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C_0-semigroups generated by second order differential operators

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Abstract. Let $W(u)(x) = \frac{1}{2}x^a(1-x)^b u''(x)$ with $a, b \geq 2$. We consider the C_0-semigroups generated by this operator on the spaces of continuous functions, respectively square integrable functions. The connection between these semigroups together with suitable approximation processes is studied. Also, some qualitative and quantitative properties are derived.

1. Introduction. Viewing differential operators as generators of C_0-semigroups is not only elegant, but has innumerable applications in exact sciences. This paper is devoted to the operator

$$W(u)(x) = \tfrac{1}{2}x^a(1-x)^b u''(x), \qquad x \in [0,1],\ a, b \geq 2, \tag{1.1}$$

acting on $\mathscr{C}([0,1],\mathbb{R})$, $\mathscr{C}([0,1],\mathbb{C})$, $L^2([0,1],\mathbb{R})$ and $L^2([0,1],\mathbb{C})$.

The operator $W(u)(x) = \frac{1}{2}x^a(1-x)^b u''(x)$ in the Banach space $\mathscr{C}([0,1],\mathbb{R})$ has been intensively investigated. It has been shown that the closure of W generates a positive C_0-semigroup of contractions. This semigroup can be approximated by iterates of a positive approximation process, and under certain conditions both $t \to 0$ and $t \to \infty$ limits can be found.

We shall denote by $\mathscr{C}^n([0,1],\mathbb{R})$ (resp. $\mathscr{C}^n([0,1],\mathbb{C})$) the space of all real-valued (resp. complex-valued) continuous functions, defined on $[0,1]$, that are n-times continuously differentiable in $[0,1]$. The spaces $\mathscr{C}([0,1],\mathbb{C})$ and $\mathscr{C}([0,1],\mathbb{R})$ will be endowed with the supremum norm, denoted by $\|\cdot\|$; we shall also consider the Hilbert spaces $L^2([0,1],\mathbb{C})$ and $L^2([0,1],\mathbb{R})$, with the norm denoted by $\|\cdot\|_2$ and the inner product $\langle\,,\rangle$. In this article it is shown that, under suitable assumptions, the operator W generates C_0-semigroups

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on all the four spaces considered, and appropriate approximation processes are proposed.

I will stand for the identity operator on the corresponding space.

L^k denotes the iterate of order $k \geq 1$ of the operator L.

If L is a linear operator defined on a subspace E of $\mathscr{C}([0,1],\mathbb{R})$ or of $L^2([0,1],\mathbb{R})$, we shall denote by $\tilde{L}$ the complexified operator defined on $E+iE$ by $\tilde{L}(f+ig) = L(f) + iL(g)$ for $f, g \in E$.

Lemma 1.1. *Let $L : \mathscr{C}([0,1],\mathbb{R}) \to \mathscr{C}([0,1],\mathbb{R})$ be linear and positive. Let $\tilde{L} : \mathscr{C}([0,1],\mathbb{C}) \to \mathscr{C}([0,1],\mathbb{C})$ be the complexified operator. Then $\|\tilde{L}\| = \|L\|$.*

Proof. Obviously $\|\tilde{L}\| \geq \|L\|$. To prove the reverse inequality, let $u = f + ig \in \mathscr{C}([0,1],\mathbb{C})$, $x \in [0,1]$ and $a + ib := \tilde{L}(u)(x) = L(f)(x) + iL(g)(x)$.

Let $\theta := \arg(a+ib)$ and

$$\alpha := (\operatorname{sgn} a)(\operatorname{sgn}\cos\theta)\cos\theta, \qquad \beta := (\operatorname{sgn} b)(\operatorname{sgn}\sin\theta)\sin\theta.$$

Then $a\sin\theta = b\cos\theta$ and $|a+ib| = \sqrt{a^2+b^2} = |a\cos\theta + b\sin\theta| = \alpha a + \beta b$. Therefore $|\tilde{L}(u)(x)| = |a+ib| = \alpha a + \beta b = L(\alpha f + \beta g)(x)$.

Since L is positive, we have

$$|\tilde{L}(u)(x)| \leq L\big(|\alpha f + \beta g|\big)(x) \leq L\big(\sqrt{f^2+g^2}\big)(x) = L(|u|)(x).$$

It follows that

$$\|\tilde{L}(u)\| \leq \|L(|u|)\| \leq \|L\|\,\|u\|, \qquad u \in \mathscr{C}([0,1],\mathbb{C}).$$

This entails $\|\tilde{L}\| \leq \|L\|$, and the proof is finished. ∎

A *Markov operator* is a positive linear operator T defined on $\mathscr{C}([0,1],\mathbb{R})$ which satisfies $T(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1}$ is the constant function 1.

2. The associated semigroups. Define an operator A with domain

$$D_V(A) := \Big\{u \in \mathscr{C}([0,1],\mathbb{R}) \cap \mathscr{C}^2((0,1),\mathbb{R}) : \lim_{x\to 0,1} W(u)(x) = 0\Big\}, \tag{2.1}$$

as

$$A(u)(x) = \begin{cases} W(u)(x) & \text{when } x \in (0,1), \\ 0 & \text{when } x \in \{0,1\}. \end{cases} \tag{2.2}$$

Then the following theorem holds [CM].

Theorem 2.1.

(i) *The operator $(A, D_V(A))$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of positive contractions on $\mathscr{C}([0,1],\mathbb{R})$.*

(ii) *$\mathscr{C}^2([0,1],\mathbb{R})$ is a core of A.*

(iii) $\lim_{t\to\infty} T(t)(f) = T(f)$ *for $f \in \mathscr{C}([0,1],\mathbb{R})$, where*

$$T(f)(x) := (1-x)f(0) + xf(1), \qquad x \in [0,1].$$

We shall consider the closures of $\tilde{W}$ on $\mathscr{C}([0,1],\mathbb{C})$ and $L^2([0,1],\mathbb{C})$ as possible generators of strongly continuous semigroups on these spaces.

As a consequence of Theorem 2.1 we obtain

PROPOSITION 2.2. *The closure of W in $\mathscr{C}([0,1],\mathbb{R})$ is $(A, D_V(A))$. Moreover, the closure of $\tilde{W}$ in $\mathscr{C}([0,1],\mathbb{C})$ is $(\tilde{A}, D_V(\tilde{A}))$, and it generates a contractive semigroup $(\tilde{T}(t))_{t\ge 0}$ on $\mathscr{C}([0,1],C)$ such that*

$$\lim_{t\to\infty} \tilde{T}(t)(f) = \tilde{T}(f), \quad f \in \mathscr{C}([0,1],\mathbb{C}). \tag{2.3}$$

PROPOSITION 2.3. *Let $M(a,b) := \frac{1}{4}\max_{x\in[0,1]}(x^a(1-x)^b)''$. Then $\tilde{W} - M(a,b)I$ is dissipative in $L^2([0,1],\mathbb{C})$.*

Proof. Apply [MR, Proposition 2.1] with $u(x) = \frac{1}{2}x^a(1-x)^b$ and $v(x) = 0$ for $x \in [0,1]$. ∎

In order to obtain estimates for $M(a,b)$, write

$$M(a,b) = \frac{1}{4}\max_{[0,1]} x^{a-2}(1-x)^{b-2}Q, \tag{2.4}$$

where

$$Q = a(a-1)(1-x)^2 - 2abx(1-x) + b(b-1)x^2 \le \max\{a(a-1), b(b-1)\}.$$

- $a = 2$ or $b = 2$ implies $x^{a-2}(1-x)^{b-2} \le 1$ for $x \in [0,1]$, so that
$$M(a,b) \le \tfrac{1}{4}\max\{a(a-1), b(b-1)\}.$$
- $a > 2$ and $b > 2$ implies $x^{a-2}(1-x)^{b-2} \le \frac{(a-2)^{a-2}(b-2)^{b-2}}{(a+b-4)^{a+b-4}}$, so that
$$M(a,b) \le \frac{1}{4}\,\frac{(a-2)^{a-2}(b-2)^{b-2}}{(a+b-4)^{a+b-4}}\max\{a(a-1), b(b-1)\}.$$

Numerical evaluation of $M(a,b)$ shows that it has a maximum value of 0.5 reached for $a = b = 2$; $M(a,b)$ tends to zero as a and b increase.

THEOREM 2.4. *$\tilde{W} - M(a,b)I$ is closable, and its closure is the infinitesimal generator of a contraction semigroup $(S(t))_{t\ge0}$ on $L^2([0,1],\mathbb{C})$. The closure of $\tilde{W}$, $\overline{(\tilde{W} - M(a,b)I)} + M(a,b)I$, thus generates a C_0-semigroup $(U(t))_{t\ge0}$ connected with $S(t)$ through $U(t) = e^{M(a,b)t}S(t)$. The norm of $U(t)$ is $\|U(t)\| = e^{M(a,b)t}$.*

Proof. Let $\mu > 0$; then the complexified operator $\mu I - \tilde{A}$ is bijective (with range $\mathscr{C}([0,1],\mathbb{C})$). Pick $f \in \mathscr{C}([0,1],\mathbb{C})$ and $u \in D_V(\tilde{A})$ with $\mu u - \tilde{A}u = f$.

Since $\mathscr{C}^2([0,1],\mathbb{C})$ is a core for $\tilde{A}$, there exists a sequence $u_n \in \mathscr{C}^2([0,1],\mathbb{C})$, $n \ge 1$, such that $u_n \to u$ and $\tilde{A}(u_n) \to \tilde{A}(u)$. Thus $u_n \in D(\tilde{W})$ and $\mu u_n - \tilde{W}(u_n) = \mu u_n - \tilde{A}(u_n) \to \mu u - \tilde{A}(u) = f$. This means that $(\mu I - \tilde{W})(u_n) \in R(\mu I - \tilde{W})$ and $(\mu I - \tilde{W})(u_n) \to f$. We conclude that

- $R(\mu I - \tilde{W})$ is uniformly dense in $\mathscr{C}([0,1],\mathbb{C})$, and therefore
- $R(\mu I - \tilde{W})$ is dense in $L^2([0,1],\mathbb{C})$.

Let $\lambda > \max\{0, -M(a,b)\}$. Then $\lambda I - (\tilde{W} - M(a,b)I) = (\lambda + M(a,b))I - \tilde{W}$ with $\lambda + M(a,b) > 0$ has dense range in $L^2([0,1],\mathbb{C})$, and $\tilde{W} - M(a,b)I$ is densely defined and dissipative in $L^2([0,1],\mathbb{C})$. By the Lumer–Phillips theorem the proof is complete. ∎

Since $\tilde{W}(1) = 0$, the semigroup generated by $\tilde{W}$ is a Markov semigroup: $U(t)(1) = 1$ for all $t \geq 0$.

By "decomplexification", the results in this subsection are also valid for $L^2([0,1],\mathbb{R})$, i.e., the closure of W generates a strongly continuous semigroup on $L^2([0,1],\mathbb{R})$.

3. Approximation processes for semigroups. A *positive approximation process* (L_n) on $\mathscr{C}([0,1],\mathbb{R})$ is a sequence of positive linear operators from $\mathscr{C}([0,1],\mathbb{R})$ in $\mathscr{C}([0,1],\mathbb{R})$ such that $\lim_{n\to\infty} L_n(f) = f$ for each continuous function f.

Assume there exists a positive approximation process L_n on $\mathscr{C}([0,1],\mathbb{R})$ such that a Voronovskaja-type relation exists between L_n and A:

$$\lim_{n\to\infty} n(L_n(f) - f) = A(f), \qquad f \in \mathscr{C}^2([0,1],\mathbb{R}). \tag{3.1}$$

Under suitable conditions [AR, CM], the semigroup generated by $(A, D_V(A))$ can be approximated by iterates of L_n:

$$T(t)(f) = \lim_{n\to\infty} L_n^{k(n)}(f), \qquad f \in \mathscr{C}([0,1],\mathbb{R}), \tag{3.2}$$

where $(k(n))$ is a sequence of positive integers such that $\lim_{n\to\infty} k(n)/n = t$. Possible types of such L_n sequences can be found in e.g. [AR, CMP2, CM, AC, R].

We shall describe a related approach. A positive approximation process for the differential operator A given by (2.1)–(2.2) can be built as in [R]. For $0 \leq s \leq 1/2$, let $F(s) : \mathscr{C}([0,1],\mathbb{R}) \to \mathscr{C}([0,1],\mathbb{R})$ be a positive linear operator defined by

$$\begin{aligned} F(s)(f)(x) := {} & \tfrac{1}{2}x^{a-1}(1-x)^b f\big((1-\sqrt{2s}x)\big) \\ & + \tfrac{1}{2}x^a(1-x)^{b-1} f\big(\sqrt{2s} + (1-\sqrt{2s})x\big) + \tfrac{1}{2}\big(2 - x^{a-1}(1-x)^{b-1}\big)f(x), \end{aligned} \tag{3.3}$$

where $f \in \mathscr{C}([0,1],\mathbb{R})$, $x \in [0,1]$. For this process, $F(s)(e_0) = e_0$, $F(s)(e_1) = e_1$ and $F(s)(e_2) = x^2 + sx^a(1-x)^b$, with $x \in [0,1]$ and $e_i(x) := x^i$, $i \in \{0,1,2\}$. Then [R, Theorem 4]

$$\lim_{n\to\infty} F(t/n)^n(f) = T(t)(f) \tag{3.4}$$

for each $t \geq 0$ and $f \in \mathscr{C}([0,1],\mathbb{R})$.

The kth iterate of the complexification of L coincides with the complexification of the kth iterate of L. Together with (3.2) and (3.4) this leads to

$$\tilde{T}(t)(f)=\lim_{n\to\infty}\tilde{L}_n^{k(n)}(f),\qquad f\in\mathscr{C}([0,1],\mathbb{C}),\tag{3.5}$$

$$\tilde{T}(t)(f)=\lim_{n\to\infty}\tilde{F}(t/n)^n(f),\qquad f\in\mathscr{C}([0,1],\mathbb{C}).\tag{3.6}$$

The aim of this section is to build a specific approximation process for L^2 functions, i.e., we want an $L_n^{k(n)}$ such that a relation of the type (3.5) holds on $L^2([0,1],\mathbb{C})$ for $(U(t))_{t\geq 0}$. We do this along the general lines presented in [CMP1].

Consider the differential operator

$$A(u)(x)=p(x)u''(x),\qquad p(x)=\tfrac{1}{2}x^a(1-x)^b,\qquad a,b\geq 2,\tag{3.7}$$

with domain $D_V(A)$ given by (2.1).

Choose a parameter $\eta\in\,]1/2,1[$, and consider an even positive function $\phi\in L^1(-1,1)$ satisfying

$$\int_{-1}^{1}\phi(t)\,dt=1\quad\text{and}\quad\int_{-1}^{1}t^2\phi(t)\,dt=\eta.\tag{3.8}$$

For $n\in\mathbb{N}$ we define the sequence

$$\sigma_n=\sqrt{\frac{2p(x)}{n\eta}}.\tag{3.9}$$

Proposition 3.1. *For sufficiently large n,*

$$0\leq x-t\sigma_n(x)\leq 1,\qquad x\in[0,1],\ t\in[-1,1].\tag{3.10}$$

Proof. It is clear from the definition of $p(x)$ that σ_n is positive, and since t is an arbitrary parameter in $[-1,1]$, it suffices to prove that

$$0\leq x-\sigma_n(x)\quad\text{and}\quad x+\sigma_n(x)\leq 1.\tag{3.11}$$

Explicitly,

$$0\leq x-\frac{x^{a/2}(1-x)^{b/2}}{\sqrt{n\eta}}\quad\text{and}\quad x+\frac{x^{a/2}(1-x)^{b/2}}{\sqrt{n\eta}}\leq 1.\tag{3.12}$$

Since $a,b\geq 2$ and $x\in[0,1]$, we have $x^{a/2}\leq x$ and $(1-x)^{b/2}\leq(1-x)$. Using these and taking n large enough such that $\sqrt{n\eta}>1$, we obtain the desired inequalities. ∎

We are interested in the operators

$$L_n(f)(x)=\int_{-1}^{1}f(x-t\sigma_n(x))\phi(t)\,dt\tag{3.13}$$

in $L^2([0,1],\mathbb{C})$. Notice that for n large enough the sequence is well defined.

To study these operators, we calculate $L_n(e_i)(x)$, where $e_i(x) = x^i$ for $i \in \{0, 1, 2\}$:

$$L_n(e_0)(x) = \int_{-1}^{1} \phi(t)\,dt = 1 = e_0(x), \tag{3.14}$$

$$L_n(e_1)(x) = \int_{-1}^{1} (x - t\sigma_n(x))\phi(t)\,dt = x = e_1(x), \tag{3.15}$$

$$\begin{aligned} L_n(e_2)(x) &= \int_{-1}^{1} (x - t\sigma_n(x))^2\phi(t)\,dt = x^2 + \frac{2p(x)}{n} \\ &= e_2(x) + \frac{2p(x)}{n}. \end{aligned} \tag{3.16}$$

It is clear that

$$\lim_{n\to\infty} L_n(e_i)(x) = e_i(x) \tag{3.17}$$

in $\mathscr{C}([0, 1], \mathbb{C})$.

Let $\omega(f, \delta)$ be the first modulus of continuity of $f \in \mathscr{C}([0, 1], \mathbb{C})$ with $\delta > 0$, defined as

$$\omega(f, \delta) := \sup\{|f(x) - f(y)| : x, y \in [0, 1],\ |x - y| \le \delta\}.$$

Proposition 3.2. *L_n is an approximation process in $\mathscr{C}([0, 1], \mathbb{C})$, i.e., there exists an absolute constant $k > 0$ such that for every $f \in \mathscr{C}([0, 1], \mathbb{C})$,*

$$\|L_n(f) - f\|_\infty \le k\omega(f, 1/\sqrt{n}), \tag{3.18}$$

hence $\|L_n(f) - f\|_\infty \to 0$. Moreover, the sequence $\|L_n\|_2$ is bounded in $L^2([0, 1], \mathbb{C})$.

Proof. To show the first property, we compute

$$\begin{aligned} L_n(f)(x) - f(x) &= \int_{-1}^{1} f(x - t\sigma_n(x))\phi(t)\,dt - f(x)\int_{-1}^{1} \phi(t)\,dt \\ &= \int_{-1}^{1} [f(x - t\sigma_n(x)) - f(x)]\phi(t)\,dt. \end{aligned} \tag{3.19}$$

Then

$$\begin{aligned} |L_n(f)(x) - f(x)| &\le \int_{-1}^{1} |f(x - t\sigma_n(x)) - f(x)|\phi(t)\,dt \\ &\le \omega\left(f, \sqrt{\frac{2\|p\|_\infty}{n\eta}}\right) \le k\omega(f, 1/\sqrt{n}). \end{aligned} \tag{3.20}$$

To show the second property, consider a function $\sigma = \sigma_n$; we want to show that the operator

$$L(f)(x) = \int_{-1}^{1} f(x - t\sigma(x))\phi(t)\,dt \tag{3.21}$$

is bounded. To this end, perform a change of variables $t \to s = x - t\sigma(x)$:

$$L(f)(x) = \int_{x-\sigma(x)}^{x+\sigma(x)} f(s)\phi\left(\frac{x-s}{\sigma(x)}\right)\frac{1}{\sigma(x)}\,ds. \tag{3.22}$$

Then

$$|L(f)(x)| \le \|\phi\|_\infty \frac{1}{\sigma(x)} \int_{x-\sigma(x)}^{x+\sigma(x)} |f(s)|\,ds. \tag{3.23}$$

Using the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(s)|\,ds \tag{3.24}$$

yields

$$|L(f)(x)| \le 2\|\phi\|_\infty Mf(x), \tag{3.25}$$

so that, by the properties of the maximal function (see e.g. [CH]),

$$\|L(f)\|_2 \le 2c_2\|\phi\|_\infty\|f\|_2, \tag{3.26}$$

where c_2 is a constant. ■

It follows that $\|L_n\|_2 \le 2c_2\|\phi\|_\infty$ for $n \ge 1$.

We will also need the fact that $\mathscr{C}^2([0,1],\mathbb{C})$ is contained in $D_V(A)$ and is a core for $(A, D_V(A))$, as shown previously.

THEOREM 3.3. *There exists $c > 0$ such that for every $f \in \mathscr{C}^2([0,1],\mathbb{C})$,*

$$\|n(L_n(f) - f) - pf''\|_\infty \le c\omega(f'', 1/\sqrt{n}), \tag{3.27}$$

hence $n(L_n(f) - f) \to pf''$ uniformly on $[0,1]$.

Proof. By using a Taylor expansion, we get

$$\begin{aligned} &f(x - t\sigma_n(x)) - f(x) \\ &\quad = (-t\sigma_n(x))f'(x) + \tfrac{1}{2}(-t\sigma_n(x))^2 f''(x) + \tfrac{1}{2}(-t\sigma_n(x))^2[f''(\zeta_{x,t}) - f''(x)]. \end{aligned}$$

Multiplying by $\phi(t)$ and integrating over $t \in [-1,1]$ we get

$$L_n(f) - f = \tfrac{1}{2} f''\eta\sigma_n^2 + R_n, \tag{3.28}$$

where

$$R_n(x) = \frac{1}{2}\int_{-1}^{1} (-t\sigma_n(x))^2[f''(\zeta_{x,t}) - f''(x)]\phi(t)\,dt. \tag{3.29}$$

We will obtain an upper bound for $R_n(x)$.

Since $\zeta_{x,t}$ is between x and $x + t\sigma_n(x)$, it follows that $|\zeta_{x,t} - x| \leq |t|\sigma_n(x) \leq \sigma_n(x)$. But $\sigma_n(x) = \sqrt{2p(x)/(n\eta)}$, where $p(x) = x^a(1-x)^b/2$. With $\sqrt{x^a(1-x)^b/\eta} \leq \theta$ (a positive constant), we get

$$|\zeta_{x,t} - x| \leq \theta/\sqrt{n}.$$

Together with the definition and properties of the modulus of continuity, we obtain

$$|f''(\zeta_{x,t}) - f''(x)| \leq \omega(f'', |\zeta_{x,t} - x|) = \omega(f'', \theta/\sqrt{n}) = (1+\theta)\omega(f'', 1/\sqrt{n}).$$

Consequently,

$$R_n(x) \leq \tfrac{1}{2}(1+\theta)\omega(f'', 1/\sqrt{n})\sigma_n^2(x) \int_{-1}^{1} t^2\phi(t)\,dt.$$

By using the definition of $\sigma_n(x)$ and the integral properties of $\phi(t)$, we get

$$R_n(x) \leq \frac{1+\theta}{n} p(x)\omega(f'', 1/\sqrt{n}).$$

With $(1+\theta)p(x) \leq c$ (some positive constant) we obtain

$$R_n(x) \leq \frac{c}{n}\omega(f'', 1/\sqrt{n}). \tag{3.30}$$

In the limit $n \to \infty$, we have $nR_n \to 0$, and thus $n(L_n(f) - f) \to pf''$. ∎

THEOREM 3.4. *With $\phi(t)$ as in* [CMP1, (57)], *successive iterations of L_n are stable, i.e.,*

$$\|L_n^k\|_2 \leq e^{Mk/n}. \tag{3.31}$$

Proof. The proof runs as the proof of [CMP1, Lemma 5.10] once the term $\tau_n = a'/n$ in that proof is set to zero. ∎

With all the concepts gathered so far, an application of Trotter's theorem [T] yields

THEOREM 3.5. *For every $t \geq 0$, if $(k(n))$ is a sequence of positive integers such that $k(n)/n \to t$ then*

$$\lim_{n\to\infty} L_n^{k(n)} = U(t) \tag{3.32}$$

strongly on $L^2([0,1], \mathbb{C})$.

Similarly, the same conclusions are valid for $L^2([0,1], \mathbb{R})$.

REMARK 3.6. A series of properties of the complexified operator $\tilde{T}(t)$ may be obtained by using known results for $T(t)$, e.g., from [R].

Let $c := \frac{(a-1)^{a-1}(b-1)^{b-1}}{(a+b-2)^{a+b-2}}$ and $q := \max\{a, b\}$, $0 < p < 1$. Moreover, suppose $|f(x) - f(0)| \leq C_f x$ and $|f(1) - f(x)| \leq K_f(1-x)$ for $x \in [0,1]$.

In [R] it is shown that

$$|T(t)(f)(x) - f(x)| \le \tfrac{1}{2}(1 - e^{-ct})x(1-x)\|f''\| \tag{3.33}$$

for $f \in \mathscr{C}^2([0,1],\mathbb{R})$, $t \ge 0$, $x \in [0,1]$, and

$$\begin{aligned} |T(t)(f)(x) - T(f)(x)| \\ \le (C_f + K_f)\big(x(1-x)e^{-tp} + (q-1)q^{q/(1-q)}t^{(p-1)/(q-1)}\big) \end{aligned} \tag{3.34}$$

for $f \in \mathscr{C}([0,1],\mathbb{R})$, $t \ge 0$, $x \in [0,1]$.

Consider these properties in a general framework of an operator L satisfying $|L(u)(x)| \le \varphi(x)\|u''\|$ for $u \in \mathscr{C}^2([0,1],\mathbb{R})$ and $x \in [0,1]$. We want to find a similar relationship for its complexification $\tilde{L}$ acting on functions $f \in \mathscr{C}^2([0,1],\mathbb{C})$, $f = u + iv$, $u, v \in \mathscr{C}^2([0,1],\mathbb{R})$. We have

$$|\tilde{L}(f)(x)| = |L(u)(x) + iL(v)(x)| = \sqrt{(L(u)(x))^2 + (L(v)(x))^2}.$$

Since $|L(u)(x)| \le \varphi(x)\|u''\|$ (and similarly for $v(x)$), we get

$$|\tilde{L}(f)(x)| \le \sqrt{\varphi^2(x)(\|u''\|^2 + \|v''\|^2)} = \varphi(x)\sqrt{\|u''\|^2 + \|v''\|^2}.$$

But

$$\begin{aligned} \|u''\| &= \max_{x\in[0,1]} |u''(x)| \le \max_{x\in[0,1]} \sqrt{(u''(x))^2 + (v''(x))^2} = \max_{x\in[0,1]} \sqrt{|f''(x)|^2} \\ &\le \max_{x\in[0,1]} |f''(x)| = \|f''\|. \end{aligned}$$

Similarly, $\|v''\| \le \|f''\|$ and thus

$$|\tilde{L}(f)(x)| \le \varphi(x)\sqrt{\|u''\|^2 + \|v''\|^2} \le \sqrt{2}\varphi(x)\|f''\|.$$

With (3.33), (3.34) and with the above by complexification we get

$$|\tilde{T}(t)(f)(x) - f(x)| \le \frac{\sqrt{2}}{2}(1 - e^{-ct})x(1-x)\|f''\| \tag{3.35}$$

for $f \in \mathscr{C}^2([0,1],\mathbb{C})$, $t \ge 0$, $x \in [0,1]$.

Let $f \in \mathscr{C}([0,1],\mathbb{C})$ with

$$|f(x) - f(0)| \le C_f x, \qquad |f(1) - f(x)| \le K_f(1-x);$$

then

$$\begin{aligned} |\tilde{T}(t)(f)(x) - \tilde{T}(f)(x)| \\ \le \sqrt{2}(C_f + K_f)\big(x(1-x)e^{-tp} + (q-1)q^{q/(1-q)}t^{(p-1)/(q-1)}\big). \end{aligned} \tag{3.36}$$

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