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Applications of differential algebra to algebraic independence of arithmetic functions

by

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1. Introduction. The Schanuel Conjecture asserts that for any $\mathbb{Q}$-linearly independent complex numbers $a_1, \ldots, a_n$ there are at least n numbers among

$$a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_n)$$

that are algebraically independent over the rational numbers. It is well-known that a number of remarkable results about transcendental numbers: the Lindemann–Weierstrass Theorem, the Gelfond–Schneider Theorem and Baker's Theorem (to name but a few) are consequences of this statement. For the state of the art on this topic, we refer the reader to Waldschmidt's paper [21].

In this article, we argue that Schanuel's insight remains valid for arithmetic functions. We improve several existing results on algebraic independence of arithmetic functions by applying an analog of the Schanuel Conjecture for differential rings. More precisely, we deduce them from the following theorem of James Ax [1, Theorem 3].

Theorem 1.1. *Let $F/C/\mathbb{Q}$ be a tower of fields. Suppose Δ is a set of derivations of F with $\bigcap_{D\in\Delta} \ker D = C$. Let $y_1, \ldots, y_n, z_1, \ldots, z_n \in F^\times$ be such that*

(a) *for all $D \in \Delta$ and $i = 1, \ldots, n$, $Dy_i = Dz_i/z_i$ and either*
(b) *no nontrivial power product of the z_j is in C, or*
(b′) *the y_i are $\mathbb{Q}$-linearly independent modulo C.*

Then

$$\operatorname{td}_C C(y_1, \ldots, y_n, z_1, \ldots, z_n) \geq n + \operatorname{rank}(Dy_i)_{D\in\Delta,\, 1\leq i\leq n}.$$

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A word about terminology. Let G be an abelian group (written multiplicatively). We say that $g_1, \dots, g_n \in G$ are (or the family $g_1, \dots, g_n$ is) *multiplicatively independent* if the equation $g_1^{k_1} \cdots g_n^{k_n} = 1$ implies that the integers $k_1, \dots, k_n$ are all zeros. The implication is vacuously true for the empty family, which is therefore multiplicatively independent. A subset X of G is multiplicatively independent if every finite family in X is. For H a subgroup of G, we say that X is *multiplicatively independent modulo H* if the image of X in the quotient group G/H is multiplicatively independent. We will use this terminology throughout. First, let us restate condition (b) in Theorem 1.1 as “the z_i are multiplicatively independent modulo $C^{\times}$”. We prefer doing so because that draws a closer analogy between (b) and (b′).

2. Arithmetic functions. In this section we introduce the notation and summarize the facts about arithmetic functions that we will use subsequently. The reader can consult [2, Chapter 2] and [18, Chapter 4] for more information. We use $\mathbb{P}$ to denote the set of primes, and p will always stand for a prime in this article.

Arithmetic functions are complex-valued functions with domain the set of natural numbers. It is beneficial at times to think of them as functions on $\mathbb{R}$ vanishing at points that are not natural numbers. The arithmetic functions form a commutative ring $\mathcal{A}$ under pointwise addition of functions $+$ and convolution product $*$ defined as

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right).$$

Identifying $\alpha \in \mathbb{C}$ with the function $1 \mapsto \alpha, n \mapsto 0$ $(n > 1)$ turns $\mathcal{A}$ into a $\mathbb{C}$-algebra. Under this identification 0 and 1 become the neutral elements for $+$ and $*$, respectively. For $A \subseteq \mathbb{N}$, we use $\mathbf{1}_A$ to denote the *indicator function* of A, i.e. $\mathbf{1}_A(k) = 1$ if $k \in A$, and $\mathbf{1}_A(k) = 0$ otherwise. We write $\mathbf{1}$ for $\mathbf{1}_{\mathbb{N}}$, $\mathbf{1}_p$ for $\mathbf{1}_{\{p^k :\, k \geq 0\}}$ and e_n for $\mathbf{1}_{\{n\}}$ $(n \in \mathbb{N})$. Since most of the time we will consider the convolution product, we often simply write fg for $f * g$ and f^k $(k \in \mathbb{N})$ for the kth power of f with respect to the convolution product. For a nonzero arithmetic function f, f^0 is understood to be 1. Unless otherwise stated, by $\mathcal{A}$ we mean the $\mathbb{C}$-algebra $(\mathcal{A}, +, *)$. However, we do also consider the structure $(\mathcal{A}, +, \cdot)$ where $\cdot$ is the pointwise multiplication of functions. This structure is also a $\mathbb{C}$-algebra but this time $\alpha \in \mathbb{C}$ is identified with the constant function $n \mapsto \alpha$ $(n \geq 1)$.

For $k \in \mathbb{N}$, let ε_k be the kth *coordinate map*, i.e. $\varepsilon_k(f) = f(k)$ $(f \in \mathcal{A})$. Among the coordinate maps only $\varepsilon := \varepsilon_1$ is a $\mathbb{C}$-algebra homomorphism from $\mathcal{A}$ to $\mathbb{C}$. For $X \subseteq \mathbb{C}$, let

$$\mathcal{A}_X = \varepsilon^{-1}(X) = \{f \in \mathcal{A} :\, f(1) \in X\}.$$

We write $\mathcal{A}_\alpha$ for $\mathcal{A}_{\{\alpha\}}$. One sees that $\mathcal{A}_0$ is the unique maximal ideal of $\mathcal{A}$ by checking that its complement is the group of units of $\mathcal{A}$.

The *support* of an arithmetic function f, denoted by $\operatorname{supp} f$, is the set of natural numbers n such that $f(n) \neq 0$. The *order* of f, denoted by $v(f)$, is the least element of its support if $f \neq 0$, and is ∞ if $f = 0$. A *prime divisor* of a set of natural numbers A is a prime that divides some member of A. Following the notation in [19], we use $[A]$ to denote the set of prime divisors of A. We say that A is (*multiplicatively*) *finitely generated* if $[A]$ is finite. We use $\mathcal{T}$ and $\mathcal{S}$ to denote the subalgebras of $\mathcal{A}$ consisting of arithmetic functions with finite support and finitely generated support, respectively. Note that $\mathcal{T}$ is the $\mathbb{C}$-subalgebra of $\mathcal{S}$ generated by the e_n ($n \in \mathbb{N}$).

LEMMA 2.1. *Let $f_1, \ldots, f_n \in \mathcal{A}$ and $a_1, \ldots, a_n$ be real numbers such that $0 < a_i \le v(f_i)$ for each $1 \le i \le n$. Then*

$$(f_1 * \cdots * f_n)\Big(\prod_{i=1}^n a_i\Big) = \prod_{i=1}^n f_i(a_i). \tag{2.1}$$

Proof. First, if some $f_i = 0$, then both sides of (2.1) are 0. So let us assume the order of each f_i is finite. For $a \in \mathbb{R}$, we have

$$(f_1 * \cdots * f_n)(a) = \sum_{\substack{d_1 \cdots d_n = a \\ d_i \in \mathbb{N}}} f_1(d_1) \cdots f_n(d_n). \tag{2.2}$$

The summand $f_1(d_1) \cdots f_n(d_n)$ appearing in (2.2) can be nonzero only if $d_i \ge v(f_i)$ $(\ge a_i)$ for each i. So by taking $a = a_1 \cdots a_n$, we see that $f_1(d_1) \cdots f_n(d_n) \neq 0$ if and only if $d_i = v(f_i) = a_i$ for each i. Thus either $a_i < v(f_i)$ for some i, in which case both sides of (2.1) are zero, or else $a_i = v(f_i)$ for each i, in which case both sides of (2.1) equal $f_1(v(f_1)) \cdots f_n(v(f_n))$. ■

PROPOSITION 2.2. *Let f_{ij} be arithmetic functions $(1 \le i, j \le n)$. Suppose a_i, b_i $(1 \le i \le n)$ are positive real numbers such that $a_i b_j \le v(f_{ij})$ for $1 \le i, j \le n$. Then*

$$\det(f_{ij})\Big(\prod_{k=1}^n a_k b_k\Big) = \det(f_{ij}(a_i b_j)). \tag{2.3}$$

Proof. For each permutation ξ of $\{1, \ldots, n\}$, by Lemma 2.1 we have

$$\begin{aligned}\Big(\operatorname{sgn}(\xi) \prod_{k=1}^n f_{k\xi(k)}\Big)\Big(\prod_{k=1}^n a_k b_k\Big) &= \Big(\operatorname{sgn}(\xi) \prod_{k=1}^n f_{k\xi(k)}\Big)\Big(\prod_{k=1}^n a_k b_{\xi(k)}\Big) \\ &= \operatorname{sgn}(\xi) \prod_{k=1}^n f_{k\xi(k)}(a_k b_{\xi(k)}).\end{aligned}$$

Equation (2.3) now follows by summing through the permutations. ■

Let $\|f\|$ denote the reciprocal of $v(f)$ with the convention $1/\infty = 0$. The assignment $f \mapsto \|f\|$ is a nonarchimedean norm on $\mathcal{A}$. In particular, $\|f*g\| = \|f\|\,\|g\|$ and consequently $\mathcal{A}$ is an integral domain. The ring operations of $\mathcal{A}$ are continuous with respect to the (ultra-)metric induced by this norm. A sequence (f_n) of arithmetic functions converges to an arithmetic function f, written $f_n \to f$, if and only if the sequence $(\|f_n - f\|)_n$ of rational numbers converges to 0. Note also that a map from $\mathcal{A}$ to itself is continuous if and only if it preserves convergence of sequences. Since the norm under consideration is nonarchimedean, the series $\sum_k^\infty f_k$ converges if and only if $f_k \to 0$. In particular, for any formal power series $\sum \alpha_k X^k$ over $\mathbb{C}$ and $g \in \mathcal{A}$, the series $\sum \alpha_k g^k$ converges if and only if $\|g\| < 1$ or equivalently $g \in \mathcal{A}_0$.

The map defined by

$$f \mapsto \operatorname{Exp}(f) = \sum_{k=0}^{\infty} \frac{f^k}{k!}$$

is a continuous isomorphism of groups from $(\mathcal{A}_0, +)$ to $(\mathcal{A}_1, *)$ [2, Theorem 2.20]. We extend it to the *exponential map* on $\mathcal{A}$ by

$$f \mapsto \exp(f(1)) * \operatorname{Exp}(f - f(1)),$$

where exp denotes the exponential map of $\mathbb{C}$. This extension is still a continuous group homomorphism from $(\mathcal{A}, +)$ to $(\mathcal{A}^\times, *)$, but no longer injective, since it extends the complex exponentiation. However, its restriction to $\mathcal{A}_\mathbb{R}$, as shown by Rearick [14], is indeed a continuous group isomorphism from $(\mathcal{A}_\mathbb{R}, +)$ to $(\mathcal{A}_+, *)$ where $\mathcal{A}_+$ is the inverse image of the set of positive reals under ε. The inverse of this group isomorphism, known as the *Rearick logarithm*, is also continuous and we denote it by Log. For convenience, we understand $\operatorname{Exp}^0 = \operatorname{Log}^0$ as the identity map of $\mathcal{A}$; and for $k \geq 1$, $\operatorname{Exp}^{-k} = \operatorname{Log}^k$. For any $f \in \mathcal{A}$, there is a largest $k \geq 0$ such that $\operatorname{Log}^k f$ is defined: $k = 0$ if $f \notin \mathcal{A}_+$, otherwise $k \geq 1$ is the integer such that $\log^k(f(1)) \leq 0$ (here log is the real logarithm). For a nonempty $W \subseteq \mathcal{A}$, let k_W be the largest nonnegative integer such that $\operatorname{Log}^{k_W} f$ is defined for each $f \in W$. We write $\operatorname{Exp}^* W$ for the set

$$\{\operatorname{Exp}^m f \colon f \in W,\ m \geq -k_W\}.$$

The ring of arithmetic functions is isomorphic, as a $\mathbb{C}$-algebra, to the ring of formal Dirichlet series [18, §4.6] via

$$f \leftrightarrow F(s) = \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}. \tag{2.4}$$

Under this isomorphism, $\mathbf{1}$ is identified with $\sum 1/n^s$ the Dirichlet series of the Riemann zeta function $\zeta(s)$. In general, for $A \subseteq \mathbb{N}$, $\mathbf{1}_A$ is identified with the Dirichlet series $\sum_{n\in\mathbb{N}} \mathbf{1}_A(n)/n^s$ which converges on a proper right halfplane and extends to a meromorphic function on $\mathbb{C}$. We call this function the *zeta function of* A and denote it by $\zeta_A(s)$.

The ring of arithmetic functions is also isomorphic, as a $\mathbb{C}$-algebra, to the formal power series ring over $\mathbb{C}$ in countably many variables t_p ($p \in \mathbb{P}$) via

$$f \leftrightarrow F(\mathbf{t}) = \sum_{n\in\mathbb{N}} f(n) \prod_p t_p^{v_p(n)}, \tag{2.5}$$

where $v_p(m)$ is the exponent of p in the prime factorization of m. Under this isomorphism e_p is mapped to the variable t_p. The isomorphism in (2.5) was utilized by Cashwell and Everett [5] to show that $\mathcal{A}$ is a unique factorization domain.

By a *derivation* of $\mathcal{A}$ we mean a $\mathbb{C}$-linear map from $\mathcal{A}$ to itself satisfying the Leibniz rule: $D(f * g) = Df * g + f * Dg$. For simplicity, we do not distinguish in notation a derivation of $\mathcal{A}$ and its unique extension to $\mathcal{F}$, the field of fractions of $\mathcal{A}$. Let Δ be a set of derivations of $\mathcal{A}$. By the *kernel* of Δ, denoted by $\ker \Delta$, we mean the intersection of the kernels of its members. By $\ker_{\mathcal{F}} \Delta$ we mean the same but when regarding the members of Δ as derivations of $\mathcal{F}$. In particular, $\ker \emptyset$ and $\ker_{\mathcal{F}} \emptyset$ are $\mathcal{A}$ and $\mathcal{F}$, respectively. It is routine to check that $\ker_{\mathcal{F}} \Delta$ is a subfield of $\mathcal{F}$ extending $\mathbb{C}$ whose intersection with $\mathcal{A}$ is $\ker \Delta$.

The *log-derivation* of $\mathcal{A}$, denoted by ∂_L, is the map sending $f \in \mathcal{A}$ to the arithmetic function defined by

$$(\partial_L f)(n) = \log(n) f(n).$$

Under the isomorphism in (2.4), ∂_L corresponds to the derivation $-d/ds$. For each prime p, the *p-basic derivation* of $\mathcal{A}$, denoted by ∂_p, is the map sending $f \in \mathcal{A}$ to the arithmetic function defined by

$$(\partial_p f)(n) = f(np) v_p(np).$$

Under the isomorphism in (2.5), ∂_p corresponds to $\partial/\partial t_p$, the partial derivation with respect to t_p. A derivation of $\mathcal{A}$ is *basic* if it is ∂_p for some p. The kernel of ∂_L is $\mathbb{C}$, and the kernel of ∂_p consists of arithmetic functions that vanish on the multiples of p. In other words,

$$f \in \ker \partial_p \;\Leftrightarrow\; p \notin [\operatorname{supp} f]. \tag{2.6}$$

Thus the kernel of the set of basic derivations is also $\mathbb{C}$. Basic derivations and the log-derivation are continuous. For a nice characterization of continuous derivations of $\mathcal{A}$, we refer the reader to [17, Section 4]. We consider continuous derivations because the derivative of a power series with respect to a continuous derivation can be computed term-by-term:

LEMMA 2.3. *Suppose D is a continuous derivation of $\mathcal{A}$ and $g \in \mathcal{A}_0$. Then for any formal power series $\sum_{k=0}^{\infty} \alpha_k X^k$ over $\mathbb{C}$,*

$$D\Big(\sum_{k=0}^{\infty} \alpha_k g^k\Big) = \Big(\sum_{k=1}^{\infty} k a_k g^{k-1}\Big) * Dg.$$

Proof. Since D is $\mathbb{C}$-linear and satisfies the Leibniz rule, for each $n \in \mathbb{N}$ we have

$$D\Big(\sum_{k=0}^{n} \alpha_k g^k\Big) = \Big(\sum_{k=1}^{n} k \alpha_k g^{k-1}\Big) * Dg. \tag{2.7}$$

The left side of (2.7) converges to $D(\sum_{k=0}^{\infty} \alpha_k g^k)$ by continuity of D. Since $g \in \mathcal{A}_0$ and the convolution product is continuous, the right side of (2.7) converges to $(\sum_{k=1}^{\infty} k\alpha_k g^{k-1}) * Dg$. The lemma now follows from the uniqueness of limits. ■

PROPOSITION 2.4. *For any continuous derivation D of $\mathcal{A}$ and $f \in \mathcal{A}$,*

$$D(\mathrm{Exp}(f)) = \mathrm{Exp}(f) * Df.$$

Proof. By applying Lemma 2.3 to the series $\sum_{k=0}^{\infty} X^k/k!$ we conclude that $D\,\mathrm{Exp}(f) = \mathrm{Exp}(f) * Df$ for any $f \in \mathcal{A}_0$. In general, since $\ker D \supseteq \mathbb{C}$, it follows that for $f \in \mathcal{A}$,

$$\begin{aligned} D\,\mathrm{Exp}(f) &= D\big(\exp(f(1)) * \mathrm{Exp}(f - f(1))\big) = \exp(f(1)) * D(\mathrm{Exp}(f - f(1))) \\ &= \exp(f(1)) * \mathrm{Exp}(f - f(1)) * D(f - f(1)) = \mathrm{Exp}(f) * Df. \ ■ \end{aligned}$$

COROLLARY 2.5. *Suppose Δ is a set of continuous derivations of $\mathcal{A}$. Then $f \in \ker \Delta$ if and only if $\mathrm{Exp}(f) \in \ker \Delta$. Moreover, if $f \in \mathcal{A}_+$ then $f \in \ker \Delta$ if and only if $\mathrm{Log}\, f \in \ker \Delta$.*

Proof. By Proposition 2.4, $D(\mathrm{Exp}(f)) = Df * \mathrm{Exp}(f)$ for any $D \in \Delta$. Since $\mathrm{Exp}(f) \neq 0$, the first assertion follows. The second assertion follows from the first because for $f \in \mathcal{A}_+$, we have $f = \mathrm{Exp}(\mathrm{Log}(f))$. ■

PROPOSITION 2.6. *Suppose $f_1, \dots, f_n \in \mathcal{A}$ and $D_1, \dots, D_n$ are continuous derivations of $\mathcal{A}$. Then for any $k \in \mathbb{Z}$ such that $\mathrm{Exp}^k f_i$ is defined for all $1 \le i \le n$,*

$$\det(D_j f_i) = 0 \ \Leftrightarrow\ \det(D_j \,\mathrm{Exp}^k f_i) = 0.$$

Proof. It suffices to show that for any $g_1, \dots, g_n \in \mathcal{A}$, $\det(D_j g_i) = 0$ if and only if $\det(D_j \,\mathrm{Exp}\, g_i) = 0$. But this follows immediately from Proposition 2.4, since

$$\det(D_j \,\mathrm{Exp}\, g_i) = \det(\mathrm{Exp}(g_i) * D_j g_i) = \det(D_j g_i) \prod_{i=1}^{n} \mathrm{Exp}(g_i)$$

and $\mathrm{Exp}\, g \neq 0$ for any $g \in \mathcal{A}$. ■

As another application of Proposition 2.4, let us compute the function $\kappa := \operatorname{Log} \mathbf{1}$. On the one hand, $\partial_L \mathbf{1} = \partial_L \kappa * \mathbf{1}$. On the other hand,

$$\partial_L \mathbf{1}(n) = \log(n) = \sum_{p|n} v_p(n) \log p = \sum_{p^j|n} \log p = (\Lambda * \mathbf{1})(n).$$

So $\partial_L \kappa = \Lambda$ is the von Mangoldt function. Thus,

$$\kappa(n) = \begin{cases} \dfrac{\Lambda(p^j)}{\log(p^j)} = \dfrac{1}{j} & \text{if } n = p^j \text{ for some prime } p \text{ and } j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \mathcal{A}$, let $\mathfrak{m}_g$ denote the $\mathbb{C}$-linear map from $\mathcal{A}$ to itself defined by $\mathfrak{m}_g(f) = g \cdot f$ (pointwise product). It is clear that $\|\mathfrak{m}_g(f)\| \leq \|f\|$. Thus, $\mathfrak{m}_g$ preserves null sequences and hence is continuous by linearity. It is also clear that $\mathfrak{m}_h$ is the compositional inverse of $\mathfrak{m}_g$ if and only if h is the pointwise multiplicative inverse of g. If g is *completely additive*, i.e. $g(nm) = g(n) + g(m)$ for all $n, m \in \mathbb{N}$, one checks that $\mathfrak{m}_g$ is a (continuous) derivation of $\mathcal{A}$, and vice versa. For example, $\mathfrak{m}_{\log}$ is simply the log-derivation ∂_L. We will use the more suggestive notation ∂_g for $\mathfrak{m}_g$ whenever it is a derivation. A completely additive function is determined by its action on the primes, and its value at 1 must be 0. Besides the real logarithm, the p-adic valuation v_p, and the function Ω, which counts (with multiplicity) the total number of prime factors of its argument, are examples of completely additive functions. If g is *completely multiplicative*, i.e. $g \neq 0$ and $g(nm) = g(n)g(m)$ for all $n, m \in \mathbb{N}$, one checks that $\mathfrak{m}_g$ is a nonzero (continuous) $\mathbb{C}$-algebra endomorphism of $\mathcal{A}$, and vice versa. If, in addition, g vanishes nowhere then its pointwise multiplicative inverse is also completely multiplicative. Thus $\mathfrak{m}_g$ is a continuous automorphism of $\mathcal{A}$. For example, $\mathfrak{m}_{\mathbf{I}}$, where $\mathbf{I}$ is the identity map of $\mathbb{N}$, is a continuous automorphism of $\mathcal{A}$. A completely multiplicative function is determined by its action on the primes, and its value at 1 must be 1. Besides the identity function, the map $n \mapsto n^\alpha$ ($\alpha \in \mathbb{C}$) and $\mathbf{1}_p$ are examples of completely multiplicative functions.

We conclude this section by an observation that will be used a number of times in Section 5.

LEMMA 2.7. *For any $f, g \in \mathcal{A}$, $p \in \mathbb{P}$ and $i \in \mathbb{Z}$ such that $\mathfrak{m}_g^i$ is defined, $v(\partial_p f) \leq v(\partial_p \mathfrak{m}_g^i(f))$. Moreover, equality holds if $g(m) \neq 0$ for all $m > 1$.*

Proof. For any $n \geq 1$,

$$\partial_p \mathfrak{m}_g^i(f)(n) = v_p(np)(g(np))^i f(np) = (g(np))^i \partial_p f(n). \tag{2.8}$$

Thus $\partial_p f(n) = 0$ implies $\partial_p \mathfrak{m}_g^i(f)(n) = 0$, and so the inequality in the lemma holds. Furthermore, if $g(np) \neq 0$ for all n, the reverse implication is also true. Thus, $\partial_p f$ and $\partial_p \mathfrak{m}_g^i f$ must have the same order. ∎

3. Ax's Theorem for $\mathcal{A}$. Our main observation is simple: Ax's Theorem holds for $(\mathcal{A}, +, *)$.

THEOREM 3.1. *Suppose* $\mathcal{C} = \ker_{\mathcal{F}} \Delta$ *for some set* Δ *of continuous derivations of* $\mathcal{A}$. *Let* $f_1, \dots, f_n$ *be arithmetic functions such that either*

(1) $\mathrm{Exp}(f_1), \dots, \mathrm{Exp}(f_n)$ *are multiplicatively independent modulo* $\mathcal{C}^\times$, *or*
(2) $f_1, \dots, f_n$ *are* $\mathbb{Q}$*-linearly independent modulo* $\mathcal{C}$.

Then

$$\mathrm{td}_{\mathcal{C}}\, \mathcal{C}(f_1, \dots, f_n, \mathrm{Exp}(f_1), \dots, \mathrm{Exp}(f_n)) \geq n + \mathrm{rank}\, (Df_i)_{D \in \Delta,\, 1 \leq i \leq n}.$$

Proof. Take the field F in Theorem 1.1 to be $\mathcal{F}$ and $C = \mathcal{C} = \ker_{\mathcal{F}} \Delta$. Let $y_i = f_i$ and $z_i = \mathrm{Exp}\, f_i$ $(i = 1, \dots, n)$. Then by Proposition 2.4, $Dy_i = Dz_i/z_i$ for all $D \in \Delta$ and $1 \leq i \leq n$. Therefore, condition (a) in Theorem 1.1 holds. Conditions (1) and (2) now translate into (b) and (b′) of Theorem 1.1, and so the inequality for the transcendence degree follows. ∎

As our first illustration of the power of Ax's Theorem, we use it to deduce the following generalization of [19, Theorem 5.3]. For $f \in \mathcal{A}_+$ and $g \in \mathcal{A}$, we write f^g as a shorthand for $\mathrm{Exp}(g * \mathrm{Log}\, f)$.

THEOREM 3.2. *Let* Δ *be a set of continuous derivations of* $\mathcal{A}$ *and* $\mathcal{C} = \ker_{\mathcal{F}} \Delta$. *Suppose* $f \in \mathcal{A}_+ \setminus \ker \Delta$ *and* $1 = c_0, c_1, \dots, c_n \in \ker \Delta$ *are linearly independent over* $\mathbb{Q}$. *Then* $\mathrm{Log}\, f, f = f^{c_0}, f^{c_1}, \dots, f^{c_n}$ *are algebraically independent over* $\mathcal{C}$.

Proof. By Corollary 2.5, $\mathrm{Log}\, f \notin \ker \Delta$. Thus $D_0 \mathrm{Log}\, f \neq 0$ for some $D_0 \in \Delta$. We claim that $f = f^{c_0}$, $f^{c_1}, \dots, f^{c_n}$ are multiplicatively independent modulo $\mathcal{C}^\times$. Suppose not; then there exist integers $k_0, \dots, k_n$ not all zero such that

$$f^{k_0} f^{k_1 c_1} \cdots f^{k_n c_n} = \mathrm{Exp}((k_0 + k_1 c_1 + \cdots + k_n c_n)\, \mathrm{Log}\, f)$$

belongs to $\mathcal{C} \cap \mathcal{A} = \ker \Delta$. An application of Corollary 2.5 yields $(k_0 + k_1 c_1 + \cdots + k_n c_n)\, \mathrm{Log}\, f \in \ker \Delta$. In particular,

$$\begin{aligned} 0 &= D_0((k_0 + k_1 c_1 + \cdots + k_n c_n)\, \mathrm{Log}(f)) \\ &= (k_0 + k_1 c_1 + \cdots + k_n c_n) D_0(\mathrm{Log}\, f). \end{aligned}$$

Since $D_0(\mathrm{Log}\, f) \neq 0$, that means $k_0 + k_1 c_1 + \cdots + k_n c_n$ must be zero, contradicting the assumption that $1, c_1, \dots, c_n$ are $\mathbb{Q}$-linearly independent. This establishes the claim. Now by applying Theorem 3.1 to the $n + 1$ functions $c_i \mathrm{Log}\, f$ $(0 \leq i \leq n)$, we conclude that the transcendence degree of the field

$$\mathcal{C}(c_i \mathrm{Log}\, f, f^{c_i})_{0 \leq i \leq n} = \mathcal{C}(\mathrm{Log}\, f, f, f^{c_1}, \dots, f^{c_n})$$

over $\mathcal{C}$ is at least

$$(n + 1) + \mathrm{rank}_{\mathcal{F}}(D\, \mathrm{Log}\, f, c_i D\, \mathrm{Log}\, f)_{D \in \Delta,\, 1 \leq i \leq n}.$$

Since $D_0 \operatorname{Log} f \neq 0$, the rank above is 1. This establishes the algebraic independence of $\operatorname{Log} f, f, f^{c_i}$ $(1 \leq i \leq n)$ over $\mathcal{C}$. ∎

COROLLARY 3.3. *With the notation of Theorem* 3.2, $\operatorname{Log} f$ *is transcendental over* $\mathcal{C}(f, f^{c_1}, \ldots, f^{c_n})$ *for any* $c_1, \ldots, c_n \in \ker \Delta$.

Proof. By re-indexing, if necessary, $1 = c_0, c_1, \ldots, c_m$ (for some $0 \leq m \leq n$) form a basis of the $\mathbb{Q}$-span of $1, c_1, \ldots, c_n$. By Theorem 3.2, $\operatorname{Log} f$ is transcendental over $\mathcal{C}(f, f^{c_1}, \ldots, f^{c_m})$. Since each c_i $(0 \leq i \leq n)$ is a $\mathbb{Q}$-linear combination of $1, c_1, \ldots, c_m$, each f^{c_i} is algebraic over $\mathcal{C}(f, f^{c_1}, \ldots, f^{c_m})$, and so the corollary follows. ∎

The following corollary is a very special case of Corollary 3.3. We refer the reader to [19, Section 5] for its consequences.

COROLLARY 3.4. *For any complex numbers* $c_1, \ldots, c_n$, $\log \zeta$ *is transcendental over* $\mathbb{C}(\zeta^{c_1}, \ldots, \zeta^{c_n})$. *In particular,* $\log \zeta$ *is transcendental over* $\mathbb{C}(\zeta)$.

Proof. By invoking the isomorphism in (2.4), it suffices to show that the function $\kappa = \operatorname{Log} \mathbf{1}$ is transcendental over $\mathbb{C}(\mathbf{1}, \mathbf{1}^{c_1}, \ldots, \mathbf{1}^{c_n})$; but that follows immediately from Corollary 3.3 by taking $\Delta = \{\partial_L\}$ and $f = \mathbf{1}$. ∎

A key result about algebraic independence of arithmetic functions is the following criterion of Shapiro and Sparer [19, Theorem 3.1]. We refer the reader to [9, 19, 16] for numerous applications of it.

JACOBIAN CRITERION. *Let* $f_1, \ldots, f_n$ *be arithmetic functions. Suppose* $D_1, \ldots, D_n$ *are derivations of* $\mathcal{A}$ *such that* $\det(D_j f_i) \neq 0$. *Then* $f_1, \ldots, f_n$ *are algebraically independent over* $\ker\{D_1, \ldots, D_n\}$.

As our second illustration of the power of Ax's Theorem, we use it to strengthen the Jacobian criterion when the derivations involved are continuous.

THEOREM 3.5. *Let* $f_1, \ldots, f_n \in \mathcal{A}$. *Suppose* $D_1, \ldots, D_n$ *are continuous derivations of* $\mathcal{A}$ *such that* $\det(D_j f_i) \neq 0$. *Then the set of arithmetic functions*

$$\operatorname{Exp}^*\{f_i \colon 1 \leq i \leq n\}$$

is algebraically independent over $\ker\{D_1, \ldots, D_n\}$.

Proof. Let $\mathcal{C} = \ker_{\mathcal{F}}\{D_1, \ldots, D_n\}$ and let $k_0 \geq 0$ be the largest integer such that for each $1 \leq i \leq n$, $g_i := \operatorname{Log}^{k_0} f_i$ is defined. It then suffices to show that for any $m \geq 1$, the set of arithmetic functions

$$\{\operatorname{Exp}^k g_i \colon 0 \leq k \leq m,\ 1 \leq i \leq n\}$$

is algebraically independent over $\mathcal{C} \supseteq \ker\{D_1, \ldots, D_n\}$. We will prove this by induction on m. First, let us argue that $g_1, \ldots, g_n$ are $\mathbb{Q}$-linearly independent modulo $\mathcal{C}$. Suppose some $\mathbb{Q}$-linear combination $\sum r_i g_i$ of the g_i's belongs to $\mathcal{C}$. Then by applying D_j $(1 \le j \le n)$ to the linear combination we obtain a system of n linear equations,

$$\sum_{i=1}^{n} r_i D_j g_i = 0 \quad (1 \le j \le n).$$

Since $\det(D_j f_i) \neq 0$, by Proposition 2.6 we have $\det(D_j g_i) \neq 0$ as well, and so the r_i $(1 \le i \le n)$ must all be zero. This establishes the claim. Now we can apply Theorem 3.1 to $g_1, \dots, g_n$ and conclude that

$$\operatorname{td}_{\mathcal{C}} \mathcal{C}\big(g_1, \dots, g_n, \operatorname{Exp}(g_1), \dots, \operatorname{Exp}(g_n)\big) \ge n + \operatorname{rank}(D_j g_i).$$

Again, since $\det(D_j g_i) \neq 0$, the $\mathcal{F}$-rank of $(D_j g_i)$ is n. This establishes the algebraic independence of $g_i, \operatorname{Exp}(g_i)$ $(1 \le i \le n)$ over $\mathcal{C}$, i.e. the case $m = 1$ of the theorem.

For the induction step, suppose the functions $\operatorname{Exp}^k(g_i)$ $(0 \le k \le m$, $1 \le i \le n)$ are algebraically independent over $\mathcal{C}$ for some $m \ge 1$. In particular, they are $\mathbb{Q}$-linearly independent modulo $\mathcal{C}$, and we conclude from Theorem 3.1 that the transcendence degree of the field

$$\mathcal{E} := \mathcal{C}(\operatorname{Exp}^k(g_i) \colon 0 \le k \le m+1,\ 1 \le i \le n)$$

over $\mathcal{C}$ is at least $n(m+1) + \operatorname{rank} V$ where V is the set of vectors

$$\{(D_j \operatorname{Exp}^k(g_i))_{1 \le j \le n} \colon 0 \le k \le m,\ 1 \le i \le n\}.$$

Again, because $\det(D_j g_i) \neq 0$, the $\mathcal{F}$-rank of V is at least (in fact exactly) n. Consequently, the transcendence degree of $\mathcal{E}$ over $\mathcal{C}$ is $(m+2)n$. This establishes the induction step and hence the theorem. ■

Theorem 3.5, strictly speaking, is not a generalization of the Jacobian criterion because it requires the derivations involved to be continuous. However, to the best of our knowledge, all existing applications of this criterion involve only the log-derivation and the basic derivations, so to all of them Theorem 3.5 is applicable. In the next two sections, we will generalize a number of results of [9, 19, 16] in various directions.

4. Algebraic independence. We begin with a very special case of Theorem 3.5 where only a single derivation is involved.

PROPOSITION 4.1. *Let D be a continuous derivation of $\mathcal{A}$ and $f \notin \ker D$. Then* $\operatorname{Exp}^*\{f\}$ *is algebraically independent over* $\ker D$. *In particular,* $\ker D$ *is algebraically closed in $\mathcal{A}$.*

This generalizes [9, Proposition 2.1]. For example, by taking $D = \partial_L$, one sees that $\mathbb{C}$ is algebraically closed in $\mathcal{A}$ and $\operatorname{Log}(f), f, \operatorname{Exp}(f)$ are algebraically independent over $\mathbb{C}$ for $f \in \mathcal{A}_+ \setminus \mathbb{C}$. We point out that the kernel of a derivation of $\mathcal{A}$, whether continuous or not, is always algebraically closed in $\mathcal{A}$. As a matter of fact, the argument given in [19, Lemma 2.1] works for any characteristic zero integral domain. From Proposition 4.1, we can also deduce the following generalization of [19, Theorem 2.1].

Theorem 4.2. *Let $f \in \mathcal{A}$ and $(g_i)_{i\in I}$ be a family of arithmetic functions. Suppose*

$$[\operatorname{supp} f] \not\subseteq \bigcup_{i\in I} [\operatorname{supp} g_i].$$

Then $\operatorname{Exp}^*\{f\}$ *is algebraically independent over the subalgebra of $\mathcal{A}$ generated by the g_i $(i \in I)$.*

Proof. By the assumption there is a prime $p \in [\operatorname{supp} f]$ that is not in the union of the $[\operatorname{supp} g_i]$ $(i \in I)$. So by Proposition 4.1, $\operatorname{Exp}^*\{f\}$ is algebraically independent over $\ker \partial_p$ which contains the subalgebra of $\mathcal{A}$ generated by the g_i $(i \in I)$. ∎

We provide a proof of one of the many consequences of [19, Theorem 2.1]. The reader can consult [19, pp. 697–699] for others.

Corollary 4.3. *$\mathcal{S}$ is algebraically closed in $\mathcal{A}$.*

Proof. Suppose $g_1, \dots, g_n \in \mathcal{S}$ and $f \in \mathcal{A} \setminus \mathcal{S}$. Then $[\operatorname{supp} f]$ is infinite, while the union of $[\operatorname{supp} g_i]$ $(1 \le i \le n)$ is finite. So it follows from Theorem 4.2 that $\operatorname{Exp}^*\{f\}$, in particular f itself, is algebraically independent over $\mathbb{C}[g_1, \dots, g_n]$. Since $g_i \in \mathcal{S}$ $(1 \le i \le n)$ are taken arbitrarily, we conclude that f is algebraically independent over any finitely generated subalgebra of $\mathcal{S}$, and hence over $\mathcal{S}$ itself. ∎

Example 4.4. The function $\mathbf{1}$ is not a member of $\mathcal{S}$, so by Corollary 4.3 it is transcendental over $\mathcal{S}$ and hence over $\mathcal{T}$. In terms of Dirichlet series, that means the Riemann zeta function is transcendental over the subalgebra of *Dirichlet polynomials* (Dirichlet series with only finitely many nonzero terms).

In contrast, $\mathcal{T}$ is not algebraically closed in $\mathcal{A}$ (in fact, not even in $\mathcal{S}$). For instance, $\mathbf{1}_2 = \sum_{k=0}^{\infty} e_2^k$ is in $\mathcal{S} \setminus \mathcal{T}$ but it is algebraic over $\mathcal{T}$ since its inverse $1 - e_2$ is in $\mathcal{T}$. This shows, in particular, that the algebra of Dirichlet polynomials is not algebraically closed in the algebra of convergent Dirichlet series.

Theorem 4.5. *Let $f_1, \dots, f_n$ be arithmetic functions. Suppose there exist continuous derivations $D_1, \dots, D_n$ of $\mathcal{A}$ such that for each $1 \le i < j \le n$,*

$$f_i \in \ker D_j \setminus \ker D_i.$$

Then $\operatorname{Exp}^*\{f_1, \dots, f_n\}$ *is algebraically independent over* $\ker\{D_1, \dots, D_n\}$.

Proof. This is an immediate consequence of Theorem 3.5, because the assumption implies $(D_j f_i)$ is a lower triangular matrix with nonzero entries on its diagonal, hence $\det (D_j f_i) \ne 0$. ∎

Corollary 4.6. *Let $f_1, \dots, f_n \in \mathcal{A}$. Suppose there exist $p_1, \dots, p_n$ such that for each $1 \le i < j \le n$,*

$$p_j \in [\operatorname{supp} f_j] \setminus [\operatorname{supp} f_i].$$

Then $\operatorname{Exp}^*\{f_1, \dots, f_n\}$ is algebraically independent over $\ker\{\partial_{p_1}, \dots, \partial_{p_n}\}$.

Proof. Take D_j in Theorem 4.5 to be ∂_{p_j} $(1 \le i \le n)$. ■

Example 4.7. Let $p_1, \dots, p_n$ be distinct primes. By taking $f_i = e_{p_i}$ $(1 \le i \le n-1)$ and $f_n = \mathbf{1}_{\mathbb{P}}$ in Corollary 4.6, the $\mathbb{C}$-algebraic independence of $\operatorname{Exp}^*\{e_{p_1}, \dots, e_{p_{n-1}}, \mathbf{1}_{\mathbb{P}}\}$ follows. Moreover, since n is arbitrary, that means the set of arithmetic functions

$$\operatorname{Exp}^*(\{e_p : p \in \mathbb{P}\} \cup \{\mathbf{1}_{\mathbb{P}}\})$$

is algebraically independent over $\mathbb{C}$.

Example 4.8. Corollary 4.6 generalizes [16, Lemma 3]: Suppose that $f_1, f_2 \in \mathcal{A} \setminus \mathbb{C}$ with $[\operatorname{supp} f_1] \ne [\operatorname{supp} f_2]$. Without loss of generality, there is a prime p_2 in $[\operatorname{supp} f_2]$ but not in $[\operatorname{supp} f_1]$. Since f_1 is not in $\mathbb{C}$, there exists a prime $p_1 \in [\operatorname{supp} f_1]$. Thus Corollary 4.6 implies $\operatorname{Exp}^*\{f_1, f_2\}$ is algebraically independent over $\mathbb{C}$. In particular, if $F(s) = \sum \alpha_n/n^s$ is a nonconstant formal Dirichlet series such that $\alpha_n = 0$ whenever n is a multiple of some fix prime p, then $F(s)$ and $\zeta(s)$ are algebraically independent over $\mathbb{C}$.

Knowing that a function does not vanish at a particular point certainly implies that it is nonzero. The following proposition is hence a corollary of Theorem 3.5. We invite the reader to prove it (or see [9, Corollary 2.3] for a proof) by checking that the left side of (4.1) expresses the value of $\det(\partial_{p_j} f_i)$ at m.

Proposition 4.9. *For any $f_1, \dots, f_n \in \mathcal{A}$, if there exist distinct primes $p_1, \dots, p_n$ such that for some $m \in \mathbb{N}$,*

$$\sum_{k_1 \cdots k_n = m} \Big(\prod_{j=1}^{n} v_{p_j}(k_j p_j) \Big) \det\left(f_i(k_j p_j)\right) \ne 0, \tag{4.1}$$

then $\operatorname{Exp}^*\{f_1, \dots, f_n\}$ is algebraically independent over $\ker\{\partial_{p_1}, \dots, \partial_{p_n}\}$.

By setting the m in Proposition 4.9 to various values, one obtains strengthened versions of Tests I–IV in [9]. These tests were used to establish algebraic independence of various Fibonacci and Lucas zeta functions [9, Propositions 2.5, 2.6]. We state here only the simplest case, i.e. $m = 1$.

Corollary 4.10. *Suppose $f_1, \dots, f_n$ are arithmetic functions such that $\det\left(f_i(p_j)\right) \ne 0$ for some primes $p_1, \dots, p_n$. Then $\operatorname{Exp}^*\{f_i : 1 \le i \le n\}$ is algebraically independent over $\ker\{\partial_{p_j} : 1 \le j \le n\}$.*

EXAMPLE 4.11. For any distinct primes $p_1, \dots, p_n$, take $f_i = \mathbf{1}_{p_i}$ $(1 \le i \le n-1)$ and $f_n = \mathbf{1}$. Then

$$\det(f_i(p_j)) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = 1.$$

Thus by Corollary 4.10, $\operatorname{Exp}^*\{\mathbf{1}_p, \mathbf{1} \colon p \in \mathbb{P}\}$ is algebraically independent over $\mathbb{C}$.

EXAMPLE 4.12. The function $\tau_* := (\mathbf{1} - 1)^2$ which counts the number of proper factors of its argument and $\mathbf{1}_{\mathbb{P}}$ are algebraically independent over $\mathbb{C}$. This is because $\partial_p \mathbf{1}_{\mathbb{P}} = 1$ for every prime p, so

$$\det \begin{pmatrix} \partial_2 \tau_* & \partial_3 \tau_* \\ \partial_2 \mathbf{1}_{\mathbb{P}} & \partial_3 \mathbf{1}_{\mathbb{P}} \end{pmatrix} = \partial_2 \tau_* - \partial_3 \tau_*,$$

and its value at 4 is $v_2(8)\tau_*(8) - v_3(12)\tau_*(12) = 2 \neq 0$. Note that Corollary 4.10 cannot be used to establish this fact since τ_*, or more generally any member of the square of the maximal ideal of $\mathcal{A}$, vanishes at every prime.

For $f_1, \dots, f_n \in \mathcal{A}$, let $\mu_d(\mathbf{f})$ be the minimum of $\|P(f_1, \dots, f_n)\|$ over all complex polynomials P of total degree d. The function $d \mapsto \mu_d(\mathbf{f})$ can be viewed as a quantitative measure of algebraic independence of $f_1, \dots, f_n$ over $\mathbb{C}$. Several results about this measure were proved in [9]. Our method, due to its nonconstructive nature, cannot produce those results. However, the nonquantitative part of both Theorems 3.2 and 3.4 of [9] can be generalized as follows.

THEOREM 4.13. *Let $f_1, \dots, f_n \in \mathcal{A}$ and $D_1, \dots, D_n$ be continuous derivations of $\mathcal{A}$. Suppose $m_1, \dots, m_n \in \mathbb{N}$ are such that $m_j \le v(D_j f_i)$ for all $1 \le i, j \le n$ and $\det(D_j f_i(m_j)) \neq 0$. Then $\operatorname{Exp}^*\{f_i \colon 1 \le i \le n\}$ is algebraically independent over $\ker\{D_1, \dots, D_n\}$.*

Proof. By taking $a_i = 1$ and $b_j = m_j$ $(1 \le i, j \le n)$ in Proposition 2.2, we conclude that the value of $\det(D_j f_i)$ at $m_1 \cdots m_n$ is $\det(D_j f_i(m_j))$, which is assumed to be nonzero. The statement now follows from Theorem 3.5. ∎

We can arrive at the same conclusion of Theorem 4.13 if $m_i \le v(D_j f_i)$ for all $1 \le i, j \le n$: the same proof goes through by taking $a_i = m_i$ and $b_j = 1$ $(1 \le i, j \le n)$. The next lemma is another easy consequence of Proposition 2.2. The same is true, more generally, for generalized Dirichlet series (see [19, Lemma 8.8]).

Lemma 4.14. *Suppose $f_1, \dots, f_n$ are nonzero arithmetic functions and $p_1, \dots, p_n$ are distinct primes such that* $\det(\partial_{p_j} f_i) = 0$. *Then* $\det(v_{p_j}(vf_i)) = 0$.

Proof. Let m_i be the order of f_i ($1 \le i \le n$). Note that for $1 \le i, j \le n$, $0 < m_i/p_j \le v(\partial_{p_j} f_i)$. So by taking $a_i = m_i$ and $b_i = 1/p_i$ ($1 \le i \le n$) in Proposition 2.2, we have

$$\det(\partial_{p_j} f_i)\bigg(\prod_{k=1}^{n} \frac{m_k}{p_k}\bigg) = \det\bigg(\partial_{p_j} f_i\bigg(\frac{m_i}{p_j}\bigg)\bigg) = \det(v_{p_j}(m_i) f_i(m_i))$$
$$= \Big(\prod_{i=1}^{n} f_i(m_i)\Big) \det(v_{p_j}(m_i)).$$

The lemma follows since $f_i(m_i)$ is nonzero for each i. ∎

Lemma 4.14 was used in [16, proof of Theorem 7]. It states that a set of nonzero noninvertible arithmetic functions is algebraically independent over $\mathbb{C}$ if the norms of its members are pairwise relatively prime. Essentially the same proof yields a more general result:

Theorem 4.15. *Suppose W is a set of nonzero arithmetic functions whose orders are multiplicatively independent. Then* $\mathrm{Exp}^* W$ *is algebraically independent over* $\mathbb{C}$.

Proof. Suppose on the contrary that $\mathrm{Exp}^* W$ is algebraically dependent over $\mathbb{C}$. Then there exist $f_1, \dots, f_n \in W$ such that $\mathrm{Exp}^*\{f_1, \dots, f_n\}$ is algebraically dependent over $\mathbb{C}$. So for any distinct primes $p_1, \dots, p_n$, $\det(\partial_{p_j} f_i) = 0$ by Theorem 3.5, and hence $\det(v_{p_j}(vf_i)) = 0$ by Lemma 4.14. That means the set of vectors

$$\left\{ \begin{pmatrix} v_p(vf_1) \\ \vdots \\ v_p(vf_n) \end{pmatrix} : p \in \mathbb{P} \right\}$$

has $\mathbb{Q}$-rank strictly less than n. Since it has the same $\mathbb{Q}$-rank as the set

$$\{(v_p(vf_i))_{p \in \mathbb{P}} : 1 \le i \le n\},$$

there exist $k_1, \dots, k_n \in \mathbb{Z}$ not all zero such that for each prime p,

$$0 = \sum_{i=1}^{n} k_i v_p(vf_i) = v_p\Big(\prod_{i=1}^{n} (vf_i)^{k_i}\Big).$$

That means $\prod_{i=1}^{n} (vf_i)^{k_i} = 1$, contradicting the assumption that the orders $v(f_i)$ ($1 \le i \le n$) are multiplicatively independent. ∎

Example 4.16. By Theorem 4.15, the set $\mathrm{Exp}^*\{e_{n_1}, \dots, e_{n_k}\}$ is algebraically independent over $\mathbb{C}$ if $v(e_{n_i}) = n_i$ ($1 \le i \le n$) are multiplicatively independent. The converse is also true and it follows easily from the fact that $e_m * e_n = e_{mn}$ for any $n, m \in \mathbb{N}$. Thus for a set N of natural numbers, the necessary and sufficient condition for $\operatorname{Exp}^*\{e_n\colon n \in N\}$ to be algebraically independent over $\mathbb{C}$ is that the elements of N are multiplicatively independent. Note that [16, Theorem 7] alone does not imply this fact since there are multiplicatively independent numbers such as 2 and 6 that are not relatively prime.

5. $\mathfrak{m}_g$-Transcendence. In this section we will establish some criteria for algebraic independence of images of a single arithmetic function under operators of the form $\mathfrak{m}_g$. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. We say that an arithmetic function f is $\mathfrak{m}_g$-*transcendental* over $\mathcal{B}$ if $\{\mathfrak{m}_g^j f\colon j \in J\}$ is algebraically independent over $\mathcal{B}$ where $J = \mathbb{N} \cup \{0\}$ if $\mathfrak{m}_g$ is not invertible, otherwise $J = \mathbb{Z}$.

THEOREM 5.1. *Let f and g be arithmetic functions. Suppose that $p_1, \ldots, p_n \in [\operatorname{supp} f]$ are such that $g(v(\partial_{p_j} f)p_j)$ $(1 \le j \le n)$ are distinct and nonzero. Then for any $k \ge 0$, the set*

$$\operatorname{Exp}^*\{\mathfrak{m}_g^i f\colon k \le i \le k+n-1\}$$

is algebraically independent over $\ker\{\partial_{p_1}, \ldots, \partial_{p_n}\}$. *Moreover, if g is nowhere vanishing then the same is true for any integer k.*

Proof. Let $f_i = \mathfrak{m}_g^i f$ $(k \le i \le k+n-1)$ and $m_j = v(\partial_{p_j} f)$ $(1 \le j \le n)$. By Lemma 2.7, $m_j \le v(\partial_{p_j} f_i)$ for all $k \le i \le k+n-1$ and $1 \le j \le n$. So by Theorem 4.13 it suffices to show that

$$\begin{aligned}\det\left(\partial_{p_j} f_i(m_j)\right) &= \det\left(v_{p_j}(m_j p_j)(g(m_j p_j))^i f(m_j p_j)\right)\\ &= \det\left((g(m_j p_j))^{i-k}\right) \prod_j \partial_{p_j} f(m_j)(g(m_j p_j))^k\end{aligned}$$

does not vanish. This is indeed the case because for each j, m_j is the order of $\partial_{p_j} f$, hence $\partial_{p_j} f(m_j) \ne 0$ and $g(m_j p_j) \ne 0$ by our assumption on g; moreover, $g(m_j p_j)$ $(1 \le j \le n)$ are assumed to be distinct, so the last determinant is Vandermonde. Finally, nothing in the argument above prevents k from being negative so long as $\mathfrak{m}_g^k$ is defined; but that precisely requires g to be nowhere vanishing. ∎

EXAMPLE 5.2. Let Q be a nonempty finite set of primes. Since for $q \in Q = [\operatorname{supp} \mathbf{1}_Q]$, the numbers

$$\log(v(\partial_q \mathbf{1}_Q)q) = \log(q)$$

are all distinct and nonzero, it follows from Theorem 5.1 (by taking g to be the real logarithm) that $\mathbf{1}_Q$ does not satisfy any differential algebraic equation with respect to ∂_L of order less than $|Q|$ over the kernel of $\{\partial_q\colon q \in Q\}$.

The assumption "$g(v(\partial_{p_j} f)p_j)$ $(1 \le j \le n)$ are distinct" in Theorem 5.1 is necessary. Consider, for example, the function e_n. It satisfies the linear differential equation

$$\partial_L X - \log(n)X = 0,$$

and its support is generated by the prime divisors of n. Therefore, the conclusion of Theorem 5.1 is false when n has more than one prime factor. Note also that for $f = e_n$ the assumption on g in Theorem 5.1 cannot be met by any arithmetic function since $v(\partial_p e_n)p = (n/p)p = n$ for all $p \in [\operatorname{supp} e_n]$. This example also shows that the assumption "$n_i p_i$ are distinct" is needed for [9, Corollary 3.5].

The following lemma is a rather simple observation about algebraic independence of arithmetic functions over $\mathcal{S}$. Since it will be called upon several times, we include it here for the record. For a set I of primes, let Δ_I be the set of basic derivations indexed by I, i.e. $\{\partial_p : p \in I\}$. We write Δ_f for $\Delta_{[\operatorname{supp} f]}$.

Lemma 5.3. *Let I be a set of primes. If E is a set of arithmetic functions that is algebraically independent over* $\ker \Delta_J$ *for any co-finite subset J of I, then E is algebraically independent over $\mathcal{S}$.*

Proof. It suffices to show that E is algebraically independent over every finitely generated subalgebra of $\mathcal{S}$. Suppose $\mathcal{H}$ is a subalgebra of $\mathcal{S}$ generated by some $h_0, \ldots, h_d \in \mathcal{S}$. Since the sets $[\operatorname{supp} h_i]$ $(0 \le i \le d)$ are finite, so is their union H. Therefore, E, by assumption, is algebraically independent over the kernel of $\Delta_{I \setminus H}$. We conclude that E is algebraically independent over $\mathcal{H}$ since each derivation in $\Delta_{I \setminus H}$ kills every h_i $(0 \le i \le d)$. ∎

Theorem 5.4. *Let $g \in \mathcal{A}$ be eventually injective and $f \in \mathcal{A} \setminus \mathcal{S}$. The set*

$$\operatorname{Exp}^*\{\mathfrak{m}_g^i f : i \ge 0\}$$

is algebraically independent over the kernel of any infinite subset of Δ_f, and hence over $\mathcal{S}$. In addition, if g is nowhere vanishing, then i can range through the integers.

Proof. Since $f \notin \mathcal{S}$, Δ_f is infinite and so are its co-finite subsets. Let J be an arbitrary infinite subset of $[\operatorname{supp} f]$. Once we establish that $E := \operatorname{Exp}^*\{\mathfrak{m}_g^i f : i \ge 0\}$ is algebraically independent over $\ker \Delta_J$, its algebraic independence over $\mathcal{S}$ follows from Lemma 5.3. Since g is eventually injective, there exists $n_0 \in \mathbb{N}$ such that g is injective and nonvanishing on $\{n \in \mathbb{N} : n \ge n_0\}$. We choose an infinite sequence from J inductively as follows: pick $p_1 \in J$ larger than n_0 and $p_{j+1} \in J$ such that

$$p_{j+1} > v(\partial_{p_j} f)p_j \quad (j \ge 1).$$

Then $v(\partial_{p_j} f)p_j$ $(j \geq 1)$ form a strictly increasing sequence, and so the $g(v(\partial_{p_j} f)p_j)$ are nonzero and distinct. Note that every finite subset of E is contained in $\operatorname{Exp}^*\{\mathfrak{m}_g^i f : k \leq i \leq k+n-1\}$ for some $k \geq 0$, $n \geq 1$. According to Theorem 5.1, the latter set is algebraically independent over $\ker\{\partial_{p_j} : 1 \leq j \leq n\} \supseteq \ker \Delta_J$. So we conclude that E is algebraically independent over $\ker \Delta_J$. In addition, if g is nowhere vanishing, then Theorem 5.1 and hence the whole argument goes through for $E := \operatorname{Exp}^*\{\mathfrak{m}_g^i f : i \in \mathbb{Z}\}$. ∎

Rather curiously, for a completely additive function g to be injective means the set of complex numbers $g(\mathbb{P})$ is $\mathbb{Q}$-linearly independent; and for a completely multiplicative g to be injective means $g(\mathbb{P})$ is multiplicatively independent. In any case, even if one requires $\mathfrak{m}_g$ in Theorem 5.4 to be a derivation or an automorphism of $\mathcal{A}$, there are still plenty arithmetic functions that satisfy the requirements.

EXAMPLE 5.5. By taking the function g in Theorem 5.4 to be the real logarithm, we conclude that $\mathbf{1}$ is ∂_L-transcendental (better known as *hyper-transcendental*) over $\mathcal{S}$. In particular, that means the Riemann zeta function $\zeta(s)$ is hyper-transcendental over $\mathbb{C}$. In fact, it is hyper-transcendental over $\mathbb{C}(s)$. If not, then the transcendence degree of $\mathbb{C}(s)(\zeta^{(k)}(s) : k \geq 0)$ over $\mathbb{C}(s)$ and hence over $\mathbb{C}$ would be finite. That would imply the transcendence degree of $\mathbb{C}(\zeta^{(k)}(s) : k \geq 0)$ over $\mathbb{C}$ to be finite as well, contradicting the fact that $\zeta(s)$ is hyper-transcendental over $\mathbb{C}$. We refer the reader to [20] for some historical remarks on this result which is usually attributed to Hilbert [7] in the literature.

EXAMPLE 5.6. For $k \in \mathbb{Z}$, let $\mathbf{I}_k$ be the arithmetic function $n \mapsto n^k$. In [4], Carlitz showed that $\mathbf{I}_k$ $(k \geq 0)$ are algebraically independent over $\mathbb{C}$. Shapiro and Sparer generalized this result to the algebraic independence of $\mathbf{I}_k$ $(k \in \mathbb{Z})$ over the kernel of any infinite set of basic derivations (and hence over $\mathcal{S}$) [19, Theorem 3.2]. By taking $g = \mathbf{I}$ the identity map of $\mathbb{N}$ and $f = \mathbf{1}$ in Theorem 5.4, we conclude more generally that $\operatorname{Log} \mathbf{I}_k, \mathbf{I}_k$ $(k \in \mathbb{Z})$ are algebraically independent over the kernel of any infinite set of basic derivations (and hence over $\mathcal{S}$).

By fixing f to be $\mathbf{1}$, one can view Theorem 5.4 as a result about algebraic independence of $g^{\langle k \rangle} := \mathfrak{m}_g^k(\mathbf{1})$ $(k \geq 0)$, the powers of g with respect to the pointwise product. In fact, since $[\operatorname{supp} \mathbf{1}] = \mathbb{P}$ and $v(\partial_p \mathbf{1}) = 1$ for each p, an assumption weaker than eventual injectivity of g is enough to guarantee algebraic independence. More precisely, we have

COROLLARY 5.7. $\operatorname{Exp}^*\{g^{\langle i \rangle} : i \geq 0\}$ *is algebraically independent over* $\mathbb{C}$ *if* $g(\mathbb{P})$ *is infinite, and is algebraically independent over* $\mathcal{S}$ *if* $g(I)$ *is infinite for every infinite set* I *of primes. Moreover, the same is true with* i *ranging through the integers if* g *is nowhere vanishing.*

Corollary 5.7, in particular, implies that if $g(\mathbb{P})$ is infinite then g does not satisfy, in the algebra $(\mathcal{A}, +, \cdot)$, any nontrivial polynomial equation over $\mathbb{C}$. We will also discuss this kind of independence in Section 6.

Let (U_n) be a linear integral recurrence of order two, by which we mean (U_n) is a sequence of integers satisfying

$$U_{n+2} = PU_{n+1} - QU_n \quad (n \geq 1)$$

for some $P, Q \in \mathbb{Z}$ with $Q \neq 0$. Suppose ρ is a ratio (the other being $1/\rho$) of the two roots of the characteristic polynomial $z^2 - Pz + Q$. Morgan Ward [22, Theorem 1] showed that the set $\{U_n : n \geq 1\}$ has infinitely many prime divisors if either (1) ρ is not a root of unity, or (2) $\rho = 1$. In the first case, the recurrence (U_n) is called *nondegenerate*. Thus, if $U \subseteq \mathbb{N}$ is the set of terms of a nondegenerate second order linear integral recurrence, then $\mathbf{1}_U \notin \mathcal{S}$. By Theorem 5.4, we conclude that $\mathbf{1}_U$ is $\mathfrak{m}_g$-transcendental over $\mathcal{S}$ for any g that is eventually injective.

EXAMPLE 5.8. The linear recurrence defining the Fibonacci numbers:

$$F_1 = 1, \quad F_2 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

is second order and nondegenerate. Thus $\mathbf{1}_F$, the indicator function of the Fibonacci numbers, is hyper-transcendental over $\mathbb{C}$. By an argument similar to the one given in Example 5.5, we conclude that the Fibonacci zeta function $\zeta_F(s)$ is hyper-transcendental over $\mathbb{C}(s)$.

Our next result generalizes both [19, Theorem 3.3] and [16, Theorem 3] by relaxing the assumption that $\operatorname{supp} f$ contains infinitely many primes to that $\operatorname{supp} f$ is not finitely generated. The proof below is a mixture of the those given in [19] and [16]. Therefore, our sole contribution here is the realization that these proofs remain valid in a more general setting. We also hope that our use of the lexicographic ordering on the index set can clarify the presentation. In the following, T^α ($\alpha \in \mathbb{C}$) stands for the operator $\mathfrak{m}_g$ where g is the function $n \mapsto n^\alpha$.

THEOREM 5.9. *For any $f \in \mathcal{A} \setminus \mathcal{S}$ and any sequence $(\alpha_i)_{i\geq 0}$ of complex numbers with distinct real parts, the set*

$$\operatorname{Exp}^*\{T^{\alpha_i}\partial_L^j f : i, j \geq 0\}$$

is algebraically independent over the kernel of any infinite subset of Δ_f, and consequently over $\mathcal{S}$.

Proof. Since $f \in \mathcal{A} \setminus \mathcal{S}$, $[\operatorname{supp} f]$ is infinite and so are its co-finite subsets. So by Lemma 5.3, we only need to show that for any $k, m \geq 0$, the set

$$\operatorname{Exp}^*\{f_{ij} : 0 \leq i \leq k,\ 0 \leq j \leq m\},$$

where $f_{ij} := T^{\alpha_i}\partial_L^j f$, is algebraically independent over the kernel of any infinite subset of Δ_f. Let

$$L = \{(a,b)\colon 0 \le a \le k,\ 0 \le b \le m\}$$

be the index set ordered lexicographically. If no confusion can arise, we follow the convention of indexing matrix entries by writing the index (a,b) as ab.

Given an infinite subset J of $[\operatorname{supp} f]$, we are going to choose a sequence of primes $(p_{uv}\colon (u,v) \in L)$ from J. Let m_{uv} be the order of $\partial_{p_{uv}} f$. By applying Lemma 2.7 twice, we conclude that $m_{uv} = v(\partial_{p_{uv}} f_{ij})$ for any $(i,j) \in L$. We claim that the determinant of the $|L| \times |L|$ matrix

$$(\partial_{p_{uv}} f_{ij}(m_{uv})) = \Big(\prod_{(u,v)\in L} \partial_{p_{uv}} f(m_{uv})\Big)\big((m_{uv}p_{uv})^{\alpha_i}(\log(m_{uv}p_{uv}))^j\big)$$

is nonzero if we impose suitable requirements on the sequence (p_{uv}). Once this is achieved, it then follows from Theorem 4.13 that $\operatorname{Exp}^*\{f_{ij}\colon (i,j) \in L\}$ is algebraically independent over $\ker\{\partial_{p_{uv}}\colon (u,v) \in L\} \supseteq \ker \Delta_J$.

Since $\partial_{p_{uv}} f(m_{uv}) \ne 0$ for each $(u,v) \in L$, it suffices to make the determinant of the matrix

$$P := \big((m_{uv}p_{uv})^{\alpha_i}(\log(m_{uv}p_{uv}))^j\big)$$

nonzero. To achieve this, first note that the entries of P are all nonzero and hence each term in the expansion of $\det P$ is nonzero. By rearranging the α_i $(1 \le i \le n)$ if necessary, we can assume their real parts form a strictly increasing sequence. Let t_{diag} denote the product of the diagonal entries of P, i.e.

$$t_{\mathrm{diag}} = \prod_{(u,v)\in L} (m_{uv}p_{uv})^{\alpha_u}(\log(m_{uv}p_{uv}))^v.$$

The key observation is that the ratio t/t_{diag}, where t is any other term in the expansion of $\det P$, has the form

$$\prod_{(u,v)\in L} (m_{uv}p_{uv})^{\gamma(u,v)}(\log(m_{uv}p_{uv}))^{d(u,v)},$$

and if $(u',v') \in L$ is the largest index such that $(\gamma(u',v'), d(u',v'))$ is not $(0,0)$, then $(\Re(\gamma(u',v')), d(u',v')) < (0,0)$ lexicographically. Therefore, if we choose an increasing sequence (p_{uv}) of primes from J such that each p_{uv} is sufficiently large compared to its predecessors, for example, pick $p_{00} \ge 3$ (to ensure $\log p_{uv} > 1$ for all $(u,v) \in L$)) and p_{uv} such that

$$\log p_{uv} > |L|! \prod_{(u',v')<(u,v)} (m_{u'v'}p_{u'v'})^{|\alpha_k|+m}.$$

Then $|t/t_{\mathrm{diag}}| < (|L|!)^{-1}$. Thus for such a choice of (p_{uv}),

$$|\det P| \geq |t_{\mathrm{diag}}|\Big(1 - \sum_{t \neq t_{\mathrm{diag}}} |t/t_{\mathrm{diag}}|\Big) > 0. \ \blacksquare$$

A couple of remarks about Theorem 5.9 are in order. First, arithmetic functions of the form $n^{\alpha_i}(\log n)^j f(n)$ $(j \in \mathbb{Z})$ were considered in both [19] and [16]. This is problematic for negative j since these functions are not defined at 1, and consequently their higher convolution powers are undefined. Second, if Theorem 5.9 admits an "algebraic" proof, by which we mean a proof similar to that of Theorem 5.4 which does not rely on the growth rate of the functions involved, then one may expect a generalization to operators of the form $\mathfrak{m}_h^i \mathfrak{m}_g^j$.

COROLLARY 5.10. *Suppose U is a set of natural numbers with an infinite set of prime divisors. Then $\zeta_U(s)$ does not satisfy any nontrivial algebraic differential difference equation over $\mathbb{C}(s)$.*

Proof. Since $\mathbf{1}_U \notin \mathcal{S}$, Theorem 5.9 implies the set of arithmetic functions $\{T^{\alpha_i}\partial_L^j \mathbf{1}_U : i, j \geq 0\}$ is algebraically independent over $\mathbb{C}$ for any complex sequence (α_i) with distinct real parts. Since $(-1)^j T^{\alpha_i}\partial_L^j \mathbf{1}_U$ corresponds to $\zeta_U^{(j)}(s - \alpha_i)$ under the isomorphism in (2.4), the corollary is true over $\mathbb{C}$. Finally, by [19, Lemma 3.1] (1), it is true for $\mathbb{C}(s)$ since the formal Dirichlet series involved are convergent. ∎

EXAMPLE 5.11. Corollary 5.10 implies a classical result of Ostrowski [13]: $\zeta(s)$ does not satisfy any nontrivial algebraic differential-difference equation over $\mathbb{C}(s)$. That means there is no nonzero polynomial $F(s, z_1, \ldots, z_k)$ over $\mathbb{C}$ such that the function

$$F\big(s, \zeta^{(m_1)}(s - r_1), \ldots, \zeta^{(m_k)}(s - r_k)\big),$$

where (m_i, r_i) are distinct pairs of integers and $m_i \geq 0$ for all $1 \leq i \leq k$, vanishes identically on its domain.

EXAMPLE 5.12. Recall that if $U \subseteq \mathbb{N}$ is the set of terms of a non-degenerate second order linear recurrence, then $\mathbf{1}_U \notin \mathcal{S}$. Thus, Corollary 5.10 also implies the Fibonacci zeta function $\zeta_F(s)$ does not satisfy any nontrivial algebraic differential-difference equation over $\mathbb{C}(s)$. Since it is not known whether the Fibonacci sequence contains infinitely many primes, this statement cannot be deduced, at least for now, from either [19, Theorem 3.3] or [16, Theorem 3].

(1) The use of this lemma can be avoided. See [19, p. 702] for an argument, or use an argument of transcendence degree slightly more elaborate than the one given in Example 5.5.

Many sequences of natural numbers, well-known to number theorists, are in fact nondegenerate second order integral linear recurrences (see [11] for a reference): Lucas sequence, Pell sequence and Pell–Lucas sequence, to name a few. Thus their zeta functions do not satisfy any nontrivial algebraic difference-differential equations over $\mathbb{C}(s)$. More generally, one can replace "algebraic" by "holomorphic" in the previous statement, if one invokes an analytic result of Reich [15, Satz 1] instead of Theorem 5.9. This is the way in which Steuding [20, Theorem 1] and Komatsu [8, Corollary 1] obtained the corresponding results for the Riemann zeta function and the Lucas zeta function, respectively. In [20], Steuding made no reference to Ward's paper [22] but did mention that his argument obviously can be extended to other Dirichlet series built from a linear recurrence.

6. Remarks. We conclude with a few observations that we made along our study of arithmetic functions. The first one is about derivations of $\mathcal{A}$. As noted above, Theorem 3.5 would be an unconditional generalization of Shapiro–Sparer's Jacobian criterion if every derivation of $\mathcal{A}$ were continuous. Unfortunately, we can neither prove that every derivation of $\mathcal{A}$ is continuous nor produce one that is not. There is indeed a construction given at the end of Section 4 in [17, pp. 309–312] which produces nonzero derivations of $\mathcal{F}$ that vanish on the e_n $(n \in \mathbb{N})$ and hence $\mathcal{T}$. Since $\mathcal{F}$ is the field of fractions of $\mathcal{A}$, any such derivation must also be nonzero on $\mathcal{A}$, but then it cannot be continuous since $\mathcal{A}$ is the closure of $\mathcal{T}$ in $\mathcal{F}$. However, it is unclear to us that any derivation constructed this way actually restricts to a map from $\mathcal{A}$ to itself.

Here we would like to offer a similar but hopefully simpler way of constructing derivations $\mathcal{F}$ that do not preserve null sequences of $\mathcal{A}$. Start with a null sequence in $\mathcal{A}$ that is algebraically independent over $\mathbb{C}$, for example $(e_p)_{p\in\mathbb{P}}$. Extend it to a transcendence base B of $\mathcal{F}$ over $\mathbb{C}$. Then db $(b \in B)$ form an $\mathcal{F}$-basis of $\Omega_{\mathcal{F}/\mathbb{C}}$ [12, Theorem 26.5]. The derivation D of $\mathcal{F}$ obtained by composing $d : \mathcal{F} \to \Omega_{\mathcal{F}/\mathbb{C}}$ with the $\mathbb{C}$-linear map determined by $db \mapsto 1$ $(b \in B)$ maps each e_p to 1, and hence cannot be a continuous derivation of $\mathcal{A}$ if it does restrict to a map from $\mathcal{A}$ to itself. The flip side of the coin is that every derivation of $\mathcal{A}$ is continuous. This will be true if the topology determined by the norm $\|\cdot\|$ is equivalent to the $\mathcal{I}$-adic topology of some ideal $\mathcal{I}$ of $\mathcal{A}$. This is because for any $n \geq 1$, and $f \in \mathcal{I}^n$, the derivative f with respect to any derivation of $\mathcal{A}$, according to the Leibniz rule, is in $\mathcal{I}^{n-1}$, and so any derivation of $\mathcal{A}$ is continuous with respect to the topology determined by any ideal of $\mathcal{A}$. We should point out, however, that in the case when $\mathcal{I}$ is the unique maximal ideal $\mathcal{A}_0$ these two topologies are inequivalent. For example, no term in the null sequence (e_p) is in $\mathcal{A}_0^2$ because members of $\mathcal{A}_0^2$ vanish on every prime.

Our second observation is about linear independence of arithmetic functions over $\mathbb{C}$. It was proved in [10, Theorems 3.2–3.4] that arithmetic functions $f_1, \ldots, f_n$ are linearly dependent over $\mathbb{C}$ if and only if their Wronskian with respect to the log-derivation, i.e.

$$\det \begin{pmatrix} f_1 & \cdots & f_n \\ \partial_L f_1 & \cdots & \partial_L f_n \\ \vdots & & \vdots \\ \partial_L^{n-1} f_1 & \cdots & \partial_L^{n-1} f_n \end{pmatrix},$$

vanishes identically.

We claim that the same is true, more generally, for elements of $\mathcal{F}$, and offer a softer proof in the sense that no formula for the values of Wronskian is needed. We take advantage of a standard result on differential fields [6, Theorem 6.3.4] which asserts that elements of a differential field (F, D) are linearly dependent over the field of constants if and only if their Wronskian with respect to D (or D-Wronskian, for short) is zero. Thus by taking the differential field to be $(\mathcal{F}, \partial_L)$, all we need to show is that the kernel of the log-derivation in $\mathcal{F}$ is still $\mathbb{C}$. Before proving that statement, it is probably worth pointing out that in general $\ker_{\mathcal{F}} D$ need not be the fraction field of $\ker D$ in $\mathcal{F}$: recall that Ω counts the total number of prime factors with multiplicity of its argument. One checks readily that $\ker \partial_\Omega = \mathbb{C}$ and $\partial_\Omega e_p = e_p$ for each prime p. Thus for distinct primes p, q we have $e_p/e_q \in \ker_{\mathcal{F}} \partial_\Omega \setminus \mathbb{C}$.

Proposition 6.1. *$\ker_{\mathcal{F}} \partial_L = \mathbb{C}$.*

Proof. First, $\ker_{\mathcal{F}} \partial_L \supseteq \ker \partial_L = \mathbb{C}$. To establish the reverse inclusion, take any $f, g \in \mathcal{A} \setminus \{0\}$ such that $\partial_L(f/g) = 0$. Then

$$\partial_L f * g = f * \partial_L g. \tag{6.1}$$

If g is invertible in $\mathcal{A}$, we have $f/g \in \mathcal{A} \cap \ker_{\mathcal{F}} \partial_L = \mathbb{C}$. So assume g is not invertible, that is, $g(1) = 0$; it then follows that $\|\partial_L g\| = \|g\|$ (> 0). Now by taking norm on both sides of (6.1), we see that $\|\partial_L f\| = \|f\|$ (> 0). Thus, by evaluating both sides of (6.1) at $v(f)v(g)$, we conclude that $\log(v(f)) = \log(v(g))$ and hence $v(f) = v(g)$. Let k be this common value and h be $f - \alpha g$ where $\alpha = f(k)/g(k)$. Then the order of h is strictly greater than k and $h/g = f/g - \alpha \in \ker_{\mathcal{F}} \partial_L$. So unless $h = 0$, i.e. $f/g = \alpha \in \mathbb{C}$, the same argument with f replaced by h will lead us to the contradicting conclusion that $v(h) = v(g) = k$. This completes the proof of the other inclusion. ∎

Viewing the linear independence result of [10] as one about differential fields frees us from focusing on the log-derivation: if the Wronskian of $f_1, \ldots, f_n$ with respect to any derivation D of $\mathcal{F}$ is nonzero, then $f_1, \ldots, f_n$

are linearly independent over $\ker_{\mathcal{F}} D$ and hence over $\mathbb{C}$. Let us give an application. Recall that $g^{\langle k\rangle}$ $(k \geq 0)$ denotes the kth power of g with respect to the pointwise product. Consider again the function Ω. The value at 1 of the ∂_2-Wronskian of $\mathbf{1} = \Omega^{\langle 0\rangle}, \Omega^{\langle 1\rangle}, \ldots, \Omega^{\langle n\rangle}$ is

$$\det(\partial_2^j \Omega^{\langle i\rangle}(1)) = \det(j!\Omega^{\langle i\rangle}(2^j)) = \det(j^i)\prod_{j=0}^{n} j!,$$

which is nonzero since the last determinant is Vandermonde. We conclude that $\Omega^{\langle k\rangle}$ $(k \geq 0)$ are linearly independent over $\mathbb{C}$. Therefore, the ∂_L-Wronskian of $\mathbf{1}, \Omega^{\langle 1\rangle}, \ldots, \Omega^{\langle n\rangle}$ must also be nonzero, but this is harder to spot since its value at 1 is 0. This also shows that Ω does not satisfy any nontrivial polynomial equation over $\mathbb{C}$ in the $\mathbb{C}$-algebra $(\mathcal{A}, +, \cdot)$. Note that this fact cannot be deduced from Corollary 5.7 since $\Omega(\mathbb{P}) = \{1\}$ is finite. Note also that this statement is stronger than asserting Ω is transcendental over $\mathbb{C}$ in the sense of Bellman and Shapiro [3]. Roughly speaking, since $(\mathcal{A}, +, \cdot)$ is not an integral domain, the "right" definition for algebraic dependence requires not just a nontrivial polynomial but an irreducible one to vanish at the functions involved.

Our last few remarks are about Theorem 3.5 and [10, Section 2]. In searching for a generalization of the Jacobian criteria, we realize that the derivations in Theorem 3.5 cannot be replaced by differential operators. More precisely, consider, for each $k \in \mathbb{N}$, the differential operator $\partial_k := \prod_p \partial_p^{v_p(k)}$ (here the product is composition of functions). One checks that for $f \in \mathcal{A}$ and $n \in \mathbb{N}$,

$$(\partial_k f)(n) = f(kn)\prod_p \prod_{j=1}^{v_p(k)} (v_p(n) + j).$$

In particular, $(\partial_k f)(1) = f(k)\prod_p (v_p(k)!)$. Thus, if we normalize ∂_k to

$$\hat{\partial}_k = \Big(\prod_p (v_p(k)!)\Big)^{-1} \partial_k,$$

then we will have $\varepsilon_1 \circ \hat{\partial}_k = \varepsilon_k$. To see that Theorem 3.5 fails if we replace the derivations by differential operators, take f_1 to be $\mathbf{1}_2$ and $f_2 = f_1 * f_1$. Note that

$$f_2(n) = \begin{cases} k+1 & \text{if } n = 2^k \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Certainly, f_1 and f_2 are algebraically dependent over $\mathbb{C}$ but

$$\det\begin{pmatrix} \hat{\partial}_2 f_1 & \hat{\partial}_4 f_1 \\ \hat{\partial}_2 f_2 & \hat{\partial}_4 f_2 \end{pmatrix}(1) = \det\begin{pmatrix} f_1(2) & f_1(4) \\ f_2(2) & f_2(4) \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = 1.$$

Incidentally, this shows that Theorem 2.2 of [10] is not true. Moreover,

$$\partial_L f_1(n) = \begin{cases} k\log(2) & \text{if } n = 2^k \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus f_1 satisfies the following differential algebraic equation over $\mathbb{C}$:

$$\partial_L X = \log(2)(X^2 - X).$$

This invalidates Corollaries 2.3–2.5 of [10]. In particular, it is not true that a Dirichlet series which is not a Dirichlet polynomial is hyper-transcendental over $\mathbb{C}$. Corollaries 2.6 and 2.7 of [10] are also problematic. Again, the pair (f_1, f_2) furnishes a counterexample to Corollary 2.7 of [10] which asserts that arithmetic functions with infinite supports are algebraically independent over $\mathbb{C}$. Since f_1, f_2 are algebraically dependent over $\mathbb{C}$, so are the arithmetic functions g_1, g_2 defined by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1(2) & f_1(4) \\ f_2(2) & f_2(4) \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Consequently, $h_1 := g_1 - 2$ and $h_2 := g_2 + 1$ are also algebraically dependent over $\mathbb{C}$. Since the first four values of h_1 and h_2 are $(0,1,0,0)$ and $(0,0,0,1)$, respectively, the pair (h_1, h_2) provides a counterexample to Corollary 2.6 of [10].

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References

[1] J. Ax, *On Schanuel conjectures*, Ann. of Math. 93 (1971), 252–268.

[2] P. T. Bateman and H. G. Diamond, *Analytic Number Theory. An Introductory Course*, Monogr. Number Theory 1, World Sci., Hackensack, NJ, 2004.

[3] R. Bellman and H. N. Shapiro, *The algebraic independence of arithmetic functions. I. Multiplicative functions*, Duke Math. J. 15 (1948), 229–235.

[4] L. Carlitz, *Independence of arithmetic functions*, Duke Math. J. 19 (1952), 65–70.

[5] E. D. Cashwell and C. J. Everett, *The ring of number-theoretic functions*, Pacific J. Math. 9 (1959), 975–985.

[6] A. Chambert-Loir, *A Field Guide to Algebra*, Undergrad. Texts in Math., Springer, New York, 2005.

[7] D. Hilbert, *Mathematische Probleme*, Arch. Math. Phys. 1 (1901), 44–63, 213–317.

[8] T. Komatsu, *On continued fraction expansions of Fibonacci and Lucas Dirichlet series*, Fibonacci Quart. 46/47 (2008/09), 268–278.

[9] T. Komatsu, V. Laohakosol and P. Ruengsinsub, *Independence measures of arithmetic functions*, J. Number Theory 131 (2011), 1–17.

[10] T. Komatsu, V. Laohakosol and P. Ruengsinsub, *Independence measures of arithmetic functions II*, Acta Arith. 153 (2012), 199–216.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure Appl. Math. (New York), Wiley-Interscience, New York, 2001.

[12] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, Cambridge, 1989.

[13] A. Ostrowski, *Über Dirichletsche Reihen und algebraische Differentialgleichungen*, Math. Z. 8 (1920), 241–298.

[14] D. Rearick, *Operators on algebras of arithmetic functions*, Duke Math. J. 35 (1968), 761–766.

[15] A. Reich, *Über Dirichletsche Reihen und holomorphe Differentialgleichungen*, Analysis 4 (1984), 27–44.

[16] P. Ruengsinsub, V. Laohakosol and P. Udomkavanich, *Observations about algebraic dependence of Dirichlet series*, J. Korean Math. Soc. 42 (2005), 695–708.

[17] H. N. Shapiro, *On the convolution ring of arithmetic functions*, Comm. Pure Appl. Math. 25 (1972), 287–336.

[18] H. N. Shapiro, *Introduction to the Theory of Numbers*, Dover, 2008.

[19] H. N. Shapiro and G. H. Sparer, *On algebraic independence of Dirichlet series*, Comm. Pure Appl. Math. 39 (1986), 695–745.

[20] J. Steuding, *The Fibonacci zeta-function is hypertranscendental*, Cubo 10 (2008), 133–136.

[21] M. Waldschmidt, *Schanuel's Conjecture: algebraic independence of transcendental numbers*, in: Colloquium De Giorgi 2013 and 2014, U. Zannier (ed.), Edizioni della Normale, 2015, 129–137.

[22] M. Ward, *Prime divisors of second order recurring sequences*, Duke Math. J. 21 (1954), 607–614.

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