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2-local Lie isomorphisms of operator algebras on Banach spaces

by

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Abstract. Let X and Y be complex Banach spaces of dimension greater than 2. We show that every 2-local Lie isomorphism ϕ of $B(X)$ onto $B(Y)$ has the form $\phi = \varphi + \tau$, where φ is an isomorphism or the negative of an anti-isomorphism of $B(X)$ onto $B(Y)$, and τ is a homogeneous map from $B(X)$ into $\mathbb{C}I$ vanishing on all finite sums of commutators.

1. Introduction and preliminaries. Let $\mathcal{A}$ be an associative algebra. A linear bijection ϕ from $\mathcal{A}$ onto another algebra is called a *Lie isomorphism* if $\phi([A, B]) = [\phi(A), \phi(B)]$ for all $A, B \in \mathcal{A}$. Here $[A, B] = AB - BA$ is the usual Lie product, also called a commutator. The study of Lie isomorphisms of associative algebras and operator algebras, primarily focusing upon their relations to associative (anti-)isomorphisms, has a long history. See [2, 3, 6, 14, 15, 16] and the references therein.

A well known direction in the study of the local action of maps is the local map problem. Let $\mathcal{A}$ be an algebra. Recall that a linear map θ of $\mathcal{A}$ is called a *local isomorphism* (respectively, *local derivation*) if for each $A \in \mathcal{A}$, there exists an isomorphism (respectively, a derivation) θ_A, depending on A, such that $\theta(A) = \theta_A(A)$. Those two notions were introduced in 1990 independently by Kadison [9] and Larson and Sourour [11]. Since then, local isomorphisms and local derivations have been studied for various algebras: see for example [17, 5, 21, 7, 8] and the references therein.

In 1997, Šemrl [19] introduced the notion of 2-local maps. A map δ of an algebra $\mathcal{A}$ (without assumption of the linearity) is called a 2-*local isomorphism* (respectively, 2-*local derivation*) if for any $A, B \in \mathcal{A}$, there exists an isomorphism (respectively, a derivation) $\delta_{A,B}$ of $\mathcal{A}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$. 2-local maps have been studied on different operator

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algebras by many authors [1, 19, 10, 12, 13]. In [19], Šemrl described 2-local derivations and 2-local isomorphisms on the algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space. A similar description for the finite-dimensional case appeared later in [10]. In [12], 2-local derivations have been described on matrix algebras over finite-dimensional division rings.

Obviously, we can define (2-)local Lie isomorphisms and Lie derivations in a natural way. In the previous paper [4], we characterized (2-)local Lie derivations of operator algebras on Banach spaces. In the present paper, we will study 2-local Lie isomorphisms. Formally, we say that a map ϕ of an algebra $\mathcal{A}$ is a *2-local Lie isomorphism* if for any $A, B \in \mathcal{A}$, there exists a Lie isomorphism $\phi_{A,B}$ of $\mathcal{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$.

Throughout, X is a complex Banach space with topological dual X^*. If $x \in X$ and $f \in X^*$, the rank at most one operator $x \otimes f$ is defined to be the map $y \mapsto f(y)x$ for $y \in X$. It is easy to see that the trace of $x \otimes f$ is $f(x)$, that is, $\mathrm{trace}(x \otimes f) = f(x)$. As usual, if X and Y are Banach spaces, $B(X, Y)$ denotes the set of all bounded linear operators from X to Y, and $B(X, X)$ is denoted simply by $B(X)$.

PROPOSITION 1.1 ([3]). *Let X and Y be complex Banach spaces of dimension greater than 2. Suppose that ϕ is a Lie isomorphism of $B(X)$ onto $B(Y)$. Then one of the following holds.*

(1) *There is an invertible operator T in $B(X, Y)$ and a linear map τ from $B(X)$ into $\mathbb{C}I$ vanishing on each commutator such that $\phi(A) = TAT^{-1} + \tau(A)$ for all $A \in B(X)$.*
(2) *There is an invertible operator S in $B(X^*, Y)$ and a linear map γ from $B(X)$ into $\mathbb{C}I$ vanishing on each commutator such that $\phi(A) = -SA^*S^{-1} + \gamma(A)$ for all $A \in B(X)$.*

LEMMA 1.2 ([20]). *Let A, B, E, F be in $B(X)$ and suppose that E and F are non-zero idempotents. If $EAETF = ETFBF$ for all $T \in B(X)$, then $EAE = \lambda E$ and $FBF = \lambda F$ for some $\lambda \in \mathbb{C}$. In particular, if $EAETF = 0$ for all $T \in B(X)$ then $EAE = 0$; and if $ETFBF = 0$ for all $T \in B(X)$ then $FBF = 0$.*

LEMMA 1.3 ([20]). *Suppose that E and F in $B(X)$ are idempotents and satisfy $EF = FE$. Then the statement "either $EF = 0$ or $(I - E)(I - F) = 0$" is true if and only if $[[E, [E, [T, F]]], F] = [E, [T, F]]$ for all $T \in B(X)$.*

2. 2-local Lie isomorphisms. Our main result reads as follows.

THEOREM 2.1. *Let X and Y be complex Banach spaces of dimension greater than 2. Let ϕ be a surjective 2-local Lie isomorphism from $B(X)$ onto $B(Y)$. Then one of the following holds.*

(1) $\phi = \varphi + \tau$, *where* φ *is an isomorphism from* $B(X)$ *onto* $B(Y)$, *and* τ *is a homogeneous map from* $B(X)$ *into* $\mathbb{C}I$ *vanishing on all finite sums of commutators.*

(2) $\phi = -\varphi + \tau$, *where* φ *is an anti-isomorphism from* $B(X)$ *onto* $B(Y)$, *and* τ *is a homogeneous map from* $B(X)$ *into* $\mathbb{C}I$ *vanishing on all finite sums of commutators.*

The proof will be given in several steps. The main idea is to divide $B(X)$ into the three-by-three block matrix algebra and to identify the behavior of ϕ on each block.

We begin with a trivial step. The proof is a direct verification and we omit it.

Lemma 2.2.

(1) ϕ *is injective and homogeneous;*
(2) ϕ^{-1} *is also a* 2*-local Lie isomorphism;*
(3) ϕ *preserves commutativity;*
(4) $\phi(\mathbb{C}I) = \mathbb{C}I$ *and* $\phi(0) = 0$.

We will make a crucial use of the following result.

Lemma 2.3.

(1) *Let* A *and* B *be in* $B(X)$*. Then* $\phi(A+B) - (\phi(A) + \phi(B)) \in \mathbb{C}I$.
(2) *Let* C *and* D *be in* $B(Y)$*. Then* $\phi^{-1}(C+D) - (\phi^{-1}(C) + \phi^{-1}(D)) \in \mathbb{C}I$.

Proof. We only prove (1); the proof of (2) is similar. Suppose that $f \in X^*$ and $x \in \ker(f)$ and set $F = x \otimes f$. We claim that $\operatorname{trace}(\phi(C)\phi(F)) = \operatorname{trace}(CF)$ for all $C \in B(X)$. Indeed, by Proposition 1.1 and noting that F is a commutator, either there is an invertible operator T in $B(X,Y)$ and a scalar λ such that

$$\phi(C) = TCT^{-1} + \lambda I \quad \text{and} \quad \phi(F) = TFT^{-1},$$

or there is an invertible operator S in $B(X^*,Y)$ and a scalar η such that

$$\phi(C) = -SC^*S^{-1} + \eta I \quad \text{and} \quad \phi(F) = -SF^*S^{-1}.$$

(In either case, we see that $\phi(F)$ is of rank one.) If the former case occurs, we have $\phi(C)\phi(F) = TCFT^{-1} + \lambda TFT^{-1}$ and then $\operatorname{trace}(\phi(C)\phi(F)) = \operatorname{trace}(CF)$; if the latter case occurs, we have $\phi(C)\phi(F) = S(FC)^*S^{-1} - \eta SF^*S^{-1}$ and then $\operatorname{trace}(\phi(C)\phi(F)) = \operatorname{trace}(FC) = \operatorname{trace}(CF)$.

Therefore

$$\operatorname{trace}(\phi(A+B)\phi(F)) = \operatorname{trace}((A+B)F) = \operatorname{trace}\big((\phi(A)+\phi(B))\phi(F)\big),$$

and so $\operatorname{trace}((\phi(A+B) - (\phi(A)+\phi(B)))\phi(F)) = 0$. Hence

$$\operatorname{trace}\big(\phi^{-1}(\phi(A+B) - (\phi(A)+\phi(B)))F\big) = 0.$$

That is,

$$f\big(\phi^{-1}(\phi(A+B)-(\phi(A)+\phi(B)))x\big)=0$$

for all $f\in X^*$ and $x\in\ker(f)$. This implies that $\phi^{-1}(\phi(A+B)-(\phi(A)+\phi(B))\in\mathbb{C}I$. So $\phi(A+B)-(\phi(A)+\phi(B))\in\mathbb{C}I$.

Since the dimension of X is greater than 2, there exist three non-trivial idempotent operators P_1,P_2,P_3 on X such that $P_1+P_2+P_3=I$ and $P_iP_j=0$ for all $i\neq j$. For each $i\in\{1,2,3\}$, by Proposition 1.1, there exists an idempotent operator Q_i in $B(Y)$ such that $\phi(P_i)-Q_i$ is a scalar multiple of I. Since P_i is non-trivial, it follows from Lemma 2.2 that Q_i is also non-trivial. Therefore, such a Q_i is unique. In the foregoing, we shall fix those P_i and Q_i.

In the rest, for $A,B\in B(X)$, the symbol $\phi_{A,B}$ stands for a Lie isomorphism from $B(X)$ onto $B(Y)$ such that $\phi(A)=\phi_{A,B}(A)$ and $\phi(B)=\phi_{A,B}(B)$.

Lemma 2.4. *Either $Q_iQ_j=0$ for all $i\neq j$, or $(I-Q_i)(I-Q_j)=0$ for all $i\neq j$.*

Proof. Since any two of $\{P_1,P_2,P_3\}$ commute, it follows that any two of $\{Q_1,Q_2,Q_3\}$ commute. Making use of the necessity of Lemma 1.3, we have $[[P_i,[P_i,[T,P_j]]],P_j]=[P_i,[T,P_j]]$ for all $T\in B(X)$, $i\neq j$. Applying the Lie isomorphism ϕ_{P_i,P_j} to both sides of this identity and noting that ϕ_{P_i,P_j} is surjective, we find that $[[Q_i,[Q_i,[S,Q_j]]],Q_j]=[Q_i,[S,Q_j]]$ for all $S\in B(Y)$. Making use of the sufficiency of Lemma 1.3, either $Q_iQ_j=0$ or $(I-Q_i)(I-Q_j)=0$. If $(I-Q_1)(I-Q_2)=(I-Q_1)(I-Q_3)=0$ but $Q_2Q_3=0$, then $I-Q_1=(I-Q_1)Q_2=(I-Q_1)(I-Q_3)Q_2=0$. This conflicts with the fact that $Q_1\neq I$, completing the proof.

In the following, we say that ϕ is 1-*type* if $Q_iQ_j=0$ for all $i\neq j$, and 2-*type* if $(I-Q_i)(I-Q_j)=0$ for all $i\neq j$. If ϕ is 1-type, we define $Q_i'=Q_i$, $i=1,2,3$; when ϕ is 2-type, we define $Q_i'=I-Q_i$, $i=1,2,3$. Note that $Q_1'+Q_2'+Q_3'$ is idempotent.

Lemma 2.5.

(1) $Q_1'+Q_2'+Q_3'=I$.
(2) *If ϕ is 1-type, then $\phi(P_i)\in Q_i'+\mathbb{C}I$ and $\phi^{-1}(Q_i')\in P_i+\mathbb{C}I$, $i=1,2,3$.*
(3) *If ϕ is 2-type, then $\phi(P_i)\in -Q_i'+\mathbb{C}I$ and $\phi^{-1}(Q_i')\in -P_i+\mathbb{C}I$, $i=1,2,3$.*

Proof. We distinguish two cases.

Case 1: *ϕ is 1-type.* Then by the definition, $\phi(P_i)\in Q_i+\mathbb{C}I=Q_i'+\mathbb{C}I$. Hence $\phi^{-1}(Q_i')\in\phi^{-1}(\phi(P_i)+\mathbb{C}I)=P_i+\mathbb{C}I$ by Lemmas 2.3(2) and 2.2.

Moreover, by Lemmas 2.3(1) and 2.2,

$$\begin{aligned} Q_1' + Q_2' + Q_3' &\in \phi(P_1) + \phi(P_2) + \phi(P_3) + \mathbb{C}I \\ &\subseteq \phi(P_1 + P_2 + P_3) + \mathbb{C}I = \phi(I) + \mathbb{C}I = \mathbb{C}I. \end{aligned}$$

It follows from idempotency that $Q_1' + Q_2' + Q_3' = I$.

CASE 2: *ϕ is 2-type.* Then $\phi(P_i) \in Q_i + \mathbb{C}I = -Q_i' + \mathbb{C}I$. Hence $\phi^{-1}(Q_i') \in \phi^{-1}(-\phi(P_i) + \mathbb{C}I) = -P_i + \mathbb{C}I$ by Lemmas 2.3(2) and 2.2. Moreover, by Lemmas 2.3(1) and 2.2,

$$\begin{aligned} Q_1' + Q_2' + Q_3' &\in \mathbb{C}I - (\phi(P_1) + \phi(P_2) + \phi(P_3)) \\ &\subseteq \mathbb{C}I - \phi(P_1 + P_2 + P_3) = \mathbb{C}I - \phi(I) = \mathbb{C}I. \end{aligned}$$

It follows from idempotency that $Q_1' + Q_2' + Q_3' = I$.

Now, let $\mathcal{A}_{ij} = P_i B(X) P_j$ and $\mathcal{B}_{ij} = Q_i' B(Y) Q_j'$, $1 \le i,j \le 3$. Then $B(X) = \sum_{i,j=1}^3 \mathcal{A}_{ij}$ and $B(Y) = \sum_{i,j=1}^3 \mathcal{B}_{ij}$. We will identify the behavior of ϕ on $\mathcal{A}_{ij}$.

LEMMA 2.6.

(1) *If ϕ is 1-type, then $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ for $i \neq j$.*
(2) *If ϕ is 2-type, then $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ji}$ for $i \neq j$.*

Proof. Let $i, j \in \{1,2,3\}$ and $i \neq j$.

(1) Suppose that ϕ is 1-type. Let $A \in \mathcal{A}_{ij}$. Then by Lemma 2.5(2),

$$\begin{aligned} \phi(A) &= \phi_{A,P_j}(A) = \phi_{A,P_j}([A, P_j]) \\ &= [\phi_{A,P_j}(A), \phi_{A,P_j}(P_j)] = [\phi(A), Q_j'], \end{aligned} \tag{2.1}$$

and for $k \neq i, j$, by Lemmas 2.2(4) and 2.5(2),

$$0 = [\phi(A), \phi(P_k)] = [\phi(A), Q_k']. \tag{2.2}$$

Combining (2.1) and (2.2), we get $\phi(A) = Q_i' \phi(A) Q_j' \in \mathcal{B}_{ij}$. Therefore, $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$. On the other hand, considering ϕ^{-1} we have $\phi(\mathcal{A}_{ij}) \supseteq \mathcal{B}_{ij}$. So $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$.

(2) Suppose that ϕ is 2-type. Let $A \in \mathcal{A}_{ij}$. Then by Lemma 2.5(3),

$$\begin{aligned} \phi(A) &= \phi_{A,P_j}(A) = \phi_{A,P_j}([A, P_j]) \\ &= [\phi_{A,P_j}(A), \phi_{A,P_j}(P_j)] = [\phi(A), -Q_j'], \end{aligned} \tag{2.3}$$

and for $k \neq i, j$, by Lemmas 2.2(4) and 2.5(3),

$$0 = [\phi(A), \phi(P_k)] = [\phi(A), Q_k']. \tag{2.4}$$

Combining (2.3) and (2.4), we get $\phi(A) = Q_j' \phi(A) Q_i' \in \mathcal{B}_{ji}$. Therefore, $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ji}$. On the other hand, considering ϕ^{-1} we have $\phi(\mathcal{A}_{ij}) \supseteq \mathcal{B}_{ji}$. So $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ji}$.

LEMMA 2.7. *For $i \in \{1,2,3\}$, there is a homogeneous map $f_i : \mathcal{A}_{ii} \to \mathbb{C}$ such that $\phi(A_{ii}) - f_i(A_{ii})I \in \mathcal{B}_{ii}$. Moreover, for each $B_{ii} \in \mathcal{B}_{ii}$ there is $A_{ii} \in \mathcal{A}_{ii}$ such that $\phi(A_{ii}) = B_{ii} + f_i(A_{ii})I$.*

Proof. We only consider the case $i = 1$. The proofs for the other cases are similar.

Let A be in $\mathcal{A}_{11}$ and write $\phi(A) = \sum_{i,j=1}^{3} B_{ij}$ corresponding to the decomposition of $B(Y)$. For each $j \in \{1,2,3\}$, since A and P_j commute, it follows that $\phi(A)$ and $\phi(P_j)$ commute. Thus, if ϕ is 1-type, we have

$$0 = [\phi(A), Q'_j] = \sum_{k\neq j}(B_{kj} - B_{jk});$$

if ϕ is 2-type, we have

$$0 = [\phi(A), -Q'_j] = -\sum_{k\neq j}(B_{kj} - B_{jk}).$$

Consequently, we always have $\sum_{k\neq j}(B_{kj} - B_{jk}) = 0$. From this, we get $B_{kj} = 0$ for all $k \neq j$. Thus $\phi(A) = B_{11} + B_{22} + B_{33}$.

For $R_{23} \in \mathcal{B}_{23}$, by Lemma 2.6 there exists $T \in \mathcal{A}_{23}$ or $T \in \mathcal{A}_{32}$ such that $\phi(T) = R_{23}$. Since A and T commute, it follows that $\phi(A)$ and $\phi(T)$ commute. Thus

$$B_{22}R_{23} - R_{23}B_{33} = \Big[\sum_{i=1}^{3} B_{ii}, R_{23}\Big] = [\phi(A), \phi(T)] = 0.$$

So, by Lemma 1.2, $B_{22} = f_1(A)Q'_2$ and $B_{33} = f_1(A)Q'_3$ for some $f_1(A) \in \mathbb{C}$. Thus

$$\phi(A) = B_{11} + f_1(A)(Q'_2 + Q'_3) = B_{11} - f_1(A)Q'_1 + f_1(A)I.$$

From this, we see that $\phi(A) - f_1(A)I \in \mathcal{B}_{11}$.

To see that f_1 is homogeneous, we let A be in $\mathcal{A}_{11}$ and λ be a scalar. Then $\phi(A) - f_1(A)I \in \mathcal{B}_{11}$ and $\phi(\lambda A) - f_1(\lambda A)I \in \mathcal{B}_{11}$. It follows from the homogeneity of ϕ that $(f_1(\lambda A) - \lambda f_1(A))I \in \mathcal{B}_{11}$. This forces that $f_1(\lambda A) - \lambda f_1(A) = 0$.

Finally, let $B_{ii} \in \mathcal{B}_{ii}$. Applying the preceding result to ϕ^{-1}, there exists an $A_{ii} \in \mathcal{A}_{ii}$ and a scalar $\lambda \in \mathbb{C}$ such that $\phi(A_{ii} + \lambda I) = B_{ii}$. By Lemmas 2.2 and 2.3, we can suppose that $\phi(A_{ii} + \lambda I) = \phi(A_{ii}) - \mu I$ for some $\mu \in \mathbb{C}$. Then $\phi(A_{ii}) = B_{ii} + \mu I$. This implies $\phi(A_{ii}) - \mu I \in \mathcal{B}_{ii}$. So $\mu = f_i(A_{ii})$, completing the proof.

Now for $\sum_{i,j=1}^{3} A_{ij} \in \sum_{i,j=1}^{3} \mathcal{A}_{ij}$, we define

$$\psi\Big(\sum_{i,j=1}^{3} A_{ij}\Big) = \sum_{i,j=1}^{3} \phi(A_{ij}) - \sum_{k=1}^{3} f_k(A_{kk})I.$$

LEMMA 2.8.

(1) $\psi(A_{ij}) = \phi(A_{ij})$, $i \neq j$.
(2) $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ *for all* $i, j \in \{1,2,3\}$ *if* ϕ *is* 1*-type;* $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ji}$ *for all* $i, j \in \{1,2,3\}$ *if* ϕ *is* 2*-type.*
(3) $\psi(\sum_{i,j=1}^{3} A_{ij}) = \sum_{i,j=1}^{3} \psi(A_{ij})$.
(4) $\psi(P_i) = Q_i$ *for all* $i \in \{1,2,3\}$.
(5) ψ *is homogeneous and bijective.*

Proof. If $i \neq j$, then $\psi(A_{ij}) = \phi(A_{ij})$ by the definition, and hence by Lemma 2.6, $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ if ϕ is 1-type and $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ji}$ if ϕ is 2-type. By the definition again, $\psi(A_{ii}) = \phi(A_{ii}) - f_i(A_{ii})I$. So $\psi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$ by Lemma 2.7 and

$$\psi\Big(\sum_{i,j=1}^{3} A_{ij}\Big) = \sum_{k=1}^{3}(\phi(A_{ii}) - f_i(A_{ii})I) + \sum_{i\neq j}^{3} \phi(A_{ij}) = \sum_{i,j=1}^{3} \psi(A_{ij}).$$

So far, we have proved the first three parts. Now the last part is an easy consequence of (2) and (3).

LEMMA 2.9. *ψ is additive on $\mathcal{A}_{ij}$ for $1 \leq i, j \leq 3$.*

Proof. Let A_{12} and B_{12} be in $\mathcal{A}_{12}$. Making use of Lemma 2.3, we see that

$$\begin{aligned}&\psi(A_{12} + B_{12}) - (\psi(A_{12}) + \psi(B_{12}))\\ &\qquad = \phi(A_{12} + B_{12}) - (\phi(A_{12}) + \phi(B_{12})) \in \mathbb{C}I.\end{aligned}$$

This together with the fact that $\psi(\mathcal{A}_{12}) = \mathcal{B}_{12}$ or $\mathcal{B}_{21}$ gives $\psi(A_{12} + B_{12}) - (\psi(A_{12}) + \psi(B_{12})) = 0$. So ψ is additive on $\mathcal{A}_{12}$.

Now let A_{11} and B_{11} be in $\mathcal{A}_{11}$. By the definition of ψ and Lemma 2.3,

$$\begin{aligned}&\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})\\ &= \phi(A_{11}+B_{11}) - \phi(A_{11}) - \phi(B_{11}) + f_1(A_{11})I + f_1(B_{11})I - f_1(A_{11}+B_{11})I\\ &\in \mathbb{C}I.\end{aligned}$$

This together with the fact that $\psi(\mathcal{A}_{11}) = \mathcal{B}_{11}$ implies that

$$\psi(A_{11} + B_{11}) - (\psi(A_{11}) + \psi(B_{11})) = 0.$$

So ψ is additive on $\mathcal{A}_{11}$.

The rest can be proved in a similar way.

PROPOSITION 2.10. *ψ is linear.*

Proof. By Lemma 2.8, it suffices to show that ψ is additive. Let

$$A = \sum_{i,j=1}^{3} A_{ij} \quad \text{and} \quad B = \sum_{i,j=1}^{3} B_{ij}$$

be in $B(X)$. Then Lemmas 2.8 and 2.9 imply that

$$\begin{aligned}\psi(A+B) &= \psi\Big(\sum_{i,j=1}^{3}(A_{ij}+B_{ij})\Big) = \sum_{i,j=1}^{3}\psi(A_{ij}+B_{ij}) \\ &= \sum_{i,j=1}^{3}(\psi(A_{ij})+\psi(B_{ij})) = \psi\Big(\sum_{i,j=1}^{3}A_{ij}\Big)+\psi\Big(\sum_{i,j=1}^{3}B_{ij}\Big) \\ &= \psi(A)+\psi(B).\end{aligned}$$

LEMMA 2.11. *One of the following holds.*

(1) *There exists an isomorphism φ of $B(X)$ onto $B(Y)$ and a linear map τ_1 from $B(X)$ into $\mathbb{C}I$ such that $\psi=\varphi+\tau_1$.*
(2) *There exists an anti-isomorphism φ of $B(X)$ onto $B(Y)$ and a linear map τ_1 from $B(X)$ into $\mathbb{C}I$ such that $\psi=-\varphi+\tau_1$.*

Proof. By the definition of ψ and Lemma 2.3, $\psi(A)-\phi(A)\in\mathbb{C}I$ for all $A\in B(X)$. Thus, if $[A,B]=0$ for $A,B\in B(X)$, then

$$[\psi(A),\psi(B)] = [\phi(A),\phi(B)] = [\phi_{A,B}(A),\phi_{A,B}(B)] = \phi_{A,B}([A,B]) = 0.$$

So ψ is a bijective linear map preserving commutativity. It follows from [3, Theorem 2] that

$$\psi=\alpha\varphi+\tau_1,$$

where α is a non-zero scalar, φ is an isomorphism or an anti-isomorphism of $B(X)$ onto $B(Y)$, and τ_1 is a linear map from $B(X)$ into $\mathbb{C}I$.

For $i\in\{1,2,3\}$, we have

$$Q_i=\psi(P_i)=\alpha\varphi(P_i)+\beta_i I \tag{2.5}$$

for some $\beta_i\in\mathbb{C}$. Since both Q_i and $\varphi(P_i)$ are idempotents, we have

$$\alpha\varphi(P_i)+\beta_i I=(\alpha^2+2\alpha\beta_i)\varphi(P_i)+\beta_i^2 I.$$

Since $\varphi(P_i)\notin\mathbb{C}I$, we have

$$\alpha^2+2\alpha\beta_i-\alpha=0 \quad\text{and}\quad \beta_i^2-\beta_i=0.$$

So either $\alpha=1$ and $\beta_i=0$ for all $i\in\{1,2,3\}$, or $\alpha=-1$ and $\beta_i=1$ for all $i\in\{1,2,3\}$.

Let A_{12} be a non-zero element in $\mathcal{A}_{12}$. Then $\psi(A_{12})=\alpha\varphi(A_{12})+\beta I$ for some scalar β. Since both $\psi(A_{12})$ and $\varphi(A_{12})$ are square-zero, it follows that

$$2\alpha\beta\varphi(A_{12})+\beta^2 I=0.$$

Hence since $\varphi(A_{12})\notin\mathbb{C}I$, we get $\beta=0$. So

$$\psi(A_{12})=\alpha\varphi(A_{12}). \tag{2.6}$$

CASE 1: *$\alpha=1$ and $\beta_i=0$ for all $i\in\{1,2,3\}$.* We will show that φ is then an isomorphism.

By (2.5), $Q_i = \varphi(P_i)$ and $Q_iQ_j = \varphi(P_i)\varphi(P_j) = 0$ for $i \neq j$. So ϕ is 1-type and so $\psi(A_{12}) \in \mathcal{B}_{12}$ by Lemma 2.8. If φ is an anti-isomorphism, then, noting $\psi(A_{12}) = \varphi(A_{12})$ by (2.6), we have

$$\psi(A_{12}) = Q_1\psi(A_{12}) = \varphi(P_1)\varphi(A_{12}) = \varphi(A_{12}P_1) = 0.$$

This contradiction shows that φ is an isomorphism.

CASE 2: $\alpha = -1$ *and* $\beta_i = 1$ *for all* $i \in \{1, 2, 3\}$. We will show that φ is then an anti-isomorphism.

By (2.5), $Q_i = -\varphi(P_i) + I$ and then

$$(I - Q_i)(I - Q_j) = \varphi(P_i)\varphi(P_j) = 0 \quad \text{for } i \neq j.$$

So ϕ is 2-type and hence $\psi(A_{12}) \in \mathcal{B}_{21}$ by Lemma 2.8. If φ is an isomorphism, then noting $\psi(A_{12}) = -\varphi(A_{12})$ by (2.6), we have

$$\psi(A_{12}) = \psi(A_{12})(I - Q_1) = -\varphi(A_{12})\varphi(P_1) = -\varphi(A_{12}P_1) = 0.$$

This contradiction shows that φ is an anti-isomorphism.

Proof of Theorem 2.1. Without loss of generality, we assume that Lemma 2.11(1) holds. For $A \in B(X)$, define $\tau(A) = \phi(A) - \varphi(A)$. Then $\phi = \varphi + \tau$. Then homogeneity of ϕ and φ gives the homogeneity of τ. Obviously, $\tau(A) \in \mathbb{C}I$ for all $A \in B(X)$. Since each isomorphism of $B(X)$ onto $B(Y)$ is spatially implemented [18], there is an invertible operator T in $B(X, Y)$ such that $\phi(A) = TAT^{-1} + \tau(A)$ for all $A \in B(X)$.

Now let P_0 be an fixed idempotent with rank one. Let B in $B(X)$ be a finite sum of commutators. Then by Proposition 1.1, either

$$TP_0T^{-1} + \tau(P_0) = S_1P_0S_1^{-1} + \lambda_1 I \quad \text{and} \quad TBT^{-1} + \tau(B) = S_1BS_1^{-1}$$

for some invertible operator S_1 in $B(X, Y)$ and saclar λ_1, or

$$TP_0T^{-1} + \tau(P_0) = -S_2P_0^*S_2^{-1} + \lambda_2 I \quad \text{and} \quad TBT^{-1} + \tau(B) = -S_2B^*S_2^{-1}$$

for some invertible operator S_2 in $B(X^*, Y)$ and scalar λ_2. If the second case occurs, we have in particular

$$TP_0T^{-1} + \tau(P_0) = -S_2P_0^*S_2^{-1} + \lambda_2 I.$$

Since the dimension of Y is greater than 2, it follows that $TP_0T^{-1} = -S_2P_0^*S_2^{-1}$. Taking the trace, we find that $1 = -1$, a contradiction. So the first case holds. Then

$$TBT^{-1} + \tau(B) = S_1BS_1^{-1}.$$

This implies that $\sigma(B) + \tau(B) = \sigma(B)$. Since the spectrum $\sigma(B)$ of B is a compact set, it follows that $\tau(B) = 0$.

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