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BRAIDED MONOIDAL CATEGORIES AND DOI–HOPF MODULES FOR MONOIDAL HOM-HOPF ALGEBRAS

BY

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Abstract. We continue our study of the category of Doi Hom-Hopf modules introduced in [Colloq. Math., to appear]. We find a sufficient condition for the category of Doi Hom-Hopf modules to be monoidal. We also obtain a condition for a monoidal Hom-algebra and monoidal Hom-coalgebra to be monoidal Hom-bialgebras. Moreover, we introduce morphisms between the underlying monoidal Hom-Hopf algebras, Hom-comodule algebras and Hom-module coalgebras, which give rise to functors between the category of Doi Hom-Hopf modules, and we study tensor identities for monodial categories of Doi Hom-Hopf modules. Furthermore, we construct a braiding on the category of Doi Hom-Hopf modules. Finally, as an application of our theory, we get a braiding on the category of Hom-modules, on the category of Hom-comodules, and on the category of Hom-Yetter–Drinfeld modules.

1. Introduction. The category ${}_A\mathcal{M}(H)^C$ of Doi–Hopf modules was introduced in [11], where H is a Hopf algebra, A a right H-comodule algebra and C a left H-module coalgebra. It is the category of those modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H. The study of ${}_A\mathcal{M}(H)^C$ turned out to be very useful: it was shown in [11] that many categories such as the module and comodule categories over bialgebras, the Hopf modules category [24], and the Yetter–Drinfeld category [22] are special cases of ${}_A\mathcal{M}(H)^C$. For a further study of Doi–Hopf modules, we refer to [3], [4]. In [2], it is proved that Yetter–Drinfeld modules are special cases of Doi–Hopf modules, therefore the category of Yetter–Drinfeld modules is a Grothendieck category.

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [18] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of multiplication is replaced by Hom-associativity, and similarly for Hom-coassociativity. They also described the

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structures of Hom-bialgebras and Hom-Hopf algebras, and extended some important results from ordinary Hopf algebras to Hom-Hopf algebras in [19] and [20]. Recently, more properties and structures of Hom-Hopf algebras have been developed: see [5]–[9], [12]–[14], [16], [25]–[28] and references therein.

Caenepeel and Goyvaerts [1] studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively; these are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. Makhlouf and Panaite [17] defined Yetter–Drinfeld modules over Hom-bialgebras and showed that Yetter–Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang–Baxter equation. Also Liu and Shen [15] studied Yetter–Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter–Drinfeld modules; they showed that the category of Hom-Yetter–Drinfeld modules is a braided monoidal category. Chen and Zhang [8] defined the category of Hom-Yetter–Drinfeld modules in a slightly different way to [15], and showed that it is a full monoidal subcategory of the left center of the left Hom-module category. We have defined in [13] the category of Doi Hom-Hopf modules and we have proved that the category of Hom-Yetter–Drinfeld modules is a subcategory of our category of Doi Hom-Hopf modules.

In this paper, we discuss the following question: how do we make the category of Doi Hom-Hopf modules into a monoidal category? We show in Section 3 that it is sufficient that (A, β) and (C, γ) are monoidal Hom-bialgebras with some extra conditions. As an example, we consider the category of Hom-Yetter–Drinfeld modules, which is well known to be a monoidal category from [15]; this category is a special case of our theory.

In Section 4, we give maps between the underlying monoidal Hom-Hopf algebras, Hom-comodule algebras and Hom-module coalgebras, which give rise to functors between the categories of Doi Hom-Hopf modules. Moreover, we study tensor identities for monoidal categories of Doi Hom-Hopf modules. As an application, we prove that the category of Doi Hom-Hopf modules has enough injective objects.

Suppose that we have a monoidal category of Doi Hom-Hopf modules. How do we define a braiding on this category? In Section 5, we point out this comes down to giving a twisted convolution inverse map $\mathscr{Q} : C \otimes C \to A \otimes A$ satisfying some complicated compatibility conditions. As an application we get a braiding on the category of Hom-modules, on the category of Hom-comodules, and on the category of Hom-Yetter–Drinfeld modules.

Throughout this paper we freely use the Hopf algebra and coalgebra terminology introduced in [10], [21], [23] and [24].

2. Preliminaries. Throughout this paper we work over a commutative ring k; we recall from [1] and [13] some information about Hom-structures, needed in what follows.

Let $\mathcal{C}$ be a category. We introduce a new category $\mathscr{H}(\mathcal{C})$ as follows: Objects are couples (M,μ) with $M \in \mathcal{C}$ and $\mu \in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f:(M,\mu)\to(N,\nu)$ is a morphism $f: M\to N$ in $\mathcal{C}$ such that $\nu\circ f = f\circ\mu$.

Let $\mathscr{M}_k$ denote the category of k-modules. Then $\mathscr{H}(\mathscr{M}_k)$ will be called the *Hom-category* associated to $\mathscr{M}_k$. If $(M,\mu)\in\mathscr{M}_k$, then $\mu: M\to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k) = (\mathscr{H}(\mathscr{M}_k),\otimes,(I,I),\widetilde{a},\widetilde{l},\widetilde{r}))$ is a monoidal category by [1, Proposition 1.1]: the tensor product of (M,μ) and (N,ν) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by $(M,\mu)\otimes(N,\nu) = (M\otimes N,\mu\otimes\nu)$.

Assume that $(M,\mu),(N,\nu),(P,\pi)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$. The associativity and unit constraints are given by the formulas

$$\begin{aligned}\widetilde{a}_{M,N,P}((m\otimes n)\otimes p) &= \mu(m)\otimes(n\otimes\pi^{-1}(p)),\\ \widetilde{l}_M(x\otimes m) = \widetilde{r}_M(m\otimes x) &= x\mu(m).\end{aligned}$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a monoidal Hom-algebra:

Definition 2.1. A *monoidal Hom-algebra* is an object $(A,\alpha)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $m_A: A\otimes A\to A$ and an element $1_A\in A$ such that

$$\begin{aligned}\alpha(ab) &= \alpha(a)\alpha(b), &\quad \alpha(1_A) &= 1_A,\\ \alpha(a)(bc) &= (ab)\alpha(c), & a1_A &= 1_Aa = \alpha(a),\end{aligned}$$

for all $a,b,c\in A$. Here we use the notation $m_A(a\otimes b) = ab$.

Definition 2.2. A *monoidal Hom-coalgebra* is an object $(C,\gamma)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with k-linear maps $\Delta: C\to C\otimes C$, $\Delta(c) = c_{(1)}\otimes c_{(2)}$ (summation implicitly understood) and $\varepsilon: C\to k$ such that

$$\Delta(\gamma(c)) = \gamma(c_{(1)})\otimes\gamma(c_{(2)}),\qquad \varepsilon(\gamma(c)) = \varepsilon(c),$$

and

$$\begin{aligned}\gamma^{-1}(c_{(1)})\otimes c_{(2)(1)}\otimes c_{(2)(2)} &= c_{(1)(1)}\otimes c_{(1)(2)}\otimes\gamma^{-1}(c_{(2)}),\\ \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)} &= \gamma^{-1}(c),\end{aligned}$$

for all $c\in C$.

Definition 2.3. A *monoidal Hom-bialgebra* $H = (H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that (H,α,m,η) is a monoidal Hom-algebra, $(H,\alpha,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra, and Δ and ε are morphisms of Hom-algebras, that is,

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(1_H) = 1_H.$$

Definition 2.4. A *monoidal Hom-Hopf algebra* is a monoidal Hom-bialgebra (H, α) together with a linear map $S : H \to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, \quad S\alpha = \alpha S.$$

Definition 2.5. Let (A, α) be a monoidal Hom-algebra. A *right (A, α)-Hom-module* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ consisting of a k-module and a linear map $\mu : M \to M$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \cdot a) = m \cdot a$, in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \quad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \to (N, \nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is called *right A-linear* if it preserves the A-action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ will denote the category of right (A, α)-Hom-modules and A-linear morphisms.

Definition 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A *right (C, γ)-Hom-comodule* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$ (notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}),$$
$$m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ)-Hom-comodules are defined in the obvious way. The category of right (C, γ)-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

Definition 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a *right (H, α)-Hom-comodule algebra* if (A, β) is a right (H, α) Hom-comodule with coaction $\rho_A : A \to A \otimes H$, $\rho_A(a) = a_{[0]} \otimes a_{[1]}$, such that

$$\rho_A(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \quad \rho_A(1_A) = 1_A \otimes 1_H,$$

for all $a, b \in A$.

Definition 2.8. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-coalgebra (C, γ) is called a *left (H, α)-Hom-module coalgebra* if

(C, γ) is a left (H, α)-Hom-module with action $\phi : H \otimes C \to C$, $\phi(h \otimes c) = h \cdot c$, such that

$$\Delta_C(h \cdot c) = h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}, \quad \varepsilon_C(h \cdot c) = \varepsilon_C(c)\varepsilon_H(h),$$

for all $c \in C$ and $g, h \in H$.

A *Doi Hom-Hopf datum* is a triple (H, A, C), where H is a monoidal Hom-Hopf algebra, A a right (H, α)-Hom comodule algebra and (C, γ) a left (H, α)-Hom module coalgebra.

DEFINITION 2.9. Given a Doi Hom-Hopf datum (H, A, C), a *Doi Hom-Hopf module* (M, μ) is a left (A, β)-Hom-module which is also a right (C, γ)-Hom-comodule with the coaction structure $\rho_M : M \to M \otimes C$ defined by $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ such that the following compatibility condition holds: for all $m \in M$ and $a \in A$,

$$\rho_M(a \cdot m) = a_{[0]} \cdot m_{[0]} \otimes a_{[1]} \cdot m_{[1]}.$$

A morphism between left-right Doi Hom-Hopf modules is a k-linear map which is a morphism in the categories ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)$ and $\widetilde{\mathscr{C}}(\mathscr{M}_k)^C$ at the same time. ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ will denote the category of left-right Doi Hom-Hopf modules and morphisms between them.

3. Making the category of Doi Hom-Hopf modules into a monoidal category. Now suppose that C and A are both monoidal Hom-bialgebras.

PROPOSITION 3.1. *Let* $(M, \mu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$, $(N, \nu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. *Then* $M \otimes N \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ *with structure maps*

$$a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n, \quad \rho_{M \otimes N}(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]}$$

if and only if

$$a_{(1)[0]} \otimes a_{(2)[0]} \otimes (a_{(1)[1]} \cdot c)(a_{(2)[1]} \cdot d) = a_{[0](1)} \otimes a_{[0](2)} \otimes a_{[1]} \circ (cd) \tag{3.1}$$

for all $a \in A$ *and* $c, d \in C$. *Furthermore,* $\mathcal{C} = {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ *is a monoidal category.*

Proof. It is easy to see that $M \otimes N$ is a left (A, β)-module and a right (C, γ)-comodule. Now we check that the compatibility condition holds:

$$\begin{aligned} \rho_{M \otimes N}(a \cdot (m \otimes n)) &= (a_{(1)} \cdot m)_{[0]} \otimes (a_{(2)} \cdot n)_{[0]} \otimes (a_{(1)} \cdot m)_{[1]} (a_{(2)} \cdot n)_{[1]} \\ &= a_{(1)[0]} \cdot m_{[0]} \otimes (a_{(2)[0]} \cdot n_{[0]}) \otimes (a_{(1)[1]} \cdot m_{[1]})(a_{(2)[1]} \cdot n_{[1]}) \\ &\overset{(3.1)}{=} a_{[0](1)} \cdot m_{[0]} \otimes (a_{[0](2)} \cdot n_{[0]}) \otimes a_{[1]} \cdot (m_{[1]} n_{[1]}) \\ &= a_{[0]} \cdot (m_{[0]} \otimes n_{[0]}) \otimes a_{[1]} \cdot (m_{[1]} n_{[1]}). \end{aligned}$$

So $M \otimes N \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$.

Conversely, one can easily check that $A\otimes C\in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$, let $m=1\otimes c$ and $n=1\otimes d$ for any $c,d\in C$ and easily get (3.2).

Furthermore, k is an object in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ with structure maps

$$a\cdot x=\varepsilon_A(a)x,\quad \rho(x)=x\otimes 1_C,$$

for all $x\in k$ if and only if

$$\varepsilon_A(a)1_C=\varepsilon_A(a_{(0)})(a_{(1)}\cdot 1_C) \tag{3.2}$$

for all $a\in A$. Then it is easy to see that $(\mathcal{C}={}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C,\otimes,k,\widetilde{a},\widetilde{l},\widetilde{r})$ is a monoidal category, where $\widetilde{a},\widetilde{l},\widetilde{r}$ are given by

$$\widetilde{a}_{M,N,P}((m\otimes n)\otimes p)=\mu(m)\otimes(n\otimes\pi^{-1}(p)),$$
$$\widetilde{l}_M(x\otimes m)=\widetilde{r}_M(m\otimes x)=x\mu(m),$$

for $(M,\mu),(N,\nu),(P,\pi)\in\mathcal{C}$. ■

We call $G=(H,A,C)$ a *monoidal Doi Hom-Hopf datum* if G is a Doi Hom-Hopf datum and A,C are Hom-bialgebras with the additional compatibility relations (3.1) and (3.2).

We will give an example of a monoidal category ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. First, we define Yetter–Drinfeld modules over a monoidal Hom-Hopf algebra; these were also introduced by Liu and Shen [15] or Guo and Zhang [13] similarly.

Definition 3.2. Let (H,α) be a monoidal Hom-Hopf algebra. A *left-right (H,α)-Hom-Yetter–Drinfeld module* is an object (M,μ) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that (M,μ) a left (H,α)-Hom-module and a right (H,α)-Hom-comodule with the following compatibility condition:

$$h_{(1)}\cdot m_{[0]}\otimes h_{(2)}m_{[1]}=\mu((h_{(2)}\cdot\mu^{-1}(m))_{[0]})\otimes(h_{(2)}\cdot\mu^{-1}(m))_{[1]}h_{(1)} \tag{3.3}$$

for all $h\in H$ and $m\in M$. We denote by ${}_H\mathscr{HYD}^H$ the category of left-right (H,α)-Hom-Yetter–Drinfeld modules, morphisms being left (H,α)-linear right (H,α)-colinear maps.

Proposition 3.3. *(3.3) is equivalent to*

$$\rho(h\cdot m)=\alpha(h_{(2)(1)})\cdot m_{[0]}\otimes(h_{(2)(2)}\alpha^{-1}(m_{[1]}))S^{-1}(h_{(1)}) \tag{3.4}$$

for all $h\in H$ and $m\in M$.

Proof. For one direction, we compute

$$\begin{aligned}
&\mu((h_{(2)}\cdot\mu^{-1}(m))_{[0]})\otimes((h_{(2)}\cdot\mu^{-1}(m))_{[1]})h_{(1)}\\
&\overset{(3.4)}{=}\mu(\alpha(h_{(2)(2)(1)})\cdot\mu^{-1}(m_{[0]}))\otimes\big((h_{(2)(2)(2)}\alpha^{-2}(m_{[1]}))S^{-1}(h_{(2)(1)})\big)h_{(1)}\\
&=\alpha(h_{(2)(1)})\cdot m_{[0]}\otimes(h_{(2)(2)}\alpha^{-1}(m_{[1]}))(S^{-1}(h_{(1)(2)})h_{(1)(1)})\\
&=h_{(1)}\cdot m_{[0]}\otimes h_{(2)}m_{[1]}.
\end{aligned}$$

Conversely, we have

$$
\begin{aligned}
&h_{(2)(1)}\cdot m_{[0]}\otimes(h_{(2)(2)}m_{[1]})S^{-1}(h_{(1)})\\
\overset{(3.3)}{=}\ & \mu((h_{(2)(2)}\cdot\mu^{-1}(m))_{[0]})\otimes((h_{(2)(2)}\cdot\mu^{-1}(m))_{[1]}h_{(2)(1)})S^{-1}(h_{(1)})\\
=\ & \mu((\alpha^{-1}(h_{(2)})\cdot\mu^{-1}(m))_{[0]})\otimes\alpha((\alpha^{-1}(h_{(2)})\cdot\mu^{-1}(m))_{[1]})(h_{(1)(2)}S^{-1}(h_{(1)(1)}))\\
=\ & \mu((\alpha^{-2}(h)\cdot\mu^{-1}(m))_{[0]})\otimes\alpha^{2}((\alpha^{-2}(h)\cdot\mu^{-1}(m))_{[1]})\\
=\ & (\alpha^{-1}(h)\cdot m)_{[0]}\otimes\alpha((\alpha^{-1}(h)\cdot m)_{[1]}),
\end{aligned}
$$

which implies (3.4). ∎

THEOREM 3.4. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode.*

(1) *H can be made into a right $H^{\mathrm{op}}\otimes H$-Hom-comodule algebra. The coaction $H\to H\otimes H^{\mathrm{op}}\otimes H$ is given by*

$$h\mapsto\alpha(h_{(2)(1)})\otimes(S^{-1}(\alpha^{-1}(h_{(1)}))\otimes h_{(2)(2)}).$$

(2) *H can be made into a left $H^{\mathrm{op}}\otimes H$-Hom-module coalgebra. The action of $H^{\mathrm{op}}\otimes H$ on H is given by*

$$(h\otimes k)\triangleright c=(k\alpha^{-1}(c))\alpha(h).$$

(3) *The category ${}_H\mathscr{HYD}^H$ of left-right Hom-Yetter–Drinfeld modules is isomorphic to a category of Doi Hom-Hopf modules, namely ${}_H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\mathrm{op}}\otimes H)^H$.*

Proof. (1) We first prove that H is a right $H^{\mathrm{op}}\otimes H$-Hom-comodule. For any $h\in H$,

$$
\begin{aligned}
(\alpha^{-1}\otimes\Delta_{H^{\mathrm{op}}\otimes H})\rho_H(h)&=h_{(2)(1)}\otimes\Delta_{H^{\mathrm{op}}\otimes H}(S^{-1}(\alpha^{-1}(h_{(1)}))\otimes h_{(2)(2)})\\
&=h_{(2)(1)}\otimes S^{-1}(\alpha^{-1}(h_{(1)(2)}))\otimes h_{(2)(2)(1)}\otimes S^{-1}(\alpha^{-1}(h_{(1)(1)}))\otimes h_{(2)(2)(2)}\\
&=\alpha(h_{(2)(1)(1)})\otimes S^{-1}(\alpha^{-1}(h_{(1)(2)}))\otimes h_{(2)(1)(2)}\\
&\quad\otimes S^{-1}(\alpha^{-1}(h_{(1)(1)}))\otimes\alpha^{-1}(h_{(2)(2)})\\
&=\alpha^{2}(h_{(2)(2)(1)(1)})\otimes S^{-1}(\alpha^{-1}(h_{(2)(1)}))\otimes\alpha(h_{(2)(2)(1)(2)})\\
&\quad\otimes S^{-1}(\alpha^{-2}(h_{(1)}))\otimes h_{(2)(2)(2)}\\
&=\alpha^{2}(h_{(2)(1)(2)(1)})\otimes S^{-1}(h_{(2)(1)(1)})\otimes\alpha(h_{(2)(1)(1)(2)})\\
&\quad\otimes S^{-1}(\alpha^{-2}(h_{(1)}))\otimes\alpha^{-1}(h_{(2)(2)})\\
&=\rho(\alpha(h_{(2)(1)}))\otimes S^{-1}(\alpha^{-2}(h_{(1)}))\otimes\alpha^{-1}(h_{(2)(2)})\\
&=(\rho_H\otimes\alpha^{-1})\rho_H(h).
\end{aligned}
$$

So H is a right $H^{\mathrm{op}} \otimes H$-Hom-comodule. We also have

$$\begin{aligned}\rho(hg) &= \alpha(h_{(2)(1)}g_{(2)(1)})\otimes\big(S^{-1}(h_{(1)}g_{(1)})\otimes\alpha^{-1}(h_{(2)(2)}g_{(2)(2)})\big)\\ &= \alpha(h_{(2)(1)})\alpha(g_{(2)(1)})\otimes\big(S^{-1}(h_{(1)})S^{-1}(g_{(1)})\otimes\alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g_{(2)(2)})\big)\\ &= \big(\alpha(h_{(2)(1)})\otimes(S^{-1}(h_{(1)})\otimes\alpha^{-1}(h_{(2)(2)}))\big)\\ &\qquad\qquad\big(\alpha(g_{(2)(1)})\otimes(S^{-1}(g_{(1)})\otimes\alpha^{-1}(g_{(2)(2)}))\big)\\ &= \rho_H(h)\rho_H(g).\end{aligned}$$

(2) Now we prove that H is an $H^{\mathrm{op}} \otimes H$-Hom-comodule. For any $h, l, k, m, c \in H$, we have

$$\begin{aligned}(\alpha(l) \otimes \alpha(m)) \triangleright [(h \otimes k) \triangleright c] &= (\alpha(l) \otimes \alpha(m)) \triangleright (k\alpha^{-1}(c))\alpha(h)\\ &= \big[\alpha(m)[(\alpha^{-1}(k)\alpha^{-2}(c))h]\big]\alpha^2(l) = \big[\alpha(m)[k(\alpha^{-2}(c))\alpha^{-1}(h)]\big]\alpha^2(l)\\ &= \alpha(mk)[[\alpha^{-1}(c))h]\alpha(l)] = \alpha(mk)[c(hl)] = mk[c\alpha(hl)]\\ &= (hl \otimes mk) \triangleright \alpha(c) = [(l \otimes m)(h \otimes k)] \triangleright \alpha(c),\end{aligned}$$

and this implies that H is an $H^{\mathrm{op}} \otimes H$-Hom-module.

Using the fact that (H, α) is an (H, α)-Hom-bimodule algebra, we can deduce that (H, α) is a left $H^{\mathrm{op}} \otimes H$-Hom-module coalgebra.

(3) Let $(M, \cdot)$ be a left (H, α)-module and (M, ρ_M) a right (H, α)-comodule. Then $M \in {}_H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\mathrm{op}} \otimes H)^H$ if and only if

$$\begin{aligned}\rho_M(h \cdot m) &= \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes \big(S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}\big) \triangleright m_{[1]}\\ &= \alpha(h_{(2)(1)}) \cdot m_{[0]} \otimes (h_{(2)(2)}\alpha^{-1}(m_{[1]}))S^{-1}(h_{(1)})\end{aligned}$$

for all $h \in H$ and $m \in M$. Thus ${}_H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\mathrm{op}} \otimes H)^H$ is isomorphic to ${}_H\mathscr{H}\mathscr{Y}\mathscr{D}^H$. ■

EXAMPLE 3.5. Let (H, α) be a monoidal Hom-Hopf algebra. We have shown that the category ${}_H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\mathrm{op}} \otimes H)^H$ of Doi Hom-Hopf modules and the category ${}_H\mathscr{H}\mathscr{Y}\mathscr{D}^H$ of Hom-Yetter–Drinfeld modules are isomorphic. Recall from [15] that the latter is a monoidal category; let us check that it is a special case of Proposition 3.3. Indeed, take $A = H$ and $C = H^{\mathrm{op}}$ as monoidal Hom-bialgebras. Let $a = h$, $c = k$ and $d = g$. Then the left-hand side amounts to

$$\begin{aligned}&h_{[0](1)} \otimes h_{[0](2)} \otimes h_{[1]} \cdot (k \bullet g)\\ &\quad= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes \big(S^{-1}(\alpha^{-1}(h_{(1)})) \otimes h_{(2)(2)}\big) \cdot (gk)\\ &\quad= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [h_{(2)(2)})\alpha^{-1}(gk)]S^{-1}(h_{(1)}).\end{aligned}$$

The right-hand side is

$$
\begin{aligned}
&h_{(1)[0]} \otimes h_{(2)[0]} \otimes (h_{[1](1)} \cdot k)(h_{[1](2)} \cdot g) \\
&= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \otimes \big((S^{-1}(\alpha^{-1}(h_{(1)(1)})) \otimes h_{(2)(2)(1)}) \cdot k\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bullet \big((S^{-1}(\alpha^{-1}(h_{(1)(2)})) \otimes h_{(2)(2)(2)}) \cdot g\big) \\
&= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \otimes ((h_{(2)(2)(2)}\alpha^{-1}(g))S^{-1}(h_{(1)(2)})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \big((h_{(2)(2)(1)}\alpha^{-1}(k))S^{-1}(h_{(1)(1)})\big) \\
&= \alpha(h_{(1)(2)(1)}) \otimes \alpha(h_{(2)(2)(1)}) \\
&\qquad \otimes \big((h_{(2)(2)(2)}\alpha^{-1}(g))[S^{-1}(\alpha^{-1}(h_{(1)(2)}))h_{(2)(2)(1)}]\big)kS^{-1}(h_{(1)(1)}) \\
&= \alpha(h_{(1)(1)(2)}) \otimes \alpha(h_{(2)(1)(2)}) \\
&\qquad \otimes \big((\alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g))[S^{-1}(h_{(1)(1)(2)})\alpha^{-1}(h_{(2)(1)})]\big)kS^{-1}(\alpha(h_{(1)(1)(1)})) \\
&= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes ((h_{(2)(2)(2)}\alpha^{-1}(g))[S^{-1}(h_{(2)(1)(1)})h_{(2)(1)(1)}]) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad kS^{-1}\big(\alpha^{-1}(h_{(1)})\big) \\
&= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes ((h_{(2)(2)}g)kS^{-1}(\alpha^{-1}(h_{(1)})) \\
&= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [(\alpha^{-1}(h_{(2)(2)})\alpha^{-1}(g))k]S^{-1}(h_{(1)}) \\
&= \alpha(h_{(2)(1)(1)}) \otimes \alpha(h_{(2)(1)(2)}) \otimes [h_{(2)(2)}\alpha^{-1}(gk)]S^{-1}(h_{(1)}).
\end{aligned}
$$

4. Tensor identities

THEOREM 4.1. *Given Doi Hom-Hopf data (H, A, C) and (H', A', C'), suppose that a morphism $\xi : (H, A, C) \to (H', A', C')$ consists of three maps $\varphi : H \to H'$, $\psi : A \to A'$ and $\phi : C \to C'$ which are respectively monoidal Hom-Hopf algebra, Hom-algebra and Hom-coalgebra maps satisfying*

$$\phi(h \cdot c) = \varphi(h) \cdot \phi(c), \tag{4.1}$$

$$\rho_A(\psi(a)) = \psi(a_{[0]}) \otimes \varphi(a_{[1]}), \tag{4.2}$$

for all $c \in C$, $h \in H$ and $a \in A$. Then we have a functor $F : {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C \to {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$, defined as follows:

$$F(M) = A' \otimes_A M,$$

where (A', β') is a left (A, β)-module via ψ and with structure maps defined by

$$b' \cdot (a' \otimes_A m) = \beta'^{-1}(b')a' \otimes_A \mu(m), \tag{4.3}$$

$$\rho_{F(M)}(a' \otimes_A m) = a'_{[0]} \otimes_A m_{[0]} \otimes a'_{[1]} \cdot \phi(m_{[1]}), \tag{4.4}$$

for all $a', b' \in A'$ and $m \in M$.

Proof. Let us show that $A' \otimes_A M$ is an object of ${}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$. It is routine to check that $F(M)$ is a left (A', β')-module. For this, we need to show that $A'\otimes_A M$ is a right (C',γ')-comodule and satisfies the compatibility condition. Indeed, for any $m\in M$ and $a',b'\in A'$, we have

$$\begin{aligned}\rho_{F(M)}(b'\cdot(a'\otimes_A m)) &= \rho_{F(M)}(\beta'^{-1}(b')a'\otimes_A \mu(m))\\ &= \beta'^{-1}(b'_{[0]})a'_{[0]}\otimes_A \mu(m_{[0]})\otimes[\beta'^{-1}(b'_{[1]})a'_{[1]}]\cdot\phi(\gamma(m_{[1]}))\\ &= b'_{[0]}[a'_{[0]}\otimes_A m_{[0]}]\otimes b'_{[1]}[a'_{[1]}\cdot\phi(m_{[1]})]\\ &= b'\cdot(a'_{[0]}\otimes_A m_{[0]}\otimes a'_{[1]}\cdot\phi(m_{[1]})) = b'\rho_{F(M)}(a'\otimes_A m),\end{aligned}$$

i.e., the compatibility condition holds. It remains to prove that $A'\otimes_A M$ is a right (C',γ')-comodule. For any $m\in M$ and $a'\in A'$, we have

$$\begin{aligned}&(\rho_{F(M)}\otimes \mathrm{id}_{C'})\rho_{F(M)}(a'\otimes_A m) = (\rho_{F(M)}\otimes \mathrm{id}'_{C})(a'_{[0]}\otimes_A m_{[0]}\otimes a'_{[1]}\cdot\phi(m_{[1]}))\\ &\quad= [a'_{[0][0]}\otimes_A m_{[0][0]}\otimes a'_{[0][1]}\cdot\phi(m_{[0][1]})]\otimes a'_{[1]}\cdot\phi(m_{[1]})\\ &\quad= [\beta'^{-1}(a'_{[0]})\otimes_A \mu^{-1}(m_{[0]})\otimes a'_{[1](1)}\cdot\phi(m_{[1](1)})]\otimes\alpha'(a'_{[1](2)})\cdot\phi(\gamma(m_{[1](2)}))\\ &\quad= a'_{[0]}\otimes_A m_{[0]}\otimes[a'_{[1](1)}\cdot\phi(m_{[1](1)})\otimes a'_{[1](2)}\cdot\phi(m_{[1](2)})]\\ &\quad= (\mathrm{id}_{F(M)}\otimes\Delta_{C'})\rho_{F(M)}(a'\otimes_A m),\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_{F(M)}\otimes\varepsilon)\rho_{F(M)}(a'\otimes_A m) &= (\mathrm{id}_{F(M)}\otimes\varepsilon)(a'_{[0]}\otimes_A m_{[0]}\otimes a'_{[1]}\cdot\phi(m_{[1]}))\\ &= a'_{[0]}\varepsilon(a'_{[1]})\otimes_A m_{[0]}\varepsilon(\phi(m_{[1]})) = a'\otimes_A m,\end{aligned}$$

as desired. ∎

THEOREM 4.2. *Under the assumptions of Theorem* 4.1*, we have a functor* $G: {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}\to {}_{A}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^{C}$ *which is right adjoint to* F*. It is defined by*

$$G(M') = M'\,\square_{C'}\,C,$$

with structure maps

$$a\cdot(m'\otimes c) = a_{[0]}\cdot m'\otimes a_{[1]}\cdot c, \tag{4.5}$$

$$\rho_{G(M')}(m'\otimes c) = \mu'^{-1}(m')\otimes c_{(1)}\otimes\gamma(c_{(2)}), \tag{4.6}$$

for all $a\in A$.

Proof. We first show that $G(M')$ is an object of ${}_{A}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^{C}$. It is not hard to check that $G(M')$ is a left (A,β)-module. Now we check that $G(M')$ is a right (C,γ)-comodule and satisfies the compatibility condition. For any $m'\in M'$ and $a\in A, c\in C$, we have

$$\begin{aligned}\rho_{G(M')}(a\cdot(m'\otimes c)) &= \rho_{G(M')}(a_{[0]}\cdot m'\otimes a_{[1]}\cdot c)\\ &= \beta^{-1}(a_{[0]})\cdot\mu'^{-1}(m')\otimes a_{[1](1)}\cdot c_{(1)}\otimes\alpha(a_{[1](2)})\cdot\gamma(c_{(2)})\\ &= a_{[0][0]}\cdot\mu'^{-1}(m')\otimes a_{[0][1]}\cdot c_{(1)}\otimes a_{[1]}\cdot\gamma(c_{(2)})\\ &= a\cdot(\mu'^{-1}(m')\otimes c_{(1)}\otimes\gamma(c_{(2)})) = a\rho_{G(M')}(m'\otimes c),\end{aligned}$$

i.e., the compatibility condition holds. It remains to prove that $M' \square_{C'} C$ is a right (C,γ)-comodule. For any $m' \in M'$ and $a \in A$, we have

$$\begin{aligned}(\rho_{G(M')} \otimes \mathrm{id}_{C'})\rho_{G(M')}(m' \otimes_A c) &= (\rho_{G(M')} \otimes \mathrm{id}_{C'})(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)}))\\ &= \mu'^{-2}(m') \otimes c_{(1)(1)} \otimes \gamma(c_{(1)(2)}) \otimes \gamma(c_{(2)})\\ &= \mu'^{-2}(m') \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(2)(1)}) \otimes \gamma^2(c_{(2)(2)})\\ &= \mu'^{-1}(m') \otimes c_{(1)} \otimes [\gamma(c_{(2)(1)}) \otimes \gamma(c_{(2)(2)})]\\ &= (\mathrm{id}_{G(M')} \otimes \Delta_C)\rho_{G(M')}(m' \otimes c),\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_{G(M')} \otimes \varepsilon)\rho_{G(M')}(m' \otimes c) &= (\mathrm{id}_{G(M')} \otimes \varepsilon)(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)}))\\ &= \mu'^{-1}(m') \otimes c_{(1)}\varepsilon(c_{(2)}) \otimes 1_C = m' \otimes c,\end{aligned}$$

as required.

We have $G(M') \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ and the functorial properties can be checked in a straightforward way. Finally, we show that G is a right adjoint to F. Take $(M,\mu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ and define $\eta_M : M \to GF(M) = (M \otimes_A A') \square_{C'} C$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes_A 1_{A'} \otimes m_{[1]}.$$

It is easy to see that $\eta_M \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Take $(M',\mu') \in {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$, and define $\delta_{M'} : FG(M') \to M'$, where

$$\delta_{M'}(m' \otimes c) \otimes_A a') = \varepsilon_C(c)m' \cdot a',$$

It is easy to check that $\delta_{M'}$ is (A,β)-linear and so $\delta_{M'} \in {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$. We can also verify η and δ defined above are natural transformations and satisfy

$$G(\delta_{M'}) \circ \eta_{G(M')} = I, \quad \delta_{F(M)} \circ F(\eta_M) = I,$$

for all $M \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ and $M' \in {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$. ∎

A morphism $\xi = (\varphi, \psi, \phi)$ between monoidal Doi Hom-Hopf data is called *monoidal* if φ and ϕ are monoidal Hom-bialgebra maps. We now consider the particular situation where $H = H'$ and $A = A'$. The following result is a generalization of [3].

THEOREM 4.3. *Let $\xi = (\mathrm{id}_H, \mathrm{id}_A, \phi) : (H,A,C) \to (H,A,C')$ be a monoidal morphism of monoidal Doi Hom-Hopf data. Then*

$$G(C') = C. \tag{4.7}$$

Let $(M,\mu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ be flat as a k-module, and take $(N,\nu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^{C'}$. If (C,γ) is a monoidal Hom-Hopf algebra, then

$$M \otimes G(N) \cong G(F(M) \otimes N) \quad \text{in } {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C. \tag{4.8}$$

If (C,γ) *has a twisted antipode* $\overline{S}$*, then*

$$G(N)\otimes M\cong G(N\otimes F(M)) \quad \textit{in } {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C. \tag{4.9}$$

Proof. We know that $\varepsilon_{C'}\otimes \mathrm{id}_C : C'\,\Box_C\, C\to C$ is an isomorphism; the inverse map is $(\phi\otimes \mathrm{id}_C)\Delta_C : C\to C'\,\Box_C\, C$. It is clear that $\varepsilon_{C'}\otimes \mathrm{id}_C$ is (A,β)-linear and (C,γ)-colinear. This proves (4.7).

Now we define a map

$$\Gamma : M\otimes G(N)=M\otimes (N\,\Box_{C'}\, C)\to G(F(M)\otimes N)=(F(M)\otimes N)\,\Box_{C'}\, C$$

by

$$\Gamma(m\otimes (n_i\otimes c_i))=(m_{[0]}\otimes n_i)\otimes m_{[1]}c_i.$$

Recall that $F(M)=M$ as an (A,β)-module, with (C',γ')-coaction given by

$$\rho_{F(M)}(m)=m_{[0]}\otimes \phi(m_{[1]}).$$

(1) Γ is well-defined. We have to show that

$$\Gamma(m_i\otimes (n_i\otimes c_i))\in (F(M)\otimes N)\,\Box'_C\, C.$$

This may be seen as follows: for any $m\in M$ and $n_i\,\Box_{C'}\, c\in N\,\Box_{C'}\, C$, we have

$$\begin{aligned}(\rho_{F(M)\otimes N}\otimes \mathrm{id}_C)((m_{[0]}\otimes n_i)\otimes m_{[1]}c_i)&=(m_{[0][0]}\otimes n_{i[0]})\otimes \phi(m_{[0][1]})n_{i[1]}\otimes m_{[1]}c_i\\&=(\mu(m_{[0]})\otimes \nu(n_i))\otimes \phi(m_{[0][1]})\phi(c_{i(1)})\otimes \gamma^{-1}(m_{[1]}c_{i(2)})\\&=(m_{[0]}\otimes n_i)\otimes [\phi(m_{[0][1]})\phi(c_{i(1)})\otimes m_{[1]}c_{i(2)}]\\&=(\mathrm{id}_{F(M)\otimes N}\otimes \rho_{C'})((m_{[0]}\otimes n_i)\otimes m_{[1]}c_i).\end{aligned}$$

(2) Γ is (A,β)-linear. Indeed, for any $a\in A, m\in M$ and $n_i\,\Box_{C'}\, c\in N\,\Box_{C'}\, C$, we have

$$\begin{aligned}\Gamma(a\cdot (m\otimes (n_i\otimes c_i)))&=\Gamma(a_{(1)}\cdot m\otimes (a_{(2)[0]}\cdot n_i\otimes a_{(2)[1]}\cdot c_i))\\&=(a_{(1)[0]}\cdot m_{[0]}\otimes a_{(2)[0]}\cdot n_i)\otimes (a_{(1)[1]}\cdot m_{[1]})(a_{(2)[1]}\cdot c_i)\\&=(a_{[0](1)}\cdot m_{[0]}\otimes a_{[0](2)}\cdot n_i)\otimes a_{(1)}\cdot (m_{[1]}c_i)\\&=a_{[0]}\cdot (m_{[0]}\otimes n_i)\otimes a_{(1)}\cdot (m_{[1]}c_i)=a\cdot \Gamma(m\otimes (n_i\otimes c_i)).\end{aligned}$$

(3) Γ is (C,γ)-colinear. Indeed, for any $m\in M$ and $n_i\,\Box_{C'}\, c\in N\,\Box_{C'}\, C$, we have

$$\begin{aligned}\rho\circ \Gamma(m\otimes (n_i\otimes c_i))&=\rho((m_{[0]}\otimes n_i)\otimes m_{[1]}c_i)\\&=(\mu^{-1}(m_{[0]})\otimes \nu^{-1}(n_i))\otimes m_{[1](1)}c_{i(1)}\otimes \gamma(m_{[1](2)}c_{i(2)})\\&=(m_{[0]}\otimes \nu^{-1}(n_i))\otimes m_{[0][1]}c_{i(1)}\otimes m_{[1]}\gamma(c_{i(2)})\\&=(\Gamma\otimes \mathrm{id}_C)(m_{[0]}\otimes (\nu^{-1}(n_i)\otimes c_{i(1)}))\otimes m_{[1]}\gamma(c_{i(2)})\\&=(\Gamma\otimes \mathrm{id}_C)\circ \rho(m\otimes (n_i\otimes c_i)).\end{aligned}$$

Assume (C,γ) has an antipode and define

$$\Psi : (F(M)\otimes N)\,\square_{C'}\, C \to M\otimes (N\,\square_{C'}\, C),$$
$$\Psi((m_i\otimes n_i)\otimes c_i) = \mu^2(m_{i[0]})\otimes (n_i\otimes S(m_{i[1]})\gamma^{-2}(c_i)).$$

We have to show that Ψ is well-defined. (M,μ) is flat, so $M\otimes (N\,\square_{C'}\, C)$ is the equalizer of the maps

$$\mathrm{id}_M\otimes \mathrm{id}_N\otimes \rho_C : M\otimes N\otimes C\to M\otimes N\otimes C'\otimes C$$

and

$$\mathrm{id}_M\otimes \rho_N\otimes \mathrm{id}_C : M\otimes N\otimes C\to M\otimes N\otimes C'\otimes C.$$

Now take $(m_i\otimes n_i)\otimes c_i \in (F(M)\otimes N)\,\square_{C'}\, C$. Then

$$(m_{i[0]}\otimes n_{i[0]})\otimes \phi(m_{i[1]})n_{i[1]}\otimes c_i = (\mu^{-1}(m_i)\otimes \nu^{-1}(n_i))\otimes \phi(c_{i(1)})\otimes \gamma(c_{i(2)}). \tag{4.10}$$

Therefore,

$$\begin{aligned}
&\mathrm{id}_M\otimes \mathrm{id}_N\otimes \rho_C(\mu^2(m_{i[0]})\otimes (n_i\otimes S(m_{i[1]})\gamma^{-2}(c_i)))\\
&\quad= \mu^2(m_{i[0]})\otimes \big(n_i\otimes \phi(S(m_{i[1](2)})\gamma^{-2}(c_{i(1)}))\otimes S(m_{i[1](1)})\gamma^{-2}(c_{i(2)})\big)\\
&\quad= m_{i[0]}\otimes \nu^{-1}(n_i)\otimes \phi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)}))\otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{id}_M\otimes \rho_N\otimes \mathrm{id}_C(\mu^2(m_{i[0]})\otimes (n_i\otimes S(m_{i[1]})\gamma^{-2}(c_i)))\\
&\quad= \mu^2(m_{i[0]})\otimes \big(n_{i[0]}\otimes n_{i[1]}\otimes S(m_{i[1]})\gamma^{-2}(c_i)\big)\\
&\quad= m_{i[0]}\otimes n_{i[0]}\otimes \gamma(n_{i[1]})\otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i).
\end{aligned}$$

Applying $(\mathrm{id}_M\otimes \phi\otimes \mathrm{id}_C)\circ(\mathrm{id}_M\otimes(\Delta_C\circ S_C))\circ\rho_M$ to the first factor of (4.10), we obtain

$$\begin{aligned}
&m_{i[0][0]}\otimes \phi(S(m_{i[0][1](2)}))\otimes S(m_{i[0][1](1)})\otimes n_{i[0]}\otimes \phi(m_{i[1]})n_{i[1]}\otimes c_i\\
&\quad= \mu^{-1}(m_{i[0]})\otimes \phi(S(\gamma^{-1}(m_{i[1](2)})))\otimes S(\gamma^{-1}(m_{i[1](1)}))\\
&\qquad\otimes \nu^{-1}(n_i)\otimes \phi(c_{i(1)})\otimes \gamma(c_{i(2)}).
\end{aligned}$$

Applying $\mathrm{id}_M\otimes\gamma^2\otimes\mathrm{id}_C\otimes\mathrm{id}_N\otimes\gamma^{-1}\otimes\gamma^{-1}$ to the above identity, we have

$$\begin{aligned}
&m_{i[0][0]}\otimes\phi(S(\gamma^2(m_{i[0][1](2)})))\otimes S(m_{i[0][1](1)})\otimes n_{i[0]}\otimes\gamma^{-1}(\phi(m_{i[1]})n_{i[1]})\otimes\gamma^{-1}(c_i)\\
&\quad= \mu^{-1}(m_{i[0]})\otimes \phi(S(\gamma(m_{i[1](2)})))\otimes S(\gamma^{-1}(m_{i[1](1)}))\otimes \nu^{-1}(n_i)\\
&\qquad\otimes \phi(\gamma^{-1}(c_{i(1)}))\otimes c_{i(2)}.
\end{aligned}$$

Multiplying the second and the fifth factor, and also the third and sixth, we have

$$\mu(m_{i[0]}) \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i)$$
$$= \mu(m_{i[0]}) \otimes \nu^{-1}(n_i) \otimes \phi\big(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})\big) \otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)},$$

and applying $\mu^{-1} \otimes \mathrm{id}_N \otimes \mathrm{id}_C \otimes \mathrm{id}_C$ to the above identity, we obtain

$$m_{i[0]} \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i)$$
$$= m_{i[0]} \otimes \nu^{-1}(n_i) \otimes \phi\big(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})\big) \otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)}$$

or

$$\mathrm{id}_M \otimes \rho_N \otimes \mathrm{id}_C \circ (\Psi((m_i \otimes n_i) \otimes c_i)) = \mathrm{id}_M \otimes \mathrm{id}_N \otimes \rho_C \circ (\Psi((m_i \otimes n_i) \otimes c_i)).$$

Let us point out that Γ and Ψ are each other's inverses. In fact,

$$\begin{aligned}
\Gamma \circ \Psi((m_i \otimes n_i) \otimes c_i) &= \Gamma(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]}\gamma^{-2}(c_i)))) \\
&= (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes \gamma^2(m_{i[0][1]})S(m_{i[1]})\gamma^{-2}(c_i) \\
&= (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes [\gamma(m_{i[0][1]})S(m_{i[1]})]\gamma^{-1}(c_i) \\
&= (\mu(m_{i[0]}) \otimes n_i) \otimes [\gamma(m_{i[1](1)})S(\gamma(m_{i[1](2)}))]\gamma^{-1}(c_i) \\
&= (m_i \otimes n_i) \otimes c_i,
\end{aligned}$$

and

$$\begin{aligned}
\Psi \circ \Gamma(m \otimes (n_i \otimes c_i)) &= \Psi((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i) \\
&= \mu^2(m_{[0][0]}) \otimes \big(n_i \otimes [S(\gamma^{-1}(m_{[0][1]}))\gamma^{-2}(m_{[1]})]\gamma^{-1}(c_i)\big) \\
&= \mu(m_{[0]}) \otimes \big(n_i \otimes [S(\gamma^{-1}(m_{[1](1)}))\gamma^{-1}(m_{[1](2)})]\gamma^{-1}(c_i)\big) \\
&= m \otimes (n_i \otimes c_i).
\end{aligned}$$

The proof of (4.9) is similar and left to the reader. ∎

COROLLARY 4.4. *Let (H, A, C) be a monoidal Doi Hom-Hopf datum, and Λ: ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C \to {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)$ the functor forgetting the (C,γ)-coaction. For any flat Doi Hom-Hopf module (M,μ), we have an isomorphism*

$$M \otimes C \cong \Lambda(M) \otimes C$$

in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. If k is a field, then ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ has enough injective objects, and any injective object in ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ is a direct summand of an object of the form $I \otimes C$, where I is an injective (A,β)-module.

We have already proved that the category of Hom-Yetter–Drinfeld modules may be viewed as the category of Doi Hom-Hopf modules corresponding to a monoidal Doi Hom-Hopf datum. Then we have the following corollary.

COROLLARY 4.5. *Let (H,α) be a monoidal Hom-Hopf algebra with the bijective antipode. Then the category of Hom-Yetter–Drinfeld modules over (H,α) is a Grothendieck category with enough injective objects.*

We continue with the dual version of Theorem 4.3.

THEOREM 4.6. *Let $\chi = (\mathrm{id}_H, \psi, \mathrm{id}_C) : (H, A, C) \to (H, A', C)$ be a monoidal morphism of monoidal Doi Hom-Hopf data. Then*

$$F(A) = A'. \tag{4.11}$$

Let $(M,\mu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ be flat as a k-module, and take $(N,\nu) \in {}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. If (A',β') is a monoidal Hom-Hopf algebra, then

$$F(M) \otimes N \cong F(M \otimes G(N)) \quad \text{in } {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C. \tag{4.12}$$

If (A',β') has a twisted antipode $\overline{S}$, then

$$N \otimes F(M) \cong F(G(N) \otimes M) \quad \text{in } {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C. \tag{4.13}$$

Proof. We only prove (4.12) and similarly for (4.11) and (4.13). Assume that (A',β') is a monoidal Hom-Hopf algebra and define

$$\Gamma : F(M \otimes G(N)) = A' \otimes_A M \otimes G(N) \to F(M) \otimes N = (A' \otimes_A M) \otimes N$$

by

$$\Gamma(a' \otimes (m \otimes n)) = (a'_{(1)} \otimes m) \otimes a'_{(2)} \cdot n$$

for all $a' \in A', m \in M$ and $n \in N$. Then Γ is well-defined since

$$\begin{aligned}
\Gamma(a'\psi(a) \otimes (m \otimes n)) &= (a'_{(1)}\psi(a_{(1)}) \otimes m) \otimes a'_{(2)}\psi(a_{(2)}) \cdot n \\
&= (a'_{(1)} \otimes a_{(1)} \cdot m) \otimes a'_{(2)}\psi(a_{(2)}) \cdot n \\
&= \Gamma(a' \otimes (a_{(1)} \cdot m \otimes \psi(a_{(2)}) \cdot n)) \\
&= \Gamma(a' \otimes a \cdot (m \otimes n)).
\end{aligned}$$

It is easy to check that Γ is (A',β')-linear. Now we shall verify that Γ is (C,γ)-colinear based on (3.1). For any $a' \in A', m \in M$ and $n \in N$, we have

$$\begin{aligned}
\rho(\Gamma(a' \otimes (m \otimes n))) &= \rho((a'_{(1)} \otimes m) \otimes a'_{(2)} \cdot n) \\
&= (a'_{(1)[0]} \otimes m_{[0]}) \otimes (a'_{(2)[0]} \cdot n_{[0]}) \otimes (a'_{(1)[1]} \otimes m_{[1]})(a'_{(2)[1]} \cdot n_{[1]}) \\
&\overset{(3.1)}{=} (a'_{[0](1)} \otimes m_{[0]}) \otimes (a'_{[0](2)} \cdot n_{[0]}) \otimes a'_{[1]}(m_{[1]}n_{[1]}) \\
&= (\Gamma \otimes \mathrm{id}_c)(a'_{[0]} \otimes (m_{[0]} \otimes n_{[0]})) \otimes a'_{[1]}(m_{[1]}n_{[1]}) \\
&= (\Gamma \otimes \mathrm{id}_c)\rho(a' \otimes (m \otimes n)).
\end{aligned}$$

The inverse of Γ is given by

$$\Psi((a' \otimes m) \otimes n) = \beta'^2(a'_{(1)}) \otimes (m \otimes S(a'_{(2)})\nu^{-2}(n))$$

for all $a' \in A', m \in M$ and $n \in N$. One can check that Ψ is well-defined similarly to Γ. Finally, we have

$$\begin{aligned}\Psi(\Gamma(a'\otimes(m\otimes n))) &= \Psi((a'_{(1)}\otimes m)\otimes a'_{(2)}\cdot n)\\ &= \beta'^2(a'_{(1)(1)})\otimes(m\otimes S(a'_{(1)(2)})\nu^{-2}(a'_{(2)}\cdot n))\\ &= \beta'(a'_{(1)})\otimes(m\otimes[S(\beta'^{-1}(a'_{(2)(1)}))\beta'^{-1}(a'_{(2)(2)})]\cdot\nu^{-1}(n))\\ &= a'\otimes a'\otimes(m\otimes n)\end{aligned}$$

and

$$\begin{aligned}\Gamma(\Psi((a'\otimes m)\otimes n)) &= \Gamma\big(\beta'^2(a'_{(1)})\otimes(m\otimes S(a'_{(2)})\nu^{-2}(n))\big)\\ &= (\beta'^2(a'_{(1)(1)})\otimes m)\otimes a'_{(2)}\cdot\beta'^2(a'_{(1)(2)})\cdot S(a'_{(2)})\nu^{-2}(n)\\ &= (\beta'(a'_{(1)})\otimes m)\otimes a'_{(2)}\cdot[\beta'(a'_{(2)(1)})\cdot S(\beta'(a'_{(2)}))]\nu^{-1}(n)\\ &= (a'\otimes m)\otimes n,\end{aligned}$$

as needed. ∎

5. Braidings on the category of Doi Hom-Hopf modules. Consider now a map $\mathscr{Q} : C\otimes C\to A\otimes A$, with twisted convolution inverse $\mathscr{R}$ such that $(\beta\otimes\beta)\mathscr{Q} = \mathscr{Q}(\gamma\otimes\gamma)$ and $(\beta\otimes\beta)\mathscr{R} = \mathscr{R}(\gamma\otimes\gamma)$. This means that

$$\begin{aligned}&\mathscr{R}\big(\mathscr{Q}^1(c_{(2)}\otimes d_{(2)})_{[1]}\cdot\gamma^{-1}(c_{(1)})\otimes\mathscr{Q}^2(c_{(2)}\otimes d_{(2)})_{[1]}\cdot\gamma^{-1}(d_{(1)})\big)\\ &\qquad\big(\beta(\mathscr{Q}^2(c_{(2)}\otimes d_{(2)})_{[0]})\otimes\beta(\mathscr{Q}^1(c_{(2)}\otimes d_{(2)})_{[0]})\big) = \varepsilon_C(c)1_A\otimes\varepsilon_C(d)1_A,\end{aligned}$$

$$\begin{aligned}&\mathscr{Q}\big(\mathscr{R}^2(c_{(2)}\otimes d_{(2)})_{[1]}\cdot\gamma^{-1}(c_{(1)})\otimes\mathscr{R}^1(c_{(2)}\otimes d_{(2)})_{[1]}\cdot\gamma^{-1}(d_{(1)})\big)\\ &\qquad\big(\beta(\mathscr{R}^2(c_{(2)}\otimes d_{(2)})_{[0]})\otimes\beta(\mathscr{R}^1(c_{(2)}\otimes d_{(2)})_{[0]})\big) = \varepsilon_C(c)1_A\otimes\varepsilon_C(d)1_A,\end{aligned}$$

for all $c, d\in C$. Sometimes, we write $\mathscr{Q}(c\otimes d) := \mathscr{Q}^1(c\otimes d)\otimes\mathscr{Q}^2(c\otimes d)$ for all $c, d\in C$.

Let $(M,\mu),(N,\nu)\in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. By Proposition 3.3 we know that $(M\otimes N,\mu\otimes\nu)\in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Define a map

$$\begin{aligned}&c_{M,N} : M\otimes N\to N\otimes M,\\ &c_{M,N}(m\otimes n) = \mathscr{Q}(n_{[1]}\otimes m_{[1]})(n_{[0]}\otimes m_{[0]}).\end{aligned}\tag{5.1}$$

We will prove that $c_{M,N}$ is an isomorphism with inverse

$$\begin{aligned}&c^{-1}_{M,N} : N\otimes M\to M\otimes N,\\ &c^{-1}_{M,N}(n\otimes m) = \mathscr{R}(n_{[1]}\otimes m_{[1]})(m_{[0]}\otimes n_{[0]}).\end{aligned}$$

For any $m \in M$ and $n \in N$, we have

$$
\begin{aligned}
&c^{-1}_{M,N} \circ c_{M,N}(m \otimes n)\\
&= c^{-1}_{M,N}(\mathscr{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]}))\\
&= \mathscr{R}\left((\mathscr{Q}^1(n_{[1]} \otimes m_{[1]}) \cdot n_{[0]})_{[1]} \otimes (\mathscr{Q}^2(n_{[1]} \otimes m_{[1]}) \cdot m_{[0]})_{[1]}\right)\\
&\qquad \left((\mathscr{Q}^2(n_{[1]} \otimes m_{[1]}) \cdot m_{[0]})_{[0]} \otimes (\mathscr{Q}^1(n_{[1]} \otimes m_{[1]}) \cdot n_{[0]})_{[0]}\right)\\
&= \mathscr{R}\left(\mathscr{Q}^1(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[1]} \cdot n_{[1](1)} \otimes \mathscr{Q}^2(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[1]} \cdot m_{[1](1)}\right)\\
&\qquad \left(\mathscr{Q}^2(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[0]} \cdot \mu^{-1}(m_{[0]}) \otimes \mathscr{Q}^1(\gamma(n_{[1](2)}) \otimes \gamma(m_{[1](2)}))_{[0]} \cdot \nu^{-1}(n_{[0]})\right)\\
&= \left(\mathscr{R}(\mathscr{Q}^1(n_{[1](2)} \otimes m_{[1](2)})_{[1]} \cdot \gamma^{-1}(n_{[1](1)}) \otimes \mathscr{Q}^2(n_{[1](2)} \otimes m_{[1](2)})_{[1]} \cdot \gamma^{-1}(m_{[1](1)}))\right.\\
&\qquad \left.(\beta(\mathscr{Q}^2(n_{[1](2)} \otimes m_{[1](2)})_{[0]}) \otimes \beta(\mathscr{Q}^1(n_{[1](2)} \otimes m_{[1](2)})_{[0]}))\right)(m_{[0]} \otimes n_{[0]})\\
&= (\varepsilon_C(m_{[1]})1_A \otimes \varepsilon_C(n_{[1]})1_A)(m_{[0]} \otimes n_{[0]}) = m \otimes n.
\end{aligned}
$$

So $c^{-1}_{M,N} \circ c_{M,N} = \mathrm{id}_{M\otimes N}$. Similarly, we can prove $c_{M,N} \circ c^{-1}_{M,N} = \mathrm{id}_{N\otimes M}$.

Our aim is now to give necessary and sufficient conditions on $\mathscr{Q}$ for $c_{M,N}$ to define a braiding on the monoidal category of Doi Hom-Hopf modules. Recall from [15] that for any $(M,\mu),(N,\nu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$, the associativity and unit constraints are given by

$$
\begin{aligned}
&a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),\\
&(m \otimes n) \otimes p \mapsto \mu(m) \otimes (n \otimes \pi^{-1}(p)),\\
&l_M : k \otimes M \to M, \qquad k \otimes m \mapsto k\mu(m),\\
&r_M : M \otimes k \to M, \qquad m \otimes k \mapsto k\mu(m).
\end{aligned}
$$

Next, we will find conditions under which $c_{M,N}$ is both an (A,β)-module map and a (C,γ)-comodule map, and satisfies the following conditions (for $P \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$):

$$a_{N,P,M} \circ c_{M,N\otimes P} \circ a_{M,N,P} = (\mathrm{id}_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes \mathrm{id}_P), \tag{5.2}$$

$$a^{-1}_{N,P,M} \circ c_{M\otimes N,P} \circ a^{-1}_{M,N,P} = (c_{M,P} \otimes \mathrm{id}_N) \circ a^{-1}_{M,P,N} \circ (\mathrm{id}_M \otimes c_{N,P}). \tag{5.3}$$

Recall from [13] that $A \otimes C$ can be made into a Doi Hom-Hopf module as follows: the (A,β)-action and (C,γ)-coaction on $A \otimes C$ are given by the formulas

$$a \cdot (b \otimes c) = \beta^{-1}(a)b \otimes \gamma(c), \qquad \rho_{A\otimes C}(b \otimes c) = (b_{[0]} \otimes c_{(1)}) \otimes b_{[1]}c_{(2)},$$

for any $a, b \in A$ and $c \in C$.

To formulate and prove our main result, we need some lemmas:

Lemma 5.1. *Let $(M,\mu),(N,\nu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Then $c_{M,N}$ is (A,β)-linear if and only if*

$$\mathscr{Q}(a_{(2)[1]} \cdot c \otimes a_{(1)[1]} \cdot d)(a_{(2)[0]} \otimes a_{(1)[0]}) = \Delta(a)\mathscr{Q}(c \otimes d) \tag{5.4}$$

for all $a \in A$ and $c, d \in C$.

Proof. If $c_{M,N}$ is (A,β)-linear then $a\triangleright c_{M,N}(m\otimes n)=c_{M,N}(a\triangleright(m\otimes n))$. We compute the two sides of the equation as follows:

$$a\triangleright c_{M,N}(m\otimes n)=(a_{(1)}\otimes a_{(2)})\mathscr{Q}(n_{[1]}\otimes m_{[1]})(n_{[0]}\otimes m_{[0]})$$

and

$$c_{M,N}(a\triangleright(m\otimes n))=\mathscr{Q}(a_{(2)[1]}\cdot n_{[1]}\otimes a_{(1)[1]}\cdot m_{[1]})(a_{(2)[0]}\cdot n_{[0]}\otimes a_{(1)[0]}\cdot m_{[0]}).$$

Conversely, considering these equations and taking $M=N=A\otimes C$ and $m=1\otimes c$ and $n=1\otimes d$ for all $c,d\in C$, we get (5.4). ∎

Recall from [7] that a *quasitriangular monoidal Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (H,α) together with an invertible element $R=R^{(1)}\otimes R^{(2)}\in H\otimes H$ such that:

(QT1) $\Delta(R^{(1)})\otimes R^{(2)}=R^{(1)}\otimes r^{(1)}\otimes R^{(2)}r^{(2)}$,
(QT2) $R^1\otimes\Delta(R^2)=R^1r^1\otimes r^2\otimes R^2$,
(QT3) $\varepsilon(R^{(1)})R^{(2)}=1_H, R^{(1)}\varepsilon(R^{(2)})=1_H$,
(QT4) $\Delta^{\mathrm{cop}}(h)R=R\Delta(h)$,
(QT5) $(\alpha\otimes\alpha)(R)=R$,

where $\Delta^{\mathrm{cop}}(h)=h_{(2)}\otimes h_{(1)}$ for all $h\in H$. Moreover, (H,α) is called *almost cocommutative* if $\Delta^{\mathrm{cop}}(h)R=R\Delta(h)$.

EXAMPLE 5.2. Suppose that $C=k$ and write $R=\mathscr{Q}(1\otimes 1)$. Then (5.4) takes the form $R\Delta_A^{\mathrm{cop}}(a)=\Delta_A(a)R$, and this means that (A,β) is almost cocommutative.

LEMMA 5.3. *Let $(M,\mu),(N,\nu)\in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Then $c_{M,N}$ is (C,γ)-colinear if and only if*

$$\mathscr{Q}(d_{(2)}\otimes c_{(2)})_{[0]}\otimes m_C\big(\mathscr{Q}(d_{(2)}\otimes c_{(2)})_{[1]}(d_{(1)}\otimes c_{(1)})\big)=\mathscr{Q}(d_{(1)}\otimes c_{(1)})\otimes c_{(2)}d_{(2)}\tag{5.5}$$

for all $c,d\in C$.

Proof. If $c_{M,N}$ is (C,γ)-colinear, then

$$\begin{aligned}\rho_{N\otimes M}c_{M,N}(m\otimes n)&=\rho_{N\otimes M}\big(\mathscr{Q}(n_{[1]}\otimes m_{[1]})(n_{[0]}\otimes m_{[0]})\big)\\&=\mathscr{Q}(n_{[1]}\otimes m_{[1]})_{[0]}(n_{[0][0]}\otimes m_{[0][0]})\otimes m_C\big(\mathscr{Q}(n_{[1]}\otimes m_{[1]})_{[1]}(n_{[0][1]}\otimes m_{[0][1]})\big)\\&=\mathscr{Q}(\gamma^{-1}(n_{[1](2)})\otimes\gamma^{-1}(m_{[1](2)}))_{[0]}(\nu(n_{[0]})\otimes\mu(m_{[0]}))\\&\qquad\otimes m_C\big(\mathscr{Q}(\gamma^{-1}(n_{[1](2)})\otimes\gamma^{-1}(m_{[1](2)}))_{[1]}(n_{[1](1)}\otimes m_{[1](2)})\big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(c_{M,N}\otimes\mathrm{id}_C)\rho_{M\otimes N}(m\otimes n)&=\mathscr{Q}(n_{[0][1]}\otimes m_{[0][1]})(n_{[0][0]}\otimes m_{[0][0]})\otimes(m_{[1]}n_{[1]})\\&=\mathscr{Q}(n_{[1](1)}\otimes m_{[1](1)})(\nu(n_{[0]})\otimes\mu(m_{[0]}))\otimes\gamma^{-1}(m_{[1](2)}n_{[1](2)}).\end{aligned}$$

Conversely, let $M = N = A \otimes C$ and take $m = 1 \otimes c$ and $n = 1 \otimes d$ for all $c, d \in C$. Then we can get (5.5). ■

Definition 5.4. A *coquasitriangular monoidal Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (H, α) together with a bilinear form σ on (H, α) (i.e. $\sigma \in \mathrm{Hom}(H \otimes H, k)$) such that:

(BR1) $\sigma(hg, l)\rangle = \sigma(h, l_{(2)})\sigma(g, l_{(1)})$,
(BR2) $\sigma(h, gl) = \sigma(h_{(1)}, g)\sigma(h_{(2)}, l)$,
(BR3) $\sigma(h_{(1)}, g_{(1)})g_{(2)}h_{(2)} = h_{(1)}g_{(1)}\sigma(h_{(2)}, g_{(2)})$,
(BR4) $\sigma(1_H, h) = \sigma(h, 1_H) = \varepsilon(h)$,
(BR5) $\sigma(\alpha(h), \alpha(g)) = \sigma(h, g)$,

for all $h, g, l \in H$. Moreover, (H, α) is called *almost commutative* if

$$\sigma(h_{(1)}, g_{(1)})g_{(2)}h_{(2)} = h_{(1)}g_{(1)}\sigma(h_{(2)}, g_{(2)}).$$

Example 5.5. Suppose $A = k$. Then (5.5) takes the form

$$\mathscr{Q}(h_{(1)}, g_{(1)})g_{(2)}h_{(2)} = h_{(1)}g_{(1)}\mathscr{Q}(h_{(2)}, g_{(2)}),$$

and this means that (A, β) is almost commutative.

Lemma 5.6. *Let $(M, \mu), (N, \nu), (P, \pi) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Then (5.2) holds if and only if, with $\mathscr{U} = \mathscr{Q}$,*

$$\begin{aligned}(5.6)\qquad \mathscr{Q}^1(e\otimes\gamma(d_{(2)}))\otimes(\mathscr{U}^1\big(\gamma^{-1}(c)\otimes\mathscr{Q}^2(e\otimes\gamma(d_{(2)}))_{[1]}d_{(1)}\big)&\\ \otimes\mathscr{U}^2\big(\gamma^{-2}(c)\otimes\mathscr{Q}^2(e\otimes c_{(2)})_{[1]}\gamma^{-1}(c_{(1)})\big)\mathscr{Q}^2(e\otimes\gamma(d_{(2)}))_{[0]}&\\ = \mathscr{Q}^1(e\gamma^{-1}(c)\otimes\gamma(m_{[1]}))_{(1)}\otimes\mathscr{Q}^1(e\gamma^{-1}(c)\otimes\gamma(c))_{(2)}\otimes\mathscr{Q}^2(\gamma^{-1}(e)\gamma^{-2}(c)\otimes d)&\end{aligned}$$

for all $c, d, e \in C$.

Proof. If (5.2) holds, then

$$\begin{aligned}&(\mathrm{id}_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes \mathrm{id}_P)((m \otimes n) \otimes p)\\
&= (\mathrm{id}_N \otimes c_{M,P}) \circ a_{N,M,P}(\mathscr{Q}^1(n_{[1]} \otimes m_{[1]})n_{[0]} \otimes \mathscr{Q}^2(n_{[1]} \otimes m_{[1]})m_{[0]} \otimes p)\\
&= (\mathrm{id}_N \otimes c_{M,P})(\beta(\mathscr{Q}^1(n_{[1]} \otimes m_{[1]}))\nu(n_{[0]}) \otimes (\mathscr{Q}^2(n_{[1]} \otimes m_{[1]})m_{[0]} \otimes \pi^{-1}(p)))\\
&= \beta(\mathscr{Q}^1(n_{[1]} \otimes m_{[1]}))\nu(n_{[0]}) \otimes \mathscr{U}(\gamma^{-1}(p_{[1]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes m_{[1]})_{[1]}m_{[0][1]})\\
&\qquad (\pi^{-1}(p_{[0]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes m_{[1]})_{[0]}m_{[0][0]})\\
&= \beta(\mathscr{Q}^1(n_{[1]} \otimes \gamma(m_{[1](2)})))\nu(n_{[0]}) \otimes \mathscr{U}(\gamma^{-1}(p_{[1]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[1]}m_{[1](1)})\\
&\qquad (\pi^{-1}(p_{[0]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[0]}\mu^{-1}(m_{[0]}))\\
&= \beta(\mathscr{Q}^1(n_{[1]} \otimes \gamma(m_{[1](2)})))\nu(n_{[0]}) \otimes \big(\mathscr{U}^1\,(\gamma^{-1}(p_{[1]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[1]}m_{[1](1)}\big)\\
&\quad \pi^{-1}(p_{[0]}) \otimes \beta^{-1}(\mathscr{U}^2(\gamma^{-1}(p_{[1]}) \otimes \mathscr{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[1]}m_{[1](1)}))\\
&\qquad \mathscr{Q}^2(n_{[1]} \otimes \gamma(m_{[1](2)}))_{[0]}m_{[0]}\big).\end{aligned}$$

Also we have

$$
\begin{aligned}
&a_{N,P,M}\circ c_{M,N\otimes P}\circ a_{M,N,P}((m\otimes n)\otimes p)\\
&= a_{N,P,M}\circ c_{M,N\otimes P}(\mu(m)\otimes(n\otimes\pi^{-1}(p)))\\
&= a_{N,P,M}\big((\Delta_A\otimes \mathrm{id}_A)(\mathscr{Q}(n_{[1]}\gamma^{-1}(p_{[1]})\otimes\gamma(m_{[1]})))((n_{[0]}\otimes\pi^{-1}(p_{[0]}))\otimes\mu(m_{[0]}))\big)\\
&= \beta\left(\mathscr{Q}^1(n_{[1]}\gamma^{-1}(p_{[1]})\otimes\gamma(m_{[1]}))_{(1)}\right)\nu(n_{[0]})\otimes\mathscr{Q}^1(n_{[1]}\gamma^{-1}(p_{[1]})\otimes\gamma(m_{[1]}))_{(2)}\\
&\qquad\qquad \pi^{-1}(p_{[0]})\otimes\beta^{-1}(\mathscr{Q}^2(n_{[1]}\gamma^{-1}(p_{[1]})\otimes\gamma(m_{[1]})))m_{[0]}.
\end{aligned}
$$

Conversely, take $M = N = P = A \otimes C$ and $m = 1 \otimes d$, $n = 1 \otimes e$, and $p = 1 \otimes c$ for all $c, d, e \in C$. Then we obtain (5.6). ■

The proof of the following lemma is similar to that of Lemma 5.6.

LEMMA 5.7. *Let $(M,\mu),(N,\nu),(P,\pi) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$. Then (5.3) holds if and only if the following condition is satisfied, with $\mathscr{U} = \mathscr{Q}$:*

$$
\begin{aligned}
(5.7)\qquad &\mathscr{U}^1\big(\mathscr{Q}^1(c_{(2)}\otimes\gamma^{-1}(e))_{[1]}\gamma^{-1}(c_{(1)})\otimes\gamma^{-2}(d)\big)\mathscr{Q}^1(\gamma(c_{(2)})\otimes e)_{[0]}\\
&\qquad\otimes\mathscr{U}^2\big(\mathscr{Q}^1(\gamma(c_{(2)})\otimes e)_{[1]}c_{(1)}\otimes\gamma^{-1}(d)\big)\otimes\mathscr{Q}^2(c\otimes e)\\
&= \mathscr{Q}^1(c\otimes\gamma^{-2}(d)\gamma^{-1}(e))\otimes\mathscr{Q}^2(\gamma(c)\otimes\gamma^{-1}(d)e)_{(1)}\otimes\mathscr{Q}^2(\gamma(c)\otimes\gamma^{-1}(d)e)_{(2)}
\end{aligned}
$$

for all $c, d, e \in C$.

Proof. If (5.3) holds, then

$$
\begin{aligned}
&(c_{M,P}\otimes\mathrm{id}_N)\circ a^{-1}_{M,P,N}\circ(\mathrm{id}_M\otimes c_{N,P})(m\otimes(n\otimes p))\\
&= (c_{M,P}\otimes\mathrm{id}_N)\circ a^{-1}_{M,P,N}\big(m\otimes\mathscr{Q}(p_{[1]}\otimes n_{[1]})(p_{[0]}\otimes n_{[0]})\big)\\
&= (c_{M,P}\otimes\mathrm{id}_N)\big((\mu^{-1}(m)\otimes\mathscr{Q}^1(p_{[1]}\otimes n_{[1]})p_{[0]})\otimes\beta(\mathscr{Q}^2(p_{[1]}\otimes n_{[1]}))\nu(n_{[0]})\big)\\
&= \mathscr{U}(\mathscr{Q}^1(p_{[1]}\otimes n_{[1]})_{[1]}p_{[0][1]}\otimes\gamma^{-1}(m_{[1]}))(\mathscr{Q}^1(p_{[1]}\otimes n_{[1]})_{[0]}p_{[0][0]}\otimes\mu^{-1}(m_{[0]}))\\
&\quad\otimes\beta(\mathscr{Q}^2(p_{[1]}\otimes n_{[1]}))\nu(n_{[0]})\\
&= \big\{\beta^{-1}\big(\mathscr{U}^1(\mathscr{Q}^1(\gamma(p_{[1](2)})\otimes n_{[1]})_{[1]}p_{[1](1)}\otimes\gamma^{-1}(m_{[1]}))\big)\mathscr{Q}^1(\gamma(p_{[1](2)})\otimes n_{[1]})_{[0]}\big\}p_{[0]}\\
&\quad\otimes\mathscr{U}^2(\mathscr{Q}^1(\gamma(p_{[1](2)})\otimes n_{[1]})_{[1]}p_{[1](1)}\otimes\gamma^{-1}(m_{[1]}))\mu^{-1}(m_{[0]})\\
&\quad\otimes\beta(\mathscr{Q}^2(p_{[1]}\otimes n_{[1]}))\nu(n_{[0]})
\end{aligned}
$$

and

$$
\begin{aligned}
&a^{-1}_{P,M,N}\circ c_{M\otimes N,P}\circ a^{-1}_{M,N,P}(m\otimes(n\otimes p))\\
&\quad= a^{-1}_{P,M,N}\circ c_{M\otimes N,P}((\mu^{-1}(m)\otimes n)\otimes\pi(p))\\
&\quad= a^{-1}_{P,M,N}\mathscr{Q}(\gamma(p_{[1]})\otimes\gamma^{-1}(m_{[1]})n_{[1]})(\pi(p_{[0]})\otimes(\mu^{-1}(m_{[0]})\otimes n_{[0]}))\\
&\quad= \beta^{-1}(\mathscr{Q}^1(\gamma(p_{[1]})\otimes\gamma^{-1}(m_{[1]})n_{[1]}))p_{[0]}\otimes\mathscr{Q}^2(\gamma(p_{[1]})\\
&\qquad\otimes\gamma^{-1}(m_{[1]})n_{[1]})_{(1)}\mu^{-1}(m_{[0]})\otimes\beta(\mathscr{Q}^2(\gamma(p_{[1]})\otimes\gamma^{-1}(m_{[1]})n_{[1]}))_{(2)}\nu(n_{[0]}).
\end{aligned}
$$

Conversely, take $M = N = P = A \otimes C$ and $m = 1 \otimes d$, $n = 1 \otimes e$, and $p = 1 \otimes c$ for all $c, d, e \in C$. Then we obtain (5.7). ∎

Therefore, we can summarize our results as follows.

THEOREM 5.8. *Let (H, A, C) be a monoidal Doi Hom-Hopf datum, and $\mathscr{Q} : C \otimes C \to A \otimes A$ a twisted convolution invertible map. For $(M, \mu), (N, \nu) \in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$, the family of maps*

$$c_{M,N} : M \otimes N \to N \otimes M, \qquad c_{M,N}(m \otimes n) = \mathscr{Q}(n_{[1]} \otimes m_{[1]})(n_{[0]} \otimes m_{[0]}),$$

defines a braiding on the category of Doi Hom-Hopf modules ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ if and only if (5.4)–(5.7) are satisfied.

EXAMPLE 5.9. (1) Take $A = k$ and write

$$R = \mathscr{Q}(1_C \otimes 1_C) = \sum R^{(1)} \otimes R^{(2)} = \sum r^{(1)} \otimes r^{(2)}.$$

Then (5.6) and (5.7) take the form

$$\Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes r^{(2)} R^{(2)},$$
$$R^{(1)} \otimes \Delta(R^{(2)}) = r^{(1)} R^{(1)} \otimes r^{(2)} \otimes R^{(2)},$$

and the braiding is

$$c_{M,N} : M \otimes N \to N \otimes M, \qquad c_{M,N}(m \otimes n) = R^{(2)} \cdot \nu^{-1}(n) \otimes R^{(1)} \cdot \mu^{-1}(m).$$

Assume that R is α-invariant (i.e. $\alpha(R^{(1)}) \otimes \alpha(R^{(2)}) = R^{(1)} \otimes R^{(2)}$). We conclude that the conditions of Theorem 5.8 are satisfied if and only if (C, R^{-1}) is a quasitriangular monoidal Hom-bialgebra.

(2) If $C = k$, then (5.6) and (5.7) take the form

$$\sigma(hg, l)\rangle = \sigma(h, l_{(1)})\sigma(g, l_{(2)}), \qquad \sigma(h, gl) = \sigma(h_{(1)}, l)\sigma(h_{(2)}, g),$$

and the braiding is

$$c_{M,N} : M \otimes N \to N \otimes M, \qquad c_{M,N}(m \otimes n) = \sigma(n_{[1]}, m_{[1]})\nu(n_{[0]}) \otimes \mu(m_{[0]}).$$

Assume that σ is α-invariant (i.e. $\sigma(\alpha(h), \alpha(g)) = \sigma(h, g)$ for all $h, g \in H$). Then the conditions of Theorem 5.8 are satisfied if and only if (A, σ) is a coquasitriangular monoidal Hom-bialgebra.

(3) Let (H, α) be a monoidal Hom-Hopf algebra with bijective antipode. We have seen that the category ${}_H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\mathrm{op}} \otimes H)^H$ of Doi Hom-Hopf modules and the category ${}_H\mathscr{H}\mathscr{Y}\mathscr{D}^H$ of Hom-Yetter–Drinfeld modules are isomorphic. Recall from [15] that ${}_H\mathscr{H}\mathscr{Y}\mathscr{D}^H$ is a braided category. The braiding is induced by

$$c_{M,N} : M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \nu(n_{[0]}) \otimes n_{[1]}\mu^{-1}(m).$$

The corresponding map $\mathscr{Q}$ is

$$\mathscr{Q} : H \otimes H \to H \otimes H, \qquad h \otimes k \mapsto \eta(\varepsilon(k)) \otimes h.$$

It is straightforward to check that $\mathscr{Q}$ satisfies the conditions of Theorem 5.8.

6. The smash product of monoidal Hom-bialgebras and the Drinfeld double. In this section, we introduce the smash product of monoidal Hom-bialgebras and prove that the Drinfeld double is a quasitriangular monoidal Hom-Hopf algebra, which generalizes [4].

Let (A,β) be a right (H,α)-Hom comodule algebra, and (B,ζ) a left (H,α)-Hom module coalgebra. Consider the smash product $A\,\#\,B$ with the multiplication given by

$$(a\,\#\,b)(c\,\#\,d)=a\beta(c_{[0]})\,\#\,(\zeta^{-1}(b)\leftharpoonup c_{[1]})d.$$

Then $A\,\#\,B$ is a monoidal Hom algebra with unit $1_A\,\#\,1_B$.

REMARK 6.1. Here the multiplication of a smash product monoidal Hom-algebra is diffierent from the one defined by Ma and Li [16].

If (C,γ) is a faithfully projective left (H,α)-Hom module coalgebra, then (C^*,γ^*) is a right (H,α)-Hom-module algebra. The right (H,α)-action is given by

$$(c^*\leftharpoonup h,c)=(c^*,h\cdot c).$$

Given $(M,\mu)\in {}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$, we define an $A\,\#\,C^*$-action on (M,μ) as follows:

$$(a\,\#\,c^*)\cdot m=\langle c^*,m_{[1]}\rangle a\cdot m_{[0]}.$$

Assume that (A,β) and (B,ζ) are both monoidal Hom-bialgebras, and consider $\Delta_{A\#B}$ and $\varepsilon_{A\#B}$ defined by

$$\Delta_{A\#B}(a\,\#\,b)=(a_{(1)}\,\#\,b_{(1)})\otimes(a_{(2)}\,\#\,b_{(2)}),\qquad \varepsilon_{A\#B}(a\,\#\,b)=\varepsilon_A(a)\varepsilon_B(b).$$

PROPOSITION 6.2. *Under the notation introduced above, we have*

$$\begin{aligned}(6.1)\qquad &\Delta_A(\beta(a_{[0]}))\otimes\Delta_A(\zeta^{-1}(b)\leftharpoonup a_{[1]})\\ &=\beta(a_{(1)[0]})\otimes\beta(a_{(2)[0]})\otimes(\zeta^{-1}(b_{(1)})\leftharpoonup a_{[1](1)})\otimes(\zeta^{-1}(b_{(2)})\leftharpoonup a_{[1](2)})\end{aligned}$$

and

$$\varepsilon_A(a_{[0]})\otimes\varepsilon_B(b\leftharpoonup a_{[1]})=\varepsilon_A(a)\varepsilon_B(b), \tag{6.2}$$

for all $a\in A$ and $b\in B$, so $A\,\#\,B$ is a monoidal Hom-bialgebra. If (A,β) and (B,ζ) are both monoidal Hom-Hopf algebras, then $A\,\#\,B$ is a monoidal Hom-Hopf algebras with antipode given by

$$S_{A\#B}(a\,\#\,b)=S(\beta(a))_{[0]}\,\#\,(S(\zeta^{-1}(b))\leftharpoonup S(a)_{[1]}).$$

Proof. We leave it to the reader to show that $\Delta_{A\#B}$ is multiplicative if and only if (6.1) holds, and $\varepsilon_{A\#B}$ is multiplicative if and only if (6.2) holds.

We show that the antipode defined above is convolution invertible. In fact,

$$
\begin{aligned}
&(a_{(1)} \# b_{(1)})S_{A\#B}(a_{(2)} \# b_{(2)}) \\
&\qquad = (a_{(1)} \# b_{(1)})(S(\beta(a_{(2)}))_{[0]} \otimes (S(\zeta^{-1}(b_{(2)})) \leftharpoonup S(a_{(2)})_{[1]})) \\
&\qquad = a_{(1)}S(\beta^2(a_{(2)}))_{[0][0]} \\
&\qquad\quad \# (\zeta^{-1}(b_{(1)}) \leftharpoonup S(\beta(a_{(2)}))_{[0][1]})(S(\zeta^{-1}(b_{(2)})) \leftharpoonup S(a_{(2)})_{[1]})) \\
&\qquad = a_{(1)}S(\beta(a_{(2)}))_{[0]} \\
&\qquad\quad \# (\zeta^{-1}(b_{(1)}) \leftharpoonup S(\beta(a_{(2)}))_{[1](1)})\big(S(\zeta^{-1}(b_{(2)})) \leftharpoonup S(\beta(a_{(2)}))_{[1](2)}\big) \\
&\qquad = a_{(1)}S(\beta(a_{(2)}))_{[0]} \# (\zeta^{-1}(b_{(1)})S(\zeta^{-1}(b_{(2)}))) \leftharpoonup S(\beta(a_{(2)}))_{[1]} \\
&\qquad = \varepsilon_A(a)\varepsilon_B(b),
\end{aligned}
$$

and

$$
\begin{aligned}
&S_{A\#B}(a_{(1)} \# b_{(1)})(a_{(2)} \# b_{(2)}) \\
&\qquad = \big(S(\beta(a_{(1)}))_{[0]} \otimes (S(\zeta^{-1}(b_{(1)})) \leftharpoonup S(a_{(1)})_{[1]})\big)(a_{(2)} \# b_{(2)}) \\
&\qquad = S(\beta(a_{(1)}))_{[0]}\beta(a_{(2)[0]}) \# \big(S(\zeta^{-1}(b_{(1)})) \leftharpoonup S(a_{(1)})_{[1]}a_{(2)[1]}\big)b_{(2)} \\
&\qquad = \varepsilon_A(a)\varepsilon_B(b),
\end{aligned}
$$

as desired. ∎

Proposition 6.3. *Let (H, A, C) be a monoidal Doi Hom-Hopf datum. Assume that (C, γ) is faithfully projective as a k-module. Then (A, β) and (C^*, γ^*) satisfy* (6.1), (6.2), *and ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ and the category of $A \# C^*$-Hom-modules are isomorphic as monoidal categories.*

Proof. Apply the arguments used in [4, p. 94]. The details are left to the reader. ∎

Inspired by [7], we have the following example.

Example 6.4. Assume that (H, α) is faithfully projective as a k-module. The monoidal Hom-algebra $A \# C^*$ is nothing else than the Drinfeld double $D(H) = H \# H^*$. Then we define multiplication by the formula

$$(h \# f)(k \# g) = h\alpha^2(h_{(2)(1)}) \# \langle \alpha^{*-2}(f), \alpha(h_{(2)(2)}) \rightharpoonup \bullet \leftharpoonup S^{-1}(\alpha^{-1}(h))\rangle g.$$

Now let (H, A, C) be a monoidal Doi Hom-Hopf datum, and $\mathscr{Q} : C \otimes C \to A \otimes A$ a twisted convolution invertible map satisfying (5.4)–(5.7). Then $\mathscr{Q}$ induces the map

$$\widetilde{\mathscr{Q}} : k \to (A \# C^*) \otimes (A \# C^*).$$

The braiding on ${}_A\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C$ translates into a braiding on $A \# C^*$-Hom-modules. This means that $A \# C^*$ is a quasitriangular monoidal Hom-Hopf-algebra. ∎

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