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Remarks on the blow-up criterion for the MHD system involving horizontal components or their horizontal gradients

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Abstract. We study the Cauchy problem for the MHD system, and provide two regularity conditions involving horizontal components (or their gradients) in Besov spaces. This improves previous results.

1. Introduction. In this paper, we consider the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$(1.1)\qquad \begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \Delta\boldsymbol{u} + \nabla\pi = \mathbf{0}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{b} = \mathbf{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \nabla\cdot\boldsymbol{b} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{b}(0) = \boldsymbol{b}_0, \end{cases}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, $\boldsymbol{b} = (b_1, b_2, b_3)$ is the magnetic field, π is a scalar pressure, and $\boldsymbol{u}_0, \boldsymbol{b}_0$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0 = \nabla\cdot\boldsymbol{b}_0 = 0$ in the distributional sense. Physically, (1.1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water. Moreover, $(1.1)_1$ reflects the conservation of momentum, $(1.1)_2$ is the induction equation, and $(1.1)_3$ specifies the conservation of mass.

Besides its physical applications, the MHD system (1.1) is also mathematically significant. Duvaut and Lions [5] constructed a global weak solution to (1.1) for initial data with finite energy. However, the issue of regularity and uniqueness of such a given weak solution remains a challenging open problem in mathematical fluid dynamics. Many sufficient conditions

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(see e.g. [2, 3, 4, 8, 9, 11, 13, 12, 17, 22, 23, 25, 24, 28, 29] and the references therein) were derived to guarantee the regularity of the weak solution. Some of them add conditions on the velocity field only (see [3, 8, 28] for example), while some others rely on some components of the velocity and magnetic fields (or their gradients).

In this paper, we are concerned with the regularity conditions in terms of horizontal components (or their gradients). In this respect, Ji–Lee [9] showed that if

$$\begin{aligned} &\boldsymbol{u}_h \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1,\ 3<q\le\infty,\\ &\boldsymbol{b}_h \in L^r(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{r}+\frac{3}{s}=1,\ 3<s\le\infty, \end{aligned} \tag{1.2}$$

then the solution is smooth on $(0,T)$. Here and in what follows, $\boldsymbol{u}_h=(u_1,u_2)$ and $\boldsymbol{b}_h=(b_1,b_2)$ are the horizontal components of $\boldsymbol{u}$ and $\boldsymbol{b}$ respectively.

Very recently, Jia [10] established the following regularity criterion:

$$\begin{aligned} &\nabla_h\boldsymbol{u}_h \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2,\ \frac{3}{2}<q\le\infty,\\ &\nabla_h\boldsymbol{b}_h \in L^r(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{r}+\frac{3}{s}=2,\ \frac{3}{2}<s\le\infty, \end{aligned} \tag{1.3}$$

where ∇_h is the horizontal gradient operator.

The motivation of this paper is to refine (1.2) and (1.3) from the Lebesgue spaces to more general Besov spaces. In fact, some improvements involving BMO spaces, multiplier spaces and Morrey–Campanato spaces have been developed in [1, 7, 26].

Now, our main result reads:

THEOREM 1.1. *Let* $(\boldsymbol{u}_0,\boldsymbol{b}_0)\in H^3(\mathbb{R}^3)$ *with* $\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{b}_0=0$, *and* $T>0$. *Assume that* $(\boldsymbol{u},\boldsymbol{b})$ *is a weak solution pair of the MHD system* (1.1) *with initial data* $(\boldsymbol{u}_0,\boldsymbol{b}_0)$ *on* $(0,T)$. *If*

$$\boldsymbol{u}_h,\boldsymbol{b}_h \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.4}$$

or

$$\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.5}$$

then the solution can be smoothly extended beyond T.

REMARK. Due to the embedding relations $L^\infty(\mathbb{R}^3)\subsetneq \mathrm{BMO}(\mathbb{R}^3)\subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$, this indeed improves (1.2) and (1.3).

The proof of Theorem 1.1 under conditions (1.4) and (1.5) will be provided in Sections 3 and 4 respectively. Before doing that, in Section 2 we introduce BMO spaces and Besov spaces, and establish some bilinear estimates in Hardy spaces.

2. Preliminaries. In this section, we introduce some function spaces which will be frequently used later.

The Hardy space $\mathcal{H}^1(\mathbb{R}^3)$ is the space of locally integrable functions f which satisfy

$$\|f\|_{\mathcal{H}^1} = \Big\|\sup_{t>0} |\phi_t * f|\Big\|_{L^1} < \infty, \tag{2.1}$$

where $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$ for a fixed $\phi \in C_0^\infty(B(0,1))$ with $\phi(x) \geq 0$ and $\int \phi(y)\,dy = 1$. It is well-known that this definition does not depend on the choice of ϕ (see [6]).

The dual of $\mathcal{H}^1(\mathbb{R}^3)$ is $\mathrm{BMO}(\mathbb{R}^3)$, the space of functions of bounded mean oscillation (see [19, Chapter 4]), with the seminorm

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |f - f_B|\,dx < \infty, \tag{2.2}$$

where the supremum is taken over all balls B in $\mathbb{R}^3$, and $f_B = |B|^{-1}\int_B f(y)\,dy$ is the mean value of f over B (one can replace f_B by any costant c_B, which does not affect the definition). Furthermore, we have

$$\Big|\int_{\mathbb{R}^3} f(x)g(x)\,dx\Big| \leq C\|f\|_{\mathrm{BMO}}\|g\|_{\mathcal{H}^1} \tag{2.3}$$

whenever the right-hand side is bounded (see [19, pp. 142–143]).

We need the following bilinear estimates in Hardy spaces.

LEMMA 2.1. *Suppose $f \in W^{1,p}(\mathbb{R}^3)$ and $g \in W^{1,q}(\mathbb{R}^3)$ with $1 < p, q < \infty$ and $1/p + 1/q = 1$. Then $\nabla(fg)$ is in $\mathcal{H}^1(\mathbb{R}^3)$. Furthermore,*

$$\|\nabla(fg)\|_{\mathcal{H}^1} \leq C\|\nabla f\|_{L^p}\|g\|_{L^q} + C\|f\|_{L^p}\|\nabla g\|_{L^q}, \tag{2.4}$$

where C is independent of f and g.

Proof. We will borrow some ideas from [20]. By a density argument, we may assume that $f, g \in C_0^\infty(\mathbb{R}^3)$. Denote

$$f_{B(x,\varepsilon)} = \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f(y)\,dy, \qquad g_{B(x,\varepsilon)} = \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} g(y)\,dy.$$

Then for any $1 \leq k \leq 3$,

$$\begin{aligned} |\phi_\varepsilon * \partial_k(fg)(x)| &= \Big|\int_{B(x,\varepsilon)} \phi_\varepsilon(x-y)\partial_k(fg - f_{B(x,\varepsilon)}g_{B(x,\varepsilon)})\,dy\Big| \\ &= \Big|\int_{B(x,\varepsilon)} \partial_k\phi_\varepsilon(x-y)[(f - f_{B(x,\varepsilon)})g + f_{B(x,\varepsilon)}(g - g_{B(x,\varepsilon)})]\,dy\Big| \end{aligned} \tag{2.5}$$

$$\leq \frac{C}{\varepsilon^4} \int\limits_{B(x,\varepsilon)} |f - f_{B(x,\varepsilon)}| \cdot |g|\, dy + \frac{C}{\varepsilon^4} \int\limits_{B(x,\varepsilon)} |f_{B(x,\varepsilon)}| \cdot |g - g_{B(x,\varepsilon)}|\, dy$$
$$\equiv I + J.$$

By the Hölder and Sobolev inequalities,

$$I_1 \leq \frac{C}{\varepsilon^4}\Big(\int\limits_{B(x,\varepsilon)} |f - f_{B(x,\varepsilon)}|^s\, dy\Big)^{1/s} \cdot \Big(\int\limits_{B(x,\varepsilon)} |g|^{\frac{s}{s-1}}\, dy\Big)^{\frac{s-1}{s}}$$
$$\leq \frac{C}{\varepsilon^4}\Big(\int\limits_{B(x,\varepsilon)} |\nabla f|^{\frac{3s}{s+3}}\, dy\Big)^{\frac{s+3}{3s}} \cdot \Big(\int\limits_{B(x,\varepsilon)} |g|^{\frac{s}{s-1}}\, dy\Big)^{\frac{s-1}{s}},$$

where we choose s so that

$$1 < s < \infty, \quad 1 \leq \frac{3s}{s+3} < p, \quad \frac{s}{s-1} < q.$$

By the definition of the Hardy–Littlewood maximal function (see [19, p. 13]),

$$Mv(x) = \sup_{\varepsilon>0} \frac{1}{|B(x,\varepsilon)|} \int\limits_{B(x,\varepsilon)} |v(y)|\, dy, \quad v \in L^1_{loc}(\mathbb{R}^3),$$

we may dominate I_1 further as

$$I_1 \leq C\Big(\frac{1}{|B(x,\varepsilon)|}\int\limits_{B(x,\varepsilon)} |\nabla f|^{\frac{3s}{s+3}}\, dy\Big)^{\frac{s+3}{3s}} \cdot \Big(\frac{1}{|B(x,\varepsilon)|}\int\limits_{B(x,\varepsilon)} |g|^{\frac{s}{s-1}}\, dy\Big)^{\frac{s-1}{s}}$$
$$\leq C[M(|\nabla f|^{\frac{3s}{s+3}})]^{\frac{s+3}{3s}} \cdot [M(|g|^{\frac{s}{s-1}})]^{\frac{s-1}{s}}.$$

Thanks to the Hardy–Littlewood maximal theorem (see [19, p. 13]),

$$\|I\|_{L^1} \leq C\big\|[M(|\nabla f|^{\frac{3s}{s+3}})\big\|^{\frac{s+3}{3s}}]_{L^p} \cdot \big\|[M(|g|^{\frac{s}{s-1}})]^{\frac{s-1}{s}}\big\|_{L^q} \tag{2.6}$$
$$\leq C\big\|M(|\nabla f|^{\frac{3s}{s+3}})\big\|^{\frac{s+3}{3s}}_{L^{\frac{p(s+3)}{3s}}} \big\|M(|g|^{\frac{s}{s-1}})\big\|^{\frac{s-1}{s}}_{L^{\frac{q(s-1)}{s}}}$$
$$\leq C\big\||\nabla f|^{\frac{3s}{s+3}}\big\|^{\frac{s+3}{3s}}_{L^{\frac{p(s+3)}{3s}}} \big\||g|^{\frac{s}{s-1}}\big\|^{\frac{s-1}{s}}_{L^{\frac{q(s-1)}{s}}}$$
$$\leq C\|\nabla f\|_{L^p}\|g\|_{L^q}.$$

We are now ready to estimate J. By the Hölder and Sobolev inequalities with

$$1 < t < \infty, \quad 1 \leq \frac{3t}{t+3} < q, \quad \frac{t}{t-1} < p,$$

we obtain

$$J \leq \frac{C}{\varepsilon^4}|f_{B(x,\varepsilon)}| \cdot \int\limits_{B(x,\varepsilon)} |g - g_{B(x,\varepsilon)}|\, dy$$
$$\leq \frac{C}{\varepsilon^7}\int\limits_{B(x,\varepsilon)} |f|\, dy \cdot \int\limits_{B(x,\varepsilon)} |g - g_{B(x,\varepsilon)}|\, dy$$

$$\le \frac{C}{\varepsilon^4}\Big(\int\limits_{B(x,\varepsilon)} |f|^{\frac{t}{t-1}}\,dy\Big)^{\frac{t-1}{t}}\Big(\int\limits_{B(x,\varepsilon)} |g-g_{B(x,\varepsilon)}|^t\,dy\Big)^{1/t}$$

$$\le \frac{C}{\varepsilon^4}\Big(\int\limits_{B(x,\varepsilon)} |f|^{\frac{t}{t-1}}\,dy\Big)^{\frac{t-1}{t}}\Big(\int\limits_{B(x,\varepsilon)} |\nabla g|^{\frac{3t}{t+3}}\,dy\Big)^{\frac{t+3}{3t}}.$$

Then, we may argue as (2.6) to conclude that

$$\|J\|_{L^1} \le C\|f\|_{L^p}\|\nabla g\|_{L^q}. \tag{2.7}$$

Plugging (2.6) and (2.7) into (2.5), we get (2.4) as desired. ∎

To introduce the definition of Besov spaces, we need to define the Littlewood–Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int\limits_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\,dx.$$

Let us choose a non-negative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \le \hat{\varphi}(\xi) \le 1, \qquad \hat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| \ge 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2), \quad \varphi_j(x) = 2^{3j}\varphi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood–Paley projection operators S_j and Δ_j are, respectively, defined by

$$S_j f = \varphi_j * f, \qquad \Delta_j f = \psi_j * f.$$

Observe that $\Delta_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \to 0 \quad \text{as } j \to -\infty; \qquad S_j f \to f \quad \text{as } j \to \infty,$$

in the L^2 sense. By telescoping the series, we have the Littlewood–Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f \tag{2.8}$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense. Notice that

$$\dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \dot{\Delta}_l \dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f,$$

so from Young's inequality, it readily follows that

$$\|\dot{\Delta}_j f\|_{L^q} \le C2^{3j(1/p-1/q)}\|\dot{\Delta}_j f\|_{L^p}, \tag{2.9}$$

where $1 \le p \le q \le \infty$, and C is an absolute constant independent of f and j.

Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as

$$\dot{B}^s_{p,q} = \big\{ f \in \mathcal{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} = \big\| \{2^{js}\|\Delta_j f\|_{L^p}\}_{j=-\infty}^{\infty} \big\|_{\ell^q} < \infty \big\},$$

where $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \{ f \in \mathcal{S}(\mathbb{R}^3);\ D^\alpha \hat{f}(0) = 0,\ \forall \alpha \in \mathbb{N}^3 \}.$$

It is well-known that (see [21] for example) for all $s \in \mathbb{R}$,

$$\dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3), \quad L^\infty(\mathbb{R}^3) \subsetneq \mathrm{BMO}(\mathbb{R}^3) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^3). \tag{2.10}$$

We end this section by collecting some nice structures of the convective terms of the MHD system (1.1) for later reference (see also [9, 10]).

Lemma 2.2. *For a smooth solution $\boldsymbol{u}, \boldsymbol{b}$ of the MHD system,*

$$\begin{aligned} &\int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx \\ &\qquad + \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\Delta\boldsymbol{b}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{b}\,dx \\ &\qquad \le \begin{cases} C\displaystyle\int_{\mathbb{R}^3} |(\boldsymbol{u}_h, \boldsymbol{b}_h)|\cdot\big|\nabla_h(|\nabla(\boldsymbol{u},\boldsymbol{b})|^2)\big|\,dx \\ C\displaystyle\int_{\mathbb{R}^3} |(\nabla_h\boldsymbol{u}_h, \nabla_h\boldsymbol{b}_h)|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,dx. \end{cases} \end{aligned} \tag{2.11}$$

Proof. The proof follows ideas from [27]. Due to the divergence-free condition $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0$ and its consequence

$$\begin{aligned} &\sum_{k=1}^{3}\int_{\mathbb{R}^3} \big\{[(\boldsymbol{b}\cdot\nabla)\partial_k\boldsymbol{b}]\cdot\partial_k\boldsymbol{u} + [(\boldsymbol{b}\cdot\nabla)\partial_k\boldsymbol{u}]\cdot\partial_k\boldsymbol{b}\big\}\,dx \\ &\qquad = \sum_{k=1}^{3}\int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\nabla)(\partial_k\boldsymbol{b}\cdot\partial_k\boldsymbol{u})\,dx = -\sum_{k=1}^{3}\int_{\mathbb{R}^3} (\nabla\cdot\boldsymbol{b})(\partial_k\boldsymbol{b}\cdot\partial_k\boldsymbol{u})\,dx = 0, \end{aligned}$$

we may integrate by parts to get

$$\begin{aligned} &\int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx \\ &\qquad = -\sum_{k=1}^{3}\int_{\mathbb{R}^3} [(\partial_k\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_k\boldsymbol{u}\,dx + \sum_{k=1}^{3}\int_{\mathbb{R}^3} [(\partial_k\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\partial_k\boldsymbol{u}\,dx \\ &\qquad\quad - \sum_{k=1}^{3}\int_{\mathbb{R}^3} [(\partial_k\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\partial_k\boldsymbol{b}\,dx + \sum_{k=1}^{3}\int_{\mathbb{R}^3} [(\partial_k\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\partial_k\boldsymbol{b}\,dx. \end{aligned} \tag{2.12}$$

Each term on the right-hand side of (2.12) can be written as

$$\pm \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k f_j \partial_j g_i \partial_k h_i \, dx \quad (\{f,g,h\} \subset \{u,b\}).$$

We classify the terms $\partial_k f_j \partial_j g_i \partial_k h_i$ $(1 \le i,j,k \le 3)$ as follows:

(1) if $k = j = 3$, or $j = i = 3$, or $i = k = 3$, then we invoke the divergence-free condition to replace $\partial_3 f_3$ (resp. $\partial_3 g_3$, $\partial_3 h_3$) by $-\partial_1 f_1 - \partial_2 f_2$ (resp. $-\partial_1 g_1 - \partial_2 g_2$, $-\partial_1 h_1 - \partial_2 h_2$);

(2) otherwise, at least two indices belong to $\{1,2\}$.

After this operation, we easily deduce (2.11) by some further integration by parts. ∎

3. Proof of Theorem 1.1 under condition (1.4). In this section, we shall prove Theorem 1.1 under condition (1.4).

It is well-known (see [18] for example) that (1.1) has a local strong solution

$$(\boldsymbol{u},\boldsymbol{b}) \in L^\infty(0,\Gamma^*; H^3(\mathbb{R}^3)) \cap L^2(0,\Gamma^*; H^4(\mathbb{R}^3)).$$

If $\Gamma^* \ge T$, then there is nothing to prove. Otherwise, we need to show that $\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$ is uniformly bounded as $t \nearrow \Gamma^*$. The standard continuity argument then shows that the solution can be extended smoothly past Γ^*, which contradicts the fact that Γ^* is the maximal existence time.

By (1.4), there exists a $\Gamma < \Gamma^*$ such that

$$\int_{\Gamma}^{\Gamma^*} \|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\dot B^0_{\infty,\infty}}^2 \, dt < \varepsilon, \tag{3.1}$$

where $0 < \varepsilon \ll 1$ is to be determined later on.

Multiplying $(1.1)_1$ with $\boldsymbol{u}$, $(1.1)_2$ with $\boldsymbol{b}$, and integrating in $\mathbb{R}^3$, we may invoke the fact that $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0$ to deduce

$$\frac{1}{2}\frac{d}{dt}\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 = 0.$$

Integrating in time, we get the fundamental energy estimate

$$\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2}(t) + 2\int_0^t \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}(s)\, ds \le \|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{L^2}^2 < \infty. \tag{3.2}$$

Taking the inner product of $(1.1)_1$ with $-\Delta \boldsymbol{u}$, $(1.1)_2$ with $-\Delta \boldsymbol{b}$ in $L^2(\mathbb{R}^3)$ respectively, adding the resulting equations together and invoking Lemma 2.2 we obtain

$$\begin{aligned}(3.3)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 &= \int_{\mathbb{R}^3}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx-\int_{\mathbb{R}^3}[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx\\ &\quad+\int_{\mathbb{R}^3}[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\Delta\boldsymbol{b}\,dx-\int_{\mathbb{R}^3}[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{b}\,dx\\ &\le C\int_{\mathbb{R}^3}|(\boldsymbol{u}_h,\boldsymbol{b}_h)|\cdot|\nabla_h(|\nabla(\boldsymbol{u},\boldsymbol{b})|^2)|\,dx\equiv I.\end{aligned}$$

By (2.3) and (2.1), the term I may be dominated as

$$\begin{aligned}(3.4)\qquad I &\le C\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\mathrm{BMO}}\|\nabla_h(|\nabla(\boldsymbol{u},\boldsymbol{b})|^2)\|_{\mathcal{H}^1}\\ &\le C\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\mathrm{BMO}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\\ &\le C\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\mathrm{BMO}}^2\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\tfrac{1}{2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.\end{aligned}$$

Plugging (3.4) into (3.3), and absorbing the diffusive term, we get

$$(3.5)\qquad \frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\le C\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\mathrm{BMO}}^2\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

To transfer the larger BMO norm to the smaller $\dot{B}^0_{\infty,\infty}$ norm, we invoke the following logarithmically improved Sobolev inequality of [16]:

$$(3.6)\qquad \|f\|_{\mathrm{BMO}}\le C\|f\|_{\dot{B}^0_{\infty,2}}\le C[1+\|f\|_{\dot{B}^0_{\infty,\infty}}\ln^{1/2}(e+\|\nabla^3 f\|_{L^2})],$$

to obtain

$$\begin{aligned}(3.7)\qquad &\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\ &\quad\le C\big[1+\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2})\big]\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.\end{aligned}$$

Applying the Gronwall inequality, we arrive at

$$\begin{aligned}(3.8)\qquad &\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t)+\int_{\Gamma}^{t}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(s)\,ds\le\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(\Gamma)\\ &\quad\cdot\exp\Big\{C\int_{\Gamma}^{t}\big[1+\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2})\big](s)\,ds\Big\}.\end{aligned}$$

Denoting

$$y(t)=\sup_{s\in[\Gamma,t]}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2},$$

and noticing the monotonicity of $y(t)$, we deduce

$$\begin{aligned}(3.9)\qquad &\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t)+\int_\Gamma^t\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(s)\,ds\\ &\le C(\Gamma)\cdot\exp\Big\{C\int_\Gamma^t\big[1+\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|^2_{\dot B^0_{\infty,\infty}}\ln(e+y(s))\big]\,ds\Big\}\\ &\le C\exp\Big[C\ln(e+y(t))\cdot\int_\Gamma^t\|(\boldsymbol{u}_h,\boldsymbol{b}_h)\|^2_{\dot B^0_{\infty,\infty}}\,ds\Big]\\ &\le C\exp[C\ln(e+y(t))\cdot\varepsilon]\le C[e+y(t)]^{C\varepsilon}.\end{aligned}$$

To get the H^3-estimate, we apply ∇^3 to $(1.1)_{1,2}$, multiply the resulting equations by $\nabla^3\boldsymbol{u}$ and $\nabla^3\boldsymbol{b}$ respectively, and sum them up to obtain

$$\begin{aligned}(3.10)\qquad &\frac12\frac{d}{dt}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\ &=-\int_{\mathbb{R}^3}\nabla^3[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\nabla^3\boldsymbol{u}\,dx-\int_{\mathbb{R}^3}\nabla^3[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\nabla^3\boldsymbol{b}\,dx\\ &\quad+\int_{\mathbb{R}^3}\{\nabla^3[(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\cdot\nabla^3\boldsymbol{u}+\nabla^3[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\nabla^3\boldsymbol{b}\}\,dx\\ &=-\int_{\mathbb{R}^3}[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\cdot\nabla^3\boldsymbol{u}\,dx-\int_{\mathbb{R}^3}[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{b}\cdot\nabla^3\boldsymbol{b}\,dx\\ &\quad+\int_{\mathbb{R}^3}\{[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{b}\cdot\nabla^3\boldsymbol{u}+[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{u}\cdot\nabla^3\boldsymbol{b}\}\,dx\equiv J\end{aligned}$$

($[f,g]=fg-gf$, and we use the incompressibility condition). To proceed further, we recall the following commutator estimate due to Kato–Ponce [15]:

$$(3.11)\qquad \|[\Lambda^s,f]g\|_{L^p}\le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}}+\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}})$$

with

$$s>0,\qquad p_2,p_3\in(1,\infty),\qquad \frac1p=\frac1{p_1}+\frac1{p_2}=\frac1{p_3}+\frac1{p_4}.$$

Consequently,

$$\begin{aligned}(3.12)\qquad J&\le\|[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\|_{L^{4/3}}\|\nabla^3\boldsymbol{u}\|_{L^4}+\|[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{b}\|_{L^{4/3}}\|\nabla^3\boldsymbol{b}\|_{L^4}\\ &\quad+\|[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{b}\|_{L^{4/3}}\|\nabla^3\boldsymbol{u}\|_{L^4}+\|[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{u}\|_{L^{4/3}}\|\nabla^3\boldsymbol{b}\|_{L^4}\\ &\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^4}\cdot\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^4}\quad\text{(by (3.11))}\\ &\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/4}\|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/4}\end{aligned}$$

(by the Gagliardo–Nirenberg inequality $\|\nabla^3 f\|_{L^4}\le C\|\nabla^2 f\|_{L^2}^{1/8}\|\nabla^4 f\|_{L^2}^{7/8}$)

$$\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^8\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\tfrac12\|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

Plugging (3.12) into (3.10), and absorbing the diffusing term, we get

$$\frac{d}{dt}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^8\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2. \tag{3.13}$$

Integrating this over (T_0,t), we find that

$$\begin{aligned}
&\|\nabla^3(\boldsymbol{u},\boldsymbol{b})(t)\|_{L^2}^2 \\
&\quad\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C\int_{T_0}^{t} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^8\|\nabla^2(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,d\tau \\
&\quad\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C\sup_{T_0<\tau<t}\|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^8\int_{T_0}^{t}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,d\tau \\
&\quad\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{4C\varepsilon}\cdot[e+y(t)]^{C\varepsilon} \quad \text{(by (3.9))} \\
&\quad\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{5C\varepsilon}.
\end{aligned}$$

Thus,

$$e+y(t) \le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{5C\varepsilon}.$$

Choosing $\varepsilon = 1/(10C)$, we deduce

$$y(t) \le C(\|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}, T_0, T) < \infty,$$

as desired. The proof of Theorem 1.1 is thus complete.

4. Proof of Theorem 1.1 under condition (1.5). In this section, we prove Theorem 1.1 under condition (1.5).

By (3.13), we only need to show that

$$\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}(t) \le C, \qquad \forall\, 0 \le t < \Gamma^*.$$

For this, we invoke (2.11) to write (3.3) as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 &+ \|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \\
&\le C\int_{\mathbb{R}^3} |(\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h)|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,dx \equiv K.
\end{aligned} \tag{4.1}$$

To estimate K, we invoke the Littlewood–Paley decomposition (2.8) to write

$$\begin{aligned}
(\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h) = \sum_{l<-N}\Delta_l(\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h) &+ \sum_{l=-N}^{N}\Delta_l(\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h) \\
&+ \sum_{l>N}\Delta_l(\nabla_h\boldsymbol{u}_h,\nabla_h\boldsymbol{b}_h),
\end{aligned}$$

where N is a positive integer to be determined. Substituting this into K, we obtain

$$\begin{aligned}K &\le C\sum_{l<-N}\int_{\mathbb{R}^3}|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,dx\\&\quad+C\sum_{l=-N}^{N}\int_{\mathbb{R}^3}|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,dx\\&\quad+C\sum_{l>N}\int_{\mathbb{R}^3}|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,dx\\&\equiv K_1+K_2+K_3.\end{aligned}$$

Using the Hölder inequality, (2.9) and the Young inequality, we obtain

$$\begin{aligned}K_1 &\le C\sum_{l<-N}\|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{L^\infty}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\le C\sum_{l<-N}2^{-3l/2}\|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{L^2}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\le 2^{-3N/2}\|(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{L^2}\cdot\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\le [C2^{-N}(\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2)]^{3/2}.\end{aligned}$$

For K_2, from the Hölder inequality,

$$\begin{aligned}K_2 &\le C\sum_{l=-N}^{N}\|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{L^\infty}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\le CN\|(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.\end{aligned}$$

And finally, by the Hölder and Gagliardo–Nirenberg inequalities, (2.9) and (2.10), K_3 can be estimated as

$$\begin{aligned}K_3 &\le C\sum_{l>N}\|\Delta_l(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{L^3}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^3}^2\\&\le C\sum_{l>N}2^{l/2}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\cdot\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\\&\le C\Big(\sum_{l>N}2^{-l}\Big)^{1/2}\Big(\sum_{l>N}2^{2l}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\Big)^{1/2}\cdot\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\\&\le [C2^{-N}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2]^{1/2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.\end{aligned}$$

Combining the bounds of K_i $(1\le i\le 3)$, and substituting into (4.1), we are led to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2+\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\quad\le [C2^{-N}(\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2)]^{3/2}+CN\|(\nabla_h \boldsymbol{u}_h,\nabla_h \boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\&\qquad+[C2^{-N}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2]^{1/2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.\end{aligned}\tag{4.2}$$

Now, we choose N as small as possible to satisfy $C2^{-N}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \leq 1/4$, that is,

$$N \geq \frac{2\ln[e + C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2]}{\ln 2} + 2,$$

and we find that (4.2) implies

$$\frac{d}{dt}\|(\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \\ \leq C + C\|(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}} \ln[e + C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2]\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

Applying the Gronwall inequality twice, we gather that

$$\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2(t) \leq C\exp\Big\{\exp\Big[C\int_0^T \|(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{b}_h)\|_{\dot{B}^0_{\infty,\infty}}(s)\,ds\Big]\Big\} < \infty$$

for any $t \in [0, \Gamma^*)$. This completes the proof of Theorem 1.1.

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