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On the Bergman distance on model domains in $\mathbb{C}^n$

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Abstract. Let P be a real-valued and weighted homogeneous plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ and let D denote the "model domain" $\{z\in\mathbb{C}^n \mid r(z) := \operatorname{Re} z_1+P(z') < 0\}$. We prove a lower estimate on the Bergman distance of D if P is assumed to be strongly plurisubharmonic away from the coordinate axes.

1. Introduction. Let $D \subset \mathbb{C}^n$ be a domain and $H^2(D)$ the Hilbert space of all holomorphic functions on D that are square-integrable with respect to the Lebesgue measure. We further suppose that the Bergman kernel $K_D : D \times D \to \mathbb{C}$ has positive values on the diagonal of $D \times D$. Then $\log K_D(z, z)$ induces a Kähler metric B_D^2, which is called the Bergman metric. The underlying Riemann structure induces a Riemann distance which is known as the Bergman distance d_D^B on D.

There exists a comprehensive literature on the question of completeness of the invariant distances of Bergman, Carathéodory, and Kobayashi on certain classes of pseudoconvex domains; for a survey see [JaPf].

The boundary behavior of the above-mentioned distance functions has been described in [BaBo] on strongly pseudoconvex domains and in [Her4] on pseudoconvex domains of finite type in dimension two; for a generalization on "Levi corank one domains" see [Her5].

In this article we want to establish suitable lower bounds for the Bergman distance on model domains, i.e. on domains of the special form

$$D = D_P := \{z \in \mathbb{C}^n \mid r(z) = \operatorname{Re} z_1 + P(z') < 0\}, \tag{M}$$

where P denotes a real-valued plurisubharmonic polynomial in the variables $z' := (z_2, \ldots, z_n)$, that is weighted homogeneous of degree one. We explain this notion in

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Definition. We fix positive integers $m_2,\dots,m_n$. Let $P:\mathbb{C}^{n-1}\to\mathbb{R}$ be plurisubharmonic (without pure terms). We call P *weighted homogeneous* of degree d (with respect to $(m_2,\dots,m_n)$) if

$$P(t^{1/(2m_2)}z_2,\dots,t^{1/(2m_n)}z_n)=t^dP(z') \tag{1.1}$$

for any $t>0$. If $d=1$, we will call P *weighted homogeneous.*

Throughout this paper we will suppose (as we did in [Her1, Her2]) that

$$P-c_0\sigma \text{ is plurisubharmonic} \tag{1.2}$$

with some small $c_0>0$, where

$$\sigma(z')=\sum_{j=2}^{n}|z_j|^{2m_j}. \tag{1.3}$$

This means that P is strictly plurisubharmonic away from the union of all coordinate axes. For later purposes we introduce another notation:

$$\widehat{\sigma}(z)=|z_1|+\sigma(z'). \tag{1.4}$$

In [Her1] the author had started to study the Bergman kernel and metric of the domains D_P, where P satisfies (1.2), under nontangential approach to the origin. Under an additional condition on the coupling terms occurring in P he also obtained in [Her2] precise estimates for the boundary behavior of the Bergman kernel and the invariant differential metrics of Bergman, Carathéodory, and Kobayashi on these domains.

The aim of this article is to give a good lower bound for the Bergman distance d_D^B of the model domain D_P. We will prove:

Theorem 1.1. *Let P denote a real-valued plurisubharmonic weighted homogeneous polynomial that satisfies* (1.1) *and* (1.2). *Let $D=D_P$ denote the model domain $D=\{r(z)=\operatorname{Re}z_1+P(z')<0\}$. For $A,Q\in D$ denote*

$$\delta(A,Q):=\frac{|h_A(Q)|}{\mathscr{R}(A)}+\sum_{\ell=2}^{n}\frac{|Q_\ell-A_\ell|}{|r(A)|^{1/(2m_\ell)}},$$

$$h_A(Q):=Q_1-A_1+2\sum_{\ell=2}^{n}\frac{\partial P(A')}{\partial z_\ell}(Q_\ell-A_\ell),$$

with $\mathscr{R}(A):=\sum_{\ell=1}^{n}\widehat{\sigma}(A)^{1-1/(2m_\ell)}|r(A)|^{1/(2m_\ell)}$ and $m_1:=1$. Further set $\mathscr{L}(A):=\log(1+\widehat{\sigma}(A)/|r(A)|)$. Then, with a universal constant $C_*>0$, for all $A,Q\in D$ we have

$$d_D^B(A,Q)\ge C_*\big(\varrho_D(A,Q)+\varrho_D(Q,A)\big),$$

where we define (*see also* (2.7) *below*)

$$\varrho_D(A,Q):=\frac{\log\big(1+\frac{\delta(A,Q)}{1+\mathscr{L}(A)}\big)}{1+\log\log\big(e^2+\frac{\delta(A,Q)}{1+\mathscr{L}(A)}\big)+\log(1+\mathscr{L}(A))}.$$

REMARKS. (i) If we keep Q fixed and let A tend to a boundary point ζ, then the above theorem gives again an estimate of the form

$$d_D^B(A,Q) \gtrsim \frac{\log(1+|r(A)|^{-1})}{\log\log(e^2+|r(A)|^{-1})},$$

in analogy to the case of bounded pseudoconvex domains treated in [Blo2].

(ii) It would be desirable to replace $\mathscr{R}(A)$ in the theorem with $|r(A)|$.

The methods that we will use in the proof of the main theorem rely on localization of the sublevel sets of the pluricomplex Green function $\mathscr{G}_{D_P}(\cdot,w)$ with pole at w (see Section 3). The most difficult part here is the proof that $\{\mathscr{G}_{D_P}(\cdot,w) < -T\}$ lies (roughly speaking) in a collar of the form $\{|r(w)| \lesssim |r| \lesssim |r(w)|\}$. (Note that D_P is unbounded.) These results enable us to give a sufficiently good lower estimate of the Bergman metric $B^2_{D_P}$. Then we modify methods from [DO, Her4, Blo2] to get a lower bound on the Bergman distance of D_P.

2. Preparatory estimates on polynomial functions

Notation. We define

$$W_{0,d}(\zeta',u') := \sum_{\ell=2}^{n} \sigma(\zeta')^{d-1/m_\ell}|u_\ell|^2 + \sigma(u')^d$$

and $W_0 := W_{0,1}$. We set $\widetilde{m} := \max\{m_2,\dots,m_n\}$.

For any set M we denote by ξ_M its characteristic function.

The Lebesgue measure in $\mathbb{C}^k$ is denoted by $d^{2k}z$.

We will need:

LEMMA 2.1. *Let $p:\mathbb{C}^{n-1}\to\mathbb{C}$ denote a polynomial which is weighted homogeneous of degree $d\geq 1$. Then, with an unimportant constant $C_{1,p}>0$,*

$$\Big|p(z')-p(\zeta')-\sum_{j=2}^{n}\frac{\partial p}{\partial z_j}(\zeta')(z_j-\zeta_j)-\sum_{j=2}^{n}\frac{\partial p}{\partial \bar z_j}(\zeta')(\bar z_j-\bar\zeta_j)\Big| \leq C_{1,p}W_{0,d}(\zeta',z'-\zeta')$$

for all $z',\zeta'\in\mathbb{C}^{n-1}$.

Proof. We note that

$$p(z')-p(\zeta')-\sum_{j=2}^{n}\frac{\partial p}{\partial z_j}(\zeta')(z_j-\zeta_j)-\sum_{j=2}^{n}\frac{\partial p}{\partial \bar z_j}(\zeta')(\bar z_j-\bar\zeta_j)$$

is a sum of terms of the form

$$T := A\prod_{\nu\in S_1}\zeta_\nu^{b_\nu}\bar\zeta_\nu^{c_\nu}\prod_{\ell\in S_2}\zeta_\ell^{a'_\ell}\bar\zeta_\ell^{b'_\ell}(z_\ell-\zeta_\ell)^{a''_\ell}(\bar z_\ell-\bar\zeta_\ell)^{b''_\ell},$$

with constants A (determined by the coefficients of p), disjoint index sets S_1,S_2 such that $S_2\neq\emptyset$ and $S_1\cup S_2=\{2,\dots,n\}$, and exponents

$b_\nu, c_\nu, a'_\ell, b'_\ell, a''_\ell$, and b''_ℓ with $a''_\ell + b''_\ell \geq 1$ for any $\ell \in S_2$ and $\sum_{\ell \in S_2}(a''_\ell + b''_\ell) \geq 2$. Moreover, they satisfy the homogeneity condition

$$\sum_{\nu \in S_1} \frac{b_\nu + c_\nu}{2m_\nu} + \sum_{\ell \in S_2} \frac{a'_\ell + b'_\ell + a''_\ell + b''_\ell}{2m_\ell} = d.$$

This leads to

$$|T| = |A| \prod_{\nu \in S_1} |\zeta_\nu|^{B_\nu} \prod_{\ell \in S_2} |\zeta_\ell|^{C_\ell} |z_\ell - \zeta_\ell|^{D_\ell}, \tag{2.5}$$

with exponents B_ν, C_ℓ, and D_ℓ that satisfy $D_\ell \geq 1$ for all $\ell \in S_2$ as well as $\sum_{\ell \in S_2} D_\ell \geq 2$, and

$$\sum_{\nu \in S_1} \frac{B_\nu}{2m_\nu} + \sum_{\ell \in S_2} \frac{C_\ell + D_\ell}{2m_\ell} = d. \tag{2.6}$$

In each case we have

$$|\zeta_\alpha| \leq \sigma(\zeta')^{1/(2m_\alpha)} \quad \text{and} \quad |z_\alpha - \zeta_\alpha| \leq \sigma(z' - \zeta')^{1/(2m_\alpha)}$$

for any $\alpha \in \{2, \ldots, n\}$.

We consider two cases:

(a) Let $\sigma(z' - \zeta') > \sigma(\zeta')$. In this case, by (2.5) and (2.6),

$$|T| \leq |A| \sigma(z' - \zeta')^d \leq |A| W_{0,d}(\zeta', z' - \zeta').$$

(b) Now assume that $\sigma(z' - \zeta') \leq \sigma(\zeta')$. In this case, again using first (2.5) and then (2.6), we get

$$\begin{aligned} |T| &\leq |A| \sigma(\zeta')^{\sum_{\nu \in S_1} \frac{B_\nu}{2m_\nu} + \sum_{\ell \in S_2} \frac{C_\ell}{2m_\ell}} \prod_{\ell \in S_2} |z_\ell - \zeta_\ell|^{D_\ell} \\ &\leq |A| \sigma(\zeta')^d \prod_{\ell \in S_2} \left(\frac{|z_\ell - \zeta_\ell|}{\sigma(\zeta')^{1/(2m_\ell)}} \right)^{D_\ell}. \end{aligned}$$

In the last product there exists $\ell_0 \in S_2$ with $D_{\ell_0} \geq 2$, or there are (at least) two $\ell_1, \ell_2 \in S_2$.

In the first case we estimate, using $\frac{|z_\ell - \zeta_\ell|}{\sigma(\zeta')^{1/(2m_\ell)}} \leq 1$ for all $\ell \in \{2, \ldots, n\}$,

$$\prod_{\ell \in S_2} \left(\frac{|z_\ell - \zeta_\ell|}{\sigma(\zeta')^{1/(2m_\ell)}} \right)^{D_\ell} \leq \left(\frac{|z_{\ell_0} - \zeta_{\ell_0}|}{(\sigma(\zeta'))^{1/(2m_{\ell_0})}} \right)^{D_{\ell_0}} \leq \left(\frac{|z_{\ell_0} - \zeta_{\ell_0}|}{(\sigma(\zeta'))^{1/(2m_{\ell_0})}} \right)^2,$$

hence

$$|T| \leq |A| \sigma(\zeta')^{d - 1/m_{\ell_0}} |z_{\ell_0} - \zeta_{\ell_0}|^2 \leq |A| W_{0,d}(\zeta', z' - \zeta').$$

In the second case we estimate

$$\prod_{\ell\in S_2}\left(\frac{|z_\ell-\zeta_\ell|}{\sigma(\zeta')^{1/(2m_\ell)}}\right)^{D_\ell}\le\left(\frac{|z_{\ell_1}-\zeta_{\ell_1}|}{(\sigma(\zeta'))^{1/(2m_{\ell_1})}}\right)\left(\frac{|z_{\ell_2}-\zeta_{\ell_2}|}{(\sigma(\zeta'))^{1/(2m_{\ell_2})}}\right)$$
$$\le\frac{1}{2}\left(\frac{|z_{\ell_1}-\zeta_{\ell_1}|^2}{\sigma(\zeta')^{1/m_{\ell_1}}}+\frac{|z_{\ell_1}-\zeta_{\ell_2}|^2}{\sigma(\zeta')^{1/m_{\ell_2}}}\right),$$

and consequently

$$|T|\le|A|\big(\sigma(\zeta')^{d-1/m_{\ell_1}}|z_{\ell_1}-\zeta_{\ell_1}|^2+\sigma(\zeta')^{d-1/m_{\ell_2}}|z_{\ell_2}-\zeta_{\ell_2}|^2\big).$$

In each case $|T|\le|A|W_{0,d}(\zeta',z'-\zeta')$. ∎

COROLLARY 2.1. *If $p:\mathbb{C}^{n-1}\to\mathbb{R}$ is weighted homogeneous with weight $(m_2,\dots,m_n)$, then*

$$\left|p(z')-p(\zeta')-2\operatorname{Re}\sum_{j=2}^{n}\frac{\partial p}{\partial z_j}(\zeta')(z_j-\zeta_j)\right|\le C_{1,p}W_0(\zeta',z'-\zeta')$$

for all $z',\zeta'\in\mathbb{C}^{n-1}$.

We consider a special case of this corollary:

LEMMA 2.2. *For an integer $m\ge 2$ we have*

$$\big||z|^{2m}-|\zeta|^{2m}\big|\le 4^m\big(|\zeta|^{2m-1}|z-\zeta|+|z-\zeta|^{2m}\big)$$

for $z,\zeta\in\mathbb{C}$.

Proof. We note that

$$|z|^{2m}-|\zeta|^{2m}=\sum_{\lambda,\mu\le m\,:\,\lambda+\mu\ge 1}\binom{m}{\lambda}\binom{m}{\mu}\zeta^{m-\lambda}(z-\zeta)^{\lambda}\overline{\zeta}^{\,m-\mu}(\overline{z}-\overline{\zeta})^{\mu},$$

and hence

$$\big||z|^{2m}-|\zeta|^{2m}\big|\le\sum_{\lambda,\mu\le m\,:\,\lambda+\mu\ge 1}\binom{m}{\lambda}\binom{m}{\mu}|\zeta|^{m-\lambda-\mu}|z-\zeta|^{\lambda+\mu}$$
$$\le 4^m(|\zeta|^{2m-1}|z-\zeta|+|z-\zeta|^{2m}).$$

The next lemma was observed by Range [Ran].

LEMMA 2.3. *There exists a constant $\gamma_1>0$ such that for any $2\le\ell\le n$ and $x,y\in\mathbb{C}$,*

$$\gamma_1(|x|^{2m_\ell-2}|y|^2+|y|^{2m_\ell})\le|x+y|^{2m_\ell}-2\operatorname{Re}\left(\frac{\partial|x|^{2m_\ell}}{\partial x}y\right)-|x|^{2m_\ell}$$
$$\le\frac{1}{\gamma_1}(|x|^{2m_\ell-2}|y|^2+|y|^{2m_\ell}).$$

The following function will be important:

$$h_\zeta(z) := z_1 - \zeta_1 + 2\sum_{\ell=2}^{n} \frac{\partial P(\zeta')}{\partial z_\ell}(z_\ell - \zeta_\ell) \tag{2.7}$$

for fixed $\zeta \in \overline{D}$.

We apply the above estimates to the defining function $r = \operatorname{Re} z_1 + P(z')$ of D and obtain

LEMMA 2.4.

(a) *For $B, x \in \mathbb{C}^n$ we have*

$$|r(x) - r(B) - \operatorname{Re} h_B(x)| \le C_2 W_0(B', x' - B').$$

(b) *Assume that $0 < \alpha < \min\{\frac{1}{4}, \frac{1}{4(1+C_2)n}\}$. Then $r(x) < r(B) + \frac{1}{2}|r(B)|$ whenever $|h_B(x)| < \alpha|r(B)|$ and*

$$|x_\ell - B_\ell| \le \alpha \min\left\{\sqrt{\frac{|r(B)|}{\sigma(B')^{1-1/m_\ell}}}, |r(B)|^{1/(2m_\ell)}\right\}$$

for $2 \le \ell \le n$.

Proof. (a) Indeed, we have

$$r(x) - r(B) - \operatorname{Re} h_B(x) = P(x') - P(B') - 2\operatorname{Re}\sum_{\ell=2}^{n} P_{z_\ell}(B')(x_\ell - B_\ell).$$

Then (a) follows by means of Corollary 2.1.

(b) follows from (a) and $r(x) \le r(B) + \operatorname{Re} h_B(x) + C_2 W_0(B', x' - B')$.

3. Sublevel sets of the Green function of D. For a domain $D \subset \mathbb{C}^n$ we denote by $\mathscr{G}_D(\cdot, w)$ the pluricomplex Green function with pole at $w \in D$. This function is defined as follows:

$$\mathscr{G}_D(z, w) := \sup\{u(z) \mid u \text{ plurisubharmonic, negative on } D,$$
$$u - \log|\cdot - w| \text{ bounded above near } w\}.$$

Its basic properties were investigated in [De, Kli]. For a bounded pseudoconvex domain Ω with smooth boundary it has been shown in [Blo2] that the sublevel sets $A_w := \{\mathscr{G}_D(\cdot, w) < -1\}$ lie, roughly speaking, within the collar $\{\frac{1}{C}\delta_\Omega(w)\big(\log\frac{1}{\delta_\Omega(w)}\big)^{-M} \le \delta_\Omega \le C\delta_\Omega(w)\big(\log\frac{1}{\delta_\Omega(w)}\big)^{M}\}$, where $C, M > 0$ are universal constants and δ_Ω denotes the boundary distance function on Ω.

We want to carry over this result to the domain $D = D_P$. How this can be done is by no means obvious, since D is unbounded.

We start with the construction of certain weight functions.

Notation. For a C^2-smooth function u we denote its Levi form by $\mathscr{L}_u(z, X)$, when evaluated at a point z in direction $X \in \mathbb{C}^n$.

Its directional derivative in direction X is denoted by $\langle \partial u(z), X\rangle$.

LEMMA 3.1. *Assume that P is a weighted homogeneous real-valued polynomial that satisfies condition* (1.2) *with some $c_0 > 0$. For $\zeta \in \overline{D}$ and $c_0 > 0$ define*

$$\begin{aligned}\Psi_{\zeta'}(z') &:= P(z') - 2\operatorname{Re}\sum_{\ell=2}^{n}\frac{\partial P}{\partial \zeta_\ell}(\zeta')(z_\ell-\zeta_\ell) - P(\zeta') \\ &\quad - \frac{c_0}{10}\sum_{\ell=2}^{n}\biggl(|z_\ell|^{2m_\ell} - 2\operatorname{Re}\biggl(\frac{\partial|\zeta_\ell|^{2m_\ell}}{\partial\zeta_\ell}(z_\ell-\zeta_\ell)\biggr) - |\zeta_\ell|^{2m_\ell}\biggr) + \frac{\gamma_1 c_0}{100}\sigma(z'-\zeta')\end{aligned}$$

and

$$\Phi_\zeta(z) := \operatorname{Re} h_\zeta(z) + \Psi_{\zeta'}(z').$$

Then:

(a) *With an unimportant constant L_0 we have $|\Psi_{\zeta'}| \le L_0 W_0(\zeta', \cdot - \zeta')$.*
(b) *$\Psi_{\zeta'}$ is plurisubharmonic, and its Levi form satisfies*

$$\mathscr{L}_{\Psi_{\zeta'}} \ge \frac{c_0\gamma_1}{100}\mathscr{L}_{\sigma(\cdot - \zeta')} + \frac{c_0}{2}\mathscr{L}_\sigma.$$

(c) *Let r be defined as in* (M). *Then*

$$r(z) - \frac{c_0}{10\gamma_1}W_0(\zeta', z'-\zeta') \le r(\zeta) + \Phi_\zeta(z) \le r(z) - \frac{9\gamma_1 c_0}{100}\sigma(z'-\zeta').$$

Proof. (a) follows from Lemma 2.1.

(b) The Levi form of $\Psi_{\zeta'}$ is given by

$$\mathscr{L}_{\Psi_{\zeta'}} = \mathscr{L}_P - \frac{c_0}{5}\mathscr{L}_\sigma + \frac{c_0\gamma_1}{20}\mathscr{L}_{\sigma(\cdot-\zeta')}.$$

This together with (1.2) proves (b).

For the proof of (c) we write

$$\begin{aligned} r(z) &= r(\zeta) + \operatorname{Re}(z_1-\zeta_1) + P(z') - P(\zeta') \\ &= r(\zeta) + \operatorname{Re}\biggl(z_1 - \zeta_1 + 2\sum_{\ell=2}^{n}\frac{\partial P}{\partial\zeta_\ell}(\zeta')(z_\ell-\zeta_\ell)\biggr) \\ &\qquad + P(z') - 2\operatorname{Re}\sum_{\ell=2}^{n}\frac{\partial P}{\partial\zeta_\ell}(\zeta')(z_\ell-\zeta_\ell) - P(\zeta') \\ &= r(\zeta) + \Phi_\zeta(z) + T(\zeta', z')\end{aligned}$$

with

$$\begin{aligned} T(\zeta', z') &:= \frac{c_0}{10}\sum_{\ell=2}^{n}\biggl(|z_\ell|^{2m_\ell} - 2\operatorname{Re}\biggl(\frac{\partial|\zeta_\ell|^{2m_\ell}}{\partial\zeta_\ell}(z_\ell-\zeta_\ell)\biggr) - |\zeta_\ell|^{2m_\ell}\biggr) \\ &\quad - \frac{\gamma_1 c_0}{100}\sigma(z'-\zeta').\end{aligned}$$

By Lemma 2.3 with $x := \zeta'$ and $y := z' - \zeta'$ we find that

$$T(\zeta', z') \geq \frac{\gamma_1 c_0}{10}\sigma(z' - \zeta') - \frac{\gamma_1 c_0}{100}\sigma(z' - \zeta') = \frac{9\gamma_1 c_0}{100}\sigma(z' - \zeta'),$$

which gives the right-hand side of the claimed estimate. The same lemma yields $T(\zeta', z') \leq \frac{c_0}{10\gamma_1} W_0(\zeta', z' - \zeta')$, hence the left-hand side of the estimate. ∎

A first lower bound on the (pluricomplex) Green function of D is provided by

LEMMA 3.2. *With a suitable constant $C_1 > 0$, for any $\ell = 2, \ldots, n$ and any $z, B \in D$,*

$$\mathscr{G}_D(z, B) \geq -\frac{1}{2}\log\biggl(1 + C_1 \frac{|r(z)|^{1/m_\ell}}{|z_\ell - B_\ell|^2}\biggr).$$

Proof. We abbreviate

$$V_{\zeta'}(z') := \sum_{\ell=2}^{n}\biggl(|z_\ell|^{2m_\ell} - 2\operatorname{Re}\biggl(\frac{\partial|\zeta_\ell|^{2m_\ell}}{\partial\zeta_\ell}(z_\ell - \zeta_\ell)\biggr) - |\zeta_\ell|^{2m_\ell}\biggr)$$

and define

$$V_\zeta(z) := \Phi_\zeta(z) - \frac{c_0}{10}V_{\zeta'}(z') + \frac{\gamma_1 c_0}{50}\sigma(z' - \zeta').$$

From Lemma 3.1(a) we then get

$$V_\zeta(z) \leq r(z) - \frac{2\gamma_1 c_0}{25}\sigma(z' - \zeta'),$$

hence

$$-(-V_\zeta(z))^{1/m_\ell} < -\biggl(\frac{2\gamma_1 c_0}{25}\biggr)^{1/m_\ell}|z_\ell - \zeta_\ell|^2.$$

The upper regularization Φ_ℓ of

$$\phi_\ell(z) := \sup_{\zeta\in\partial D}\biggl(-(-V_\zeta(z))^{1/m_\ell} + \biggl(\frac{2\gamma_1 c_0}{25}\biggr)^{1/m_\ell}|z_\ell - \zeta_\ell|^2\biggr)$$

is plurisubharmonic and negative on D. This implies that for any $B \in \mathbb{C}^n$ the function

$$\widetilde{\phi}_\ell(z) := \Phi_\ell(z) - \biggl(\frac{2\gamma_1 c_0}{25}\biggr)^{1/(2m_\ell)}|z_\ell - B_\ell|^2$$

is also plurisubharmonic, and the negative function

$$-\log\biggl(1 + \frac{-\Phi_\ell(z)}{(\frac{2\gamma_1 c_0}{25})^{1/(2m_\ell)}|z_\ell - B_\ell|^2}\biggr)$$
$$= \log\biggl(\biggl(\frac{2\gamma_1 c_0}{25}\biggr)^{1/(2m_\ell)}|z_\ell - B_\ell|^2\biggr) - \log(-\widetilde{\phi}_\ell(z))$$

is a candidate for the supremum that defines $\mathscr{G}_D(z,B)$. Thus we get

$$\mathscr{G}_D(z,B) \ge -\frac{1}{2}\log\biggl(1+\frac{-\Phi_\ell(z)}{\bigl(\frac{2\gamma_1 c_0}{25}\bigr)^{1/(2m_\ell)}|z_\ell - B_\ell|^2}\biggr).$$

Given $z \in D$ we choose $\zeta := z - r(z)e_1$. This is a point in ∂D with $V_{\zeta'}(z) = 0$, $\Psi_{\zeta'}(z') = 0$, and $h_\zeta(z) = r(z)$. Therefore $V_\zeta(z) = r(z)$ and $\Phi_\ell(z) \ge -|r(z)|^{1/m_\ell}$. From this the desired estimate follows if we choose $C_1 \ge \max_{2\le\ell\le n}\bigl(\frac{25}{2\gamma_1 c_0}\bigr)^{1/(2m_\ell)}$. ∎

LEMMA 3.3. *On D there exists a zero-free holomorphic function F_∞ and constants $L_* > 0$ and $N \in \mathbb{N}$ such that:*

(i) $-\pi/8 \le \arg \sqrt[N]{F_\infty} \le \pi/8$,
(ii) $L_*^{-1}\widehat{\sigma}(z) \le |F_\infty(z)| \le L_*\widehat{\sigma}(z)$,
(iii) $\frac{1}{2}(L_*^{-1}\widehat{\sigma}(z))^{1/N} \le \frac{1}{2}|F_\infty(z)|^{1/N} \le \operatorname{Re} \sqrt[N]{F}_\infty(z) \le (L_*\widehat{\sigma}(z))^{1/N}$,

where $\widehat{\sigma}$ is as in (1.4).

Proof. We argue as in [AGK, proof of Thm. 4.2]. Let $\Pi(z) := (z_1, z_2^{\mu_2}, \ldots, z_n^{\mu_n})$, where $\mu_j := m_0/m_j$ for $j = 2,\ldots,n$ and $m_0 := m_2\cdot\ldots\cdot m_n$. Then $\Pi : \Omega := \Pi^{-1}(D) \to D$ is a covering map. By [BeFo, Thm. 4.1] one can find a holomorphic zero-free function f_∞ and constants $L_* > 0$ and $N \in \mathbb{N}$ such that $-\pi/(8\nu_0) \le \arg \sqrt[N]{f_\infty} \le \pi/(8\nu_0)$, where ν_0 is the number of sheets of Π, and

$$L_*^{-1}\widehat{\sigma}_0(w) \le |f_\infty(w)| \le L_*\widehat{\sigma}_0(w) \quad \text{for } w \in \Omega,$$

where $\widehat{\sigma}_0(w) := |w_1| + \sum_{\ell=2}^n |w_\ell|^{2m_0}$. Then we let

$$\widetilde{F}_\infty(z) := \prod_{w\in\Omega\,:\,\Pi(w)=z} f_\infty(w).$$

This function is holomorphic on $\mathbb{C}^n \setminus \Pi(X)$, where X denotes the branching locus of Π. From $\widehat{\sigma}_0 = \widehat{\sigma}\circ\Pi$ we obtain $(L_*^{-1}\widehat{\sigma})^{\nu_0} \le |\widetilde{F}_\infty| \le (L_*\widehat{\sigma})^{\nu_0}$ on $D \setminus \Pi(X)$, hence $\widetilde{F}_\infty$ is locally bounded at the points of $\Pi(X)$ and can in particular be extended holomorphically to all of D, by Riemann's theorem on removable singularities. Note also that $-\pi/8 \le \arg \sqrt[N]{\widetilde{F}_\infty} \le \pi/8$, and so $F_\infty := \widetilde{F}_\infty^{1/\nu_0}$ has the desired properties. ∎

Our first application of the function F_∞ is

LEMMA 3.4. *On the sublevel set $\{\mathscr{G}_D(\cdot,B) < -1\}$ we have $\widehat{\sigma} \le C_5\widehat{\sigma}(B)$ with an unimportant constant $C_5 > 0$.*

Proof. If $g : D \to \mathbb{D}$ is holomorphic, then

$$\mathscr{G}_D(z,B) \ge \mathscr{G}_{\mathbb{D}}(g(z),g(B)) = \log\biggl|\frac{g(z)-g(B)}{1-\overline{g(B)}g(z)}\biggr|.$$

For z with $\mathscr{G}_D(z,B)\le -1$ this means that $\bigl|\frac{g(z)-g(B)}{1-\overline{g(B)}g(z)}\bigr|\le e^{-1}$. From this and an elementary computation we get

$$\left|g(z)-\frac{1-e^{-2}}{1-e^{-2}|g(B)|^2}g(B)\right|\le\frac{1-|g(B)|^2}{1-e^{-2}|g(B)|^2},$$

and hence

$$|g(z)|\le\varrho:=\frac{e^{-1}+|g(B)|}{1+e^{-1}|g(B)|}. \tag{3.1}$$

We apply this to $g:=(1-\sqrt[N]{F_\infty})/(1+\sqrt[N]{F_\infty})$, where F_∞ is the function from Lemma 3.3. We obtain

$$\begin{aligned}|\sqrt[N]{F_\infty}(z)|&\le\frac{1+\varrho}{1-\varrho}=\frac{1+e^{-1}}{1-e^{-1}}\,\frac{\bigl(|1+\sqrt[N]{F_\infty}(B)|+|1-\sqrt[N]{F_\infty}(B)|\bigr)^2}{4\operatorname{Re}\sqrt[N]{F_\infty}(B)}\\&\le\frac{1+e^{-1}}{1-e^{-1}}\,\frac{1+|\sqrt[N]{F_\infty}(B)|^2}{\operatorname{Re}\sqrt[N]{F_\infty}(B)}.\end{aligned}$$

Let us consider two cases.

Case (1): $\widehat{\sigma}(B)\ge 1$. Then $|\sqrt[N]{F_\infty}(B)|\ge L_*^{-1/N}\widehat{\sigma}(B)^{1/N}\ge L_*^{-1/N}$. In conjunction with $\operatorname{Re}\sqrt[N]{F_\infty}(B)\ge\frac12|\sqrt[N]{F_\infty}(B)|$ this yields

$$\begin{aligned}L_*^{-1/N}\widehat{\sigma}(z)^{1/N}&\le\bigl|\sqrt[N]{F_\infty}(z)\bigr|\\&\le 8\frac{1+e^{-1}}{1-e^{-1}}L_*^{1/N}\bigl|\sqrt[N]{F_\infty}(B)\bigr|\le 8\frac{1+e^{-1}}{1-e^{-1}}L_*^{2/N}\widehat{\sigma}(B)^{1/N},\end{aligned}$$

which implies the conclusion with $C_5:=\bigl(8\frac{1+e^{-1}}{1-e^{-1}}\bigr)^N L_*^3$.

Case (2): $\widehat{\sigma}(B)\le 1$. In this case we define the scaling map

$$S(x):=\left(\frac{x_1}{\widehat{\sigma}(B)},\frac{x_2}{\widehat{\sigma}(B)^{1/(2m_2)}},\ldots,\frac{x_n}{\widehat{\sigma}(B)^{1/(2m_n)}}\right)$$

and obtain $\mathscr{G}_D(S(z),S(B))=\mathscr{G}_D(z,B)\le -1$. But $\widehat{\sigma}(S(B))=1$, so the result of Case (1) yields

$$\widehat{\sigma}(S(z))\le\left(8\frac{1+e^{-1}}{1-e^{-1}}\right)L_*^2.$$

Since $\widehat{\sigma}(S(z))=\widehat{\sigma}(z)/\widehat{\sigma}(B)$, we obtain the desired estimate also in this case. ∎

Lemma 3.5. *For $\delta>0$ let Φ_ℓ denote the upper regularization of the function*

$$\phi_\ell(z):=\sup_{\zeta\in\partial D}\left(-(-V_\zeta(z))^{1/m_\ell}+\left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell}|z_\ell-\zeta_\ell|^2\right).$$

(a) *Then the function*

$$U_\delta := -\log(1 - r/\delta) + \sum_{\ell=2}^{n} \delta^{-1/m_\ell} \Phi_\ell$$

is plurisubharmonic on D.

Further there exist constants $c_2, M_* > 0$ such that the following holds:

(b) *For $B \in D$ and $\delta > 0$ let*

$$\mathscr{R}_\delta(B) := \delta + \sum_{\ell=2}^{n} \sigma(B')^{1-1/(2m_\ell)} \delta^{1/(2m_\ell)}, \tag{3.2}$$

$$W_{B,\delta}(z) := \frac{|h_B(z)|^2}{\mathscr{R}_\delta(B)^2} + \sum_{\ell=2}^{n} \frac{|z_\ell - B_\ell|^2}{\delta^{1/m_\ell}}.$$

If $\delta \ge |r(B)|$, then also $U_\delta - c_2 W_{B,\delta}$ is plurisubharmonic on $S_{B,\delta} := \{W_{B,\delta} \le 1\}$.

(c) *Suppose that $B \in D$ and $\delta \ge |r(B)|$. Let χ be a smooth increasing function such that $\chi(x) = x$ for $x \le 1/2$ and $\chi(x) = 3/4$ for $x \ge 1$. Then the function $\varphi_{1,B,\delta} := \frac{1}{2} \log \chi \circ W_{B,\delta} + M_* U_\delta$ is also plurisubharmonic.*

(d) *In particular if $B \in D$ and δ are as in* (c)*, then $\mathscr{G}_D(\cdot, B) \ge \varphi_{1,B,\delta}$.*

Proof. (a) Clearly the function $z \mapsto \Phi_\ell(z) - \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} |z_\ell - B_\ell|^2$ is plurisubharmonic for any $B \in \mathbb{C}^n$. We set

$$\widetilde{U}_\delta(z) := -\log(1 - r/\delta) + \sum_{\ell=2}^{n} \delta^{-1/m_\ell} \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} |z_\ell - B_\ell|^2.$$

We only need to prove (b) for $\widetilde{U}_\delta$ instead of U_δ.

For $z \in D$ and $X \in \mathbb{C}^n$, we have

$$\mathscr{L}_{\widetilde{U}_\delta}(z; X) \ge \frac{|\langle \partial r(z), X\rangle|^2}{(\delta - r)^2} + \sum_{\ell=2}^{n} \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} \frac{|X_\ell|^2}{\delta^{1/m_\ell}}.$$

We consider two cases:

Case (1): $|\langle \partial h_B(z), X\rangle| \ge 4|\langle \partial P(z'), X'\rangle|$. Then

$$|\langle \partial r(z), X\rangle| \ge \tfrac{1}{4} |\langle \partial h_B(z), X\rangle|,$$

and therefore

$$\mathscr{L}_{\widetilde{U}_\delta}(z; X) \ge \frac{|\langle \partial h_B(z), X\rangle|^2}{16(\delta - r(z))^2} + \sum_{\ell=2}^{n} \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} \frac{|z_\ell - B_\ell|^2}{\delta^{1/m_\ell}}.$$

Case (2): $|\langle \partial h_B(z), X\rangle| \le 4|\langle \partial P(z'), X'\rangle|$. In this case we choose $\ell_z \in \{2, \ldots, n\}$ such that $\sigma(z')^{1-1/(2m_{\ell_z})} |X_{\ell_z}| = \max_{2\le\ell\le n} \sigma(z')^{1-1/(2m_\ell)} |X_\ell|$.

Then
$$|\langle \partial h_B(z), X\rangle| \le 4M_0 n\sigma(z')^{1-1/(2m_{\ell_z})}|X_{\ell_z}|.$$
This gives
$$\begin{aligned}\mathscr{L}_{\widetilde{U}_\delta}(z;X) &\ge \sum_{\ell=2}^n \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} \frac{|X_\ell|^2}{\delta^{1/m_\ell}} \ge \frac{1}{2}\,\frac{|X_{\ell_z}|^2}{\delta^{1/m_{\ell_z}}} + \frac{1}{2}\sum_{\ell=2}^n \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} \frac{|X_\ell|^2}{\delta^{1/m_\ell}} \\ &\ge \frac{1}{32M_0^2 n^2 \sigma(z')^{2-1/m_{\ell_z}}}\,\frac{|\langle \partial h_B(z), X\rangle|^2}{\delta^{1/m_{\ell_z}}} + \frac{1}{2}c_2''\sum_{\ell=2}^n \frac{|X_\ell|^2}{\delta^{1/m_\ell}}\end{aligned}$$
with $c_2'' := \min_{2\le\ell\le n}(2\gamma_1 c_0/25)^{1/m_\ell}$. To estimate the Levi form of $\widetilde{U}_\delta$ in terms of $\mathscr{L}_{W_{B,\delta}}$ we note that
$$\mathscr{L}_{W_{B,\delta}}(z;X) = \frac{|\langle \partial h_B(z), X\rangle|^2}{\mathscr{R}_\delta(B)^2} + \sum_{\ell=2}^n \frac{|X_\ell|^2}{\delta^{1/m_\ell}}.$$
On the set $S_{B,\delta}$ we have
$$\begin{aligned}\sigma(z') &\le \sigma(B') + 4^{\widetilde{m}}\sum_{\ell=2}^n |B_\ell|^{2m_\ell-1}|z_\ell - B_\ell| + 4^{\widetilde{m}}\sigma(z'-B') \\ &\le \sigma(B') + 4^{\widetilde{m}}\sum_{\ell=2}^n \sigma(B')^{1-1/(2m_\ell)}\delta^{1/(2m_\ell)} + 4^{\widetilde{m}} n\delta \\ &\le (2\cdot 4^{\widetilde{m}} n + 1)(\sigma(B') + \delta),\end{aligned} \tag{3.3}$$
hence
$$\begin{aligned}\sigma(z')^{1-1/(2m_{\ell_z})}\delta^{1/(2m_{\ell_z})} &\le (2\cdot 4^{\widetilde{m}} n + 1)(\sigma(B') + \delta)^{1-1/(2m_{\ell_z})}\delta^{1/(2m_{\ell_z})} \\ &\le (2\cdot 4^{\widetilde{m}} n + 1)\mathscr{R}_\delta(B).\end{aligned} \tag{3.4}$$
Moreover (by Lemma 2.4),
$$\begin{aligned}\delta + |r(z)| &\le \delta + |r(B)| + |h_B(z)| + C_2 W_0(B', z'-B') \\ &\le 2\delta + \mathscr{R}_\delta(B) + C_2\sum_{\ell=2}^n \sigma(B')^{1-1/m_\ell}\delta^{1/m_\ell} + n\delta \\ &\qquad \text{(since } |r(B)|\le\delta) \\ &\le (3 + (C_2+1)n)\mathscr{R}_\delta(B).\end{aligned} \tag{3.5}$$
In Case (1) the outcome of this is $\mathscr{L}_{\widetilde{U}_\delta}(z;X) \ge c_2'\mathscr{L}_{W_{B,\delta}}$ on $S_{B,\delta}$ with $c_2' := \min\{\frac{1}{16(3+(C_2+1)n)^2}, c_2''\}$.

In Case (2), from (3.4) we obtain
$$\begin{aligned}\mathscr{L}_{\widetilde{U}_\delta}(z;X) &\ge \frac{1}{32M_0^2 n^2 \sigma(z')^{2-1/m_{\ell_z}}}\,\frac{|\langle \partial h_B(z), X\rangle|^2}{\delta^{1/m_{\ell_z}}} + \frac{1}{2}\sum_{\ell=2}^n \left(\frac{2\gamma_1 c_0}{25}\right)^{1/m_\ell} \frac{|X_\ell|^2}{\delta^{1/m_\ell}} \\ &\ge c_2''\mathscr{L}_{W_{B,\delta}}(z;X)\end{aligned}$$
with $c_2''' := \min\{\frac{1}{32M_0^2 n^2(2\cdot 4^{\widetilde{m}} n+1)^2}, c_2''\}$. This proves (b) with $c_2 := \min\{c_2', c_2'''\}$.

(c) Let $M_* > 1$ to be chosen later. Then it is enough to check the plurisubharmonicity of $\widetilde{\varphi}_{1,B,\delta} := \frac{1}{2}\log\chi\circ W_{B,\delta} + M_*\widetilde{U}_\delta$. On $\{W_{B,\delta} \le 1/2\} \cup \{W_{B,\delta} \ge 1\}$ this is clear. On $\{1/2 \le W_{B,\delta} \le 1\}$ we have

$$\begin{aligned}\mathscr{L}_{\widetilde{\varphi}_{1,B,\delta}} &\ge \tfrac{1}{2}(\chi'/\chi)'\circ W_{B,\delta}\cdot\partial W_{B,\delta}\cdot\overline{\partial W_{B,\delta}} + M_*\mathscr{L}_{\widetilde{U}_\delta}\\ &\ge -\max_{1/2\le x\le 1}|(\chi'/\chi)'(x)|W_{B,\delta}\cdot\mathscr{L}_{W_{B,\delta}} + M_*\mathscr{L}_{\widetilde{U}_\delta}\\ &\ge \Big(-\max_{1/2\le x\le 1}|(\chi'/\chi)'| + M_*c_2\Big)\mathscr{L}_{W_{B,\delta}} \ge 0\end{aligned}$$

if M_* is suitably chosen.

(d) This is now clear, since $\varphi_{1,B,\delta}$ belongs to the family whose supremum is just $\mathscr{G}_D(\cdot,B)$. ∎

Lemma 3.6. *For suitable constants $C_4,\eta_0>0$ we have, for any $B\in D$ and $\eta\ge\eta_0$,*

$$\{\mathscr{G}_D(\cdot,B) < -\eta\} \subset \{|h_B| \le C_4\widehat{\sigma}(B)e^{-\eta}\} \cap \{\sigma(\cdot - B') < C_4\widehat{\sigma}(B)e^{-\eta}\}.$$

Proof. We intend to apply Lemma 3.5(c) and (d). For this we need some preparations. Let $\eta\ge\eta_0>0$ with η_0 to be chosen later.

Now we choose $\delta := M_0\widehat{\sigma}(B)$ in Lemma 3.5. For $z\in\{\mathscr{G}_D(\cdot,B) < -\eta\}$ we find in conjunction with $|\varPhi_\ell(z)| \le |r(z)|^{1/m_\ell}$ and $|r(z)| \le M_0\widehat{\sigma}(z) \le C_5\delta$ that

$$\begin{aligned}-\eta_0 > -\eta \ge \mathscr{G}_D(z,B) \ge \varphi_{1,B,\delta}(z) &\ge \tfrac{1}{2}\log\chi\circ W_{B,\delta}(z) + M_*U_\delta\\ &\ge \tfrac{1}{2}\log\chi\circ W_{B,\delta}(z) - (n+1)C_5M_*.\end{aligned}$$

For $\eta_0 > (\log 2)/2 + (n+1)C_5M_*$ no z in the set $\{\mathscr{G}_D(\cdot,B) < -\eta_0\}$ has $W_{B,\delta}(z) > 1/2$, hence we even obtain $-\eta \ge \frac{1}{2}\log\chi\circ W_{B,\delta}(z) - (n+1)C_5M_* = \frac{1}{2}\log W_{B,\delta}(z) - (n+1)C_5M_*$ for z in that set. This implies

$$|h_B(z)| \le e^{(n+1)C_5M_*-\eta}\mathscr{R}_\delta(B) \le C_4e^{-\eta}\widehat{\sigma}(B), \quad \sigma(z'-B') \le C_4e^{-\eta}\widehat{\sigma}(B),$$

which proves the lemma. ∎

Now we intend to estimate the Green function $\mathscr{G}_D(z,B)$ in terms of $r(z)$ and $r(B)$. We will even need a slight generalization:

Lemma 3.7. *Let $B\in D$ and $\delta\ge|r(B)|$. Define*

$$\begin{aligned}a = a(\delta,B) &:= \frac{|r(B)|}{2(1+C_2)\mathscr{R}_\delta(B)},\\ M_\delta(B) &:= \frac{1}{C_6}\bigg(2(n+1) + C_2 + \log\bigg(1+\frac{\mathscr{R}_\delta(B)}{|r(B)|}\bigg)\bigg),\end{aligned}$$

with $\mathscr{R}_\delta(B)$ as in (3.2), *and the constant*

$$C_6 := \frac{1}{2(2L_*)^{1/N}2L_*(n+1)4^{\widetilde{m}}M_0}.$$

If φ denotes one of the functions r or $\rho := \max\{r, \rho'\}$, where $\rho' := -\operatorname{Re} \frac{1}{\sqrt[N]{F_\infty}+1}$, then the function

$$w_B(z) := \begin{cases} \max\{\frac{M_\delta(B)}{|\varphi(B)|}\varphi(z), \varphi_{1,B,\delta}(z)\} & \text{if } W_{B,\delta}(z) \geq a, \\ \varphi_{1,B,\delta}(z) & \text{if } W_{B,\delta}(z) \leq a, \end{cases}$$

is plurisubharmonic on D.

Proof. It suffices to show that

$$\frac{M_\delta(B)}{|\varphi(B)|} \cdot \varphi(z) \leq \varphi_{1,B,\delta}(z) \quad \text{for } z \in \{W_{B,\delta} = a\}.$$

From Lemma 2.4 we obtain $|r(z) - r(B)| \leq |h_B(z)| + C_2 W_0(B', z' - B')$. On the set $\{W_{B,\delta} = a\}$ this yields

$$|r(z) - r(B)| \leq a(1 + C_2)\mathscr{R}_\delta(B) \leq \tfrac{1}{2}|r(B)|$$

and

$$\tfrac{3}{2}r(B) \leq r(z) \leq \tfrac{1}{2}r(B). \tag{3.6}$$

We further know from Lemma 2.2 that

$$\widehat{\sigma}(z) \leq |h_B(z)| + (1 + M_0(n+1))\widehat{\sigma}(B) + 4^{\widetilde{m}}\sigma(z' - B'),$$

which implies that

$$\begin{aligned} \widehat{\sigma}(z) &\leq a\mathscr{R}_\delta(B) + (1 + M_0(n+1))\widehat{\sigma}(B) + 4^{\widetilde{m}}na\delta \\ &\leq 2(n+1)4^{\widetilde{m}}M_0\widehat{\sigma}(B) \end{aligned} \tag{3.7}$$

on the set $\{W_{B,\delta} = a\}$. Lemma 3.3(iii) yields

$$\rho'(z) = -\operatorname{Re} \frac{\sqrt[N]{F_\infty}(z) + 1}{|\sqrt[N]{F_\infty}(z) + 1|^2} \leq -\frac{1}{2}\frac{1}{1 + \operatorname{Re}\sqrt[N]{F_\infty}(z)} \leq -\frac{1}{2}\frac{1}{1 + (L_*\widehat{\sigma}(z))^{1/N}}$$

and likewise

$$\rho'(z) \geq -\frac{1}{1 + (2L_*)^{-1/N}\widehat{\sigma}(z)^{1/N}}.$$

This, combined with (3.7), leads to

$$\begin{aligned} \rho'(z) &\leq -\frac{1}{2}\frac{1}{1 + (L_*\widehat{\sigma}(z))^{1/N}} \leq -\frac{1}{2}\frac{1}{1 + (2L_*(n+1)4^{\widetilde{m}}M_0\widehat{\sigma}(B))^{1/N}} \\ &\leq -C_6\frac{1}{1 + (2L_*)^{-1/N}\widehat{\sigma}(B))^{1/N}} \leq C_6\rho'(B). \end{aligned} \tag{3.8}$$

Together with (3.6) we obtain $\varphi \leq C_6\varphi(B)$. Now we estimate $\varphi_{1,B,\delta}$ on $\{W_{B,\delta} = a\}$ from below:

$$\begin{aligned}\varphi_{1,B,\delta}(z) &\geq \frac{1}{2}\log a - \log\left(1+\frac{|r(z)|}{\delta}\right) + \sum_{\ell=2}^{n}\frac{\Phi_\ell(z)}{\delta^{1/(2m_\ell)}}\\ &\geq \frac{1}{2}\log a - \log\left(1+\frac{3|r(B)|}{2\delta}\right) - \sum_{\ell=2}^{n}\left(\frac{3|r(B)|}{2\delta}\right)^{1/(2m_\ell)}\\ &\geq \frac{1}{2}\Big(\log a - 3(n+1)\Big) \geq -C_6 M_\delta(B) \geq M_\delta(B)\frac{\varphi(z)}{|\varphi(B)|}.\end{aligned}$$

This proves that w_B is well-defined; its plurisubharmonicity is clear. ∎

We are now ready to estimate $\mathscr{G}_D$ in terms of r and ρ.

LEMMA 3.8. *Let $B \in D$ and*

$$\widehat{\mathscr{L}_1}(B) := C_6' \log\left(1+\frac{2nM_0\widehat{\sigma}(B)}{|r(B)|}\right) \quad \textit{with} \quad C_6' := 2\frac{2(n+1)+C_2}{C_6}.$$

Then:

(a) *For $z \in D$ with $|r(z)| \leq \frac{1}{2}|r(B)|$ we have*

$$\mathscr{G}_D(z,B) \geq \widehat{\mathscr{L}_1}(B)\frac{r(z)}{|r(B)|}.$$

(b) *Suppose that $|\rho(z)| \leq C_6|\rho(B)|$. Then*

$$\mathscr{G}_D(z,B) \geq \widehat{\mathscr{L}_1}(B)\frac{\rho(z)}{|\rho(B)|}.$$

Proof. (a) We choose $\delta := M_0\widehat{\sigma}(B)$ in Lemma 3.7. If a is as in that lemma, we see that $W_{B,\delta}(z) \geq a$ once $|r(z)| \leq \frac{1}{2}|r(B)|$, by (3.6). Thus

$$\mathscr{G}_D(z,B) \geq w_B(z) \geq \frac{M_\delta(B)}{|r(B)|}r(z).$$

This, combined with $M_\delta(B) \leq \widehat{\mathscr{L}_1}(B)$, gives the claim.

(b) In the same way we find $W_{B,\delta}(z) \geq a$ whenever $|\rho(z)| \leq C_6|\rho(B)|$, by (3.8). As in (a) the claim follows from Lemma 3.7. ∎

The pluricomplex Green function is not symmetric in general, but we can compare $\mathscr{G}_D(z,Q)$ and $\mathscr{G}_D(Q,z)$ for certain points $z, Q \in D$. For this purpose we need Lemma 3.9 below that does the same service in our situation as Lemma 4.1 of [Her3] did for bounded domains.

We fix a smooth function $\alpha_1 \geq 0$ with compact support in the unit ball $\mathbb{B}_n$ such that $\int_{\mathbb{B}_n}\alpha_1\, d^{2n}z = 1$. Let $\alpha_t(z) := t^{-2n}\alpha_1(z/t)$ for $t > 0$. With a constant $\widehat{C}$ to be chosen later we set

$$\varepsilon := \left(\widehat{C}\frac{|\rho(Q)|}{2+\widehat{\sigma}(Q)}\right)^2, \qquad \widehat{\varepsilon} := \frac{1}{\log(1/\varepsilon)}, \tag{3.9}$$

where ρ is as in Lemma 3.7, and define, for $Q \in D$, the function

$$\phi_Q(x) := (1-\widehat{\varepsilon})(\log\varepsilon + \varphi_{1,Q,\delta}(x)) + \varepsilon, \tag{3.10}$$

where we choose $\delta := M_0\widehat{\sigma}(Q)$.

The following estimates for ε will be useful: From

$$|\rho'(Q)| \le \frac{1}{1+\mathrm{Re}\,\sqrt[N]{F_\infty}(Q)} \le \frac{4L_*^{1/N}}{(2+\widehat{\sigma}(Q))^{1/N}}$$

we obtain

$$\sqrt{\varepsilon} \le \widehat{C}\frac{|\rho'(Q)|}{2+\widehat{\sigma}(Q)} \le \frac{4\widehat{C}}{(2+\widehat{\sigma}(Q))^{1+1/N}} \tag{3.11}$$

and

$$\varepsilon \le \widehat{C}^2\frac{|r(Q)|}{2+\widehat{\sigma}(Q)}. \tag{3.12}$$

Further, we define on the set $D_t := \{\delta_D > t\}$ the function

$$\psi_{t,Q}(x) := \int\limits_{|y-x|<t} \mathscr{G}_D(y,Q)\alpha_t(x-y)\,d^{2n}y. \tag{3.13}$$

Our crucial lemma is now

Lemma 3.9. *The constant $\widehat{C}$ can be chosen independently of Q in such a way that the function*

$$v(x) := \begin{cases} \varepsilon^{-2}\rho(x) & \text{if } \rho(x) > -\varepsilon^3, \\ \max\{\psi_{t,Q}(x)-\varepsilon, \varepsilon^{-2}\rho(x)\} & \text{if } -\varepsilon \le \rho(x) \le -\varepsilon^3, \\ \max\{\psi_{t,Q}(x)-\varepsilon, \phi_Q(x)\} & \text{if } \rho(x) < -\varepsilon,\ |x-Q| \ge t, \\ \phi_Q(x) & \text{if } \rho(x) < -\varepsilon,\ |x-Q| < t, \end{cases}$$

is plurisubharmonic on D if we define

$$t := \widehat{C}\exp(-3\log^2(1/\varepsilon)). \tag{3.14}$$

Proof. There exists a constant $C_* > 0$ such that for any $x \in D$ the ball around x with radius $\frac{|r(x)|}{C_*(1+\widehat{\sigma}(x))}$ is contained in D, and consequently

$$\delta_D(x) \ge \frac{|r(x)|}{C_*(1+\widehat{\sigma}(x))}. \tag{3.15}$$

This implies that

$$\delta_D(Q) \ge \frac{|r(Q)|}{C_*(1+\widehat{\sigma}(Q))} \ge \frac{|\rho(Q)|}{C_*(1+\widehat{\sigma}(Q))} \ge \frac{\sqrt{\varepsilon}}{\widehat{C}} > t$$

if $\widehat{C} \ll 1$.

(A) Assume that $|\rho(x)| \geq \varepsilon^3$. Then

$$\delta_D(x) \geq \frac{|r(x)|}{C_*(1+\widehat{\sigma}(x))} \geq \frac{|\rho(x)|}{C_*(1+\widehat{\sigma}(x))} \geq \frac{\varepsilon^3}{C_*(1+\widehat{\sigma}(x))}.$$

On the other hand,

$$\varepsilon^3 \leq |\rho'(x)| \leq \operatorname{Re} \frac{1}{\sqrt[N]{F_\infty(x)}} \leq \frac{2L_*^{1/N}}{\widehat{\sigma}(x)^{1/N}}.$$

From both estimates we get

$$\delta_D(x) \geq \frac{\varepsilon^3}{C_*(1+2^N L_* \varepsilon^{-3N})} = \frac{\varepsilon^{3(N+1)}}{C_*(\varepsilon^{3N}+2^N L_*)}.$$

But this last term is greater than t if $\widehat{C}$ is chosen sufficiently small.

So far we have shown that $\psi_{t,Q}$ is well-defined on $\{\rho \leq -\varepsilon^3\}$.

Clearly $\max\{\psi_{t,Q}(x)-\varepsilon, \varepsilon^{-2}\rho(x)\} = \varepsilon^{-2}\rho(x)$ whenever $\rho(x) = -\varepsilon^3$.

Next we show $\max\{\psi_{t,Q}(x)-\varepsilon, \varepsilon^{-2}\rho(x)\} = \psi_{t,Q}(x)-\varepsilon$ for $x \in \{\rho = -\varepsilon\}$. First we observe that $|\rho(Q)| \geq (2/\widehat{C})\sqrt{\varepsilon} \geq (1/C_6)\varepsilon$, so that we can apply Lemma 3.8. For $x \in D$ with $\rho(x) = -\varepsilon$ we find that

$$\begin{aligned}\psi_{t,Q}(x) - \varepsilon &\geq \mathscr{G}_D(x,Q) - \varepsilon \\ &\geq \frac{\widehat{\mathscr{L}_1}(Q)}{|\rho(Q)|}\rho(x) - \varepsilon = -\varepsilon\left(1 + \frac{\widehat{\mathscr{L}_1}(Q)}{|\rho(Q)|}\right) > -1,\end{aligned}$$

because by our choice of ε,

$$\varepsilon\left(1+\frac{\widehat{\mathscr{L}_1}(Q)}{|\rho(Q)|}\right) \leq \varepsilon + C_6' \frac{|\rho(Q)|}{(2+\widehat{\sigma}(Q))^2}\log\left(1+\frac{2nM_0\widehat{\sigma}(Q)}{|r(Q)|}\right) \leq \varepsilon + C_7\widehat{C}^2,$$

with some unimportant constant $C_7 > 0$. But $\varepsilon + C_7\widehat{C}^2 < 1$ once we choose $\widehat{C} \ll 1$. In particular, $\psi_{t,Q}(x) - \varepsilon \geq -1/\varepsilon = \varepsilon^{-2}\rho(x)$.

We will show that $|x-Q| > t$ for $x \in \{\rho = -\varepsilon\}$. Two cases have to be considered:

Case (I): *$x \in \{\rho = -\varepsilon\}$ and $\rho(x) = r(x)$.* Then, if we assume that $|x-Q| \leq t$, we would find

$$\begin{aligned}\varepsilon = |r(x)| &\geq |r(Q)| - |h_Q(x)| - C_2 W_0(Q', x'-Q') \\ &\geq |r(Q)| - (1+n(1+C_2))(2+\widehat{\sigma}(Q))t \\ &\geq \frac{\varepsilon}{2\widehat{C}} - 4\widehat{C}(1+n(1+C_2))\frac{t}{\sqrt{\varepsilon}} \quad \text{(by (3.11) and (3.12))},\end{aligned}$$

which yields a contradiction if we choose $\widehat{C} \ll 1$.

Case (II): *$x \in \{\rho = -\varepsilon\}$ and $\rho(x) = \rho'(x)$, but $|x-Q| \leq t$.* Then it would follow that

$$\begin{aligned}\varepsilon = |\rho'(x)| &= \frac{1 + \operatorname{Re} \sqrt[N]{F_\infty(x)}}{(1 + \operatorname{Re} \sqrt[N]{F_\infty(x)})^2 + (\operatorname{Im} \sqrt[N]{F_\infty(x)})^2} \\ &\geq \frac{1 + \operatorname{Re} \sqrt[N]{F_\infty(x)}}{(1 + \operatorname{Re} \sqrt[N]{F_\infty(x)})^2 + (\operatorname{Re} \sqrt[N]{F_\infty(x)})^2} \\ &\geq \frac{1 + \sqrt[N]{L_*\widehat{\sigma}(x)}}{(1 + \sqrt[N]{L_*\widehat{\sigma}(x)})^2 + (\sqrt[N]{L_*\widehat{\sigma}(x)})^2}.\end{aligned}$$

This gives $L_*\widehat{\sigma}(x) \geq (4\varepsilon)^{-N}$. On the other hand, at the same time we have

$$\widehat{\sigma}(x) \leq (1 + \widehat{\sigma}(Q))(1 + t) \leq 4\widehat{C}/\sqrt{\varepsilon}.$$

For $\widehat{C} \ll 1$ again a contradiction arises.

Since

$$\phi_Q(x) \leq \varepsilon + (1 - \widehat{\varepsilon}) \log \varepsilon < -1,$$

we find that $\psi_{t,Q}(x) - \varepsilon - \phi_Q(x) \geq -1 - \varepsilon + (1 - \widehat{\varepsilon}) \log(1/\varepsilon) > 0$ for $\widehat{C} \ll 1$.

So far we have shown that v is well-defined at the "border" $\{\rho = -\varepsilon\}$.

It remains to show that $\psi_{t,Q}(x) - \varepsilon - \phi_Q(x) \leq 0$ on the set $\{|x - Q| = t\}$.

Now we have $|x - y - Q| < 2t < \delta_D(Q)$, hence

$$\mathscr{G}_D(x - y, Q) \leq \log \frac{|x - y - Q|}{\delta_D(Q)} \leq \log \frac{2t}{\delta_D(Q)}$$

if $|y| \leq t$. This yields the estimate

$$\begin{aligned}\psi_{t,Q}(x) &= \int_{|y|\leq t} \mathscr{G}_D(x - y, Q)\alpha_t(y)\, d^n y \\ &\leq \int_{|y|\leq t} \left(\log \frac{2t}{\delta_D(Q)}\right)\alpha_t(y)\, d^{2n} y = \log \frac{2t}{\delta_D(Q)} \\ &\leq \log t + \log(2C_*\widehat{C}) + \tfrac{1}{2}\log(1/\varepsilon),\end{aligned} \tag{3.16}$$

using (3.12) and (3.15). This implies

$$\begin{aligned}\phi_Q(x) - \psi_{t,Q}(x) + \varepsilon \geq 2\varepsilon &+ (1 - \widehat{\varepsilon})\varphi_{1,Q,\delta}(x) \\ &+ \log(1/t) - \log(2C_*) - \tfrac{3}{2}\log(1/\varepsilon).\end{aligned} \tag{3.17}$$

We next estimate $\varphi_{1,Q,\delta}(x)$. For this we observe that

$$\begin{aligned}|r(x)| &\leq |r(Q)| + |h_Q(x)| + C_2 W_0(Q', x' - Q') \\ &\leq |r(Q)| + (1 + 2(M_0 + C_2))\mathscr{R}_t(Q),\end{aligned}$$

and hence

$$|r(x)|/\delta \leq 4nM_0(C_2 + 1).$$

This gives

$$-\log\left(1+\frac{|r(x)|}{\delta}\right)+\sum_{\ell=2}^{n}\frac{\Phi_\ell(x)}{\delta^{1/m_\ell}}\ge -C_8$$

with $C_8 = 4n(n+1)M_0(C_2+1)$. From the choice of t it follows that

$$\sqrt{W_{Q,\delta}(x)}\ge \frac{t}{n^2}\sqrt{\varepsilon},$$

so that

$$\varphi_{1,Q,\delta}(x)\ge -\log(1/t)-2\log n-\tfrac12\log(1/\varepsilon)-C_8,$$

and, by (3.17),

$$\phi_Q(x)-\psi_{t,Q}(x)+\varepsilon\ge\widehat{\varepsilon}\log(1/t)-2\log n-2\log(1/\varepsilon)-C_8>0$$

because of our choice of t.

This completes the proof of Lemma 3.9. ■

In the next step we want to compare $\mathscr{G}_D(z,Q)$ with $\mathscr{G}_D(Q,z)$ for points $z\in\{\mathscr{G}_D(\cdot,Q)<-T\}$ with a large positive T. For this we prove

LEMMA 3.10. *Let $Q\in D$ and ε be defined as in* (3.9) *and t as in* (3.14). *Let $\eta>0$ be such that*

$$\eta\ge 2(\widetilde{m}+1)\left(\log^4\left(\frac{1}{t}\right)+\log\left(4+\frac{\widehat{\sigma}(Q)}{|\rho(Q)|}\right)\right). \tag{3.18}$$

If $z\in D$ and $\mathscr{G}_D(z,Q)\le -1$, then, after another shrinking of the constant $\widehat{C}$ from Lemma 3.9, *one of the following statements holds:*

(i) *For any $\ell\in\{2,\dots,n\}$ one has $|z_\ell-Q_\ell|\le|r(Q)|^{1/(2m_\ell)}$, and $|h_Q(z)|\le|r(Q)|$.*
(ii) *$|\mathscr{G}_D(x,Q)|\ge(1-\widehat{\varepsilon})|\mathscr{G}_D(z,Q)|-\varepsilon-1$ for all $x\in D$ with $\mathscr{G}_D(x,z)=-\eta$.*

Proof. We first make some reduction steps.

(1) If $|z-Q|\le t$, then

$$|z_\ell-Q_\ell|^{2m_\ell}\le t^{2m_\ell}\le(\widehat{C}\varepsilon)^{4m_\ell}\le|r(Q)|$$

for any $\ell\in\{2,\dots,n\}$, and

$$|h_Q(z)|\le(2+\sigma(Q'))t\le|\rho(Q)|\le|r(Q)|,$$

so that (i) holds. Therefore we will assume that $|z-Q|>t$.

(2) If $\mathscr{G}_D(z,Q)\le-1$, then

$$\rho(z)<-\varepsilon^3,$$

for otherwise we would have $|\rho(z)| \leq \varepsilon^3 \leq C_6|\rho(Q)|$ (after shrinking $\widehat{C}$), hence, by Lemma 3.8,

$$\begin{aligned}(3.19)\qquad 1 \leq |\mathscr{G}_D(z,Q)| &\leq C_6' \log\left(1 + \frac{2nM_0\widehat{\sigma}(Q)}{|r(Q)|}\right)\frac{\varepsilon^3}{|\rho(Q)|} \\ &\leq C_6'\widehat{C}^6 \frac{|\rho(Q)|^4}{(2+\widehat{\sigma}(Q))^5} \leq C_6'\widehat{C}^6 M_0^4 < 1\end{aligned}$$

if $\widehat{C}$ is chosen small enough, a contradiction.

(3) From $\mathscr{G}_D(x,z) = -\eta$ it follows that $\delta_D(x) > t$. We can argue as follows: From (2) we know that $|\rho(z)| \geq \varepsilon^3$. In the case where $|\rho(x)| \leq C_6\varepsilon^3$, it would follow that $|\rho(x)| \leq C_6|\rho(z)|$, hence by means of Lemma 3.8 also

$$\begin{aligned}(3.20)\qquad \eta = |\mathscr{G}_D(x,z)| &\leq C_6C_6' \log\left(1 + \frac{2nM_0\widehat{\sigma}(z)}{|r(z)|}\right) \\ &\leq C_6C_6' \log\left(1 + \frac{2nM_0C_4\widehat{\sigma}(Q)}{\varepsilon^3}\right) \qquad \text{(by Lemma 3.4)} \\ &\leq C_6C_6' \log\left(1 + \frac{8nM_0C_4\widehat{C}}{\varepsilon^{7/2}}\right).\end{aligned}$$

But $\eta \geq (\log(1/\varepsilon))^8$. After shrinking $\widehat{C}$ we again obtain a contradiction. Hence

$$\frac{2L_*^{1/N}}{\widehat{\sigma}(x)^{1/N}}|\rho'(x)| \geq |\rho(x)| > C_6\varepsilon^3,$$

and further

$$\delta_D(x) \geq \frac{C_6}{C_*(1+\widehat{\sigma}(x))} \geq \frac{C_6}{C_*(2^NL_*+1)}\varepsilon^{3(N+1)} > t$$

if $\widehat{C}$ is small enough (which can be arranged uniformly in Q).

Before we can start proving the lemma, we need the following sublemma:

SUBLEMMA. *There exists a constant $C_9 > 0$ (independent of Q and $\widehat{C}$) such that:*

(a) *If $y \in B(x,t)$ and there is an $\ell_1 \in \{2,\ldots,n\}$ such that $|y_{\ell_1} - Q_{\ell_1}|^2 \geq \frac{1}{4}|r(Q)|^{1/m_{\ell_1}}$, then*

$$|\mathscr{G}_D(y,Q)| \leq \log\left(1 + C_9\frac{\widehat{\sigma}(Q)}{|r(Q)|}\right).$$

(b) *Suppose that there is an $\ell_1 \in \{2,\ldots,n\}$ such that $|z_{\ell_1} - Q_{\ell_1}|^{2m_{\ell_1}} > |r(Q)|$. Then*

$$|\mathscr{G}_D(y,Q)| \leq \log\left(1 + C_9\frac{\widehat{\sigma}(Q)}{|r(Q)|}\right) \qquad \textit{for any } y \in B(x,t).$$

(c) *If* $|h_Q(z)| > |r(Q)|$, *then*

$$|\mathscr{G}_D(y,Q)| \le C_9 + \log\left(1 + C_9 \frac{\widehat{\sigma}(Q)}{|r(Q)|}\right) \quad \textit{for any } y \in B(z,t).$$

Assume that (i) of Lemma 3.10 does not hold and the Sublemma is true. Then we can make use of the function v from Lemma 3.9 and find that

$$\begin{aligned}\mathscr{G}_D(x,Q) &\le \psi_{t,Q}(x) = \psi_{t,Q}(z) + \psi_{t,Q}(x) - \psi_{t,Q}(z)\\ &\le v(z) + \varepsilon + \psi_{t,Q}(x) - \psi_{t,Q}(z)\\ &\le (1-\widehat{\varepsilon})\mathscr{G}_D(z,Q) + \varepsilon + |\psi_{t,Q}(x) - \psi_{t,Q}(z)|,\end{aligned}$$

hence

$$|\mathscr{G}_D(x,Q)| \ge (1-\widehat{\varepsilon})|\mathscr{G}_D(z,Q)| - \varepsilon - |\psi_{t,Q}(x) - \psi_{t,Q}(z)|.$$

We next estimate $|\psi_{t,Q}(x) - \psi_{t,Q}(z)|$. The Sublemma allows us to carry over the methods from [Her3, Lemma 3.5]. We denote by $M_t(x,z)$ the symmetric difference of the balls $B(x,t)$ and $B(z,t)$. Analogously to [Her3, p. 520] we have, with some constant $C_{10} > 0$,

$$\begin{aligned}|\psi_{t,Q}(x) - \psi_{t,Q}(z)| \le C_{10} t^{-2n}\bigg(\frac{|x-z|}{t} &\int\limits_{y\in B(x,t)} |\mathscr{G}_D(y,Q)|\, d^{2n}y\\ &+ \int\limits_{y\in M_t(x,z)} |\mathscr{G}_D(y,Q)|\, d^{2n}y\bigg).\end{aligned} \tag{3.21}$$

Parts (b) and (c) of the sublemma apply, and we obtain $|\mathscr{G}_D(y,Q)| \le C_9 + \log\big(1 + C_9 \frac{\widehat{\sigma}(Q)}{|r(Q)|}\big)$ on $B(x,t) \cup B(z,t)$. In conjunction with

$$\int\limits_{y\in B(x,t)} |\mathscr{G}_D(y,Q)|\, d^{2n}y \le c_n t^{2n}\left(C_9 + \log\left(1 + C_9\frac{\widehat{\sigma}(Q)}{|r(Q)|}\right)\right)$$

and

$$\int\limits_{y\in M_t(x,z)} |\mathscr{G}_D(y,Q)|\, d^{2n}y \le c_n t^{2n-1}|x-z|\left(C_9 + \log\left(1 + C_9\frac{\widehat{\sigma}(Q)}{|r(Q)|}\right)\right),$$

where $c_n = n\,\mathrm{Vol}(B(0,1))$, we eventually obtain

$$\begin{aligned}|\psi_{t,Q}(x) - \psi_{t,Q}(z)| &\le c_n C_{10}\frac{|x-z|}{t}\left(C_9 + \log\left(1 + C_9\frac{\widehat{\sigma}(Q)}{|r(Q)|}\right)\right)\\ &\le c_n C_{10}\frac{|x-z|}{t}\left(C_9 + \log\left(1 + C_9\frac{1}{\varepsilon}\right)\right).\end{aligned} \tag{3.22}$$

By Lemma 3.6 (first for $B = z$ and then for $B = Q$) we have

$$\begin{aligned}|x-z| &\le |x_1-z_1| + \sum_{\ell=2}^{n} \sigma(x'-z')^{1/(2m_\ell)}\\ &\le |h_z(x)| + \sum_{\ell=2}^{n}(2+\widehat{\sigma}(z)^{1-1/(2m_\ell)})\sigma(x'-z')^{1/(2m_\ell)}\\ &\le (4n+1)C_4\widehat{\sigma}(z)e^{-\eta/(2\widetilde{m})}\\ &\le (4n+1)C_4^2\widehat{\sigma}(Q)e^{-\eta/(2\widetilde{m})} \le \frac{nC_4^2}{\varepsilon}e^{-\eta/(2\widetilde{m})}.\end{aligned}$$

Plugging this into (3.22) we obtain statement (ii) of the lemma (after possibly shrinking $\widehat{C}$ and hence ε once more).

Proof of the Sublemma. (a) We use the estimate

$$|\mathscr{G}_D(y,Q)| \le \frac{1}{2}\log\biggl(1 + C_1\frac{|r(y)|^{1/m_{\ell_1}}}{|y_{\ell_1}-Q_{\ell_1}|^2}\biggr). \tag{3.23}$$

The right-hand side is less than or equal to $\frac{1}{2}\log\bigl(1+4C_1\frac{|r(y)|^{1/m_{\ell_1}}}{|r(Q)|^{1/m_{\ell_1}}}\bigr)$. So we must compare $|r(y)|$ with $|\widehat{\sigma}(Q)|$.

First we note that, using first Lemma 2.3 and then Lemma 3.6 with the pair (x,z) and then with the pair (z,Q), we get

$$\begin{aligned}\widehat{\sigma}(y) &\le \widehat{\sigma}(x) + |y_1-x_1| + (4\widetilde{m}+\gamma_1^{-1})\sum_{\ell=2}^{n}\sigma(x')^{1-1/(2m_\ell)}|y_\ell-x_\ell| \\ &\le C_4\widehat{\sigma}(z)e^{-\eta/(2\widetilde{m})} + (1+(4\widetilde{m}+\gamma_1^{-1}))t + C_4(4\widetilde{m}+\gamma_1^{-1})\widehat{\sigma}(z)e^{-\eta/(2\widetilde{m})}\\ &\le C_4^2(1+(4\widetilde{m}+\gamma_1^{-1}))e^{-\eta/\widetilde{m}}\widehat{\sigma}(Q) + (4\widetilde{m}+\gamma_1^{-1})\widehat{C}t \le 2\widehat{\sigma}(Q).\end{aligned} \tag{3.24}$$

In conjunction with $|r(y)| \le M_0\widehat{\sigma}(y)$ we get the claim.

(b) Assume that $y \in B(x,t)$ and there is an $\ell_1 \in \{2,\dots,n\}$ such that $|z_{\ell_1}-Q_{\ell_1}|^{2m_{\ell_1}} \ge |r(Q)|$. Then

$$\begin{aligned}|y_{\ell_1}-Q_{\ell_1}| &\ge |z_{\ell_1}-Q_{\ell_1}| - |z_{\ell_1}-x_{\ell_1}| - |y_{\ell_1}-x_{\ell_1}|\\ &\ge |r(Q)|^{1/(2m_{\ell_1})} - C_4e^{-\eta/2\widetilde{m}} - t \ge \tfrac{1}{2}|r(Q)|^{1/(2m_{\ell_1})}.\end{aligned}$$

Now we plug this and $|r(y)| \le M_0\widehat{\sigma}(y) \le 2M_0\widehat{\sigma}(Q)$ into (3.23).

(c) Let $y \in B(z,t)$. From Lemma 3.7 we know that (recall $\delta := M_0\widehat{\sigma}(Q)$)

$$\begin{aligned}|\mathscr{G}_D(y,Q)| &\le |\varphi_{1,Q,\delta}(y)|\\ &= -\frac{1}{2}\log\chi(W_{Q,\delta}(y)) + \log\biggl(1+\frac{|r(y)|}{\delta}\biggr) + \sum_{\ell=2}^{n}\frac{|\Phi_\ell(y)|}{\delta^{1/m_\ell}}\\ &\le -\frac{1}{2}\log\chi(W_{Q,\delta}(y)) + \log\biggl(1+\frac{|r(y)|}{\delta}\biggr) + \sum_{\ell=2}^{n}\biggl(\frac{|r(y)|}{\delta}\biggr)^{1/m_\ell}.\end{aligned} \tag{3.25}$$

Further we have

$$|h_Q(y)-h_Q(z)| \le t + \sum_{\ell=2}^{n} \sigma(Q')^{1-1/(2m_\ell)} t$$
$$\le (2+\widehat{\sigma}(Q))t \le \widehat{C}t/\varepsilon < \widehat{C}\varepsilon \le \tfrac{1}{2}|r(Q)|.$$

This implies

$$|h_Q(y)| \ge |h_Q(z)| - |h_Q(y)-h_Q(z)| \ge \tfrac{1}{2}|r(Q)|,$$

and therefore $W_{Q,\delta}(y) \ge \frac{|r(Q)|}{2(n+1)M_0\widehat{\sigma}(Q)}$. Since

$$\widehat{\sigma}(y) \le \widehat{\sigma}(z) + (2+\widehat{\sigma}(z))t = (1+t)\widehat{\sigma}(z) + 2t \le 4C_4\widehat{\sigma}(Q)$$

by Lemma 3.5, hence $|r(y)| \le 4C_4\delta$, we find, in conjunction with (3.25), that

$$|\mathscr{G}_D(y,Q)| \le C_9 + \log\left(1 + C_9 \frac{\widehat{\sigma}(Q)}{|r(Q)|}\right)$$

with a suitable universal constant $C_9 > 0$.

The sublemma is proved. ∎

We are now ready for the proof of the main lemma.

Main Lemma 3.11. For a point $Q \in D$ we define

$$\mathscr{L}(Q) := \log\left(1 + \frac{K\widehat{\sigma}(Q)}{|r(Q)|}\right) \quad \text{with} \quad K := 1 + 8(M_0^N L_*)^{1/N+1},$$
$$\mathscr{R}(Q) := \sqrt{\widehat{\sigma}(Q)|r(Q)|} + \sum_{\ell=2}^{n} \widehat{\sigma}(Q)^{1-1/(2m_\ell)} |r(Q)|^{1/(2m_\ell)}.$$

Then there are constants $T_0, C_{11} > 1$ such that for all $T \ge T_0$ and all $z, Q \in D$ with $\mathscr{G}_D(z,Q) \le -T$,

$$|z_\ell - Q_\ell| \le C_{11}\mathscr{L}(Q)^{4(n-1)} T^{-n/2} |r(Q)|^{1/(2m_\ell)}, \quad 2 \le \ell \le n, \tag{3.26}$$
$$\frac{1}{C_{11}} \frac{1}{1+\mathscr{L}(Q)} |r(Q)| \le |r(z)| \le C_{11}(1+\mathscr{L}(Q))^{8n} |r(Q)|, \tag{3.27}$$
$$|h_Q(z)| \le C_{11} e^{-T} (1+\mathscr{L}(Q))^{8n} \mathscr{R}(Q). \tag{3.28}$$

Proof. Let $\widehat{\mathscr{L}}(Q) := \log\big(1 + \frac{4+\widehat{\sigma}(Q)}{|\rho(Q)|}\big)$. It suffices to show

$$|z_\ell - Q_\ell| \le C_{11}\widehat{\mathscr{L}}(Q)^{4(n-1)} T^{-n/2} |r(Q)|^{1/(2m_\ell)}, \quad 2 \le \ell \le n, \tag{3.29}$$
$$\frac{1}{C_{11}} \frac{1}{1+\widehat{\mathscr{L}}(Q)} |r(Q)| \le |r(z)| \le C_{11}(1+\widehat{\mathscr{L}}(Q))^{8n} |r(Q)|, \tag{3.30}$$

whenever $T \ge T_0$.

Suppose we have shown (3.29) and (3.30). For $x > 0$ we define the scaling map $S_x(u) := \big(\frac{u_1}{x}, \frac{u_2}{x^{1/(2m_2)}}, \dots, \frac{u_n}{x^{1/(2m_n)}}\big)$ on D. If $z, Q \in D$ are as in the Main

Lemma, then also $\mathscr{G}_D(S_x(z), S_x(Q)) < -T$, hence, by (3.29),

$$\begin{aligned}|z_\ell - Q_\ell| &= x^{1/(2m_\ell)}|(S_x(z) - S_x(Q))_\ell| \\ &\le C_{11}\widehat{\mathscr{L}}(S_x(Q))^{4(n-1)}T^{-n/2}x^{1/(2m_\ell)}|r(S_x(Q))|^{1/(2m_\ell)} \\ &= C_{11}\widehat{\mathscr{L}}(S_x(Q))^{4(n-1)}T^{-n/2}|r(Q)|^{1/(2m_\ell)}, \quad 2 \le \ell \le n.\end{aligned}$$

But if we choose $x := K - 1$, we get $|\rho(S_x(Q))| = |r(S_x(Q))| = x^{-1}|r(Q)|$, and hence $\widehat{\mathscr{L}}(S_x(Q)) = \mathscr{L}(Q)$. This proves (3.26). In a similar manner we obtain (3.27) from (3.30).

In Lemma 3.10 we choose

$$\eta = 3(\widetilde{m} + 1)\biggl(\log^4\biggl(\frac{1}{t}\biggr) + \log\biggl(4 + \frac{\widehat{\sigma}(Q)}{|\rho(Q)|}\biggr)\biggr).$$

If statement (i) of the lemma holds for z and Q, there is nothing to be done.

Suppose now that it does not hold. Let $D_\nu := D \cap B(0, \nu)$. We exhaust D by a sequence $(D_\nu)_{\nu\ge1}$ of bounded domains. From an inequality in [Blo1] (see also [Her3, p. 513]) we get, for $z \in D_\nu$,

$$\begin{aligned}&\int_{D_\nu} |\max\{\mathscr{G}_D(\cdot, z), -k(\nu - |z|)\}|^n \bigl(dd^c(\max\{\mathscr{G}_{D_\nu}(\cdot, z), -\eta\})\bigr)^n \\ &\quad\le n!\eta^{n-1} \int_{D_\nu} |\max\{\mathscr{G}_{D_\nu}(\cdot, z), -\eta\}| \bigl(dd^c(\max\{\mathscr{G}_D(\cdot, Q), -k(\nu - |z|)\})\bigr)^n.\end{aligned}$$

B. Levi's theorem, combined with results of Bedford–Taylor [BeT], allows letting $k \to \infty$ (see [Her3, proof of Lemma 3.3]), which leads to

$$\begin{aligned}&\int_{D_\nu} |\mathscr{G}_D(\cdot, z)|^n \bigl(dd^c(\max\{\mathscr{G}_{D_\nu}(\cdot, z), -\eta\})\bigr)^n \\ &\qquad\le n!\eta^{n-1} \int_{D_\nu} |\max\{\mathscr{G}_{D_\nu}(\cdot, z), -\eta\}|(dd^c\mathscr{G}_D(\cdot, Q))^n \\ &\qquad\le n!\eta^{n-1}|\max\{\mathscr{G}_{D_\nu}(Q, z), -\eta\}| \le n!\eta^{n-1}|\mathscr{G}_{D_\nu}(Q, z)|.\end{aligned}$$

Now we can let ν tend to infinity by a similar reasoning and get

$$\int_D |\mathscr{G}_D(\cdot, z)|^n \bigl(dd^c(\max\{\mathscr{G}_D(\cdot, z), -\eta\})\bigr)^n \le n!\eta^{n-1}|\mathscr{G}_D(Q, z)|.$$

All the measures $(dd^c(\max\{\mathscr{G}_D(\cdot, z), -\eta\}))^n$ have the same total mass $(2\pi)^n$, hence, by Hölder's inequality, we obtain

$$\begin{aligned}&\int_D |\mathscr{G}_D(\cdot, z)|\bigl(dd^c(\max\{\mathscr{G}_D(\cdot, z), -\eta\})\bigr)^n \\ &\qquad\qquad\le (2\pi)^n \sqrt[n]{n!}\,\eta^{1-1/n}|\mathscr{G}_D(Q, z)|^{1/n}.\end{aligned} \tag{3.31}$$

Now, the measure $(dd^c(\max\{\mathscr{G}_D(\cdot, z), -\eta\}))^n$ has support in $\{\mathscr{G}_D(\cdot, z) = -\eta\}$, hence by statement (ii) we see that

$$\int_D |\mathscr{G}_D(\cdot,z)|\big(dd^c(\max\{\mathscr{G}_D(\cdot,z),-\eta\})\big)^n$$
$$\geq (1-\widehat{\varepsilon})|\mathscr{G}_D(z,Q)| - \varepsilon - 1 \geq \tfrac{1}{2}|\mathscr{G}_D(z,Q)| - \varepsilon - 1.$$

So we obtain

$$T \leq |\mathscr{G}_D(z,Q)| \leq 2\big((2\pi)^n \sqrt[n]{n!}\,\eta^{1-1/n}|\mathscr{G}_D(Q,z)|^{1/n} + \varepsilon + 1\big). \tag{3.32}$$

In conjunction with Lemma 3.2 it follows that

$$T \leq 2\bigg((2\pi)^n \sqrt[n]{n!}\,\eta^{1-1/n}\bigg(C_1\frac{|r(Q)|^{1/m_\ell}}{|z_\ell - Q_\ell|^2}\bigg)^{1/n} + \varepsilon + 1\bigg),$$

or equivalently

$$|z_\ell - Q_\ell| \leq \sqrt{(2\pi)^{n^2} n!}\,C_1\bigg(\frac{2}{T-4}\bigg)^{n/2}|r(Q)|^{1/(2m_\ell)}\eta^{(n-1)/2}$$
$$\leq C_{11}T^{-n/2}|r(Q)|^{1/(2m_\ell)}\eta^{(n-1)/2}$$

if $T \geq 12$, with some unimportant constant $C_{11} > 0$.

Next we prove the lower bound for $|r(z)|$. If we assume that $|r(z)| \leq |r(Q)|/2\mathscr{L}(Q)$, then again by Lemma 3.8 we obtain (recalling the definition of $\widehat{\mathscr{L}_1}(Q)$)

$$T \leq |\mathscr{G}_D(z,Q)| \leq \widehat{\mathscr{L}_1}(Q)\frac{|r(z)|}{|r(Q)|} \leq \frac{\widehat{\mathscr{L}_1}(Q)}{2\mathscr{L}(Q)} \leq C_6'(1+2nM_0),$$

which gives a contradiction if $T \geq T_0 > C_6'(1+2nM_0)$. Hence we obtain the desired lower estimate for $|r(z)|/|r(Q)|$.

The claimed upper bound for $|r(z)|$ is harder to prove. If we had $|r(Q)| \leq |r(z)|/\eta^n$, it would follow, by Lemma 3.8, that

$$\eta^{1-1/n}|\mathscr{G}_D(Q,z)|^{1/n} \leq \bigg(\widehat{\mathscr{L}_1}(Q)\frac{|r(Q)|}{|r(z)|}\bigg)^{1/n}\frac{1}{\eta^{1/n}}$$
$$\leq \bigg(C_6'(1+2nM_0)\log\bigg(1+\frac{\widehat{\sigma}(Q)}{|r(Q)|}\bigg)\bigg)^{1/n}\frac{1}{\eta^{1/n}}$$
$$\leq C_{12} := C_6'(1+2nM_0).$$

We plug this into (3.32) and obtain

$$T \leq 2((2\pi)^n \sqrt[n]{n!}\,C_{12} + 2).$$

For $T_0 > 2((2\pi)^n \sqrt[n]{n!}\,C_{12}+2)$ we get a contradiction whenever $T \geq T_0$. Thus we must have

$$|r(z)| \leq \eta^n |r(Q)|$$

for all $z \in \{\mathscr{G}_D(\cdot,Q) < -T\}$.

The estimates (3.29) and (3.30) now follow from the estimate

$$\eta \leq C_{13}(1+\widehat{\mathscr{L}}(Q))^{8n}$$

with some universal constant C_{13}.

We now turn to the estimate (3.28). We will make use of Lemma 3.5(c) and (d) for $B := Q$ and $\delta := C_{11}(1+\mathscr{L}(Q))^{8n}|r(Q)|$. This is allowed because of (3.27), and hence

$$-T_0 \geq -T \geq \varphi_{1,Q,\delta} \geq \tfrac{1}{2}\log\chi\circ W_{Q,\delta} - (n+1)M_*.$$

For $T_0 > 2+(n+1)M_*$ we see that $W_{Q,\delta}(z) \leq 1/2$ whenever $\mathscr{G}_D(z,Q) < -T$. This shows $W_{Q,\delta} \leq e^{2(n+1)M_*}e^{-2T}$ on $\{\mathscr{G}_D(\cdot,Q) < -T\}$. In particular we find (after a possible enlargement of C_{11}) that

$$|h_Q| \leq e^{(n+1)M_*}e^{-T}\mathscr{R}_\delta(Q) \leq C_{11}(1+\widehat{\mathscr{L}}(Q))^{8n}\mathscr{R}(Q)$$

on the sublevel set $\{\mathscr{G}_D(\cdot,Q) < -T\}$. ∎

4. Estimation of the Bergman distance on D. We want to prove Theorem 1.1. By symmetry it suffices to show $d_D^B(A,Q) \geq C_*\varrho_D(A,Q)$, with some unimportant constant $C_* > 0$.

To begin with we clarify what to do.

The following lemma follows for instance from [Blo2, Prop. 2.3]:

LEMMA 4.1. *Assume that $\Omega \subset \mathbb{C}^n$ is a domain for which the Bergman kernel K_Ω has positive values on the diagonal of $\Omega\times\Omega$. For $q \in \Omega$ set $M_q := K_\Omega(\cdot,q)/\sqrt{K_\Omega(q,q)}$. If $p,q \in \Omega$ and $f \in H^2(\Omega)$ with $f(q) = 0$ and $f(p) = M_q(p)$, then the Bergman distance of p and q can be estimated by*

$$d_\Omega^B(p,q) \geq \frac{1}{\sqrt{1+\|f\|^2}}.$$

The second tool for the estimation of the Bergman distance is the following version of [Blo2, Theorem 4.4] which does without the assumption that the domain is bounded.

LEMMA 4.2. *Assume that $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain that admits a negative strongly plurisubharmonic function Ψ. Let $T > 0$. Then there is a constant c_{00} such that for $p,q\in\Omega$ with $\{\mathscr{G}_\Omega(\cdot,p)<-T\}\cap\{\mathscr{G}_\Omega(\cdot,q)<-T\} = \emptyset$ we have $d_\Omega^B(p,q) \geq c_{00}$.*

Proof. We exhaust Ω by an increasing sequence $(\Omega_t)_{t>0}$ of strongly pseudoconvex domains. Let $\mathscr{G}_\Omega^t(\cdot,p)$ and $\mathscr{G}_\Omega^t(\cdot,q)$ denote regularizations of $\mathscr{G}_\Omega(\cdot,p)$ and $\mathscr{G}_\Omega(\cdot,q)$ on Ω_t that decrease to $\mathscr{G}_\Omega(\cdot,p)$ and $\mathscr{G}_\Omega(\cdot,q)$, respectively. Then also $\{\mathscr{G}_\Omega^t(\cdot,p) < -T\}\cap\{\mathscr{G}_\Omega^t(\cdot,q) < -T\} = \emptyset$. We choose $t > 0$ so small that $\mathscr{G}_\Omega^t(p,p) < -2T$. Next we choose a smooth cut-off function $\chi : (-\infty,0) \to [0,1]$ such that $\chi(x) = 1$ for $x \leq -2T$ and $\chi(x) = 0$ for $x \geq -T$. Then $v := \bar\partial\chi(\mathscr{G}_\Omega^t(\cdot,p))M_q$ is a smooth $\bar\partial$-closed $(0,1)$-form.

For any plurisubharmonic function φ on Ω Hörmander's theory [Hoer] gives on Ω_t a solution u_t to the equation $\bar\partial u_t = v$ with $\int_\Omega |u_t|^2e^{-t\Psi-\varphi}\,d^{2n}z \leq 2\int_{\Omega_t}|v|^2_{\partial\bar\partial(t\Psi+\varphi)}e^{-t\Psi-\varphi}\,d^{2n}z$. If we choose $\varphi := 2n\mathscr{G}_\Omega^t(\cdot,p) + 2n\mathscr{G}_\Omega^t(\cdot,q) +$

$e^{\mathscr{G}_\Omega^t(\cdot,p)}$, we see that the length $|v|_{\partial\bar\partial(t\Psi+\varphi)}$ of v measured with respect to the Kähler metric with potential $t\Psi+\varphi$ is dominated by

$$|v|^2_{\partial\bar\partial(t\Psi+\varphi)} \le \max(\chi')^2 e^{(2+4n)T+t\sup_{\{\mathscr{G}_\Omega(\cdot,p)<-T\}}|\Psi|},$$

hence (since $-t\Psi-\varphi > -2n\mathscr{G}_\Omega^t(\cdot,p)-1$ and $\mathscr{G}_\Omega^t(\cdot,q)\ge -T$ on $\operatorname{supp}(v)$)

$$\int_\Omega |u_t|^2 e^{-2n\mathscr{G}_\Omega^t(\cdot,p)-2n\mathscr{G}_\Omega^t(\cdot,q)}\,d^{2n}z \le 2e\int_{\Omega_t}|v|^2_{\partial\bar\partial(\Psi+\varphi)}e^{-t\Psi-\varphi}\,d^{2n}z \le 2e\max(\chi')^2e^{(2+4n)T}e^{t\sup_{\{\mathscr{G}_\Omega(\cdot,p)<-T\}}|\Psi|}, \tag{4.1}$$

since M_q is normalized. The function

$$f_t := \chi(\mathscr{G}_\Omega^t(\cdot,p))M_q - u_t$$

is holomorphic with L^2-norm $\le c_t := 1+2e\max(\chi')^2e^{(2+4n)T}m(t,T)$, where $m(t,T) := e^{t\sup_{\{\mathscr{G}_\Omega(\cdot,p)<-T\}}|\Psi|}$. We apply the Alaoglu–Bourbaki theorem to select a weakly*-convergent subsequence $(u_{t_k})_k$ with a limit $u\in L^2(\Omega)$ which satisfies

$$\int_\Omega |u|^2e^{-2n\mathscr{G}_\Omega(\cdot,p)-2n\mathscr{G}_\Omega(\cdot,q)}\,d^{2n}z \le 2e\max(\chi')^2e^{(2+4n)T}.$$

From this we obtain $u(p)=u(q)=0$. Further, the function

$$f := \chi(\mathscr{G}_\Omega(\cdot,p))M_q - u$$

belongs to $H^2(\Omega)$ and satisfies $f(p)=M_q(p)$, $f(q)=0$, and $\|f\|\le c' := 1+2e\max(\chi')^2e^{(2+4n)T}$. From Lemma 4.1 the claim follows with $c_{00} := 1/\sqrt{1+c'^2}$. ∎

For $Q\in D$ and constants $c,\widetilde{c}>0$ we let

$$S_-(Q) := \left\{z\in D \;\middle|\; |r(z)|(1+\mathscr{L}(z))^{8n} > c\frac{|r(Q)|}{1+\mathscr{L}(Q)}\right\},$$

$$S_+(Q) := \left\{z\in D \;\middle|\; \frac{|r(z)|}{1+\mathscr{L}(z)} < c^{-1}(1+\mathscr{L}(Q))^{8n}|r(Q)|\right\}.$$

We introduce suitable "pseudoballs", using the same notation as in the preceding sections:

$$\begin{aligned}\mathscr{B}(Q) := \{z\in D \mid{} & |h_Q(z)| < \widetilde{c}^{-1}(1+\mathscr{L}(Q))^{q_2}\mathscr{R}(Q),\\ & |z_\ell - Q_\ell| < \widetilde{c}^{-1/2}|r(Q)|^{1/(2m_\ell)}\mathscr{L}(Q)^{q_1},\ 2\le\ell\le n\}\\ & \cap S_-(Q)\cap S_+(Q)\end{aligned} \tag{4.2}$$

with constants $c,\widetilde{c}>0$ and integers $q_1,q_2>0$ to be chosen later. We abbreviate $\mathscr{S}_T(Q) := \{\mathscr{G}_D(\cdot,Q) < -T\}$ for $T>0$ and prove

LEMMA 4.3. *For sufficiently small $c,\widetilde{c}$ and suitably chosen q_1,q_2 and $T\ge T_0$ we have $\mathscr{S}_T(Q)\cap\mathscr{S}_T(\widetilde{Q})=\emptyset$ whenever $\widetilde{Q}\notin\mathscr{B}(Q)$.*

Proof. Suppose first that $\widetilde{Q} \notin S_-(Q)$, but there exists $z_0 \in \mathscr{S}_T(Q) \cap \mathscr{S}_T(\widetilde{Q})$. Then, if we recall (3.27), we get

$$\begin{aligned}\frac{1}{C_{11}} \frac{|r(Q)|}{1+\mathscr{L}(Q)} &\overset{z_0 \in \mathscr{S}_T(Q)}{\leq} |r(z_0)| \overset{z_0 \in \mathscr{S}_T(\widetilde{Q})}{\leq} C_{11}(1+\mathscr{L}(\widetilde{Q}))^{8n}|r(\widetilde{Q})| \\ &\overset{\widetilde{Q}\notin S_-(Q)}{<} cC_{11}\frac{|r(Q)|}{1+\mathscr{L}(Q)},\end{aligned}$$

a contradiction if $c < C_{11}^{-2}$.

The case of $\widetilde{Q} \notin S_+(Q)$ is treated in a similar manner.

So far we know that $\widetilde{Q} \in S_+(Q) \cap S_-(Q)$ if $\mathscr{S}_T(Q) \cap \mathscr{S}_T(\widetilde{Q}) \neq \emptyset$ and $T \geq T_0$. Before we go on with the proof, we have to estimate $\mathscr{L}(\widetilde{Q})$ and $\mathscr{R}(\widetilde{Q})$ in terms of $\mathscr{L}(Q)$ and $\mathscr{R}(Q)$.

For this we note that

$$\begin{aligned}\widehat{\sigma}(\widetilde{Q}) &\leq 2^{\widetilde{m}}(\widehat{\sigma}(z_0) + \widehat{\sigma}(\widetilde{Q} - z_0)) \leq 4^{\widetilde{m}}\big(\widehat{\sigma}(Q) + \widehat{\sigma}(Q - z_0) + \widehat{\sigma}(\widetilde{Q} - z_0)\big) \\ &\leq 4^{\widetilde{m}}\big((1 + C_4 e^{-T})\widehat{\sigma}(Q) + C_4 e^{-T}\widehat{\sigma}(\widetilde{Q})\big) \quad \text{(by Lemma 3.6)} \\ &\leq 2 \cdot 4^{\widetilde{m}}\widehat{\sigma}(Q) + \tfrac{1}{2}\widehat{\sigma}(\widetilde{Q})\end{aligned}$$

if $T \geq T_0 \geq 2 \cdot 4^{\widetilde{m}} C_4$. Hence

$$\widehat{\sigma}(\widetilde{Q}) \leq C_{14}\widehat{\sigma}(Q)$$

with some unimportant constant $C_{14} > 0$. In conjunction with $|r(\widetilde{Q})| \leq c^{-1}(1 + \mathscr{L}(Q))^{8n+1}|r(Q)|$ this implies

$$\mathscr{R}(\widetilde{Q}) \leq C_{15}(1 + \mathscr{L}(Q))^{8n+1}\mathscr{R}(Q).$$

Next we estimate

$$\begin{aligned}\mathscr{L}(\widetilde{Q}) &= \log\left(1 + \frac{\widehat{\sigma}(\widetilde{Q})}{|r(\widetilde{Q})|}\right) \\ &\leq \log\left(1 + \frac{C_{14}(1+\mathscr{L}(\widetilde{Q}))^{8n+1}\widehat{\sigma}(Q)}{c|r(Q)|}\right) \quad (\text{since } \widetilde{Q} \notin S_-(Q)) \\ &\leq (8n+1)\log(1 + \mathscr{L}(\widetilde{Q})) + \log(1 + C_{14}/c) + \mathscr{L}(Q).\end{aligned}$$

We claim that $\mathscr{L}(\widetilde{Q}) \leq C_{15}\mathscr{L}(Q)$ with some unimportant constant C_{15}. If $\mathscr{L}(\widetilde{Q}) \leq 2\mathscr{L}(Q)$, this is clear. So assume that $\mathscr{L}(\widetilde{Q}) \geq 2\mathscr{L}(Q)$. Then from the last estimate we obtain

$$\mathscr{L}(\widetilde{Q}) \leq 2(8n+1)\log(1 + \mathscr{L}(\widetilde{Q})) + \log(1 + C_{14}/c),$$

hence $\mathscr{L}(\widetilde{Q}) \leq C_{16}$ with a universal constant C_{16}, since we have $\mathscr{L}(Q) \geq \log(1 + M_0^{-1})$; this again gives the claim.

We are ready to continue the proof of the lemma. Let us assume that $\widetilde{Q}\in S_+(Q)\cap S_-(Q)$ and $|\widetilde{Q}_\ell - Q_\ell| < \widetilde{c}^{-1/2}|r(Q)|^{1/(2m_\ell)}\mathscr{L}(Q)^{q_1}$, $2\le\ell\le n$, but $|h_Q(\widetilde{Q})|\ge\widetilde{c}^{-1}(1+\mathscr{L}(Q))^{q_2}\mathscr{R}(Q)$. We show again that no $z_0\in\mathscr{S}_T(Q)\cap\mathscr{S}_T(\widetilde{Q})$ can exist. Otherwise we could write

$$
\begin{aligned}
(4.3)\qquad |h_Q(\widetilde{Q})| &\le \Big|h_Q(z_0)-h_{\widetilde{Q}}(z_0)+2\sum_{\ell=2}^{n}(P_{z_\ell}(\widetilde{Q}')-P_{z_\ell}(Q'))(z_{0,\ell}-\widetilde{Q}_\ell)\Big|\\
&\le e^{-T}C_{11}\bigg((1+\mathscr{L}(Q))^{8n}\mathscr{R}(Q)+(1+\mathscr{L}(\widetilde{Q}))^{8n}\mathscr{R}(\widetilde{Q})\\
&\quad + 2nM_0\frac{\mathscr{L}(Q)^{q_1}}{T^{n/2}}\sum_{k,\ell=2}^{n}\sigma(Q')^{1-1/(2m_k)-1/(2m_\ell)}|\widetilde{Q}_k-Q_k|\,|r(\widetilde{Q})|^{1/(2m_\ell)}\bigg)\\
&\quad\text{(by (3.28)}\\
&\le e^{-T}C_{11}\bigg((1+\mathscr{L}(Q))^{8n}\mathscr{R}(Q)+(1+\mathscr{L}(\widetilde{Q}))^{8n}\mathscr{R}(\widetilde{Q})\\
&\quad+\frac{2nM_0}{\widetilde{c}^{1/2}T^{n/2}}(1+\mathscr{L}(Q))^{q_1}\sum_{k,\ell=2}^{n}\sigma(Q')^{1-1/(2m_k)-1/(2m_\ell)}|r(Q)|^{1/(2m_k)+1/(2m_\ell)}\bigg).
\end{aligned}
$$

The last term is at most $(2nM_0^2/\sqrt{\widetilde{c}})(1+\mathscr{L}(Q))^{q_1}T^{-n/2}\mathscr{R}(Q)$. This, combined with (4.3), gives

$$
\begin{aligned}
|h_Q(\widetilde{Q})| &\le e^{-T}C_{11}\bigg((1+\mathscr{L}(Q))^{8n}\mathscr{R}(Q)+(1+\mathscr{L}(\widetilde{Q}))^{8n}\mathscr{R}(\widetilde{Q})\\
&\qquad\qquad+\frac{2nM_0^2}{\sqrt{\widetilde{c}}}(1+\mathscr{L}(Q))^{q_1}T^{-n/2}\mathscr{R}(Q)\bigg)\\
&\le \frac{C_{11}}{\sqrt{\widetilde{c}}}(1+\mathscr{L}(Q))^{q_1+8n+1}\mathscr{R}(Q)
\end{aligned}
$$

if $T\ge T_0$, which can be done uniformly in $Q,\widetilde{Q},\widetilde{c}$. But the left-hand side is supposed to be greater than $\widetilde{c}^{-1}(1+\mathscr{L}(Q))^{q_2}\mathscr{R}(Q)$. For $q_2>q_1+8n+1$ this yields a contradiction if $\widetilde{c}<C_{11}^{-2}$.

Finally, we show that no $z_0\in\mathscr{S}_T(Q)\cap\mathscr{S}_T(\widetilde{Q})$ can exist if there exists an $\ell_1\in\{2,\dots,n\}$ such that $|Q_{\ell_1}-\widetilde{Q}_{\ell_1}| > \widetilde{c}^{-1/2}|r(Q)|^{1/(2m_{\ell_1})}\mathscr{L}(Q)^{q_1}$. Otherwise we would obtain

$$
\begin{aligned}
|Q_{\ell_1}-\widetilde{Q}_{\ell_1}| &\le |Q_{\ell_1}-z_{0,\ell_1}|+|z_{0,\ell_1}-\widetilde{Q}_{\ell_1}|\\
&\le C_{11}T_0^{-n/2}\big(|r(Q)|^{1/(2m_{\ell_1})}\mathscr{L}(Q)^{4(n-1)}+|r(\widetilde{Q})|^{1/(2m_{\ell_1})}\mathscr{L}(\widetilde{Q})^{4(n-1)}\big)\\
&\le C_{11}|r(Q)|^{1/(2m_{\ell_1})}\big(\mathscr{L}(Q)^{4(n-1)}+(1+\mathscr{L}(Q))^{8n}(1+\mathscr{L}(\widetilde{Q}))^{4(n-1)+1}\big)\\
&\le C_{11}C_{15}|r(Q)|^{1/(2m_{\ell_1})}(1+\mathscr{L}(Q))^{12n}.
\end{aligned}
$$

But the left-hand side is $> \widetilde{c}^{-1/2}|r(Q)|^{1/(2m_{\ell_1})}\mathscr{L}(Q)^{q_1}$. If we choose $q_1 = 12n+1$ and then $\widetilde{c} \le C_{11}^{-2}C_{15}^{-2}$, this yields a contradiction. ■

COROLLARY 4.1. *Let $T_0 > 0$ be as before and c_{00} as in Lemma 4.1. Then we have:*

(a) $d_B^D(\widetilde{Q}, Q) \ge c_{00}$ *if* $\widetilde{Q} \notin \mathscr{B}(Q)$,
(b) $\mathscr{S}_T(w) \subset \mathscr{B}(w)$ *for* $T \ge T_0$,
(c) *for the Bergman differential metric B_D of D,*

$$B_D(w,X) \ge C\mathscr{L}(w)^{-q_2}\left(\frac{|\langle \partial h_w(w), X\rangle|}{\mathscr{R}(w)} + \sum_{\ell=2}^{n}\frac{|X_\ell|}{|r(w)|^{1/(2m_\ell)}}\right)$$

with some constant $C > 0$.

Proof. (a) is clear. Observe that D admits a negative strongly plurisubharmonic function, namely

$$\Psi := \sum_{\ell=2}^{n}\gamma_\ell|z_\ell|^2 - \left(-r + \frac{c_0}{2}\sigma\right)^{1/m_\ell}$$

for suitably chosen constants $\gamma_2, \ldots, \gamma_n > 0$.

(b) A point $\widetilde{w} \notin \mathscr{B}(w)$ cannot lie in $\mathscr{S}_T(w) \cap \mathscr{S}_T(\widetilde{w})$, hence not in $\mathscr{S}_T(w)$.

(c) Since a strictly plurisubharmonic function Ψ is available on D, we get in view of [DH]

$$\begin{aligned} B_D(w,X) &\ge C'_{T_0}B_{\mathscr{S}_{T_0}(w)}(w,X) \\ &\ge C'_{T_0}F^{\mathrm{Cara}}_{\mathscr{S}_{T_0}(w)}(w,X) \\ &\ge C\mathscr{L}(w)^{-q_2}\left(\frac{|\langle \partial h_w(w), X\rangle|}{\mathscr{R}(w)} + \sum_{\ell=2}^{n}\frac{|X_\ell|}{|r(w)|^{1/(2m_\ell)}}\right) \end{aligned}$$

by part (b), where $F^{\mathrm{Cara}}_{\bullet}$ stands for the Carathéodory differential metric. ■

We further need to know how the quantities $\widehat{\sigma}(z)$, $\mathscr{L}(z)$, and $\mathscr{R}(z)$ vary within $\mathscr{B}(w)$.

LEMMA 4.4. *There exist constants $K, K_* > 1$ such that for any pair $(z,w) \in D \times D$ with $z \in \overline{\mathscr{B}(w)}$ we have:*

(a) $\mathscr{L}(z) \le K_*\mathscr{L}(w)$,
(b) $|r(z)| \le c^{-1}(1 + K_*\mathscr{L}(w))^{8n+1}|r(w)|$, *where c is as in the definition of $\mathscr{B}(w)$,*
(c) $\mathscr{L}(z) \le K + \mathscr{L}(w) + k_0\log(1 + K_*\mathscr{L}(w))$ *with* $k_0 := 8n + \widetilde{m}q_1 + q_2 + 1$,
(d) $\mathscr{R}(z) \le K_*(1 + \mathscr{L}(w))^{8n+\widetilde{m}q_1+1}\mathscr{R}(w)$.

Proof. (a) We write

$$\tag{4.4} \begin{aligned} \sigma(z') &\le \sigma(w') + 4^{\widetilde{m}}\Big(\sum_{\lambda=2}^{n}\sigma(w')^{1-1/(2m_\lambda)}|z_\lambda - w_\lambda| + \sigma(z'-w')\Big) \\ &\le \sigma(w') + \frac{4^{\widetilde{m}}}{\widetilde{c}^{1/2}}\mathscr{L}(w)^{\widetilde{m}q_1}\Big(\sum_{\lambda=2}^{n}\sigma(w')^{1-1/(2m_\lambda)}|r(w)|^{1/(2m_\lambda)} + |r(w)|\Big) \\ &\le C_{17}\mathscr{L}(w)^{\widetilde{m}q_1}\widehat{\sigma}(w) \end{aligned}$$

with some constant $C_{17} > 0$ (independent of z and w). In the same way,

$$\tag{4.5} \begin{aligned} |z_1 - w_1| &= \Big|h_w(z) - 2\sum_{\ell=2}^{n} P_{z_\ell}(w')(z_\ell - w_\ell)\Big| \\ &\le |h_w(z)| + M_0\sum_{\ell=2}^{n}\sigma(w')^{1-1/(2m_\ell)}|z_\ell - w_\ell| \\ &\le \frac{1}{\widetilde{c}^{1/2}}\mathscr{L}(w)^{q_2}\mathscr{R}(w) + M_0\sum_{\ell=2}^{n}\sigma(w')^{1-1/(2m_\ell)}|z_\ell - w_\ell|. \end{aligned}$$

Combining this with (4.4) we get

$$\widehat{\sigma}(z) \le C_{18}\mathscr{L}(w)^{\widetilde{m}q_1+q_2}\widehat{\sigma}(w)$$

with some universal constant $C_{18} > 1$. From $z \in S_-(w)$ we obtain

$$|r(z)|^{-1} \le \frac{1+\mathscr{L}(w)}{c|r(w)|}(1+\mathscr{L}(z))^{8n},$$

and therefore

$$\tag{4.6} \begin{aligned} \mathscr{L}(z) &= \log\Big(1 + \frac{\widehat{\sigma}(z)}{|r(z)|}\Big) \\ &\le \log\Big(1 + C_{18}\mathscr{L}(w)^{\widetilde{m}q_1+q_2}\frac{\widehat{\sigma}(w)}{c|r(w)|}(1+\mathscr{L}(z))^{8n}(1+\mathscr{L}(w))\Big) \\ &\le \log(1 + C_{18}/c) + \mathscr{L}(w) + 8n\log(1+\mathscr{L}(z)) \\ &\quad + (\widetilde{m}q_1 + q_2 + 1)\log(1+\mathscr{L}(w)). \end{aligned}$$

If $\mathscr{L}(z) \le 2\mathscr{L}(w)$, part (a) is clear. If $\mathscr{L}(w) \le \frac{1}{2}\mathscr{L}(z)$, we get

$$\mathscr{L}(z) \le 2\log(1 + C_{18}/c) + (8n + \widetilde{m}q_1 + q_2 + 1)\log(1+\mathscr{L}(z)).$$

This implies that

$$\begin{aligned} \mathscr{L}(z) \le C'_{19} := \sup\{y \mid y - (8n + \widetilde{m}q_1 + q_2 + 1)\log(1+y) \\ \le 2\log(1+C_{18}/c)\} \le C_{19}\mathscr{L}(w) \end{aligned}$$

with $C_{19} := C'_{19}/\log(1+M_0^{-1})$. So we find that $\mathscr{L}(z) \le (2 + C_{19})\mathscr{L}(w)$. This proves (a).

(b) We obtain, by definition of $\mathscr{B}(w)$,

$$|r(z)| \le \frac{1}{c}(1+\mathscr{L}(z))^{8n}|r(w)|,$$

which, in conjunction with (a), gives the claimed estimate.

(c) follows from (a) and (4.6) if we let $K := \log(1 + C_{18}/c)$.

(d) follows from the preceding parts and the definition of $\mathscr{R}(z)$. ∎

In the next steps we adopt an idea from [Her4, Sec. 6].

Let $A, Q \in D$. Then there exists a smooth curve $c : [0,1] \to D$ from A to Q with

$$L(c) \le 2d_D^B(A,Q). \tag{4.7}$$

We consider two cases:

Case (I): $Q \in \mathscr{B}(A)$. If $c([0,1]) \subset \overline{\mathscr{B}(A)}$, by the preceding lemma (with $T \gg 1$) we have

$$\begin{aligned}
2d_D^B(A,Q) \ge L(c) &= \int_0^1 B_D(c(t);\dot c(t))\,dt \\
&\ge C\int_0^1 \frac{1}{\mathscr{L}(c(t))^{q_2}}\bigg(\frac{|\langle \partial h_w(c(t)), \dot c(t)\rangle|}{\mathscr{R}(c(t))} + \sum_{\ell=2}^n \frac{|\dot c_\ell|}{|r(c(t))|^{1/(2m_\ell)}}\bigg)\,dt \\
&\ge C\frac{1}{(1+\mathscr{L}(A))^{k_0}}\bigg(\frac{1}{\mathscr{R}(A)}\int_0^1 |\langle \partial h_w(c(t)), \dot c(t)\rangle|\,dt \\
&\qquad\qquad + K_*^{-q_2}\sum_{\ell=2}^n \frac{1}{|r(A)|^{1/(2m_\ell)}}\int_0^1 |\dot c_\ell(t)|\,dt\bigg) \\
&\ge CK_*^{-q_2}\frac{1}{(1+\mathscr{L}(A))^{k_0}}\bigg(\frac{|h_A(Q)|}{\mathscr{R}(A)} + \sum_{\ell=2}^n \frac{|Q_\ell - A_\ell|}{|r(A)|^{1/(2m_\ell)}}\bigg) \\
&= CK_*^{-q_2}\frac{\delta(A,Q)}{(1+\mathscr{L}(A))^{k_0}}.
\end{aligned} \tag{4.8}$$

Assume that there exists a smallest $t_0 \in (0,1)$ with $c(t_0) \notin \mathscr{B}(A)$. Now we have $L(c) \ge d_D^B(A, c(t_0)) \ge c_{00}$, and since $Q \in \mathscr{B}(A)$, also

$$\frac{\delta(A,Q)}{(1+\mathscr{L}(A))^{k_0}} \le \frac{n}{\widetilde{c}} \le \frac{n}{c_{00}\widetilde{c}}L(c).$$

So from (4.7) we obtain

$$d_D^B(A,Q) \ge c_* \log\bigg(1 + \frac{\delta(A,Q)}{(1+\mathscr{L}(A))^{k_0}}\bigg)$$

in Case (I).

CASE (II): $Q \notin \overline{\mathscr{B}(A)}$. We introduce the nonempty set

$$\mathscr{S} := \{j \in \mathbb{N} \mid j \geq 1,\ \exists t_0 = 0 < t_1 < \cdots < t_j < 1 \text{ such that } c([t_{\nu-1}, t_\nu)) \subset \mathscr{B}(c(t_{\nu-1}) \text{ and } c(t_\nu) \in \partial\mathscr{B}(c(t_{\nu-1}), 1 \leq \nu \leq j\}.$$

By Corollary 4.1(a) we get $2d_D^B(A, Q) \geq L(c) \geq c_{00} j$ for any $j \in \mathscr{S}$. So $\mathscr{S}$ is finite and has a well-defined maximum m_*. Let now $(t_0 = 0, t_1, \ldots, t_{m_*})$ be a chain that corresponds to m_*. We abbreviate $A_\nu := c(t_\nu)$ for $0 \leq \nu \leq m_*$, and set $A_{m_*+1} := Q$ and $t_{m_*+1} := 1$. Since $c([t_{\nu-1}, t_\nu)) \subset \mathscr{B}(A_\nu)$ we obtain, in analogy to (4.8),

$$\begin{aligned}
(4.9)\qquad L(c) &\geq \sum_{\nu=0}^{m_*} \int_{t_\nu}^{t_\nu+1} B_D(c(t); \dot{c}(t))\, dt \\
&\geq C K_*^{-q_2} \sum_{\nu=0}^{m_*} \frac{\frac{|h_{A_\nu}(A_{\nu+1})|}{\mathscr{R}(A_\nu)} + \sum_{\ell=2}^{n} \frac{|A_{\nu+1,\ell} - A_{\nu,\ell}|}{|r(A_\nu)|^{1/(2m_\ell)}}}{(1 + \mathscr{L}(A_\nu))^{k_0}} \\
&\geq \frac{C}{2M_0 K_*^{q_2}} \sum_{\nu=0}^{m_*} \frac{\frac{|A_{\nu+1,1} - A_{\nu,1})|}{\mathscr{R}(A_\nu)} + \sum_{\ell=2}^{n} \frac{|A_{\nu+1,\ell} - A_{\nu,\ell}|}{|r(A_\nu)|^{1/(2m_\ell)}}}{(1 + \mathscr{L}(A_\nu))^{k_0}},
\end{aligned}$$

because

$$(4.10)\qquad \frac{1}{2M_0} \frac{|h_{A_\nu}(A_{\nu+1})|}{\mathscr{R}(A_\nu)} \geq \frac{1}{2M_0} \frac{|A_{\nu+1,1} - A_{\nu,1})|}{\mathscr{R}(A_\nu)} - \frac{1}{4} \sum_{\ell=2}^{n} \frac{|A_{\nu+1,\ell} - A_{\nu,\ell}|}{|r(A_\nu)|^{1/(2m_\ell)}}.$$

We have to estimate $\mathscr{R}(A_\nu)$ and $|r(A_\nu)|$ in terms of $\mathscr{R}(A)$ and $|r(A)|$, respectively. By Lemma 4.4 we find

$$\mathscr{R}(A_\nu) \leq K_*(1 + \mathscr{L}(A_{\nu-1}))^{k_0} \mathscr{R}(A_{\nu-1})$$

and, inductively on ν,

$$\mathscr{R}(A_\nu) \leq K_*^\nu \mathscr{R}(A) \prod_{k=1}^{\nu-1} (1 + \mathscr{L}(A_k))^{k_0}.$$

Likewise we have

$$|r(A_\nu)| \leq c^{-1}(1 + \mathscr{L}(A_{\nu-1}))^{8n+1} |r(A_{\nu-1})|$$

and, inductively on ν,

$$|r(A_\nu)| \leq c^{-\nu} |r(A)| \prod_{k=1}^{\nu-1} (1 + \mathscr{L}(A_k))^{8n+1}.$$

By Lemma 4.4(c) we get

$$\begin{aligned}
\mathscr{L}(A_k) &\leq K + \mathscr{L}(A_{k-1}) + k_0 \log(1 + K_* \mathscr{L}(A_{k-1})) \\
&\leq K + \mathscr{L}(A_{k-1}) + k_0 \log(1 + K_*^k \mathscr{L}(A)) \\
&\leq K + \mathscr{L}(A_{k-1}) + k_0\big(k \log(1 + K_*) + \log(1 + \mathscr{L}(A))\big)
\end{aligned}$$

and inductively

$$\mathscr{L}(A_k) \leq k\big(K + k_0 \log(1 + \mathscr{L}(A))\big) + k_0 \frac{k(k+1)}{2} \log(1 + K_*) + \mathscr{L}(A).$$

From this we find that, with some universal constant $C_{20} > 1$,

$$\mathscr{L}(A_k) \leq C_{20} m_*^2 \mathscr{L}(A)$$

for all $k \leq m_*$. This implies

$$\begin{aligned} |r(A_\nu)| &\leq (1/c)^\nu (1 + K_* C_{20} m_*^2)^\nu \mathscr{L}(A)^\nu |r(A)| \\ &\leq \left(\frac{1 + K_* C_{20}}{c}\right)^{m_*} m_*^{2m_*} \mathscr{L}(A)^{m_*} |r(A)|, \end{aligned}$$

and in an analogous way

$$\mathscr{R}(A_\nu) \leq (K_*(1 + K_* C_{20}))^{m_*} m_*^{2m_*} (1 + \mathscr{L}(A))^{m_*} \mathscr{R}(A)$$

for any $\nu \leq m_*$. We plug this into (4.9) and obtain, with universal constants $C_{21}, C_{22} > 1$,

$$\begin{aligned} L(c) &\geq \frac{\sum_{\nu=0}^{m_*} \frac{|A_{\nu+1,1} - A_{\nu,1}|}{\mathscr{R}(A)} + \sum_{\ell=2}^{n} \frac{|A_{\nu+1,\ell} - A_{\nu,\ell}|}{|r(A)|^{1/(2m_\ell)}}}{C_{21}^{m_*} m_*^{2m_* + 2k_0} (1 + \mathscr{L}(A))^{2m_* + 2k_0}} \\ &\geq \frac{1}{C_{21}^{m_*} m_*^{2m_* + 2k_0} (1 + \mathscr{L}(A))^{2m_* + 2k_0}} \left(\frac{|Q_1 - A_1|}{\mathscr{R}(A)} + \sum_{\ell=2}^{n} \frac{|Q_\ell - A_\ell|}{|r(A)|^{1/(2m_\ell)}} \right) \\ &\geq \frac{\delta(A, Q)}{C_{22}^{m_*} m_*^{2m_* + 2k_0} (1 + \mathscr{L}(A))^{2m_* + 2k_0}} \end{aligned}$$

in analogy with (4.10). Let $S > 1$ to be chosen later. We next consider two cases:

Case (I): $m_* \leq S$. Then, in conjunction with $d_D^B(A, Q) \geq c_{00} m_*$, we have

$$d_D^B(A, Q) \geq C_{23} f(m_*)$$

with

$$f(x) := x + \frac{\delta(A, Q)}{(C_{22} S^2 (1 + \mathscr{L}(A)))^{k_0 x}}.$$

Since $Q \notin \mathscr{B}(A)$ we have $\delta(A, Q) > 1$. If now $C_{22} S^2 (1 + \mathscr{L}(A)) > e$, we get $f(x) \geq f(x_0)$, where x_0 is the only zero of f', explicitly

$$x_0 = \frac{\log \delta(A, Q) + \log\log(C_{22} S^2 (1 + \mathscr{L}(A)))}{k_0 \log(C_{22} S^2 (1 + \mathscr{L}(A)))}.$$

This gives

$$d_D^B(A, Q) \geq C_{23} \frac{\log \delta(A, Q)}{k_0 \log(C_{22} S^2 (1 + \mathscr{L}(A)))}.$$

Case (II): $m_* \geq S$. Then $d_D^B(A,Q) \geq C_{23} m_* \geq C_{23} S$.

We now choose

$$S := \log\left(e^2 + \frac{\delta(A,Q)}{1+\mathscr{L}(A)}\right).$$

Then, since $Q \notin \mathscr{B}(A)$, we can see that $C_{22} S^2 (1+\mathscr{L}(A)) > e$.

This concludes the proof of the main theorem.

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