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Rank, trace and determinant in Banach algebras: generalized Frobenius and Sylvester theorems

by

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Abstract. As a follow-up to a paper of Aupetit and Mouton (1996), we consider the spectral definitions of rank, trace and determinant applied to elements in a general Banach algebra. We prove a generalization of Sylvester's Determinant Theorem to Banach algebras and thereafter a generalization of the Frobenius inequality.

1. Introduction. Determinants of infinite matrices were first investigated by astronomers, more than 100 years ago. The notions of rank, trace and determinant are well-established in operator theory. Recently, in their paper entitled *Trace and determinant in Banach algebras* [2], Aupetit and Mouton managed to show that these notions can be developed, without the use of operators, in a purely spectral and analytic manner. This paper is fundamental to our discussion here, which can be viewed as a follow-up to their pioneering work. We briefly summarize some of the theory before we proceed.

Throughout, A will denote a semisimple Banach algebra, with identity $\mathbf{1}$. We will use $\sigma_A(a)$ or simply $\sigma(a)$ (if the context is clear) to denote the spectrum of a; furthermore $\sigma'(a)$ will denote the nonzero spectrum of a. Finally, $\#K$ will denote the number of elements in a set $K \subset \mathbb{C}$.

Let $a \in A$, and suppose $\sigma(ax)$ is finite for all x in some open set $U \subset A$. Fix $x \in U$ and $y \in A$. If we consider the function $\lambda \mapsto a[(1-\lambda)x + \lambda y]$ then, by the Scarcity Principle, we deduce that $\sigma(ay)$ is also finite. The category argument in [2, p. 117] then shows that $\sigma(ay)$ is uniformly finite as y runs through A.

For each nonnegative integer m, let

$$\mathcal{F}_m = \{a \in A : \#\sigma'(xa) \leq m \text{ for every } x \in A\}.$$

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The *rank* of an element a of A is the smallest integer m such that $a \in \mathcal{F}_m$, if it exists; otherwise the rank is infinite. In other words,

$$\operatorname{rank}(a) = \sup_{x \in A} \#\sigma'(xa).$$

Obviously, by Jacobson's Lemma [2, Lemma 3.1.2] ($\sigma'(ab) = \sigma'(ba)$), we also have $\operatorname{rank}(a) = \sup_{x \in A} \#\sigma'(ax)$.

The *socle*, denoted $\operatorname{Soc} A$, is the sum of all minimal left ideals, or minimal right ideals, of A, if they exist, otherwise it is zero. The socle of A is a two-sided ideal of A and all of its elements are algebraic. Furthermore, it coincides with the set of finite-rank elements [2, Corollary 2.9].

We mention a few elementary properties of the rank of an element [2, p. 117]. Firstly, $\#\sigma'(a) \leq \operatorname{rank}(a)$ for all $a \in A$. Furthermore, $\operatorname{rank}(xa) \leq \operatorname{rank}(a)$ and $\operatorname{rank}(ax) \leq \operatorname{rank}(a)$ for all $x, a \in A$, with equality if x is invertible. If p is a projection of A, then p has rank one if and only if p is a minimal projection, that is, $pAp = \mathbb{C}p$. Moreover, the rank is lower semicontinuous on $\operatorname{Soc} A$. Finally, if ϕ is an isomorphism of A onto a Banach algebra B, then $\operatorname{rank}_A(a) = \operatorname{rank}_B(\phi(a))$ for every $a \in A$.

If $a \in A$ is a finite-rank element, then

$$E(a) = \{x \in A : \#\sigma'(xa) = \operatorname{rank}(a)\}$$

is a dense open subset of A [2, Theorem 2.2]. A finite-rank element a of A is said to be a *maximal finite-rank element* if $\operatorname{rank}(a) = \#\sigma'(a)$.

The following two results are fundamental to the theory developed in [2]:

Scarcity Theorem for Rank ([2, Theorem 2.3]). *Let f be an analytic function from a domain D of $\mathbb{C}$ into A. Then either the set of λ for which the rank of $f(\lambda)$ is finite has zero capacity, or there exist an integer N and a closed discrete subset E of D such that* $\operatorname{rank}(f(\lambda)) = N$ *on $D - E$ and* $\operatorname{rank}(f(\lambda)) < N$ *on E.*

Diagonalization Theorem ([2, Theorem 2.8]). *Let $a \in A$ be a nonzero maximal finite-rank element and denote by $\lambda_1, \ldots, \lambda_n$ its nonzero distinct spectral values. Then there exist n orthogonal minimal projections $p_1, \ldots, p_n$ such that*

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n.$$

If $a \in \operatorname{Soc} A$ we define the *trace* of a by

$$\operatorname{Tr}(a) = \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a),$$

and the *determinant* of $\mathbf{1} + a$ by

$$\operatorname{Det}(\mathbf{1} + a) = \prod_{\lambda \in \sigma(a)} (1 + \lambda)^{m(\lambda, a)},$$

where $m(\lambda, a)$ is the *multiplicity of a at λ*. A brief description of the notion of multiplicity in the abstract case goes as follows (for details consult [2]): Let $a \in \operatorname{Soc} A$ and $\lambda \in \sigma(a)$ and let $B(\lambda, r)$ be an open disk centered at λ such that $B(\lambda, r)$ contains no other points of $\sigma(a)$. It can be shown [2, pp. 119–120] that there exists an open ball, say $U \subset A$, centered at $\mathbf{1}$ such that $\#[\sigma(ax) \cap B(\lambda, r)]$ is constant as x runs through $E(a) \cap U$. This constant integer is the multiplicity of a at λ.

Let f_λ be the holomorphic function which takes the value 1 on $B(\lambda, r)$ and the value 0 elsewhere. Let $\Gamma_0 = \partial B(\lambda, 3r/4)$. Then

$$p(\lambda, a) = f_\lambda(a) = \frac{1}{2\pi i} \int_{\Gamma_0} f_\lambda(\alpha)(\alpha\mathbf{1} - a)^{-1}\, d\alpha$$

is referred to as the *Riesz projection* associated with a and λ. Moreover, for $\lambda \neq 0$ we have $m(\lambda, a) = \operatorname{rank}(p(\lambda, a))$ [2, Theorem 2.6], and

$$p(\lambda, a) = \frac{a}{2\pi i} \int_{\Gamma_0} \frac{f_\lambda(\alpha)}{\alpha}(\alpha\mathbf{1} - a)^{-1}\, d\alpha.$$

In the operator case, $A = \mathcal{L}(X)$ (bounded linear operators on a Banach space X), the "spectral" rank, trace (and determinant) all coincide with the respective classical operator definitions. It should be noted that the Aupetit–Mouton approach is not merely an alternative to the long established theory of rank, trace and determinant for $A = \mathcal{L}(X)$—it is an improvement. The Aupetit-Mouton definition simultaneously takes care of the matter in subalgebras of $\mathcal{L}(X)$ as well, since the notions of rank, trace and determinant are clearly relative concepts.

2. Frobenius' inequality and Sylvester's theorem. We begin this section with some preliminary results that will aid us in what follows:

Let p be a finite-rank projection of A. Then pAp is a closed semisimple subalgebra of A with identity p and

$$\sigma'_{pAp}(pxp) = \sigma'_A(pxp) \tag{1}$$

for each $x \in A$ (see [1, Chapter 3, Exercise 6]). The following lemma shows that rank is also preserved here.

LEMMA 2.1. *Let p be a finite-rank projection of A. Then*

$$\operatorname{rank}_{pAp}(pxp) = \operatorname{rank}_A(pxp)$$

for each $x \in A$.

Proof. Let $x \in A$ be arbitrary. From (1), Jacobson's Lemma and the fact that $p = p^2$, it readily follows that

$$\begin{aligned}\operatorname{rank}_{pAp}(pxp) &= \sup_{y\in A} \#\sigma'_{pAp}((pyp)(pxp)) = \sup_{y\in A} \#\sigma'_A((pyp)(pxp))\\ &= \sup_{y\in A} \#\sigma'_A(y(pxp)) = \operatorname{rank}_A(pxp),\end{aligned}$$

as desired. ■

Lemma 2.2 gives us an invariance result for multiplicity under positive scalar multiplication:

LEMMA 2.2. *Let a be a finite-rank element of A and let $\alpha > 0$ be a real number. Then $m(\lambda, a) = m(\alpha\lambda, \alpha a)$ for each $\lambda \in \sigma'(a)$.*

Proof. Set $\lambda_0 = 0$ and denote by $\lambda_1, \dots, \lambda_n$ the distinct nonzero spectral values of a. Choose $r > 0$ so that the open disks $B(\lambda_0, r), B(\lambda_1, r), \dots, B(\lambda_n, r)$ are all disjoint. By the Spectral Mapping Theorem it follows that $\sigma'(\alpha a) = \{\alpha\lambda_1, \dots, \alpha\lambda_n\}$. Notice that $B(\alpha\lambda_0, \alpha r), B(\alpha\lambda_1, \alpha r), \dots, B(\alpha\lambda_n, \alpha r)$ are also all disjoint, and for each $i \in \{0, 1, \dots, n\}$, we have $\beta \in B(\lambda_i, r)$ if and only if $\alpha\beta \in B(\alpha\lambda_i, \alpha r)$. Fix $i \in \{1, \dots, n\}$. For $j \in \{1, \alpha\}$ we let U_j be an open disk centered at $\mathbf{1}$ such that $x \in U_j \cap E(ja)$ implies that

$$\#(\sigma(jxa) \cap \Delta_0^j) = m(j\lambda_i, ja),$$

where Δ_0^j is the interior of $\partial B(j\lambda_i, jr)$. Since

$$\#\sigma'(xa) = \#\sigma'(\alpha xa)$$

by the Spectral Mapping Theorem, and since $\operatorname{rank}(a) = \operatorname{rank}(\alpha a)$, it follows that $E(a) \subseteq E(\alpha a)$. Moreover, since $E(a)$ is open and dense in A, we have $U_1 \cap U_\alpha \cap E(a) \neq \emptyset$. Pick an $x_0 \in U_1 \cap U_\alpha \cap E(a)$. Then $x_0 \in U_1 \cap E(a)$ and $x_0 \in U_\alpha \cap E(\alpha a)$, so by the Spectral Mapping Theorem and our choice of contours we obtain

$$m(\lambda_i, a) = \#(\sigma(x_0 a) \cap \Delta_0^1) = \#(\sigma(\alpha x_0 a) \cap \Delta_0^\alpha) = m(\alpha\lambda_i, \alpha a).$$

Since i was arbitrary, this completes the proof. ■

The following lemma gives us Sylvester's Determinant Theorem as an easy consequence:

LEMMA 2.3. *Let $a \in \operatorname{Soc} A$. Then $m(\lambda, ab) = m(\lambda, ba)$ for all $b \in A$ and for each $\lambda \in \sigma'(ab)$.*

Proof. Let $\lambda \in \sigma'(ab)$ and let $\alpha \in (0, 1]$. By the Spectral Mapping Theorem we have

$$\alpha\lambda \in \sigma'(\alpha ab) = \sigma'(\alpha ba).$$

Consider the Riesz projections $p(\alpha\lambda, \alpha ab)$ and $p(\alpha\lambda, \alpha ba)$. Let

$$\Gamma = \bigcup_{\beta \in \sigma(\alpha ab) \cup \{0\}} \partial B(\beta, r/2),$$

where $r > 0$ is chosen sufficiently small to ensure that $B(\beta_1, r) \cap B(\beta_2, r) = \emptyset$ for all $\beta_1, \beta_2 \in \sigma(\alpha ab) \cup \{0\}$ with $\beta_1 \neq \beta_2$. Moreover, let g be the holomorphic function which takes on the value 1 on $B(\alpha\lambda, r)$ and the value 0 elsewhere on $\bigcup_{\beta \in \sigma(\alpha ab) \cup \{0\}} B(\beta, r)$. Notice that r, Γ and g all depend on α, but the contours of integration can be chosen such that their lengths are not longer than that of the contour surrounding $\sigma(ab) \cup \{0\}$. Then

$$p(\alpha\lambda, \alpha ab) = \frac{1}{2\pi i} \int_\Gamma g(z)(z\mathbf{1} - \alpha ab)^{-1}\, dz,$$

$$p(\alpha\lambda, \alpha ba) = \frac{1}{2\pi i} \int_\Gamma g(z)(z\mathbf{1} - \alpha ba)^{-1}\, dz.$$

Now observe that for each $z \in \Gamma$ we have

$$(z\mathbf{1} - \alpha ab)^{-1} - (z\mathbf{1} - \alpha ba)^{-1} = (z\mathbf{1} - \alpha ab)^{-1}(\alpha ab - \alpha ba)(z\mathbf{1} - \alpha ba)^{-1}.$$

Moreover, by the continuity of the mappings $z \mapsto \|(z\mathbf{1} - \alpha ab)^{-1}\|$ and $z \mapsto \|(z\mathbf{1} - \alpha ba)^{-1}\|$, and the compactness of Γ, we can find positive real numbers K_1 and K_2 such that $\|(z\mathbf{1} - \alpha ab)^{-1}\| \leq K_1$ and $\|(z\mathbf{1} - \alpha ba)^{-1}\| \leq K_2$ for all $z \in \Gamma$ and $\alpha \in [0, 1]$. Consequently, if we choose α sufficiently small, then

$$\begin{aligned}
&\|p(\alpha\lambda, \alpha ab) - p(\alpha\lambda, \alpha ba)\| \\
&\quad = \left\| \frac{1}{2\pi i} \int_\Gamma g(z)[(z\mathbf{1} - \alpha ab)^{-1} - (z\mathbf{1} - \alpha ba)^{-1}]\, dz \right\| \\
&\quad = \left\| \frac{1}{2\pi i} \int_\Gamma g(z)[(z\mathbf{1} - \alpha ab)^{-1}(\alpha ab - \alpha ba)(z\mathbf{1} - \alpha ba)^{-1}]\, dz \right\| \\
&\quad \leq \frac{1}{2\pi} \int_\Gamma |g(z)| \cdot \|(z\mathbf{1} - \alpha ab)^{-1}\| \cdot \alpha \cdot \|ab - ba\| \cdot \|(z\mathbf{1} - \alpha ba)^{-1}\|\, d|z| \\
&\quad \leq \frac{1}{2\pi} \int_\Gamma K_1 \cdot \alpha \cdot \|ab - ba\| \cdot K_2\, d|z| < 1.
\end{aligned}$$

For two nonzero projections p and q, we have $\operatorname{rank}(p) = \operatorname{rank}(q)$ whenever $\|p - q\| < 1$ (see [3]). This together with Lemma 2.2 gives us

$$\begin{aligned}
m(\lambda, ab) = m(\alpha\lambda, \alpha ab) &= \operatorname{rank}(p(\alpha\lambda, \alpha ab)) = \operatorname{rank}(p(\alpha\lambda, \alpha ba)) \\
&= m(\alpha\lambda, \alpha ba) = m(\lambda, ba).
\end{aligned}$$

Since λ was arbitrary, the theorem is proved. ■

If $a \in \operatorname{Soc} A$ and ab is quasinilpotent then, from Jacobson's Lemma, clearly $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$ and $\operatorname{Det}(\mathbf{1} + ab) = \operatorname{Det}(\mathbf{1} + ba)$ for all $b \in A$. If $\sigma(ab) \neq \{0\}$, then as an immediate consequence of Lemma 2.3, we obtain the same. Hence, we provide a generalization of Sylvester's Determinant Theorem for matrices:

Theorem 2.4. *Let* $a \in \operatorname{Soc} A$. *Then* $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$ *and* $\operatorname{Det}(\mathbf{1} + ab) = \operatorname{Det}(\mathbf{1} + ba)$ *for all* $b \in A$.

We now proceed to prove a generalization of the well-known Frobenius inequality for matrices. Firstly, however, we verify the finite-dimensional case.

Lemma 2.5. *Let* A *be finite-dimensional. For all* $a, b, c \in A$ *we have*

$$\operatorname{rank}(ab) + \operatorname{rank}(bc) \leq \operatorname{rank}(b) + \operatorname{rank}(abc).$$

Proof. Because A is semisimple and finite-dimensional, we apply [1, Lemma 5.4.1] and conclude that A is isomorphic as an algebra to

$$B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Hence, since isomorphisms preserve rank, it suffices to prove that the inequality holds in B. Let $x = (x_1, \ldots, x_k)$, where $x_j \in B_j = M_{n_j}(\mathbb{C})$ for each $j \in \{1, \ldots, k\}$. Firstly,

$$\sigma_B((x_1, \ldots, x_k)) = \bigcup_{j=1}^{k} \sigma_{B_j}(x_j).$$

Hence, if $y = (y_1, \ldots, y_k) \in B$, then

$$\#\sigma'_B(yx) = \#\sigma'_B((y_1x_1, \ldots, y_kx_k)) \leq \sum_{j=1}^{k} \#\sigma'_{B_j}(y_jx_j) \leq \sum_{j=1}^{k} \operatorname{rank}_{B_j}(x_j),$$

so that

$$\operatorname{rank}_B(x) \leq \sum_{j=1}^{k} \operatorname{rank}_{B_j}(x_j).$$

To see that the other inequality also holds true, consider any x_j. Let u_j be an element in B_j such that $\#\sigma'_{B_j}(u_jx_j) = \operatorname{rank}_{B_j}(x_j) = m_j$. Then, from the properties of rank, we see that u_jx_j is a maximal finite-rank element. Thus, by the Diagonalization Theorem we can write $u_jx_j = \lambda_{j,1}p_{j,1} + \cdots + \lambda_{j,m_j}p_{j,m_j}$, where $p_{j,1}, \ldots, p_{j,m_j}$ are orthogonal minimal projections in B_j and $\lambda_{j,1}, \ldots, \lambda_{j,m_j} \in \mathbb{C} - \{0\}$. Hence, by the orthogonality of the $p_{j,r}$, if we let

$$v_j = \frac{1}{\lambda_{j,1}}p_{j,1} + \frac{2}{\lambda_{j,2}}p_{j,2} + \cdots + \frac{m_j}{\lambda_{j,m_j}}p_{j,m_j},$$

then

$$v_ju_jx_j = p_{j,1} + 2p_{j,2} + \cdots + m_jp_{j,m_j}.$$

Moreover, $\sigma'_{B_j}(v_ju_jx_j) = \{1, \ldots, m_j\}$ (see [1, Chapter 3, Exercise 9]). Finally, let α_j be any nonzero complex number with $\operatorname{Arg}(\alpha_j) = \pi/j$. Now, if we take $z = (\alpha_1v_1u_1, \ldots, \alpha_kv_ku_k)$, then, in particular, $\sigma'_{B_j}(\alpha_jv_ju_jx_j) \cap \sigma'_{B_r}(\alpha_rv_ru_rx_r) = \emptyset$ for $j \neq r$, so

$$\#\sigma'_B(zx) = \#\Big(\bigcup_{j=1}^{k} \sigma'_{B_j}(\alpha_j v_j u_j x_j)\Big) = \sum_{j=1}^{k} \#\sigma'_{B_j}(\alpha_j v_j u_j x_j) = \sum_{j=1}^{k} \operatorname{rank}_{B_j}(x_j),$$

and the result follows. This shows that

$$\operatorname{rank}_B(x) = \sum_{j=1}^{k} \operatorname{rank}_{B_j}(x_j) \tag{2}$$

for each $x = (x_1, \ldots, x_k) \in B$. Let $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$ be arbitrary elements from B. Then, from (2) and Frobenius' inequality for matrices, it follows that

$$\begin{aligned}\operatorname{rank}_B(ab) + \operatorname{rank}_B(bc) &= \sum_{j=1}^{k} \operatorname{rank}_{B_j}(a_j b_j) + \sum_{j=1}^{k} \operatorname{rank}_{B_j}(b_j c_j) \\ &\le \sum_{j=1}^{k} \operatorname{rank}_{B_j}(b_j) + \sum_{j=1}^{k} \operatorname{rank}_{B_j}(a_j b_j c_j) \\ &= \operatorname{rank}_B(b) + \operatorname{rank}_B(abc),\end{aligned}$$

so the lemma is proved. ■

THEOREM 2.6 (Frobenius inequality for Banach algebras). *For any $a, b, c \in A$, we have*

$$\operatorname{rank}(ab) + \operatorname{rank}(bc) \le \operatorname{rank}(b) + \operatorname{rank}(abc).$$

Proof. Let $a, b, c \in A$. We may assume that $\operatorname{rank}_A(b) = n$, where $1 \le n < \infty$, for otherwise the inequality is trivially true. Consequently, ab, bc and abc are all of finite rank in A.

We firstly assume that b is a maximal finite-rank element. By the Diagonalization Theorem there are orthogonal minimal projections $p_1, \ldots, p_n$ in A and nonzero complex numbers $\lambda_1, \ldots, \lambda_n$ so that $b = \lambda_1 p_1 + \cdots + \lambda_n p_n$. Because the projections are all orthogonal, it follows that $p = p_1 + \cdots + p_n$ is a finite-rank projection, and moreover

$$pbp = pb = bp = b. \tag{3}$$

Let $B = pAp$. Then B is a semisimple, finite-dimensional and closed subalgebra of A with identity p. Since $E(a)$ is open and dense for every finite-rank a, there exist invertible $x, y \in A$ such that

$$\#\sigma'_A(xab) = \operatorname{rank}_A(ab) \quad \text{and} \quad \#\sigma'_A(bcy) = \operatorname{rank}_A(bc).$$

Moreover, by Jacobson's Lemma, (1) and (3) we have

$$\sigma'_A(xab) = \sigma'_A(xabp) = \sigma'_A(pxabp) = \sigma'_B(pxabp),$$

and similarly

$$\sigma'_A(bcy) = \sigma'_B(pbcyp).$$

Consequently, $\mathrm{rank}_A(ab) \leq \mathrm{rank}_B(pxabp)$ and $\mathrm{rank}_A(bc) \leq \mathrm{rank}_B(pbcyp)$. Thus, by Lemmas 2.1 and 2.5, we obtain

$$\begin{aligned}\mathrm{rank}_A(ab) + \mathrm{rank}_A(bc) &\leq \mathrm{rank}_B(pxabp) + \mathrm{rank}_B(pbcyp)\\ &= \mathrm{rank}_B(pxab) + \mathrm{rank}_B(bcyp)\\ &\leq \mathrm{rank}_B(b) + \mathrm{rank}_B(pxabcyp)\\ &= \mathrm{rank}_A(b) + \mathrm{rank}_A(pxabcyp)\\ &\leq \mathrm{rank}_A(b) + \mathrm{rank}_A(abc),\end{aligned}$$

where the last inequality follows from the properties of rank.

If b is not a maximal finite-rank element, then let u be an invertible element in A such that ub is a maximal finite-rank element. We then apply the preceding argument to the elements au^{-1}, ub and c, and use the rank properties to obtain

$$\begin{aligned}\mathrm{rank}_A(ab) + \mathrm{rank}_A(bc) &= \mathrm{rank}_A(au^{-1}ub) + \mathrm{rank}_A(ubc)\\ &\leq \mathrm{rank}_A(ub) + \mathrm{rank}_A(au^{-1}ubc)\\ &= \mathrm{rank}_A(b) + \mathrm{rank}_A(abc),\end{aligned}$$

as desired. ■

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