], \\ v(0) = 0, \quad \alpha_1 v(1) + \beta_1 v'(1) = 0, \end{cases} \tag{2.3}$$

where

$$a(t) = \frac{r^{2(N-1)}(t)}{(N-2)^2 {r_0}^{2(N-2)}} K(r(t)), \qquad t \in (0,1], \tag{2.4}$$

$$\alpha_1 = \alpha, \qquad \beta_1 = (N-2)\beta/r_0. \tag{2.5}$$

Clearly, if $v \in C[0,1]\cap C^2(0,1]$ is a solution of (2.3), then $u(r) = v(t(r))$ is a solution of (2.1) and $u(|x|) \in C(\overline{\Omega})\cap C^2(\Omega)$ is a classical radial solution. We will discuss (2.3) in order to obtain positive radial solutions of (1.1). Note that (2.3) is an ordinary differential equation boundary value problem with singularity at $t = 0$. We will use the fixed point index theory in cones to obtain the existence of positive solutions for (2.3).

We first consider the corresponding linear boundary value problem (LBVP)

$$\begin{cases} -v''(t) = a(t)h(t), \quad t \in (0,1], \\ v(0) = 0, \quad \alpha_1 v(1) + \beta_1 v'(1) = 0, \end{cases} \tag{2.6}$$

where $h \in C(I)$ is a given function.

Lemma 2.1. *For every $h \in C(I)$, LBVP (2.6) has a unique solution $v := Sh \in C(I)\cap C^2(0,1]$. Moreover,*

(a) *the solution operator $S : C(I) \to C(I)$ is a completely continuous linear operator;*
(b) *if $h \in C^+(I)$, then $v = Sh$ satisfies $v(t) \ge t(1-\sigma t)\,\|v\|_C$ for all $t \in I$.*

Proof. Let $G(t,s)$ be the Green function of the homogeneous linear boundary value problem

$$\begin{cases} -v''(t) = 0, & t \in [0,1], \\ v(0) = 0, & \alpha_1 v(1) + \beta_1 v'(1) = 0, \end{cases} \tag{2.7}$$

which is explicitly expressed by

$$G(t,s) = \begin{cases} t(1-\sigma s), & 0 \le t \le s \le 1, \\ s(1-\sigma t), & 0 \le s \le t \le 1, \end{cases} \tag{2.8}$$

where $\sigma = \alpha_1/(\alpha_1+\beta_1) \in [0,1]$ is a constant. From (2.8) we can verify that:

(1) $G(t,s) > 0$ for all $t, s \in (0,1)$;
(2) $G(t,s) \le G(s,s)$ for all $t, s \in I$;
(3) $G(t,s) \ge G(t,t)G(s,s)$ for all $t, s \in I$.

Indeed, properties (1) and (2) are evident. Since

$$\frac{G(t,s)}{G(t,t)G(s,s)} > 1 \qquad \forall t, s \in (0,1),$$

it follows by continuity and positivity that (3) holds.

By (2.4) and assumption (A1), the coefficient a is in $C^+(0,1]$ and satisfies

$$\int_0^1 sa(s)\,ds = \frac{1}{N-2}\int_{r_0}^{\infty} rK(r)\,dr < \infty. \tag{2.9}$$

Hence $sa(s) \in L(I)$. Given $h \in C(I)$, we will verify that

$$v(t) = \int_0^1 G(t,s)a(s)h(s)\,ds := Sh(t), \qquad t \in I, \tag{2.10}$$

belongs to $C(I) \cap C^2(0,1]$ and is a solution of (2.6). In fact, for every $t \in I$, since

$$|G(t,s)a(s)h(s)| \le G(s,s)a(s)\|h\|_\infty \le \|h\|_\infty sa(s), \qquad s \in I, \tag{2.11}$$

by (2.11) and the Lebesgue dominated convergence theorem, the function

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)a(s)h(s)\,ds \\ &= (1-\sigma t)\int_0^t sa(s)h(s)\,ds + t\int_t^1 (1-\sigma s)a(s)h(s)\,ds \end{aligned}$$

is well defined on I and $v \in C(I)$. Differentiating, we obtain

$$v'(t) = -\sigma\int_0^t sa(s)h(s)\,ds + \int_t^1 (1-\sigma s)a(s)h(s)\,ds, \qquad t \in (0,1], \tag{2.12}$$

and

$$v''(t) = -a(t)h(t), \qquad t \in (0,1]. \tag{2.13}$$

By (2.12) and (2.13), $v \in C(I) \cap C^2(0,1]$ is a solution of (2.6). Since the homogeneous equation (2.7) has only the zero solution, it follows that $v(t)$ is a unique solution of (2.6).

By (2.11), $S : C(I) \to C(I)$ is a bounded linear operator. Let $H \subset C(I)$ be a bounded set. Then there exists a positive constant $M > 0$ such that for every $h \in H$ we have $\|h\|_C \le M$. For all $h \in H$, by (2.11),

$$|Sh(t_2) - Sh(t_2)| \le M \int_{t_1}^{t_1} sa(s)\,ds, \qquad 0 \le t_1 < t_2 \le 1.$$

Hence by the absolute continuity of the Lebesgue integral, $S(H)$ is a bounded equicontinuous set in $C(I)$. By the Ascoli–Arzelà theorem, $S(H)$ is a precompact subset of $C(I)$. Therefore, $S : C(I) \to C(I)$ is completely continuous. Hence, the conclusion of Lemma 2.1(a) holds.

Let $h \in C^+(I)$ and $v = Sh$. For every $t \in I$, by (2.10) and property (2) of G we have

$$v(t) = \int_0^1 G(t,s)a(s)h(s)\,ds \le \int_0^1 G(s,s)a(s)h(s)\,ds.$$

Hence

$$\|v\|_C \le \int_0^1 G(s,s)a(s)h(s)\,ds.$$

By property (3) of G and the above inequality,

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)a(s)h(s)\,ds \ge G(t,t)\int_0^1 G(s,s)a(s)h(s)\,ds \\ &\ge G(t,t)\|v\|_C = t(1-\sigma t)\|v\|_C, \qquad t \in I. \end{aligned}$$

Hence, the conclusion of Lemma 2.1(b) holds. ■

Consider the weighted linear eigenvalue problem (EVP)

$$\begin{cases} -v''(t) = \lambda a(t)v(t), & t \in (0,1], \\ v(0) = 0, \quad \alpha_1 v(1) + \beta_1 v'(1) = 0. \end{cases} \tag{2.14}$$

LEMMA 2.2. *EVP* (2.14) *has a minimal positive real eigenvalue* λ_1 *satisfying*

$$\lambda_1 > 1\Big/\int_0^1 sa(s)\,ds = (N-1)\Big/\int_{r_0}^{\infty} rK(r)\,dr. \tag{2.15}$$

Moreover, λ_1 *has a positive unit eigenfunction, namely there exists* ϕ_1 *in* $C^+(I) \cap C^2(0,1]$ *with* $\|\phi_1\|_C = 1$ *that satisfies the equation*

$$\begin{cases} -\phi_1''(t) = \lambda_1 a(t)\phi_1(t), \quad t \in (0,1], \\ \phi_1(0) = 0, \quad \alpha_1\phi_1'(1) + \beta_1\phi_1'(1) = 0. \end{cases} \tag{2.16}$$

Proof. Clearly, $\lambda > 0$ is an eigenvalue of (2.14) if and only if $\mu = 1/\lambda > 0$ is an eigenvalue of the solution operator S of (2.6).

By Lemma 2.1(b), $S : C(I) \to C(I)$ is a positive linear operator and it satisfies the strongly positive estimate

$$Sh(t) \geq t(1 - \sigma t)\|Sh\|_C, \quad \forall h \in C^+(I),\ t \in I. \tag{2.17}$$

Set $h_0 = t(1 - \sigma t)$. Then clearly $h_0 \in C^+(I)$, and by (2.15) we have

$$Sh_0 \geq \|Sh_0\|_C h_0.$$

Applying S to both sides, by the positivity of S we have

$$S^2 h_0 \geq (\|Sh_0\|_C)^2 h_0.$$

Hence for any $n \in \mathbb{N}$, by recurrence,

$$S^n h_0 \geq (\|Sh_0\|_C)^n h_0.$$

Consequently,

$$\|S^n h_0\|_C \geq (\|Sh_0\|_C)^n \|h_0\|_C,$$

so that

$$\|S^n\| \geq \frac{\|S^n h_0\|_C}{\|h_0\|_C} \geq (\|Sh_0\|_C)^n.$$

By this inequality and the Gelfand formula for the spectral radius, we obtain

$$r(S) = \lim_{k\to\infty} \sqrt[n]{\|S^n\|} \geq \|Sh_0\|_C > 0. \tag{2.18}$$

Hence by the well-known Krein–Rutman theorem, $r(S)$ is the maximal positive real eigenvalue of S, which has a positive eigenfunction. Thus, there is an eigenfunction $\phi_1 \in C^+(I)$ with $\|\phi_1\|_C = 1$ such that $S\phi_1 = r(S)\phi_1$. Set $\lambda_1 = 1/r(S)$. Then λ_1 is the minimal positive real eigenvalue of (1.14) and $\phi_1 = S(\lambda_1\phi_1) \in C(I) \cap C^2(0,1]$ is the corresponding positive unit eigenfunction.

For every $h \in C(I)$, by property (3) of G,

$$\begin{aligned} |Sh(t)| &= \int_0^1 G(t,s)a(s)|h(s)|\,ds \leq \|h\|_C \int_0^1 G(s,s)a(s)\,ds \\ &= \|h\|_C \int_0^1 a(s)h_0(s)\,ds, \quad t \in I. \end{aligned}$$

Hence,

$$\|S\| \le \int_0^1 a(s)h_0(s)\,ds < \int_0^1 aa(s)\,ds. \tag{2.19}$$

Since $r(S) \le \|S\|$, from (2.19) and (2.9) it follows that

$$\lambda_1 \ge 1/\|S\| > 1/\int_0^1 sa(s)\,ds = (N-1)/\int_{r_0}^{\infty} rK(r)\,dr.$$

Hence (2.15) holds. ∎

We consider the radially symmetric elliptic eigenvalue problem (1.2). Making the variable transformations of (2.2) converts (1.2) into (2.14). Clearly, $\lambda \in \mathbb{R}$ is an eigenvalue of (1.2) if and only if it is an eigenvalue of (2.14). Hence, by Lemma 2.2 we have

Lemma 2.3. *λ_1 is a minimal positive real eigenvalue of* (1.2), *and* $\psi_1(x) = \phi_1({r_0}^{N-2}/|x|^{N-2})$ *is the corresponding eigenfunction.*

Now we consider the nonlinear ordinary differential equation BVP (2.3). Define a closed convex cone K in the Banach space $E = C(I)$ by

$$K = \{v \in C(I) : v(t) \ge t(1-\sigma t)\|v\|_C,\ t \in I\}. \tag{2.20}$$

By Lemma 2.1(b), $S(C^+(I)) \subset K$. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous. For every $v \in K$, set

$$F(v)(t) := f(r(t)), \qquad t \in I. \tag{2.21}$$

Then $F : K \to C^+(I)$ is continuous and it maps every bounded subset in K into a bounded subset in $C^+(I)$. Consider the composite mapping

$$A = S \circ F. \tag{2.22}$$

Then $A : K \to K$ is completely continuous since $S : C(I) \to C(I)$ is. By the definitions of S and K, a positive solution of (2.3) is equivalent to a nonzero fixed point of A.

To find a nonzero fixed point of A defined by (2.22), we recall some concepts and conclusions on the fixed point index in [2, 3]. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume D is a bounded open subset of E with boundary ∂D, and $K \cap D \ne \emptyset$. Let $A : K \cap \overline{D} \to K$ be a completely continuous mapping. If $Av \ne v$ for every $v \in K \cap \partial D$, then the fixed point index $i(A, K \cap D, K)$ is well defined. If $i(A, K \cap D, K) \ne 0$, then A has a fixed point in $K \cap D$. The following two lemmas of [2, 3] are needed in our argument.

Lemma 2.4. *Let D be a bounded open subset of E with $\theta \in D$, and $A : K \cap \overline{D} \to K$ a completely continuous mapping. If $\mu A v \ne v$ for every $v \in K \cap \partial D$ and $0 < \mu \le 1$, then $i(A, K \cap D, K) = 1$.*

LEMMA 2.5. *Let D be a bounded open subset of E and $A : K \cap \overline{D} \to K$ a completely continuous mapping. If there exists $w_0 \in K \setminus \{\theta\}$ such that $v - Av \neq \tau w_0$ for every $v \in K \cap \partial D$ and $\tau \geq 0$, then $i(A, K \cap D, K) = 0$.*

3. Proof of Theorem 1.1. Let $K \subset C(I)$ be the closed convex cone defined by (2.20) and $A : K \to K$ be the completely continuous mapping defined by (2.22). If $v \in K$ is a nontrivial fixed point of A, then by the definitions of S and A, $v(t)$ is a positive solution of (2.3) and $u = v(r_0^{N-2}/|x|^{N-2})$ is a classical positive radial solution of (1.1). Hence we only need to prove that A has a nontrivial fixed point.

We only consider the case that (H1) holds; the case of (H2) can be handled similarly.

Let $0 < R_1 < R_2 < \infty$ and set

$$D_1 = \{v \in C(I) : \|v\|_C < R_1\}, \qquad D_2 = \{v \in C(I) : \|v\|_C < R_2\}. \tag{3.1}$$

We prove that A has a fixed point in $K \cap (D_2 \setminus \overline{D}_1)$ when R_1 is small enough and R_2 large enough. To do so, we need to calculate the fixed point index $i(A, K \cap (D_2 \setminus \overline{D}_1), K)$.

Firstly, since $f^0 < \lambda_1$, by the definition of f^0, there exist $\varepsilon \in (0, \lambda_1)$ and $\delta > 0$ such that

$$f(u) \leq (\lambda_1 - \varepsilon)u, \qquad 0 \leq u \leq \delta. \tag{3.2}$$

Choosing $R_1 \in (0, \delta)$, we prove that A satisfies the condition of Lemma 2.4 in $K \cap \partial D_1$, namely

$$\mu Av \neq v, \qquad \forall v \in K \cap \partial D_1,\ 0 < \mu \leq 1. \tag{3.3}$$

In fact, if (3.3) does not hold, there exist $v_0 \in K \cap \partial D_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0 A v_0 = v_0$. Since $v_0 = S(\mu_0 F(v_0))$, by the definition of S, v_0 is the unique solution of (2.6) for $h = \mu_0 F(v_0) \in C^+(I)$. Hence, $v_0 \in C^2(0,1]$ satisfies the differential equation

$$\begin{cases} -v_0''(t) = \mu_0 a(t) f(v_0(t)), & t \in (0,1], \\ v_0(0) = 0, \quad \alpha_1 v_0(1) + \beta_1 v_0'(1) = 0. \end{cases} \tag{3.4}$$

Since $v_0 \in K \cap \partial D_1$, by the definitions of K and D_1 we have

$$0 \leq v_0(t) \leq \|v_0\|_C = R_1 < \delta, \qquad t \in I.$$

Hence by (3.2),

$$f(v_0(t)) \leq (\lambda_1 - \varepsilon) a(t) v_0(t), \qquad t \in (0,1].$$

From this inequality and (3.4) we have

$$-v_0''(t) \leq (\lambda_1 - \varepsilon) a(t) v_0(t), \qquad t \in I.$$

Multiplying this inequality by $\phi_1(t)$ and integrating on I, then using integration by parts for the left side, we obtain

$$(3.5)\qquad -\int_0^1 v_0''(t)\phi_1(t)\,dt = v_0(1)\phi_1'(1) - v_0'(1)\phi_1(1) + \lambda_1 \int_0^1 a(t)v_0(t)\phi_1(t)\,dt$$
$$\le (\lambda_1 - \varepsilon)\int_0^1 a(t)v_0(t)\phi_1(t)\,dt.$$

Since $\alpha_1, \beta_1 \ge 0$ and $\alpha_1 + \beta_1 > 0$, from the boundary conditions

$$\alpha_1 v_0(1) + \beta_1 v_0'(1) = 0, \qquad \alpha_1 \phi_1(1) + \beta_1 \phi_1'(1) = 0,$$

we easily see that

$$(3.6)\qquad v_0(1)\phi_1'(1) - v_0'(1)\phi_1(1) = 0.$$

Combining this with (3.5), we obtain

$$(3.7)\qquad \lambda_1 \int_0^1 a(t)v_0(t)\phi_1(t)\,dt \ge b\int_0^1 a(t)v_0(t)\phi_1(t)\,dt.$$

By Lemma 2.1(b),

$$(3.8)\qquad v_0(t) \ge t(1-\sigma t)\|v_0\|_C, \quad \phi_1(t) \ge t(1-\sigma t)\|\phi_1\|_C = t(1-\sigma t), \quad t \in I,$$

and hence

$$\int_0^1 a(t)v_0(t)\phi_1(t)\,dt \ge \|v_0\|_C \int_0^1 t^2(1-\sigma t)^2 a(t)\,dt > 0.$$

From this and (3.7) it follows that $\lambda_1 \le \lambda_1 - \varepsilon$, which is a contradiction.

Hence (3.3) holds, so A satisfies the condition of Lemma 2.4 in $K \cap \partial D_1$. By Lemma 2.4, we have

$$(3.9)\qquad i(A, K \cap D_1, K) = 1.$$

Secondly, since $f_\infty > \lambda_1$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $M > 0$ such that

$$(3.10)\qquad f(u) \ge (\lambda_1 + \varepsilon_1)u, \quad u \ge M.$$

Define a positive constant by

$$C_0 = \max_{0\le u\le M} |f(u) - (\lambda_1 + \varepsilon_1)u| + 1.$$

Then from (3.10) it follows that

$$(3.11)\qquad f(u) \ge (\lambda_1 + \varepsilon_1)u - C_0, \quad u \ge 0.$$

Let ϕ_1 be the positive eigenvalue function of (2.14) in Lemma 2.2. Since $\phi_1 = S(\lambda_1\phi_1)$, from Lemma 2.1(b) it follows that $\phi_1 \in K \setminus \{\theta\}$. Choosing

$w_0 = \phi_1$, we will show that A satisfies the condition of Lemma 2.5 in $K \cap \partial D_2$ when R_2 is large enough, namely

$$v - Av \neq \tau\phi_1, \quad \forall v \in K \cap \partial D_2,\ \tau \geq 0. \tag{3.12}$$

Indeed, if (3.12) does not hold, there exist $v_1 \in K \cap \partial D_2$ and $\tau_1 \geq 0$ such that $v_1 - A_1 v_1 = \tau_1\phi_1$. Since $v_1 = Av_1 + \tau_1\phi_1 = S(F(v_1) + \tau_1\lambda_1\phi_1)$, by definition of S, v_1 is the unique solution of (2.6) for $h = F(v_1) + \tau_1\lambda_1\phi_1 \in C^+(I)$. Hence, $v_1 \in C^2(0,1]$ satisfies the differential equation

$$\begin{cases} -v_1''(t) = a(t)[f(v_1(t)) + \tau_1\lambda_1\phi_1(t)], & t \in (0,1], \\ v_1(0) = 0, \quad \alpha_1 v_1(1) + \beta_1 v_1'(1) = 0. \end{cases} \tag{3.13}$$

Since $v_1(t) \geq 0$ for all $t \in I$, by (3.11) and (3.13) we have

$$\begin{aligned} -v_1''(t) &= a(t)[f(v_1(t)) + \tau_1\lambda_1\phi_1(t)] \\ &\geq a(t)[(\lambda_1 + \varepsilon_1)v_1(t) - C_0 + \tau_1\lambda_1\phi_1(t)] \\ &\geq (\lambda_1 + \varepsilon_1)a(t)v_1(t) - C_0 a(t), \quad t \in (0,1]. \end{aligned}$$

Multiplying this inequality by $\phi_1(t)$ and integrating on I, then using integration by parts for the left side, we have

$$\lambda_1 \int_0^1 a(t)v_1(t)\phi_1(t)\,dt \geq (\lambda_1 + \varepsilon_1)\int_0^1 a(t)v_1(t)\phi_1(t)\,dt - C_0\int_0^1 a(t)\,dt.$$

From this inequality it follows that

$$\int_0^1 a(t)v_1(t)\phi_1(t)\,dt \leq \frac{C_0}{\varepsilon_1}\int_0^1 a(t)\,dt. \tag{3.14}$$

By Lemma 2.1(b),

$$v_1(t) \geq t(1 - \sigma t)\|v_1\|_C, \quad \phi_1(t) \geq t(1 - \sigma t), \quad t \in I,$$

so we have

$$\int_0^1 a(t)v_1(t)\phi_1(t)\,dt \geq \|v_2\|_C \int_0^1 t^2(1 - \sigma t)^2 a(t)\,dt.$$

From this inequality and (3.14) we obtain

$$\|v_1\|_C \leq \frac{C_0\int_0^1 a(t)\,dt}{\varepsilon_1\int_0^1 t^2(1 - \sigma t)^2 a(t)\,dt} =: R_0. \tag{3.15}$$

Let $R_2 > \max\{R_0, \delta\}$. Since $v_2 \in K \cap \partial D_2$, by the definition of D_2, we have $\|v_2\|_C = R_2 > R_0$. This contradicts (3.15).

Hence, (3.12) holds, so A satisfies the condition of Lemma 2.5 in $K \cap \partial D_2$. By Lemma 2.5, we have

$$i(A, K \cap D_2, K) = 0. \tag{3.16}$$

Now, by the additivity of fixed point index, (3.9) and (3.16), we have

$$i(A, K \cap (D_2 \setminus \overline{D}_1), K) = i(A, K \cap D_2, K) - i(A,\ K \cap D_1, K) = -1.$$

Hence A has a fixed point $v^* \in K \cap (D_2 \setminus \overline{D}_1)$. By the definition of A, $v^* \in C(I) \cap C^2(0,1]$ is a positive solution of (2.3). Hence $v^*(r_0^{N-2}/|x|^{N-2})$ is a classical positive radial solution of (1.1).

The proof of Theorem 1.1 is complete. ∎

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