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APPROXIMATION IN WEIGHTED GENERALIZED GRAND LEBESGUE SPACES

BY

DANIYAL M. ISRAFILOV (Balikesir and Baku) and AHMET TESTICI (Balikesir)

Abstract. The direct and inverse problems of approximation theory in the subspace of weighted generalized grand Lebesgue spaces of 2π-periodic functions with the weights satisfying Muckenhoupt's condition are investigated. Appropriate direct and inverse theorems are proved. As a corollary some results on constructive characterization problems in generalized Lipschitz classes are presented.

1. Introduction. Let $\omega : \mathbb{T} := [0, 2\pi] \to [0,\infty]$ be a *weight function*, that is, an integrable function, positive almost everywhere on $\mathbb{T}$. The usual weighted Lebesgue space $L^p_\omega(\mathbb{T})$ is the set of all measurable functions on $\mathbb{T}$ for which

$$\left\{\frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^p \omega(x)\, dx\right\}^{1/p} < \infty, \qquad 1 < p < \infty.$$

We denote by $L^{p),\theta}_\omega(\mathbb{T})$, $\theta > 0$, the *weighted generalized grand Lebesgue space* which consists of the measurable functions on $\mathbb{T}$ such that

$$\sup_{0<\varepsilon<p-1} \left\{\frac{\varepsilon^\theta}{2\pi}\int_{\mathbb{T}} |f(x)|^{p-\varepsilon}\omega(x)\, dx\right\}^{1/(p-\varepsilon)} < \infty.$$

$L^{p),\theta}_\omega(\mathbb{T})$ becomes a Banach space when equipped with the norm

$$\|f\|_{L^{p),\theta}_\omega(\mathbb{T})} = \sup_{0<\varepsilon<p-1} \left\{\varepsilon^\theta \frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^{p-\varepsilon}\omega(x)\, dx\right\}^{1/(p-\varepsilon)}.$$

If $\theta = 0$ then $L^{p),\theta}_\omega(\mathbb{T})$ turns into the grand Lebesgue space $L^{p)}_\omega(\mathbb{T})$. In the case of $\theta = 1$, the non-weighted space $L^{p)}(\mathbb{T}) := L^{p),1}(\mathbb{T})$ is a grand Lebesgue space, introduced in [12] and later in [11], for $\theta > 1$. The space $L^{p)}(\mathbb{T})$ is a rearrangement invariant Banach function space, but is not reflexive. It is clear that $L^p(\mathbb{T}) \subset L^{p)}(\mathbb{T}) \subset L^{p-\varepsilon}(\mathbb{T})$, but $L^p(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$ (see

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for example [6]). Denote by $\mathcal{L}^{p)}(\mathbb{T})$ the closure of $L^p(\mathbb{T})$, $1<p<\infty$, with respect to the norm of $L^{p)}(\mathbb{T})$. From [10], [2] we know that $\mathcal{L}^{p)}(\mathbb{T})$ consists of all functions f such that

$$\lim_{\varepsilon\to 0}\left(\varepsilon\frac{1}{2\pi}\int_{\mathbb{T}}|f(x)|^{p-\varepsilon}\,dx\right)=0.$$

In general, $L_\omega^{p),\theta}(\mathbb{T})$ is not a rearrangement invariant space. Embedding relations similar to those above hold in the case of weighted generalized grand Lebesgue spaces: if $\theta_1<\theta_2$ and $1<p<\infty$, then

$$L_\omega^p(\mathbb{T})\subset L_\omega^{p),\theta_1}(\mathbb{T})\subset L_\omega^{p),\theta_2}(\mathbb{T})\subset L_\omega^{p-\varepsilon}(\mathbb{T}).$$

$L_\omega^p(\mathbb{T})$ is not dense in $L_\omega^{p),\theta}(\mathbb{T})$. We denote by $\mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ the closure of $L_\omega^p(\mathbb{T})$ with respect to the norm of $L_\omega^{p),\theta}(\mathbb{T})$; by [18], it is the set of functions f satisfying

$$\lim_{\varepsilon\to 0}\left(\varepsilon^\theta\frac{1}{2\pi}\int_{\mathbb{T}}|f(x)|^{p-\varepsilon}\omega(x)\,dx\right)=0.$$

Definition 1. Let $1<p<\infty$ and let ω be a weight function on $\mathbb{T}$. Then ω is said to satisfy the *Muckenhoupt A_p-condition* on $\mathbb{T}$ if

$$\sup_I\left(\frac{1}{|I|}\int_I\omega(x)^p\,dx\right)^{1/p}\left(\frac{1}{|I|}\int_I\omega(x)^{-1/(p-1)}\,dx\right)^{p-1}<\infty,$$

where the supremum is taken over all subintervals I of $\mathbb{T}$.

Let us denote by $A_p(\mathbb{T})$ the set of all weight functions ω satisfying the Muckenhoupt A_p-condition on $\mathbb{T}$. Let $I\subset\mathbb{T}$ be an interval and let

$$Mf(x):=\sup_{I\ni x}\frac{1}{|I|}\int_I f(y)\,dy,\qquad x\in\mathbb{T},$$

be the Hardy–Littlewood maximal function. The following theorem holds.

Theorem A ([14]). *Let $1<p<\infty$ and $\theta>0$. Then Mf is a bounded operator in $L_\omega^{p),\theta}(\mathbb{T})$ if and only if $\omega\in A_p(\mathbb{T})$.*

Let $f\in L^1(\mathbb{T})$ and let $\widetilde{f}$ be its conjugate function, with Fourier series

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^\infty(a_k\cos kx+b_k\sin kx),\qquad \widetilde{f}(x)\sim\sum_{k=1}^\infty(a_k\sin kx-b_k\cos kx).$$

Let

$$S_n(f,x)=\frac{a_0}{2}+\sum_{k=1}^n(a_k\cos kx+b_k\sin kx),\qquad n=1,2,\dots,$$

be the nth partial sum of the Fourier series of f.

To date, grand Lebesgue spaces have been considered in various fields; in particular in PDE theory (see for example [19], [17], [13]), where they are the right spaces to consider some nonlinear equations, in the study of maximal operators and, more generally, quasilinear operators, and in interpolation theory (see for example [11], [9], [8], [1]). There are also some pioneering results [3], [4], [20] on approximation in subspaces of grand Lebesgue spaces. In particular, in [3] the authors stated direct and inverse theorems of approximation theory in non-weighted spaces $\mathcal{L}^{p),\theta}(\mathbb{T})$, and later in [4], in weighted spaces $\mathcal{L}^{p),\theta}_{\omega}(\mathbb{T})$.

THEOREM B ([20]). *Let $1 < p < \infty$ and $\theta > 0$. Then*

$$\Big\|\sup_n |S_n(f,\cdot)|\Big\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \le c\|f\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \;\Leftrightarrow\; \omega \in A_p(\mathbb{T}),$$

where c is a constant independent of f.

THEOREM C ([20]). *Let $1 < p < \infty$ and $\theta > 0$. If $f \in \mathcal{L}^{p),\theta}_{\omega}(\mathbb{T})$ with $\omega \in A_p(\mathbb{T})$, then*

$$\lim_{n\to\infty} \|S_n(f,\cdot) - f\|_{L^{p),\theta}_{\omega}(\mathbb{T})} = 0.$$

THEOREM D ([20]). *Let $1 < p < \infty$ and $\theta > 0$. Then*

$$\|\widetilde{f}\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \le c\|f\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \;\Leftrightarrow\; \omega \in A_p(\mathbb{T}),$$

where c is a constant independent of f.

Let $f \in L^{p),\theta}_{\omega}(\mathbb{T})$ with $1 < p < \infty$ and $\theta > 0$, and let

$$\Delta^r_t f(x) := \sum_{s=0}^{r} (-1)^{r+s+1} \binom{r}{s} f(x+st), \qquad t > 0,$$

for a given $r \in \mathbb{N}$. We define the mean value operator

$$\sigma^r_h f(x) := \frac{1}{h}\int_0^h |\Delta^r_t f(x)|\,dt.$$

If $\omega \in A_p(\mathbb{T})$ and $0 < \delta < \infty$, then by Theorem A we have

$$\sup_{|h|\le\delta} \|\sigma^r_h f(x)\|_{L^{p),\theta}_{\omega}(\mathbb{T})} \le c\|f\|_{L^{p),\theta}_{\omega}(\mathbb{T})} < \infty,$$

which implies the correctness of the following definition.

DEFINITION 2. Let $1 < p < \infty$ and $\theta > 0$, and let $f \in \mathcal{L}^{p),\theta}_{\omega}(\mathbb{T})$ with $\omega \in A_p(\mathbb{T})$. The function $\Omega_r(f,\cdot)_{p),\theta,\omega} : [0,\infty) \to [0,\infty)$ defined by

$$\Omega_r(f,\delta)_{p),\theta,\omega} := \sup_{|h|\le\delta} \|\sigma^r_h f(x)\|_{L^{p),\theta}_{\omega}(\mathbb{T})}, \qquad r \in \mathbb{N},$$

is called the *rth mean modulus* of f.

The modulus $\Omega_r(f,\delta)_{p),\theta,\omega}$ has the following properties:

(i) $\Omega_r(f,\delta)_{p),\theta,\omega}$ is a non-negative and non-decreasing function of $\delta > 0$.
(ii) $\Omega_r(f_1+f_2,\cdot)_{p),\theta,\omega} \le \Omega_r(f_1,\cdot)_{p),\theta,\omega} + \Omega_r(f_2,\cdot)_{p),\theta,\omega}$.
(iii) $\lim_{h\to 0} \Omega_r(f,\delta)_{p),\theta,\omega} = 0$.

Denote by Π_n the set of trigonometric polynomials of degree not exceeding n and by $E_n(f)_{p),\theta,\omega}$ the best approximation number of $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$, defined as $E_n(f)_{p),\theta,\omega} := \inf\{\|f - T_n\|_{L_\omega^{p),\theta}(\mathbb{T})} : T_n \in \Pi_n\}$. By Theorems B and D we infer that

$$\|f - S_n(f,\cdot)\|_{L_\omega^{p),\theta}(\mathbb{T})} \le c_1 E_n(f)_{p),\theta,\omega}, \qquad E_n(\widetilde{f})_{p),\theta,\omega} \le c_2 E_n(f)_{p),\theta,\omega}, \tag{1.1}$$

with the constants independent of n.

Let $r \in \mathbb{N}$ and let $W_{r,\omega}^{p),\theta}(\mathbb{T})$ (resp. $\mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$) be the space of functions f such that $f^{(r-1)}$ is absolutely continuous on $\mathbb{T}$ and $f^{(r)} \in L_\omega^{p),\theta}(\mathbb{T})$ (resp. $f^{(r)} \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$). Then $W_{r,\omega}^{p),\theta}(\mathbb{T})$ becomes a Banach space with the norm

$$\|f\|_{W_{r,\omega}^{p),\theta}(\mathbb{T})} := \|f\|_{L_\omega^{p),\theta}(\mathbb{T})} + \|f^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})}.$$

Our main results are the following:

Theorem 1. *Let $1 < p < \infty$, $\theta > 0$ and $r \in \mathbb{N}$. If $f \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$ with $\omega \in A_p(\mathbb{T})$, then*

$$E_n(f)_{p),\theta,\omega} \le \frac{c}{n^r} E_n(f^{(r)})_{p),\theta,\omega}, \quad n \in \mathbb{N},$$

with a constant $c > 0$ independent of n.

Theorem 2. *Let $1 < p < \infty$, $\theta > 0$ and $r \in \mathbb{N}$. If $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ with $\omega \in A_p(\mathbb{T})$, then*

$$E_n(f)_{p),\theta,\omega} \le c\Omega_r(f,\delta)_{p),\theta,\omega}, \quad n \in \mathbb{N},$$

with a constant $c > 0$ independent of n.

Theorem 3. *Let $1 < p < \infty$, $\theta > 0$ and $r \in \mathbb{N}$. If $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ with $\omega \in A_p(\mathbb{T})$, then*

$$\Omega_r(f,\delta)_{p),\theta,\omega} \le \frac{c}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{p),\theta,\omega}, \quad n \in \mathbb{N},$$

with a constant $c > 0$ independent of n.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $H^1(\mathbb{D})$ be the Hardy space of analytic functions in $\mathbb{D}$. It is known that every function $f \in H^1(\mathbb{D})$ has non-tangential boundary limit values almost everywhere on the unit cirle and the limit function belongs to $L^1(\mathbb{T})$ (see, for example, [15] and [7]). Let $1 < p < \infty$ and $\theta > 0$, and let $H_\omega^{p),\theta}(\mathbb{D})$ be the *weighted generalized grand Hardy space* defined as

$$H_\omega^{p),\theta}(\mathbb{D}) := \{f \in H^1(\mathbb{D}) : f \in L_\omega^{p),\theta}(\mathbb{T})\}.$$

Denote by $\mathcal{H}_\omega^{p),\theta}(\mathbb{D})$ the closure of $H^p(\mathbb{D})$ in $L_\omega^{p),\theta}(\mathbb{T})$. Then we obtain the following theorems.

THEOREM 4. *Let* $f \in \mathcal{H}_{\omega}^{p),\theta}(\mathbb{D})$ *with* $\omega \in A_p(\mathbb{T})$, *and let* $1 < p < \infty$ *and* $\theta > 0$. *If* $\sum_{k=0}^{\infty} \beta_k(f) z^k$ *is the Taylor series of* f *at the origin, then*

$$\Big\| f(z) - \sum_{k=0}^{n} \beta_k(f) z^k \Big\|_{L_{\omega}^{p),\theta}(\mathbb{T})} \leq c \Omega_r(f, \delta)_{p),\theta,\omega}, \qquad r \in \mathbb{N},$$

with a constant $c > 0$ *independent of* n.

THEOREM 5. *Let* $1 < p < \infty$, $\theta > 0$ *and let* $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ *with* $\omega \in A_p(\mathbb{T})$. *If*

$$\sum_{k=1}^{\infty} k^{r-1} E_k(f)_{p),\theta,\omega} < \infty$$

for some $r \in \mathbb{N}$, *then* $f \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$.

COROLLARY 1. *Let* $1 < p < \infty$ *and* $\theta > 0$, *and let* $\omega \in A_p(\mathbb{T})$. *If* $f \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$ *for some* $r \in \mathbb{N}$, *then*

$$E_n(f)_{p),\theta,\omega} \leq \frac{c}{n^r} \| f^{(r)} \|_{L_{\omega}^{p),\theta}(\mathbb{T})}$$

with a constant $c > 0$ *independent of* n.

COROLLARY 2. *Let* $1 < p < \infty$, $\theta > 0$ *and let* $\omega \in A_p(\mathbb{T})$. *If* $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ *and* $E_n(f)_{p),\theta,\omega} = \mathcal{O}(n^{-\alpha})$, $n \in \mathbb{N}$, *for some* $\alpha > 0$, *then*

$$\Omega_r(f, \delta)_{p),\theta,\omega} = \begin{cases} \mathcal{O}(\delta^{\alpha}), & r > \alpha, \\ \mathcal{O}(\delta^{\alpha} \log(1/\delta)), & r = \alpha, \\ \mathcal{O}(\delta^{r}), & r < \alpha, \end{cases}$$

for $\delta > 0$ *and* $r \in \mathbb{N}$.

Hence if we define the *generalized grand Lipschitz class* $\mathrm{Lip}_{\alpha}^{p),\theta}(\mathbb{T})$ for $\alpha > 0$ and $r := [\alpha] + 1$ as

$$\mathrm{Lip}_{\omega}^{p),\theta}(\mathbb{T}, \alpha) = \{ f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T}) : \Omega_r(f, \delta)_{p),\theta,\omega} = \mathcal{O}(\delta^{\alpha}) \text{ for } \delta > 0 \},$$

then we have

COROLLARY 3. *Let* $1 < p < \infty$, $\theta > 0$ *and let* $\omega \in A_p(\mathbb{T})$. *If* $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ *and* $E_n(f)_{p),\theta,\omega} = \mathcal{O}(n^{-\alpha})$ *for some* $\alpha > 0$, *then* $f \in \mathrm{Lip}_{\omega}^{p),\theta}(\mathbb{T}, \alpha)$.

COROLLARY 4. *Let* $1 < p < \infty$ *and* $\theta > 0$. *If* $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ *for some* $\alpha > 0$, *then* $E_n(f)_{p),\theta,\omega} = \mathcal{O}(n^{-\alpha})$, $n \in \mathbb{N}$.

Combining Corollaries 3 and 4 we obtain the following theorem.

THEOREM 6. *Let* $\omega \in A_p(\mathbb{T})$ *with* $1 < p < \infty$ *and* $\theta > 0$. *Then for* $\alpha > 0$ *the following assertions are equivalent:*

(i) $f \in \mathrm{Lip}_{\omega}^{p),\theta}(\mathbb{T}, \alpha)$,
(ii) $E_n(f)_{p),\theta,\omega} = \mathcal{O}(n^{-\alpha})$.

2. Auxiliary results. We shall denote by $c, c_1, c_2, \ldots$ constants (in general, different in different relations) depending only on numbers that are not important for the questions of our interest.

LEMMA 1. *Let $1 < p < \infty$, $\theta > 0$ and $\omega \in A_p(\mathbb{T})$. If $f \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$, $r \in \mathbb{N}$, then*

$$\Omega_r(f,\delta)_{p),\theta,\omega} \leq c\delta^r \|f^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})}$$

with a constant $c > 0$ independent of n.

Proof. Let $f \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$. Since $f^{(r)} \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ we have

$$\Delta_t^r f(x) = \int_0^t \ldots \int_0^t f^{(r)}(x + t_1 + \cdots + t_r)\, dt_1 \ldots dt_r.$$

Now by Theorem A and the substitution $t := t_1 + \cdots + t_r$ we get

$$\begin{aligned}
\Omega_r(f,\delta)_{p),\theta,\omega} &= \sup_{|h|\leq\delta} \left\| \frac{1}{h}\int_0^h |\Delta_t^r f(x)|\, dt \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\
&\leq c_3 \left\| \int_0^\delta \ldots \int_0^\delta |f^{(r)}(x + t_1 + \cdots + t_r)|\, dt_1 \ldots dt_r \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\
&= c_3\delta^r \left\| \frac{1}{\delta}\int_0^\delta \left\{ \frac{1}{\delta^{r-1}} \int_0^\delta \ldots \int_0^\delta |f^{(r)}(x + t_1 + \cdots + t_r)|\, dt_1 \ldots dt_{r-1} \right\} dt_r \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\
&\leq c_4\delta^r \left\| \frac{1}{\delta^{r-1}} \int_0^\delta \ldots \int_0^\delta |f^{(r)}(x + t_1 + \cdots + t_{r-1})|\, dt_1 \ldots dt_{r-1} \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\
&\leq \cdots \leq c_5\delta^r \left\| \frac{1}{\delta} \int_0^{r\delta} |f^{(r)}(x+t)|\, dt \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\
&= c_5 r\delta^r \left\| \frac{1}{\delta r} \int_0^{r\delta} |f^{(r)}(x+t)|\, dt \right\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c\delta^r \|f^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})}. \quad \blacksquare
\end{aligned}$$

Let $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$ and $r \in \mathbb{N}$. We define

$$K_r(f,\delta)_{p),\theta,\omega} := \inf\left\{ \|f-g\|_{L_\omega^{p),\theta}(\mathbb{T})} + \delta^r \|g^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})} : g \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T}),\ \delta > 0 \right\}.$$

THEOREM 7. *Let $1 < p < \infty$, $\theta > 0$ and $\omega \in A_p(\mathbb{T})$. If $f \in \mathcal{L}_\omega^{p),\theta}(\mathbb{T})$, then*

$$c_6\Omega_r(f,\delta)_{p),\theta,\omega} \leq K_r(f,\delta)_{p),\theta,\omega} \leq c_7\Omega_r(f,\delta)_{p),\theta,\omega}$$

with some constants c_6 and c_7 independent of δ.

Proof. Let $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ and $g \in \mathcal{W}_{r,\omega}^{p),\theta}(\mathbb{T})$. By Lemma 1,

$$\begin{aligned}\Omega_r(f,\delta)_{p),\theta,\omega} &\le \Omega_r(f-g,\delta)_{p),\theta,\omega} + \Omega_r(g,\delta)_{p),\theta,\omega}\\ &\le c\big(\|f-g\|_{L_{\omega}^{p),\theta}(\mathbb{T})} + \delta^r\|g^{(r)}\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\big),\end{aligned}$$

and taking the infimum in the last inequality we obtain

$$\Omega_r(f,\delta)_{p),\theta,\omega} \le cK_r(f,\delta)_{p),\theta,\omega}. \tag{2.1}$$

To prove the reverse inequality we define, for $r \ge 1$ and $\delta > 0$,

$$\begin{aligned}f_{r,\delta}(x) := \frac{2}{\delta}\int_{\delta/2}^{\delta}\bigg\{\frac{1}{h^r}\int_0^h\cdots\int_0^h\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\\ \times f\bigg(x+\frac{r-s}{r}(t_1+\cdots+t_r)\bigg)\,dt_1\ldots\,dt_r\bigg\}\,dh.\end{aligned}$$

Then

$$\Delta^r_{(t_1+\cdots+t_r)/r}f(x) = \sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}f\bigg(x+\frac{r-s}{r}(t_1+\cdots+t_r)\bigg) - f(x),$$

therefore

$$\begin{aligned}&\|f_{r,\delta}(\cdot)-f\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ &\le \frac{2}{\delta}\int_{\delta/2}^{\delta}\bigg\|\frac{1}{h^r}\int_0^h\cdots\int_0^h\Delta^r_{(t_1+\cdots+t_r)/r}f(\cdot)\,dt_1\ldots dt_r\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}dh\\ &\le \frac{2}{\delta}\int_{\delta/2}^{\delta}\sup_{\delta/2\le h\le\delta}\bigg\|\frac{1}{h^r}\int_0^h\cdots\int_0^h\Delta^r_{(t_1+\cdots+t_r)/r}f(\cdot)\,dt_1\ldots dt_r\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}dh\\ &= \sup_{\delta/2\le h\le\delta}\bigg\|\frac{1}{h^r}\int_0^h\cdots\int_0^h\Delta^r_{(t_1+\cdots+t_r)/r}f(\cdot)\,dt_1\ldots dt_r\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ &= \sup_{\delta/2\le h\le\delta}\bigg\|\frac{1}{h^r}\int_0^h\cdots\int_0^h\bigg(\int_{t_2+\cdots+t_r}^{t_2+\cdots+t_r+h}|\Delta^r_{t/r}f(\cdot)|\,dt\bigg)\,dt_2\ldots dt_r\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ &\le c_8\sup_{\delta/2\le h\le\delta}\bigg\|\frac{1}{h^{r-1}}\int_0^h\cdots\int_0^h\bigg(\int_0^{rh}|\Delta^r_{t/r}f(\cdot)|\,dt\bigg)\,dt_2\ldots dt_r\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ &\le\cdots\le c_9\sup_{\delta/2\le h\le\delta}\bigg\|\frac{1}{h}\int_0^{rh}|\Delta^r_{t/r}f(\cdot)|\,dt\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ &= c_9 r\sup_{0\le h\le\delta}\bigg\|\frac{1}{h}\int_0^h|\Delta^r_m f(\cdot)|\,dm\bigg\|_{L_{\omega}^{p),\theta}(\mathbb{T})} = c_{10}\Omega_r(f,\delta)_{p),\theta,\omega}.\end{aligned} \tag{2.2}$$

On the other hand,

$$f_{r,\delta}^{(r)}(x) = \frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\left(\frac{1}{h^r}\sum_{s=0}^{r-1}(-1)^{r+s}\binom{r}{s}\left(\frac{r}{r-s}\right)^r \varDelta^r_{(r-s)h/r} f(x)\right)dh,$$

and hence

$$\begin{aligned}|f_{r,\delta}^{(r)}(x)| &\leq \frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\frac{2^r}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r |\varDelta^r_{(r-s)h/r} f(x)|\,dh \\ &\leq 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r \frac{1}{\delta}\int\limits_{0}^{\delta}|\varDelta^r_{(r-s)h/r} f(x)|\,dh.\end{aligned}$$

Therefore,

$$\begin{aligned}(2.3)\qquad &\|f_{r,\delta}^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})} \\ &\leq 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left\|\frac{1}{\delta}\int\limits_0^\delta|\varDelta^r_{(r-s)h/r} f(\cdot)|\,dh\right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\ &= 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left\|\frac{1}{(r-s)\delta/r}\int\limits_0^{(r-s)\delta/r}|\varDelta^r_m f(x)|\,dm\right\|_{L_\omega^{p),\theta}(\mathbb{T})} \\ &\leq 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r \varOmega_r(f,\delta)_{p),\theta,\omega} = 2^{2r}\delta^{-r}\varOmega_r(f,\delta)_{p),\theta,\omega}.\end{aligned}$$

Inequalities (2.2) and (2.3) imply that

$$\begin{aligned}K_r(f,\delta)_{p),\theta,\omega} &\leq \|f_{r,\delta}-f\|_{L_\omega^{p),\theta}(\mathbb{T})} + \delta^r\|f_{r,\delta}^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})} \\ &\leq 2^{r-1}r\varOmega_r(f,\delta)_{p),\theta,\omega} + 2^{2r}\delta^{-r}\varOmega_r(f,\delta)_{p),\theta,\omega} \leq c\ \varOmega_r(f,\delta)_{p),\theta,\omega}.\end{aligned}$$

Together with (2.1) this gives the desired inequality of Theorem 7. ■

Let

$$D_n(t) := \frac{1}{2}+\sum_{k=1}^{n}\cos kt \quad \text{and} \quad F_n(t) := \frac{1}{n+1}+\sum_{k=0}^{n}D_k(t)$$

be the Dirichlet and Fejér kernels of order n, respectively. Consider the sequence $\{K_n(f,x)\}$ of the arithmetic means of $S_n(f,x)$ defined as

$$K_n(f,x) := \frac{S_0(f,x)+S_1(f,x)+\cdots+S_n(f,x)}{n+1}, \qquad n\in\mathbb{N},$$

and having [5, p. 3] the representation

$$K_n(f,x) = \frac{1}{\pi}\int\limits_{\mathbb{T}} f(t)F_n(x-t)\,dt.$$

Lemma 2. *Let $1 < p < \infty$, $\theta > 0$ and $\omega \in A_p(\mathbb{T})$. If T_n is a trigonometric polynomial of degree n, then*

$$\|T'_n\|_{L^{p),\theta}_\omega(\mathbb{T})} \le cn\|T_n\|_{L^{p),\theta}_\omega(\mathbb{T})}.$$

Proof. We use the technique from [21, Vol. I, p. 118]. Since

$$T_n(x) = S_n(T_n, x) = \frac{1}{\pi}\int_{\mathbb{T}} T_n(u)D_n(u-x)\,du,$$

we have

$$\begin{aligned} T'_n(x) &= -\frac{1}{\pi}\int_{\mathbb{T}} T_n(u)D'_n(u-x)\,du \\ &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\Big(\sum_{k=1}^{n} k\sin ku\Big)\,du. \end{aligned}$$

Taking into account that

$$\int_{\mathbb{T}} T_n(u+x)\sum_{k=1}^{n-1} k\sin(2n-k)u\,du = 0,$$

we get

$$\begin{aligned} T'_n(x) &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\Big(\sum_{k=1}^{n} k\sin ku + \sum_{k=1}^{n-1} k\sin(2n-k)u\Big)\,du \\ &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)2n\sin nu\Big(\frac{1}{2} + \sum_{k=1}^{n-1}\frac{n-k}{n}\cos ku\Big)\,du \\ &= 2n\frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)F_{n-1}(u)\sin nu\,du. \end{aligned}$$

Since F_{n-1} is non-negative,

$$\begin{aligned} |T'_n(x)| &\le 2n\frac{1}{\pi}\int_{\mathbb{T}} |T_n(u+x)|F_{n-1}(u)\,du \\ &= 2n\frac{1}{\pi}\int_{\mathbb{T}} |T_n(u)|F_{n-1}(u-x)\,du = 2nK_{n-1}(|T_n|, x). \end{aligned}$$

Now, using Theorem B and taking into account the definition of $K_n(f,x)$ we obtain the desired inequality. ∎

3. Proof of main results

Proof of Theorem 1. Let $f \in \mathcal{W}^{p),\theta}_{r,\omega}(\mathbb{T})$. For the Fourier coefficients of f, denoted by a_k and b_k, $k = 1, 2, \ldots$, we set

$$A_0(f,x)=a_0/2, \qquad A_k(f,x)=a_k\cos kx+b_k\sin kx,$$
$$A_k(\widetilde{f},x)=a_k\sin kx-b_k\cos kx.$$

Since

$$A_k(f,x)=A_k\Big(f,x+\frac{r\pi}{2k}\Big)\cos\frac{r\pi}{2}+A_k\Big(\widetilde{f},x+\frac{r\pi}{2k}\Big)\sin\frac{r\pi}{2},$$
$$A_k(f^{(r)},x)=k^rA_k\Big(f,x+\frac{r\pi}{2k}\Big),$$
$$A_k(\widetilde{f^{(r)}},x)=k^rA_k\Big(\widetilde{f},x+\frac{r\pi}{2k}\Big),$$

we have

$$\begin{aligned}\sum_{k=n+1}^{\infty}&\frac{1}{k^r}A_k(f^{(r)},x)\\ &=\sum_{k=n+1}^{\infty}\frac{1}{k^r}\big([S_k(f^{(r)},x)-f^{(r)}(x)]-[S_{k-1}(f^{(r)},x)-f^{(r)}(x)]\big)\\ &=\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)[S_k(f^{(r)},x)-f^{(r)}(x)]\\ &\quad-\frac{1}{(n+1)^r}[S_n(f^{(r)},x)-f^{(r)}(x)].\end{aligned}$$

A similar relation holds for $\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(\widetilde{f^{(r)}},x)$. By (1.1),

$$\begin{aligned}\|f-S_n(f,\cdot)\|_{L_\omega^{p),\theta}(\mathbb{T})}&=\Big\|\sum_{k=n+1}^{\infty}A_k(f,\cdot)\Big\|_{L_\omega^{p),\theta}(\mathbb{T})}\\ &=\Big\|\cos\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(f^{(r)},\cdot)+\sin\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(\widetilde{f^{(r)}},\cdot)\Big\|_{L_\omega^{p),\theta}(\mathbb{T})}\\ &\le\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)\|S_k(f^{(r)},\cdot)-f^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})}\\ &\quad+\frac{1}{(n+1)^r}\|S_n(f^{(r)},\cdot)-f^{(r)}\|_{L_\omega^{p),\theta}(\mathbb{T})}\\ &\quad+\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)\big\|S_k(\widetilde{f^{(r)}},\cdot)-\widetilde{f^{(r)}}\big\|_{L_\omega^{p),\theta}(\mathbb{T})}\\ &\quad+\frac{1}{(n+1)^r}\big\|S_n(\widetilde{f^{(r)}},\cdot)-\widetilde{f^{(r)}}\big\|_{L_\omega^{p),\theta}(\mathbb{T})}\end{aligned}$$

$$\leq c_{11}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)E_k(f^{(r)})_{p),\theta,\omega}+\frac{1}{(n+1)^r}E_n(f^{(r)})_{p),\theta,\omega}\Big\}$$
$$+c_{12}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)E_k(\widetilde{f^{(r)}})_{p),\theta,\omega}+\frac{1}{(n+1)^r}E_n(\widetilde{f^{(r)}})_{p),\theta,\omega}\Big\}$$
$$\leq c_{13}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)E_k(f^{(r)})_{p),\theta,\omega}+\frac{1}{(n+1)^r}E_n(f^{(r)})_{p),\theta,\omega}\Big\}$$
$$\leq c_{13}\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}+\frac{1}{(n+1)^r}\Big)E_n(f^{(r)})_{p),\theta,\omega}$$
$$\leq\frac{c}{(n+1)^r}E_n(f^{(r)})_{p),\theta,\omega}.\ \blacksquare$$

Proof of Theorem 2. Let $f\in\mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$. For a given $g\in\mathcal{W}_{r,\omega}^{p),\theta}$ and $\delta:=1/n$, by Theorem 1 and Corollary 1 we have

$$\begin{aligned}E_n(f)_{p),\theta,\omega}&\leq E_n(f-g)_{p),\theta,\omega}+E_n(g)_{p),\theta,\omega}\\&\leq c\big\{\|f-g\|_{L_{\omega}^{p),\theta}(\mathbb{T})}+\delta^r\|g^{(r)}\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\big\}\\&\leq c_{14}\Big\{\|f-g\|_{L_{\omega}^{p),\theta}(\mathbb{T})}+\frac{1}{n^r}\|g^{(r)}\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\Big\},\end{aligned}$$

which by Theorem 7 implies that

$$E_n(f)_{p),\theta,\omega}\leq c_{15}K_r(f,1/n)_{p),\theta,\omega}\leq c\Omega_r(f,1/n)_{p),\theta,\omega}.\ \blacksquare$$

Proof of Theorem 3. Let $f\in\mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ and let $T_n\in\Pi_n$ $(n\in\mathbb{N})$ be the polynomial of best approximation to f. For a given $n\in\mathbb{N}$ we choose $m\in\mathbb{N}$ such that $2^m\leq n\leq 2^{m+1}$. Using the subadditivity property of $\Omega_r(f,\delta)_{p),\theta,\omega}$ we have

$$\Omega_r(f,\delta)_{p),\theta,\omega}\leq\Omega_r(f-T_{2^{m+1}},\delta)_{p),\theta,\omega}+\Omega_r(T_{2^{m+1}},\delta)_{p),\theta,\omega}.\tag{3.1}$$

Using the inequality [5, p. 209]

$$2^{(\nu+1)r}E_{2^\nu}(f)_{p),\theta,\omega}\leq 2^{2r}\sum_{k=2^{\nu-1}+1}^{2^\nu}k^{r-1}E_k(f)_{p),\theta,\omega}\tag{3.2}$$

and setting $\delta:=1/n$ we have

$$\begin{aligned}\Omega_r(f-T_{2^{m+1}},\delta)_{p),\theta,\omega}&\leq c\|f-T_{2^{m+1}}\|_{L_{\omega}^{p),\theta}(\mathbb{T})}=cE_{2^{m+1}}(f)_{p),\theta,\omega}\\&\leq\frac{c}{n^r}2^{(m+1)r}E_{2^m}(f)_{p),\theta,\omega}\leq c\delta^r2^{2r}\sum_{k=2^{m-1}+1}^{2^m}k^{r-1}E_k(f)_{p),\theta,\omega}.\end{aligned}\tag{3.3}$$

Now, by Lemmas 1 and 2 and by (3.2),

$$\begin{aligned}
\Omega_r(T_{2^{m+1}},\delta)_{p),\theta,\omega} &\le c\delta^r \|T^{(r)}_{2^{m+1}}\|_{L^{p),\theta}_\omega(\mathbb{T})}\\
&= c\delta^r\Big\{\|T^{(r)}_1 - T^{(r)}_0\|_{L^{p),\theta}_\omega(\mathbb{T})} + \sum_{\nu=0}^{m}\|T^{(r)}_{2^{\nu+1}} - T^{(r)}_{2^\nu}\|_{L^{p),\theta}_\omega(\mathbb{T})}\Big\}\\
&\le c_{16}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + \sum_{\nu=0}^{m}2^{(\nu+1)r}\|T_{2^{\nu+1}} - T_{2^\nu}\|_{L^{p),\theta}_\omega(\mathbb{T})}\Big\}\\
&\le c_{17}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + \sum_{\nu=0}^{m}2^{(\nu+1)r}E_{2^\nu}(f)_{p),\theta,\omega}\Big\}\\
&\le c_{17}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + 2^rE_1(f)_{p),\theta,\omega} + \sum_{\nu=1}^{m}2^{(\nu+1)r}E_{2^\nu}(f)_{p),\theta,\omega}\Big\}\\
&\le c_{18}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + \sum_{\nu=1}^{m}\sum_{k=2^{\nu-1}+1}^{2^\nu}k^{r-1}E_k(f)_{p),\theta,\omega}\Big\}.
\end{aligned}$$

Therefore,

$$\Omega_r(T_{2^{m+1}},\delta)_{p),\theta,\omega} \le c_{18}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + \sum_{k=1}^{2^m}k^{r-1}E_k(f)_{p),\theta,\omega}\Big\}, \tag{3.4}$$

and hence by (3.1), (3.3) and (3.4) we conclude that

$$\begin{aligned}
\Omega_r(f,\delta)_{p),\theta,\omega} &\le \delta^r 2^{2r}\sum_{k=2^{m-1}+1}^{2^m}k^{r-1}E_k(f)_{p),\theta,\omega}\\
&\quad + c_{18}\delta^r\Big\{E_0(f)_{p),\theta,\omega} + \sum_{k=1}^{2^m}k^{r-1}E_k(f)_{p),\theta,\omega}\Big\}\\
&\le \frac{c_{19}}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p),\theta,\omega}. \quad \blacksquare
\end{aligned}$$

Proof of Theorem 4. Let $f \in \mathcal{H}^{p),\theta}_\omega(\mathbb{D})$ and let $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be the Fourier series of f with the nth partial sum $S_n(f,t)$. Since $f \in H^1(\mathbb{D})$ we have [7, p. 38]

$$c_k = \begin{cases} \beta_k(f), & k \ge 0,\\ 0, & k < 0.\end{cases}$$

Let $T_n \in \Pi_n$ be the polynomial of best approximation to $f \in \mathcal{H}^{p),\theta}_\omega(\mathbb{D})$. By

Theorem 2,

$$\begin{aligned}\Big\|f(z)-\sum_{k=0}^{n}\beta_k(f)z^k\Big\|_{L^{p),\theta}_\omega(\mathbb{T})} &= \|f(e^{it})-S_n(f,t)\|_{L^{p),\theta}_\omega(\mathbb{T})}\\ &\le \|f(e^{it})-T_n(t)\|_{L^{p),\theta}_\omega(\mathbb{T})}+\|S_n(f,t)-T_n(t)\|_{L^{p),\theta}_\omega(\mathbb{T})}\\ &= \|f(e^{it})-T_n(t)\|_{L^{p),\theta}_\omega(\mathbb{T})}+\|S_n(f-T_n,t)\|_{L^{p),\theta}_\omega(\mathbb{T})}\\ &\le c_{20}\|f(e^{it})-T_n(t)\|_{L^{p),\theta}_\omega(\mathbb{T})}\\ &= c_{21}E_n(f)_{p),\theta,\omega}\le c\Omega_r(f,1/n)_{p),\theta,\omega}. \ \blacksquare\end{aligned}$$

Proof of Theorem 5. Let T_n, $n\in\mathbb{N}$, be the polynomials of best approximation to f. By Lemma 2,

$$\begin{aligned}\|T^{(r)}_{2^{m+1}}-T^{(r)}_{2^m}\|_{L^{p),\theta}_\omega(\mathbb{T})} &\le c2^{(m+1)r}\|T_{2^{m+1}}-T_{2^m}\|_{L^{p),\theta}_\omega(\mathbb{T})}\\ &\le c2^{(m+1)r}\big\{\|T_{2^{m+1}}-f\|_{L^{p),\theta}_\omega(\mathbb{T})}+\|f-T_{2^m}\|_{L^{p),\theta}_\omega(\mathbb{T})}\big\}\\ &\le c_{21}2^{(m+1)r}E_{2^m}(f)_{p),\theta,\omega}.\end{aligned}\tag{3.5}$$

By (3.5), (3.2) and using the definition of the norm in $\mathcal{W}^{p),\theta}_{r,\omega}$ we have

$$\begin{aligned}\sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{W^{p),\theta}_{r,\omega}} &= \sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{L^{p),\theta}_\omega}+\sum_{m=1}^{\infty}\|T^{(r)}_{2^{m+1}}-T^{(r)}_{2^m}\|_{L^{p),\theta}_\omega}\\ &\le c_{22}\sum_{m=1}^{\infty}2^{(m+1)r}E_{2^m}(f)_{p),\theta,\omega}\\ &\le c_{23}2^{2r}\sum_{m=1}^{\infty}\sum_{k=2^{m-1}+1}^{2^m}k^{r-1}E_k(f)_{p),\theta,\omega}\\ &\le c\sum_{m=1}^{\infty}k^{r-1}E_k(f)_{p),\theta,\omega}<\infty.\end{aligned}$$

Hence $\|T_{2^{m+1}}-T_{2^m}\|_{L^{p),\theta}_\omega(\mathbb{T})}\to 0$ as $m\to\infty$, which implies that $\{T_{2^m}\}$ is a Cauchy sequence converging to some $f\in\mathcal{W}^{p),\theta}_{r,\omega}(\mathbb{T})$. $\blacksquare$

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Daniyal M. Israfilov
Department of Mathematics
Balikesir University
Balikesir, Turkey
and
Institute of Mathematics and Mechanics of ANAS
Baku, Azerbaijan
E-mail: mdaniyal@balikesir.edu.tr

Ahmet Testici
Department of Mathematics
Balikesir University
Balikesir Turkey
E-mail: testiciahmet@hotmail.com