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NUMBER OF SOLUTIONS IN A BOX OF A LINEAR EQUATION IN AN ABELIAN GROUP

BY

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Abstract. For every finite Abelian group Γ and for all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^{k} a_i x_i = g$ in non-negative integers $x_i \le b_i$, where b_i are positive integers, then the number of such solutions is estimated from below in the best possible way.

1. Introduction. We have proved in [3] the following conjecture of A. Schinzel [2]: For every finite Abelian group Γ and all $a_1, \ldots, a_k \in \Gamma$, the number of solutions of the equation $a_1x_1 + \cdots + a_kx_k = 0$ in non-negative integers $x_i \le b_i$, where the b_i are positive integers, is at least $2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1)$, where $D(\Gamma)$ is the Davenport constant of the group Γ (see Definition 2.1 below).

The present paper is a sequel to [3], and the notation of that paper is used throughout. The aim of this paper is to deal with the inhomogeneous case. We shall prove the following two statements.

THEOREM 1.1. *For every finite Abelian group Γ and all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^{k} a_i x_i = g$ in non-negative integers $x_i \le b_i$, where the b_i are positive integers, then the number of such solutions is at least*

$$3^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1). \tag{1.1}$$

THEOREM 1.2. *For every finite Abelian group Γ and all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^{k} a_i x_i = g$ in non-negative integers $x_i \le b_i$, where $b_i \in \{2^s - 1 : s \in \mathbb{N}\}$, then the number of such solutions is at least*

$$2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1). \tag{1.2}$$

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REMARK 1.3. Let $\Gamma = n\mathbb{Z}_2$ be the direct product of n cyclic groups of order two, and $a_1, \ldots, a_n$ a basis for Γ. Then the number of solutions of the equation $\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i$ in non-negative integers $x_i \le b_i = 2$ equals 1. Since $D(\Gamma) = n+1$ (see Olson [1]) and $1 = 3^{1-D(\Gamma)} \prod_{i=1}^n (2+1)$, we see that in Theorem 1.1, $3^{1-D(\Gamma)}$ is the best possible coefficient independent of a_i, b_i, g and depending only on Γ.

2. Lemmas and definitions. Let Γ be a finite Abelian group with multiplicative notation.

DEFINITION 2.1. Define the *Davenport constant* $D(\Gamma)$ to be the smallest positive integer n such that for any sequence $g_1, \ldots, g_n$ of group elements there exist indices $1 \le i_1 < \cdots < i_t \le n$ for which

$$g_{i_1} \cdot \ldots \cdot g_{i_t} = 1.$$

DEFINITION 2.2. For an element $\sum_{g\in\Gamma} N_g g$ of the group ring $\mathbb{Q}[\Gamma]$ and a number $n \in \mathbb{Q}$ we write

$$\sum_{g\in\Gamma} N_g g \succeq n \quad \text{iff} \quad N_1 \ge n.$$

LEMMA 2.3. *For every finite Abelian group Γ and all $a_1, \ldots, a_k, g \in \Gamma$, the number of solutions of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \le b_i$ is equal to N_1, where*

$$g^{-1} \prod_{i=1}^{k} (1 + a_i + \cdots + a_i^{b_i}) = \sum_{h\in\Gamma} N_h h$$

is an identity in $\mathbb{Q}[\Gamma]$.

Proof. We interpret the equation $g^{-1}\prod_{i=1}^k (1+a_i+\cdots+a_i^{b_i}) = \sum_{h\in\Gamma} N_h h$ combinatorially. For $g \in \Gamma$ look at all sequences $a_1^{x_1}, \ldots, a_k^{x_k}$ whose product is g, where $x_i \le b_i$ are non-negative integers. Then N_1 counts these sequences. Therefore the number of solutions of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \le b_i$ is equal to N_1. ■

LEMMA 2.4. *Theorem 1.1 with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ and all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \le b_i$, where the b_i are positive integers, then*

$$g^{-1} \prod_{i=1}^{k} (1 + a_i + \cdots + a_i^{b_i}) \succeq 3^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1), \tag{2.1}$$

where $D(\Gamma)$ is the Davenport constant of the group Γ.

Proof. This follows from Lemma 2.3 and Definition 2.2. ■

LEMMA 2.5. *Theorem 1.2 with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ, all $g, a_1, \ldots, a_k \in \Gamma$ and all positive integers $b_1, \ldots, b_k \in \{2^s - 1 : s \in \mathbb{N}\}$, if there exists a solution of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \le b_i$, then*

$$g^{-1} \prod_{i=1}^{k} (1 + a_i + \cdots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1). \tag{2.2}$$

Proof. This follows from Lemma 2.3 and Definition 2.2. ■

LEMMA 2.6. *For every finite Abelian group Γ and all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \le 1$, then*

$$g^{-1} \prod_{i=1}^{k} (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \tag{2.3}$$

Proof. We may assume that $\prod_{i=1}^t a_i = g$, where $1 \le t \le k$. We have the identities

$$\begin{aligned} g^{-1} \prod_{i=1}^{k} (1 + a_i) &= g^{-1} \prod_{i=1}^{t} a_i \prod_{i=1}^{t} (1 + a_i^{-1}) \prod_{i=t+1}^{k} (1 + a_i) \\ &= \prod_{i=1}^{t} (1 + a_i^{-1}) \prod_{i=t+1}^{k} (1 + a_i). \end{aligned}$$

By [3, Theorem 1.1],

$$\prod_{i=1}^{t} (1 + a_i^{-1}) \prod_{i=t+1}^{k} (1 + a_i) \succeq 2^{1-D(\Gamma)} 2^k.$$

This implies

$$g^{-1} \prod_{i=1}^{k} (1 + a_i) \succeq 2^{1-D(\Gamma)} 2^k. \blacksquare$$

LEMMA 2.7. *If $0 \le t < b$, where t, b are integers, then*

$$b - t + 1 \ge \left(\frac{2}{3}\right)^t (b + 1).$$

Proof. We verify by differentiation that the function $f(x) = 2\left(\frac{3}{2}\right)^x - x - 2$ is increasing in the interval $(1, \infty)$. Since $f(0) = f(1) = 0$ and $f(2) = 1/2$ we get $2\left(\frac{3}{2}\right)^t \ge t + 2$ for non-negative integers t. Hence

$$1 - \frac{t}{b+1} \ge 1 - \frac{t}{t+2} \ge \left(\frac{2}{3}\right)^t,$$

which yields the desired conclusion. ■

LEMMA 2.8. *For $s \geq 1$ we have the following identity in $\mathbb{Q}[\Gamma]$:*

$$1 + x + x^2 + \cdots + x^{2^s-1} = \prod_{j=1}^{s}(1 + x^{2^{j-1}}). \tag{2.4}$$

Proof. We proceed by induction on s. The case $s = 1$ is obvious. Assume the identity is true for $s - 1$, where $s > 1$. Then

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{2^s-1} &= (1 + x^{2^{s-1}})(1 + x + x^2 + \cdots + x^{2^{s-1}-1}) \\ &= (1 + x^{2^{s-1}}) \prod_{j=1}^{s-1}(1 + x^{2^{j-1}}) = \prod_{j=1}^{s}(1 + x^{2^{j-1}}). \quad \blacksquare \end{aligned}$$

3. Proofs of theorems

Proof of Theorem 1.1. We may find $0 \leq t_i \leq b_i$, where $1 \leq i \leq k$, such that $a_1^{t_1} \cdot \ldots \cdot a_k^{t_k} = g$. By definition of the Davenport constant we may assume that

$$\sum_{i=1}^{k} t_i \leq D(\Gamma) - 1. \tag{3.1}$$

Suppose $t_i = b_i$ for $1 \leq i \leq s \leq k$, and $t_i < b_i$ for $s + 1 \leq i \leq k$; if $t_i < b_i$ for $1 \leq i \leq k$, then we take $s = 0$. We have the identities

$$\begin{aligned} &g^{-1} \prod_{i=1}^{s}(1 + a_i + \cdots + a_i^{b_i}) \prod_{i=s+1}^{k} (a_i^{t_i} + a_i^{t_i+1} + \cdots + a_i^{b_i}) \\ &\quad = \prod_{i=1}^{s} a_i^{-b_i} \prod_{i=s+1}^{k} a_i^{-t_i} \prod_{i=1}^{s}(1 + a_i + \cdots + a_i^{b_i}) \prod_{i=s+1}^{k} (a_i^{t_i} + a_i^{t_i+1} + \cdots + a_i^{b_i}) \\ &\quad = \prod_{i=1}^{s}(1 + a_i^{-1} + \cdots + (a_i^{-1})^{b_i}) \prod_{i=s+1}^{k} (1 + a_i + \cdots + a_i^{b_i-t_i}). \end{aligned}$$

By [3, Theorem 1.1],

$$\begin{aligned} &\prod_{i=1}^{s}(1 + a_i^{-1} + \cdots + (a_i^{-1})^{b_i}) \prod_{i=s+1}^{k} (1 + a_i + \cdots + a_i^{b_i-t_i}) \\ &\qquad\qquad \succeq 2^{1-D(\Gamma)} \prod_{i=1}^{s}(b_i + 1) \prod_{i=s+1}^{k} (b_i - t_i + 1). \end{aligned}$$

Lemma 2.7 yields

$$\begin{aligned} 2^{1-D(\Gamma)} \prod_{i=1}^{s}(b_i + 1) \prod_{i=s+1}^{k} (b_i - t_i + 1) &\geq 2^{1-D(\Gamma)} \prod_{i=1}^{s}(b_i + 1) \prod_{i=s+1}^{k} \left(\tfrac{2}{3}\right)^{t_i}(b_i + 1) \\ &\geq 2^{1-D(\Gamma)} \prod_{i=1}^{k}\left(\tfrac{2}{3}\right)^{t_i}(b_i + 1). \end{aligned}$$

From (3.1) it follows that

$$2^{1-D(\Gamma)}\prod_{i=1}^{k}\left(\tfrac{2}{3}\right)^{t_i}(b_i+1)\geq 2^{1-D(\Gamma)}\left(\tfrac{2}{3}\right)^{D(\Gamma)-1}\prod_{i=1}^{k}(b_i+1)=3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Hence

$$g^{-1}\prod_{i=1}^{s}(1+a_i+\dots+a_i^{b_i})\prod_{i=s+1}^{k}(a_i^{t_i}+a_i^{t_i+1}+\dots+a_i^{b_i})\succeq 3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Finally

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\dots+a_i^{b_i})\succeq 3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Proof of Theorem 1.2. Let $b_i=2^{s_i}-1$, where $s_i\in\mathbb{N}$. We take $0\leq t_i\leq b_i$, where $1\leq i\leq k$, such that $a_1^{t_1}\cdot\ldots\cdot a_k^{t_k}=g$. Since $0\leq t_i\leq 2^{s_i}-1$ we may find $\epsilon_{ji}\in\{0,1\}$ such that

$$t_i=\sum_{j=1}^{s_i}\epsilon_{ji}2^{j-1}$$

for $1\leq i\leq k$. Using (2.4) we obtain

$$\begin{aligned}a_i^{-t_i}(1+a_i+\dots+a_i^{b_i})&=a_i^{-t_i}\prod_{j=1}^{s_i}(1+a_i^{2^{j-1}})\\&=\prod_{j=1}^{s_i}a_i^{-\epsilon_{ji}2^{j-1}}(1+a_i^{2^{j-1}})=\prod_{j=1}^{s_i}(a_i^{-\epsilon_{ji}2^{j-1}}+a_i^{(1-\epsilon_{ji})2^{j-1}})\\&=\prod_{j=1}^{s_i}(1+a_i^{\eta_{ji}2^{j-1}}),\end{aligned}$$

where $\eta_{ji}=1-2\epsilon_{ji}\in\{-1,1\}$. Thus

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\dots+a_i^{b_i})=\prod_{i=1}^{k}a_i^{-t_i}(1+a_i+\dots+a_i^{b_i})=\prod_{i=1}^{k}\prod_{j=1}^{s_i}(1+a_i^{\eta_{ji}2^{j-1}}).$$

By [3, Theorem 1.1],

$$\prod_{i=1}^{k}\prod_{j=1}^{s_i}(1+a_i^{\eta_{ji}2^{j-1}})\succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}\prod_{j=1}^{s_i}2=2^{1-D(\Gamma)}\prod_{i=1}^{k}2^{s_i}=2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1),$$

which implies

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\dots+a_i^{b_i})\succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

References

[1] J. E. Olson, *A combinatorial problem on finite Abelian groups. I*, J. Number Theory 1 (1969), 8–10.

[2] A. Schinzel, *The number of solutions of a linear homogeneous congruence*, in: Diophantine Approximation: Festschrift for Wolfgang Schmidt (H. P. Schlickewei et al., eds.), Developments Math. 16, Springer, 2008, 363–370.

[3] M. Zakarczemny, *Number of solutions in a box of a linear homogeneous equation in an Abelian group*, Acta Arith. 155 (2012), 227–231.

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