STUDIA MATHEMATICA 229 (1) (2015)

Infinitesimal generators for a class of polynomial processes

by

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Abstract. We study the infinitesimal generators of evolutions of linear mappings on the space of polynomials, which correspond to a special class of Markov processes with polynomial regressions called quadratic harnesses. We relate the infinitesimal generator to the unique solution of a certain commutation equation, and we use the commutation equation to find an explicit formula for the infinitesimal generator of free quadratic harnesses.

1. Introduction. In this paper we study properties of evolutions of degree-preserving linear mappings on the space of polynomials. We analyze these mappings in a self-contained algebraic language assuming a number of algebraic properties which have been abstracted out from some special properties of a family of Markov processes called quadratic harnesses. We therefore begin with a review of the relevant theory of Markov processes that motivates our assumptions.

The relevant Markov processes have polynomial regressions with respect to the past σ-fields; they enjoy many interesting properties and appeared in numerous references [2, 3, 10, 12, 15–17, 22–31]. As observed by Cuchiero [14] and Szabłowski [32], the transition probabilities $P_{s,t}(x,dy)$ of such a process on an infinite state space define a family $(\mathsf{P}_{s,t})_{0\le s\le t}$ of linear transformations that map the linear space $\mathcal{P}=\mathcal{P}(\mathbb{R})$ of all real polynomials in the variable x into itself. The crucial property that holds in many interesting examples is that the transformations $\mathsf{P}_{s,t}$ do not increase the degree of a polynomial, that is, $\mathsf{P}_{s,t}:\mathcal{P}_{\le k}\to\mathcal{P}_{\le k}$, where $\mathcal{P}_{\le k}$ denotes the linear space of polynomials of degree $\le k$, $k=0,1,\dots$. This will be the basis for our algebraic approach. Cuchiero [14] (see also [15]) introduced the term “polynomial process” for

2010 *Mathematics Subject Classification*: Primary 60J35; Secondary 47D06, 46L53.
Key words and phrases: infinitesimal generators, quadratic conditional variances, polynomial processes.
Received 14 June 2015; revised 20 November 2015.
Published online 7 December 2015.

DOI: 10.4064/sm8330-12-2015

such a process in the time-homogeneous case; we will use this term more broadly to denote the family of operators rather than a Markov process. That is, we adopt the point of view that the linear mappings $\mathsf{P}_{s,t}$ of $\mathcal{P}$ can be analyzed "in the abstract" without explicit reference to the underlying Markov process and the transition operators.

We note that the operators $\mathsf{P}_{s,t}$ are well defined whenever the support of X_s is infinite. So, strictly speaking, the operators $\mathsf{P}_{0,t}$ are well defined only for the so called Markov families that can be started at an infinite number of values of X_0. However, it will turn out that in the cases we are interested in, even if a Markov process starts with $X_0 = 0$ we can pass to the limit in $\mathsf{P}_{s,t}$ as $s \to 0$ and in this way define a unique degree-preserving mapping $\mathsf{P}_{0,t} : \mathcal{P} \to \mathcal{P}$.

With the above in mind, we introduce the following definition.

Definition 1.1. Let $\mathcal{P}$ be the linear space of real polynomials in one variable x. A *polynomial process* is a family $\{\mathsf{P}_{s,t} : \mathcal{P} \to \mathcal{P},\ 0 \le s \le t\}$ of linear maps with the following properties:

(i) for $k = 0, 1, \dots$ and $0 \le s \le t$,
$$\mathsf{P}_{s,t}(\mathcal{P}_{\le k}) = \mathcal{P}_{\le k},$$
(ii) $\mathsf{P}_{s,t}(1) = 1$,
(iii) for $0 \le s \le t \le u$,
$$\mathsf{P}_{s,t} \circ \mathsf{P}_{t,u} = \mathsf{P}_{s,u}. \tag{1.1}$$

Our next task is to abstract out the properties of polynomial processes that correspond to a special class of Markov processes with linear regressions and quadratic conditional variances under two-sided conditioning. The two-sided linearity of regression, which is sometimes called the *harness property*, takes the following form:
$$\mathrm{E}(X_t \mid X_s, X_u) = \frac{u-t}{u-s}X_s + \frac{t-s}{u-s}X_u, \quad 0 \le s < t < u \tag{1.2}$$
(see e.g. [19, formula (2)]).

We will also assume that the conditional second moment of X_t given X_s, X_u is a second degree polynomial in the variables X_s and X_u. The conclusion of [8, Theorem 2.2] says that there are five numerical constants $\eta, \theta \in \mathbb{R}$, $\sigma, \tau \ge 0$, and $\gamma \le 1 + 2\sqrt{\sigma\tau}$ such that for all $0 \le s < t < u$,
$$\begin{aligned}(1.3)\quad & \mathrm{Var}(X_t \mid X_s, X_u) \\ &= \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-\gamma s}\biggl(1 + \eta\frac{uX_s - sX_u}{u-s} + \theta\frac{X_u - X_s}{u-s} \\ &\quad + \sigma\frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau\frac{(X_u - X_s)^2}{(u-s)^2} - (1-\gamma)\frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2}\biggr).\end{aligned}$$

Such processes, when standardized, are called *quadratic harnesses.* Typically they are uniquely determined by the five constants η, θ, σ, τ, γ from (1.3). We will also assume that (X_t) is a martingale in the natural filtration. This is a consequence of (1.2) when $\mathrm{E}(X_t)$ does not depend on t (cf. [10, p. 417].

The martingale property is easily expressed in the language of polynomial processes, as it just says that $\mathsf{P}_{s,t}(x) = x$. Somewhat more generally, if polynomials $m_k(x;t)$ in x for $k \geq 0$ are *martingale polynomials* for a Markov process (X_t), that is, $\mathrm{E}(m_k(X_t;t) \,|\, X_s) = m_k(X_s, s)$, $k \geq 0$, then $\mathsf{P}_{s,t}(m_k(\cdot;t))(x) = m_k(x;s)$ for $s < t$.

It is harder to deduce the properties of $\mathsf{P}_{s,t}$ that correspond to (1.2) and (1.3). We will find it useful to describe linear mappings on polynomials in the algebraic language, and we will rely on martingale polynomials to complete this task.

1.1. The algebra $\mathcal{Q}$ of sequences of polynomials. We consider the linear space $\mathcal{Q}$ of all infinite sequences of polynomials in x with a nonstandard multiplication defined as follows. For $\mathbb{P} = (p_0, p_1, \dots)$ and $\mathbb{Q} = (q_0, q_1, \dots)$ in $\mathcal{Q}$, we define the product $\mathbb{R} = \mathbb{P}\mathbb{Q}$ as the sequence $\mathbb{R} = (r_0, r_1, \dots) \in \mathcal{Q}$ of polynomials given by

$$r_k(x) = \sum_{j=0}^{\deg(q_k)} [q_k]_j p_j(x), \qquad k = 0, 1, \dots, \tag{1.4}$$

where $[q_k]_j$ is the coefficient of x^j in q_k. It is easy to check that the algebra $\mathcal{Q}$ has the identity

$$\mathbb{E} = (1, x, x^2, \dots).$$

We will frequently use two special elements of $\mathcal{Q}$:

$$\mathbb{F} = (x, x^2, x^3, \dots), \tag{1.5}$$

and

$$\mathbb{D} = (0, 1, x, x^2, \dots), \tag{1.6}$$

which is the left inverse of $\mathbb{F}$.

The algebra $\mathcal{Q}$ is isomorphic to the algebra of all linear mappings $\mathcal{P} \to \mathcal{P}$ under composition: to each $\mathsf{P} : \mathcal{P} \to \mathcal{P}$ we associate a sequence $\mathbb{P} = (p_0, p_1, \dots)$ of polynomials in x by setting $p_k = \mathsf{P}(x^k)$. Of course, if P does not increase the degree, then p_k is of degree at most k.

This is indeed an algebra isomorphism: the composition $\mathsf{R} = \mathsf{P} \circ \mathsf{Q}$ of linear operators P and Q on $\mathcal{P}$ induces the product $\mathbb{R} = \mathbb{P}\mathbb{Q}$ of the corresponding sequences $\mathbb{P} = (p_0, p_1, \dots)$ and $\mathbb{Q} = (q_0, q_1, \dots)$, defined in (1.4).

It is clear that degree-preserving linear mappings of $\mathcal{P}$ are invertible.

Proposition 1.2. *If for every n the polynomial p_n is of degree n then $\mathbb{P} = (p_0, p_1, \dots)$ has a multiplicative inverse $\mathbb{Q} = (q_0, q_1, \dots)$ and each polynomial q_n is of degree n.*

In this paper we study $\mathsf{P}_{s,t}$ through the corresponding elements $\mathbb{P}_{s,t}$ of the algebra $\mathcal{Q}$. We therefore rewrite Definition 1.1 in terms of $\mathcal{Q}$.

Definition 1.3 (Equivalent form of Definition 1.1). A *polynomial process* is a family $\{\mathbb{P}_{s,t} \in \mathcal{Q} : 0 \le s \le t\}$ with the following properties:

(i) for $0 \le s \le t$ and $n = 0, 1, \dots$, the nth component of $\mathbb{P}_{s,t}$ is a polynomial of degree n,

(ii) $\mathbb{P}_{s,t}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}$,

(iii) for $0 \le s \le t \le u$,

$$\mathbb{P}_{s,t}\mathbb{P}_{t,u} = \mathbb{P}_{s,u}. \tag{1.7}$$

Since the special elements (1.5) and (1.6) satisfy $(1, 0, 0, \dots) = \mathbb{E} - \mathbb{F}\mathbb{D}$, property (ii) is $\mathbb{P}_{s,t}(1, 0, 0, \dots) = (1, 0, 0, \dots)$ (see Definition 1.1(ii)).

1.2. Martingale polynomials. In this section we rederive [32, Theorem 1]. We note that by Proposition 1.2 each $\mathbb{P}_{s,t}$ is an invertible element of $\mathcal{Q}$. So from (1.7) we see that $\mathbb{P}_{t,t} = \mathbb{E}$ for all $t \ge 0$. In particular, $\mathbb{P}_{0,t}$ is invertible and $\mathbb{M}_t = \mathbb{P}_{0,t}^{-1}$ consists of polynomials $m_k(x; t)$ in x of degree k. From (1.7), we get $\mathbb{P}_{0,s}\mathbb{P}_{s,t}\mathbb{M}_t = \mathbb{P}_{0,t}\mathbb{M}_t = \mathbb{E}$. Multiplying this on the left by $\mathbb{M}_s = \mathbb{P}_{0,s}^{-1}$, we see that

$$\mathbb{M}_s = \mathbb{P}_{s,t}\mathbb{M}_t, \tag{1.8}$$

i.e., $\mathbb{M}_t$ is a sequence of martingale polynomials for $\{\mathbb{P}_{s,t}\}$ and in addition $\mathbb{M}_0 = \mathbb{E}$. Conversely,

$$\mathbb{P}_{s,t} = \mathbb{M}_s\mathbb{M}_t^{-1}. \tag{1.9}$$

Ref. [32] points out that in general martingale polynomials are not unique. However, any sequence $\widetilde{\mathbb{M}}_t = (m_0(x; t), m_1(x; t), \dots)$ of martingale polynomials (with $m_k(x; t)$ of degree k for $k = 0, 1, \dots$) still uniquely determines $\mathbb{P}_{s,t}$ via $\mathbb{P}_{s,t} = \widetilde{\mathbb{M}}_s\widetilde{\mathbb{M}}_t^{-1}$.

1.3. Quadratic harnesses. Recall that our goal is to abstract out the properties of the (algebraic) polynomial process $\{\mathbb{P}_{s,t}\}$ that correspond to relations (1.2) and (1.3). Since this correspondence is not direct, we first give an algebraic definition, and then a motivation for it.

Definition 1.4. We say that a polynomial process $\{\mathbb{P}_{s,t} : 0 \le s \le t\}$ is a *quadratic harness* with parameters $\eta, \theta, \sigma, \tau, \gamma$ if the following three conditions hold:

(i) (martingale property) $\mathbb{P}_{s,t}(\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2) = \mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2$,

(ii) (harness property) there exists $\mathbb{X} \in \mathcal{Q}$ such that for all $t \geq 0$,

$$\mathbb{P}_{0,t}\mathbb{F} = (\mathbb{F} + t\mathbb{X})\mathbb{P}_{0,t}, \tag{1.10}$$

(iii) (quadratic harness property) the element $\mathbb{X} \in \mathcal{Q}$ in (1.10) satisfies the quadratic equation

$$\mathbb{X}\mathbb{F} - \gamma\mathbb{F}\mathbb{X} = \mathbb{E} + \eta\mathbb{F} + \theta\mathbb{X} + \sigma\mathbb{F}^2 + \tau\mathbb{X}^2. \tag{1.11}$$

REMARK 1.5. Using the martingale polynomials $\mathbb{M}_t = \mathbb{P}_{0,t}^{-1}$ from Section 1.2, we can rewrite assumption (1.10) as

$$\mathbb{F}\mathbb{M}_t = \mathbb{M}_t(\mathbb{F} + t\mathbb{X}). \tag{1.12}$$

We now give a brief explanation of how the algebraic properties in conditions (i)–(iii) of Definition 1.4 are related to the corresponding properties of a (polynomial) Markov process.

1.3.1. *Motivation for the martingale property.* For a Markov process (X_t) with the natural filtration $(\mathcal{F}_t)$ the martingale property takes a more familiar form, $\mathrm{E}(X_t \mid \mathcal{F}_s) = X_s$. Translated back into the language of linear operators on polynomials, this is $\mathsf{P}_{s,t}(x) = x$. In the language of the algebra $\mathcal{Q}$ this is $\mathbb{P}_{s,t}(0, x, 0, 0, \dots) = (0, x, 0, 0, \dots)$. To get the final form of condition (i) we note that $\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2 = (0, x, 0, 0, \dots)$.

1.3.2. *Motivation for the harness property* (1.10). For a Markov processes (X_t) with martingale polynomials $m_n(x; t)$, property (1.2) implies that

$$\begin{aligned}
\mathrm{E}(X_t m_n(X_t; t) \mid X_s) &= \mathrm{E}(X_t m_n(X_u; u) \mid X_s) \\
&= \mathrm{E}\big(\mathrm{E}(X_t \mid X_s, X_u) m_n(X_u; u) \mid X_s\big) \\
&= \frac{u-t}{u-s} X_s \mathrm{E}(m_n(X_u; u) \mid X_s) + \frac{t-s}{u-s}\mathrm{E}(X_u m_n(X_u; u) \mid X_s) \\
&= \frac{u-t}{u-s} X_s m_n(X_s; s) + \frac{t-s}{u-s}\mathrm{E}(X_u m_n(X_u; u) \mid X_s).
\end{aligned}$$

The resulting identity

$$\mathrm{E}(X_t m_n(X_t; t) \mid X_s) = \frac{u-t}{u-s} X_s m_n(X_s; s) + \frac{t-s}{u-s}\mathrm{E}(X_u m_n(X_u; u) \mid X_s)$$

in the language of the algebra $\mathcal{Q}$ with $\mathbb{M}_t = (m_0(x; t), m_1(x; t), \dots)$ becomes

$$\mathbb{P}_{s,t}\mathbb{F}\mathbb{M}_t = \frac{u-t}{u-s}\mathbb{F}\mathbb{M}_s + \frac{t-s}{u-s}\mathbb{P}_{s,u}\mathbb{F}\mathbb{M}_u. \tag{1.13}$$

Since each polynomial $x m_n(x; t)$ can be written as a linear combination of $m_0(x; t), \dots, m_{n+1}(x; t)$, one can find $\mathbb{J}_t \in \mathcal{Q}$ such that $\mathbb{F}\mathbb{M}_t = \mathbb{M}_t\mathbb{J}_t$. Inserting this into (1.13), we see that the martingale property eliminates $\mathbb{P}_{s,t}$ and after left-multiplication by $\mathbb{M}_s^{-1}$ we get

$$(u-s)\mathbb{J}_t = (u-t)\mathbb{J}_s + (t-s)\mathbb{J}_u.$$

In particular, $\mathbb{J}_t$ depends linearly on t and thus

$$\mathbb{J}_t = \mathbb{J}_0 + t(\mathbb{J}_1 - \mathbb{J}_0).$$

This shows that for any martingale polynomials the harness property implies that there exist $\mathbb{Y}, \mathbb{X} \in \mathcal{Q}$ such that

$$\mathbb{F}\mathbb{M}_t = \mathbb{M}_t(\mathbb{Y} + t\mathbb{X}). \tag{1.14}$$

In our special case of $\mathbb{M}_t = \mathbb{P}_{0,t}^{-1}$, we get $\mathbb{Y} = \mathbb{J}_0 = \mathbb{F}\mathbb{M}_0 = \mathbb{F}\mathbb{E} = \mathbb{F}$. This establishes (1.12), which is of course equivalent to (1.10).

1.3.3. *Motivation for the quadratic harness property* (1.11). Suppose the polynomial process $\{\mathbb{P}_{s,t}\}$ in the sense of Definition 1.3 arises from a Markov process with polynomial conditional moments which is a harness and in addition has quadratic conditional variances (1.3). Then from the previous discussion, (1.14) holds, and under mild technical assumptions, [8, Theorem 2.3] shows that $\mathbb{X}, \mathbb{Y}$ satisfy a commutation equation which reduces to (1.11) when $\mathbb{Y} = \mathbb{F}$. This motivates condition (iii).

In fact, it is known that under some additional assumptions on the growth of moments, (1.10) and (1.11) imply (1.2) and (1.3) (see [31, Section 4.1]); this equivalence is also implicit in the proof of [8, Theorem 2.3] and is explicitly used in [12, p. 1244]. However, this has no direct bearing on our paper, as in this paper we simply adopt the algebraic Definition 1.4.

1.4. Infinitesimal generator. A polynomial process $\{\mathbb{P}_{s,t}\}$ with the harness property (1.10) is uniquely determined by $\mathbb{X}$. Indeed, the nth element of the sequence on the left hand side of (1.10) is the $(n+1)$th polynomial in $\mathbb{P}_{0,t}$, while the nth element of the sequence on the right hand side of (1.10) depends only on the first n polynomials in $\mathbb{P}_{0,t}$. In fact, one can check that

$$\mathbb{P}_{0,t} = \sum_{k=0}^{\infty} (\mathbb{F} + t\mathbb{X})^k (\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{D}^k. \tag{1.15}$$

Since the kth element of $\mathbb{F}+t\mathbb{X}$ has degree $k+1$, it follows from (1.12) that $\mathbb{M}_t$ is a rational function of t. Formula (1.9) shows that the left infinitesimal generator

$$\mathbb{A}_t = \lim_{h\to 0^+} \frac{1}{h}(\mathbb{P}_{t-h,t} - \mathbb{E}), \qquad t > 0, \tag{1.16}$$

is a well defined element of $\mathcal{Q}$, and that

$$\mathbb{A}_t \mathbb{M}_t = -\frac{\partial}{\partial t}\mathbb{M}_t \tag{1.17}$$

with differentiation defined componentwise.

Since $\mathbb{P}_{s,t}$ is continuous in t, the right infinitesimal generator exists and is given by the same expression. To see this, we compute $\lim_{h\to 0^+} \frac{1}{h}(\mathbb{P}_{t,t+h} - \mathbb{E})$ on $\mathbb{M}_t$. We have

$$\lim_{h\to 0^+}\frac{1}{h}(\mathbb{P}_{t,t+h}\mathbb{M}_t-\mathbb{M}_t)=\lim_{h\to 0^+}\mathbb{P}_{t,t+h}\frac{1}{h}(\mathbb{M}_t-\mathbb{M}_{t+h})$$
$$=\lim_{h\to 0^+}\mathbb{P}_{t,t+h}\lim_{h\to 0^+}\frac{1}{h}(\mathbb{M}_t-\mathbb{M}_{t+h})=-\frac{\partial}{\partial t}\mathbb{M}_t.$$

The infinitesimal generator $\mathbb{A}_t$ determines $\mathbb{P}_{s,t}$ uniquely; for an algebraic proof see Proposition 2.3. Clearly, $\mathbb{M}_t=\mathbb{E}-\int_0^t\mathbb{A}_s\mathbb{M}_s\,ds$.

The infinitesimal generator $\mathbb{A}_t$ and its companion operator $\mathsf{A}_t:\mathcal{P}\to\mathcal{P}$ are the main objects of interest in this paper.

Remark 1.6. We note that $\mathbb{A}_t=(0,a_1(x;t),a_2(x;t),\dots)$ always starts with a 0, since from $\mathsf{P}_{s,t}(1)=1$ it follows that $\mathsf{A}_t(1)=0$. It is clear that $a_n(x;t)$ is a polynomial in x of degree at most n. The martingale property implies that $a_1(x;t)=0$, as $\mathsf{A}_t(x)=0$. The infinitesimal generator considered as an element of $\mathcal{Q}$ in the latter case starts with two zeros, $\mathbb{A}_t=(0,0,a_2(x;t),a_3(x;t),\dots)$.

2. Basic properties of infinitesimal generators for quadratic harnesses. Our first result introduces an auxiliary element $\mathbb{H}_t\in\mathcal{Q}$ that is related to $\mathbb{A}_t$ by a commutation equation.

Theorem 2.1. *Suppose that* $\{\mathbb{P}_{s,t}\in\mathcal{Q}:0\le s\le t\}$ *is a quadratic harness as in Definition* 1.4 *with generator* $\mathbb{A}_t$. *For* $t>0$, *let*

$$\mathbb{H}_t=\mathbb{A}_t\mathbb{F}-\mathbb{F}\mathbb{A}_t \tag{2.1}$$

and denote $\mathbb{T}_t=\mathbb{F}-t\mathbb{H}_t$. *Then*

$$\mathbb{H}_t\mathbb{T}_t-\gamma\mathbb{T}_t\mathbb{H}_t=\mathbb{E}+\theta\mathbb{H}_t+\eta\mathbb{T}_t+\tau\mathbb{H}_t^2+\sigma\mathbb{T}_t^2. \tag{2.2}$$

Proof. Differentiating (1.12) and then using (1.17) we get

$$-\mathbb{F}\mathbb{A}_t\mathbb{M}_t=-\mathbb{A}_t\mathbb{M}_t(\mathbb{F}+t\mathbb{X})+\mathbb{M}_t\mathbb{X}.$$

Multiplying this from the right by $\mathbb{M}_t^{-1}$ and using (1.12) to replace $\mathbb{M}_t(\mathbb{F}+t\mathbb{X})\mathbb{M}_t^{-1}$ by $\mathbb{F}$ we get

$$-\mathbb{F}\mathbb{A}_t=-\mathbb{A}_t\mathbb{F}+\mathbb{M}_t\mathbb{X}\mathbb{M}_t^{-1}.$$

Comparing this with (2.1) we see that

$$\mathbb{H}_t=\mathbb{M}_t\mathbb{X}\mathbb{M}_t^{-1}. \tag{2.3}$$

We now use this together with (1.12) in the definition of $\mathbb{T}_t=\mathbb{F}-t\mathbb{H}_t=\mathbb{M}_t(\mathbb{F}+t\mathbb{X})\mathbb{M}_t^{-1}-t\mathbb{H}_t$ to get

$$\mathbb{T}_t=\mathbb{M}_t\mathbb{F}\mathbb{M}_t^{-1}.$$

To derive equation (2.2) we now multiply (1.11) by $\mathbb{M}_t$ from the left and by $\mathbb{M}_t^{-1}$ from the right. ∎

Remark 2.2. As observed in Remark 1.6, the nth element of $\mathbb{A}_t$ is a polynomial of degree at most n. Thus writing $\mathbb{H}_t=(h_0(x),h_1(x),\dots)$, from

(2.1) we see that h_n is of degree at most $n+1$. The martingale property implies that $h_0(x)=0$.

We will need several uniqueness results that follow from the more detailed elementwise analysis of sequences of polynomials. We first show that the generator determines the polynomial process uniquely, at least under the harness condition (1.10).

Proposition 2.3. *A polynomial process* $\{\mathbb{P}_{s,t} : 0 \le s < t\}$ *with harness property* (1.10) *is uniquely determined by its generator* $\mathbb{A}_t$.

Proof. Fix $s<t$. Combining (1.9) with (1.12) we get

$$\begin{aligned}\mathbb{P}_{s,t}\mathbb{F} - \mathbb{F}\mathbb{P}_{s,t} &= \mathbb{M}_s\mathbb{M}_t^{-1}\mathbb{F} - \mathbb{F}\mathbb{M}_s\mathbb{M}_t^{-1}\\ &= \mathbb{M}_s(\mathbb{Y}+t\mathbb{X})\mathbb{M}_t^{-1} - \mathbb{M}_s(\mathbb{Y}+s\mathbb{X})\mathbb{M}_t^{-1}\\ &= (t-s)\mathbb{M}_s\mathbb{M}_t^{-1}(\mathbb{M}_t\mathbb{X}\mathbb{M}_t^{-1}).\end{aligned}$$

Therefore, from (2.3) we get

$$\mathbb{P}_{s,t}\mathbb{F} = \mathbb{F}\mathbb{P}_{s,t} + (t-s)\mathbb{P}_{s,t}\mathbb{H}_t.$$

With $s=0$, this implies

$$\mathbb{M}_t\mathbb{F} = (\mathbb{F} - t\mathbb{H}_t)\mathbb{M}_t.$$

This equation is similar to (1.10) and again uniqueness follows from consideration of the degrees of the polynomials. The solution is

$$\mathbb{M}_t = \sum_{k=0}^{\infty}(\mathbb{F}-t\mathbb{H}_t)^k(\mathbb{E}-\mathbb{F}\mathbb{D})\mathbb{D}^k$$

(cf. (1.15)). Thus $\mathbb{M}_t$ is uniquely determined, and (1.9) shows that the operators $\mathbb{P}_{s,t}$ are uniquely determined. Since $\mathbb{H}_t$ is expressed in terms of $\mathbb{A}_t$ by (2.1), this ends the proof. ∎

Next, we show that $\mathbb{H}_t = (h_0, h_1, \dots)$ is uniquely determined by the commutation equation (2.2) with the "initial condition" $h_0 = 0$. From the proof of Proposition 2.3 it then follows that the entire quadratic harness $\{\mathbb{P}_{s,t}\}$ as well as its generator are also uniquely determined by (2.2).

Proposition 2.4. *If* $\sigma, \tau \ge 0$, *and* $\sigma\tau \ne 1$ *then equation* (2.2) *has a unique solution among* $\mathbb{H}_t \in \mathcal{Q}$ *such that* $h_0(x) = 0$.

Proof. Eliminating $\mathbb{T}_t = \mathbb{F} - t\mathbb{H}_t$ from (2.2) we can rewrite it in the following equivalent form:

$$\mathbb{H}_t\mathbb{F} - \gamma\mathbb{F}\mathbb{H}_t = \mathbb{E} + \theta\mathbb{H}_t + \eta(\mathbb{F}-t\mathbb{H}_t) + \tau\mathbb{H}_t^2 + \sigma(\mathbb{F}-t\mathbb{H}_t)^2 + (1-\gamma)t\mathbb{H}_t^2. \tag{2.4}$$

Write $\mathbb{H}_t = \mathbb{H} = (h_n(x))_{n=0,1,\dots}$ for a fixed $t>0$, with $h_0 = 0$. We will simultaneously prove by induction on n that the polynomial $h_n(x)$ is uniquely determined and that its degree is at most $n+1$. With $h_0 = 0$ the assertion is clear. We therefore assume that $h_0, h_1, \dots, h_n$ are given polynomials of degrees at most $1, 2, \dots, n+1$, respectively.

Looking at the nth element of (2.4) for $n \geq 0$, we get the following equation for $h_{n+1}(x)$:

$$\begin{aligned}(2.5)\qquad &\big(1 + \sigma t - [h_n]_{n+1}(\sigma t^2 + (1-\gamma)t + \tau)\big)h_{n+1}(x) \\ &\quad = x^n + \eta x^{n+1} + \sigma x^{n+2} + (\theta - t\eta)h_n(x) \\ &\qquad + (\gamma - \sigma t)x h_n(x) + (\sigma t^2 + (1-\gamma)t + \tau)\sum_{j=0}^{n}[h_n]_j h_j(x).\end{aligned}$$

The degree of the polynomial on the right hand side is at most $n+2$, as the highest degree term is $(\sigma + (\gamma - \sigma t)[h_n]_{n+1})x^{n+2}$. To complete the proof, it suffices to verify that for all $n \geq 0$, the coefficient $1 + \sigma t - [h_n]_{n+1}(\sigma t^2 + (1-\gamma)t + \tau)$ on the left hand side of (2.5) does not vanish.

We consider separately two cases.

Case $\gamma + \sigma\tau \neq 0$. Suppose that for some $n \geq 0$ the coefficient of $h_{n+1}(x)$ on the left hand side of (2.5) is 0. Since $1 + \sigma t > 0$, this implies that $\sigma t^2 + (1-\gamma)t + \tau$ cannot be zero. So we get

$$[h_n]_{n+1} = \frac{1 + \sigma t}{\sigma t^2 + (1-\gamma)t + \tau}.$$

We now use this value to compute the coefficient of x^{n+2} on the right hand side of (2.5). We get

$$\sigma + (\gamma - \sigma t)[h_n]_{n+1} = \frac{\gamma + \sigma\tau}{\sigma t^2 + (1-\gamma)t + \tau} \neq 0.$$

Since the left hand side of (2.5) is 0, and the degree of the right hand side of (2.5) is $n+2$, this is a contradiction.

This shows that the coefficient of $h_{n+1}(x)$ on the left hand side of (2.5) is non-zero for all $n \geq 0$. So each polynomial $h_{n+1}(x)$ is uniquely determined and has degree at most $n+2$ for all $n \geq 0$.

Case $\gamma + \sigma\tau = 0$. In this case $\sigma t^2 + (1-\gamma)t + \tau = (1+\sigma t)(\tau + t)$. Equating the coefficients of x^{n+2} on both sides of (2.5) we get

$$(2.6)\qquad (1+\sigma t)(1 - [h_n]_{n+1}(\tau + t))[h_{n+1}]_{n+2} = \sigma(1 - [h_n]_{n+1}(\tau + t)).$$

Since $h_0(x) = 0$, this gives $[h_1]_2 = \sigma/(1+\sigma t)$. So for $n = 1$ we get $1 - [h_n]_{n+1}(\tau + t) = (1 - \sigma\tau)/(1+\sigma t) \neq 0$. Dividing both sides of (2.6) by this expression we get recursively $[h_n]_{n+1} = \sigma/(1+\sigma t)$ for all $n \geq 1$. Therefore, for $n \geq 1$, the left hand side of (2.5) simplifies to $(1 - \sigma\tau)h_{n+1}(x)$. Using again $1 - \sigma\tau \neq 0$, this shows that the polynomial h_{n+1} is uniquely determined and its degree is at most $n+2$. Of course, (2.5) determines $h_1(x)$ uniquely too, as $h_0(x) = 0$. ∎

Our main result is the identification of the infinitesimal generator for quadratic harnesses with $\gamma = -\sigma\tau$. Such processes were called "free quadratic harnesses" in [8, Section 4.1]. Markov processes with the free quadratic harness property were constructed in [9] but the construction required more restrictions on the parameters than what we impose here.

In this section we represent the infinitesimal generators using the auxiliary power series $\varphi_t(\mathbb{D})$ with $\mathbb{D}$ defined by (1.6). In Section 4 we will use the results of this section to derive the integral representation (4.3) for the operator A_t under a more restricted range of parameters $\eta, \theta, \sigma, \tau$.

For a formal power series $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$ we shall write $\varphi(\mathbb{D})$ for the series $\sum_k c_k \mathbb{D}^k$. We note that since the sequence $\mathbb{D}^k$ begins with k zeros, $\sum_k c_k \mathbb{D}^k$ is a sequence of finite sums:

$$\varphi(\mathbb{D}) = \Big(c_0, c_0 x + c_1, c_0 x^2 + c_1 x + c_2, \dots, \sum_{j=0}^{n} c_j x^{n-j}, \dots\Big).$$

So $\varphi(\mathbb{D})$ is a well defined element of $\mathcal{Q}$.

We will also need $\mathbb{D}_1 = \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{D}^{k+1} = (0, 1, 2x, 3x^2, \dots)$ which represents the derivative.

THEOREM 2.5. *Fix $\eta, \theta \in \mathbb{R}$ and $\sigma, \tau \geq 0$ such that $\sigma\tau \neq 1$. Then the infinitesimal generator of the quadratic harness with the above parameters and with $\gamma = -\sigma\tau$ is given by*

$$\mathbb{A}_t = \frac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1 \varphi_t(\mathbb{D})\mathbb{D}, \qquad t > 0, \tag{2.7}$$

where $\varphi_t(z) = \sum_{k=1}^{\infty} c_k(t) z^{k-1}$ for small enough z solves the quadratic equation

$$(z^2 + \eta z + \sigma)(t+\tau)\varphi_t^2 + ((\theta - t\eta)z - 2t\sigma - \sigma\tau - 1)\varphi_t + t\sigma + 1 = 0 \tag{2.8}$$

and the solution is chosen so that $\varphi_t(0) = 1$.

(*For $\sigma\tau < 1$ this solution is written explicitly in formula* (4.4) *below.*)

We note that for each fixed t equation (2.8) has two real roots for z close enough to 0. We let $\varphi_t(z)$ be the smaller root when $\sigma\tau < 1$ and the larger root when $\sigma\tau > 1$. For $z = 0$, equation (2.8) becomes

$$\sigma(t+\tau)\varphi_t^2(0) - (1 + \sigma\tau + 2t\sigma)\varphi_t(0) + 1 + t\sigma = 0,$$

so this procedure ensures that $\varphi_t(0) = 1$.

3. Proof of Theorem 2.5. The plan of the proof is to solve equation (2.2) for $\mathbb{H}_t$, and then to use equation (2.1) to determine $\mathbb{A}_t$.

3.1. Part I: solution of (2.2) when $\gamma = -\sigma\tau$. Equation (2.2) takes the form

$$\begin{aligned}&\mathbb{H}_t\mathbb{F} - t\mathbb{H}_t^2 + \sigma\tau\mathbb{F}\mathbb{H}_t - \sigma\tau t\mathbb{H}_t^2\\ &\quad = \mathbb{E} + \theta\mathbb{H}_t + \eta(\mathbb{F} - t\mathbb{H}_t) + \tau\mathbb{H}_t^2 + \sigma(\mathbb{F}^2 - t\mathbb{H}_t\mathbb{F} - t\mathbb{F}\mathbb{H}_t + t^2\mathbb{H}_t^2).\end{aligned}$$

So after simplifications, the equation to solve for the unknown $\mathbb{H}_t$ is

$$\begin{aligned}&(1+\sigma t)\mathbb{H}_t\mathbb{F}\\ &\quad = \mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2 + (\theta - \eta t)\mathbb{H}_t - \sigma(t+\tau)\mathbb{F}\mathbb{H}_t + (t+\tau)(1+\sigma t)\mathbb{H}_t^2.\end{aligned} \tag{3.1}$$

LEMMA 3.1. *The solution of* (3.1) *with the initial element $h_0 = 0$ is*

$$\mathbb{H}_t = \frac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\varphi_t(\mathbb{D})\mathbb{D}, \tag{3.2}$$

where φ_t satisfies equation (2.8) *and* $\varphi_t(0) = 1$.

Proof. Since $t > 0$ is fixed, we suppress the dependence on t and we use Remark 2.2 to write $\mathbb{H}_t = \mathbb{H} = (0, h_1(x), \dots)$. From (3.1) we find that $h_1(x) = \frac{1}{1+\sigma t}(1 + \eta x + \sigma x^2)$.

From Proposition 2.4 we see that (3.1) has a unique solution. In view of the uniqueness, we seek the solution in a special form

$$\mathbb{H} = \frac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=1}^{\infty} c_k\mathbb{D}^k \tag{3.3}$$

with $c_1 = 1$ and $c_k = c_k(t) \in \mathbb{R}$. Note that

$$\begin{aligned}\mathbb{H}^2 &= \frac{1}{(1+\sigma t)^2}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=1}^{\infty} c_k\mathbb{D}^k(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{j=1}^{\infty} c_j\mathbb{D}^j\\ &= \frac{1}{(1+\sigma t)^2}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=1}^{\infty} c_k\mathbb{D}^k\sum_{j=1}^{\infty} c_j\mathbb{D}^j\\ &\quad + \frac{\eta}{(1+\sigma t)^2}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=1}^{\infty} c_k\mathbb{D}^{k-1}\sum_{j=1}^{\infty} c_j\mathbb{D}^j\\ &\quad + \frac{\sigma}{(1+\sigma t)^2}(\mathbb{F} + \eta\mathbb{F}^2 + \sigma\mathbb{F}^3)\sum_{j=1}^{\infty} c_j\mathbb{D}^j\\ &\quad + \frac{\sigma}{(1+\sigma t)^2}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=2}^{\infty} c_k\mathbb{D}^{k-2}\sum_{j=1}^{\infty} c_k\mathbb{D}^j\,.\end{aligned}$$

Inserting this into (3.1) we get

$$
\begin{aligned}
&\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2+(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\sum_{k=2}^\infty c_k\mathbb{D}^{k-1}\\
&\quad=\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2\\
&\qquad+\frac{\theta-t\eta}{1+\sigma t}(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\sum_{k=1}^\infty c_k\mathbb{D}^k-\frac{\sigma(t+\tau)}{1+\sigma t}(\mathbb{F}+\eta\mathbb{F}^2+\sigma\mathbb{F}^3)\sum_{k=1}^\infty c_k\mathbb{D}^k\\
&\qquad+\frac{t+\tau}{1+\sigma t}\Big[(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\sum_{k=1}^\infty c_k\mathbb{D}^k\sum_{j=1}^\infty c_j\mathbb{D}^j\\
&\qquad+\eta(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\sum_{k=1}^\infty c_k\mathbb{D}^{k-1}\sum_{j=1}^\infty c_j\mathbb{D}^j\\
&\qquad+\sigma(\mathbb{F}+\eta\mathbb{F}^2+\sigma\mathbb{F}^3)\sum_{j=1}^\infty c_j\mathbb{D}^j+\sigma(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\sum_{k=2}^\infty c_k\mathbb{D}^{k-2}\sum_{j=1}^\infty c_j\mathbb{D}^j\Big].
\end{aligned}
$$

The terms with $\mathbb{F}+\eta\mathbb{F}^2+\sigma\mathbb{F}^3$ cancel out, so $\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2$ factors out. We further restrict our search for the solution by requiring that the remaining factors match, i.e.

$$
\begin{aligned}
&\sum_{k=1}^\infty c_{k+1}\mathbb{D}^k=\frac{\theta-t\eta}{1+\sigma}\sum_{k=1}^\infty c_k\mathbb{D}^k\\
&+\frac{t+\tau}{1+\sigma t}\Big(\sum_{k=1}^\infty c_k\mathbb{D}^k\sum_{j=1}^\infty c_j\mathbb{D}^j+\eta\sum_{k=1}^\infty c_k\mathbb{D}^{k-1}\sum_{j=1}^\infty c_j\mathbb{D}^j+\sigma\sum_{k=2}^\infty c_k\mathbb{D}^{k-2}\sum_{j=1}^\infty c_j\mathbb{D}^j\Big).
\end{aligned}
$$

Collecting the coefficients of the powers of $\mathbb{D}$ we get

$$
\begin{aligned}
&\sum_{k=1}^\infty c_{k+1}\mathbb{D}^k=\frac{\theta-t\eta}{1+\sigma t}\sum_{k=1}^\infty c_k\mathbb{D}^k\\
&+\frac{t+\tau}{1+\sigma t}\Big(\sum_{k=2}^\infty\sum_{j=1}^{k-1}c_jc_{k-j}\mathbb{D}^k+\eta\sum_{k=1}^\infty\sum_{j=0}^{k-1}c_{j+1}c_{k-j}\mathbb{D}^k+\sigma\sum_{k=1}^\infty\sum_{j=0}^{k-1}c_{j+2}c_{k-j}\mathbb{D}^k\Big).
\end{aligned}
$$

We now equate the coefficients of the powers of $\mathbb{D}$. Since $\sigma\tau\neq1$, for $k=1$ we get

$$c_2=\frac{\theta-\eta t}{1+\sigma t}+\frac{t+\tau}{1+\sigma t}\eta+\frac{t+\tau}{1+\sigma t}\sigma c_2.$$

So $c_2=(\theta+\eta\tau)/(1-\sigma\tau)=\beta$ (say).

For $k\ge2$, we have the recurrence

$$
\begin{aligned}
c_{k+1}&=\frac{\theta-\eta t}{1+\sigma t}c_k\\
&\quad+\frac{t+\tau}{1+\sigma t}\Big(\sum_{j=1}^{k-1}c_jc_{k-j}+\eta\sum_{j=0}^{k-1}c_{j+1}c_{k-j}+\sigma\sum_{j=0}^{k-1}c_{j+2}c_{k-j}\Big).
\end{aligned}\tag{3.4}
$$

We solve it by the method of generating functions. One can proceed here with a formal power series, and then invoke uniqueness to verify that the power series has positive radius of convergence. Or one can use the a priori bound from Lemma 3.2 below and restrict the argument of the generating function to small enough $|z|$. Note that in this argument, t is fixed.

Let $\varphi(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$. Then

$$\begin{aligned}\varphi(z) &= 1 + \beta z + \sum_{k=2}^{\infty} c_{k+1} z^k \\ &= \frac{\theta - \eta t}{1+\sigma t} z(\varphi(z) - 1) + \frac{t+\tau}{1+\sigma t} z^2\varphi^2(z) + \eta z \frac{t+\tau}{1+\sigma t}(\varphi^2(z) - 1) \\ &\quad + \sigma \frac{t+\tau}{1+\sigma t}\big(\varphi(z)(\varphi(z) - 1) - \beta z\big).\end{aligned}$$

This gives the quadratic equation (2.8) for $\varphi = \varphi_t$. ∎

The following technical lemma ensures that the series $\varphi(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$ converges for all small enough $|z|$.

LEMMA 3.2. *Suppose $\sigma\tau \neq 1$ and $\{c_k\}$ is a solution of the recursion* (3.4) *with $c_1 = 1$ for a fixed t. Then for every $p > 1$ there exists a constant M such that*

$$|c_k| \le \frac{M^{k-2}}{k^p} \quad \text{for all } k \ge 3. \tag{3.5}$$

Proof. We will prove the case $p = 2$ only, as this suffices to justify the convergence of the series.

Solving (3.4) for c_{k+1} (which also appears on the right hand side), we get

$$\begin{aligned}&\frac{|1-\sigma\tau|}{1+\sigma t}|c_{k+1}| \\ &\le \frac{|\theta - \eta t|}{1+\sigma t}|c_k| + \frac{t+\tau}{1+\sigma t}\Big(\sum_{j=1}^{k-1}|c_j c_{k-j}| + |\eta|\sum_{j=0}^{k-1}|c_{j+1}c_{k-j}| + \sigma\sum_{j=0}^{k-2}|c_{j+2}c_{k-j}|\Big).\end{aligned}$$

Since we are not going to keep track of the constants, we simplify this as

$$|c_{n+1}| \le A|c_n| + B\Big(\sum_{k=1}^{n-1}|c_k c_{n-k}| + \sum_{k=0}^{n-1}|c_{k+1}c_{n-k}| + \sum_{k=0}^{n-2}|c_{k+2}c_{n-k}|\Big) \tag{3.6}$$

with $A = \frac{|\theta-\eta t|}{|1-\sigma\tau|}$ and $B = \frac{t+\tau}{|1-\sigma\tau|}(1 + |\eta| + \sigma)$. Here, $n \ge 2$.

We now choose $M \ge 1$ large enough so that (3.5) holds for $k = 3, 4, 5, 6$, and we also require that

$$4A + (24|c_2| + 175)B \le M. \tag{3.7}$$

We now proceed by induction and assume that (3.5) holds for all indices k between 3 and n for some $n \ge 6$.

To complete the induction step we provide bounds for the sums on the right hand side of (3.6). The first sum is handled as follows:

$$\sum_{k=1}^{n-1} |c_k c_{n-k}| = 2|c_1 c_{n-1}| + 2|c_2 c_{n-2}| + \sum_{k=3}^{n-3} |c_k c_{n-k}|.$$

We apply the induction bound (3.5) to $c_3, \ldots, c_{n-1}$. Noting that $c_1 = 1$ we get

$$\begin{aligned}
\sum_{k=1}^{n-1} |c_k c_{n-k}| &\le 2\frac{M^{n-3}}{(n-1)^2} + 2|c_2|\frac{M^{n-4}}{(n-2)^2} + M^{n-4}\sum_{k=3}^{n-3} \frac{1}{k^2(n-k)^2} \\
&\le 8\frac{M^{n-2}}{(n+1)^2} + 8|c_2|\frac{M^{n-2}}{(n+1)^2} + M^{n-2}\sum_{k=3}^{[n/2]} \frac{1}{k^2(n-k)^2} \\
&\quad + M^{n-2}\sum_{k=[n/2]+1}^{n-3} \frac{1}{k^2(n-k)^2} \\
&\le 8\frac{M^{n-2}}{(n+1)^2} + 8|c_2|\frac{M^{n-2}}{(n+1)^2} + \frac{4M^{n-2}}{n^2}\sum_{k=3}^{[n/2]} \frac{1}{k^2} \\
&\quad + \frac{4M^{n-2}}{n^2}\sum_{k=[n/2]+1}^{n-2} \frac{1}{(n-k)^2} \\
&< \frac{M^{n-2}}{(n+1)^2}(8 + 8|c_2| + 16\pi^2/3) < (61 + 8|c_2|)\frac{M^{n-2}}{(n+1)^2}.
\end{aligned}$$

Here we have used several times the bound $(n+1)/(n-2) \le 2$ for $n \ge 6$, the inequality $M \ge 1$, and we have estimated two finite sums by the infinite series $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$.

We proceed similarly with the second sum:

$$\begin{aligned}
\sum_{k=0}^{n-1} |c_{k+1} c_{n-k}| &= 2|c_1 c_n| + 2|c_2 c_{n-1}| + \sum_{k=2}^{n-3} |c_{k+1} c_{n-k}| \\
&\le 2\frac{M^{n-2}}{n^2} + 2|c_2|\frac{M^{n-3}}{(n-1)^2} + \frac{4M^{n-3}}{n^2}\sum_{k=1}^{[n/2]} \frac{1}{k^2} \\
&\quad + \frac{4M^{n-3}}{n^2}\sum_{k=[n/2]+1}^{n-3} \frac{1}{(n-k)^2} \\
&\le \frac{M^{n-2}}{(n+1)^2}(8 + 8|c_2| + 16\pi^2/3) \le \frac{M^{n-2}}{(n+1)^2}(61 + 8|c_2|).
\end{aligned}$$

Finally, we bound the third sum on the right hand side of (3.6):

$$\begin{aligned}\sum_{k=0}^{n-2}|c_{k+2}c_{n-k}| &= 2|c_2c_n| + \sum_{k=1}^{n-3}|c_{k+2}c_{n-k}| \\ &\le 2|c_2|\frac{M^{n-2}}{n^2} + \frac{4M^{n-2}}{n^2}\sum_{k=1}^{[n/2]}\frac{1}{k^2} + \frac{4M^{n-2}}{n^2}\sum_{k=[n/2]+1}^{n-3}\frac{1}{(n-k)^2} \\ &\le \frac{M^{n-2}}{(n+1)^2}(8|c_2| + 16\pi^2/3) \le \frac{M^{n-2}}{(n+1)^2}(8|c_2| + 53).\end{aligned}$$

Combining these bounds and applying the induction assumption to the first term on the right hand side of (3.6) we get

$$|c_{n+1}| \le \frac{M^{n-2}}{(n+1)^2}\big(4A + (24|c_2| + 175)B\big).$$

In view of (3.7), this shows that $|c_{n+1}| \le M^{n-1}/(n+1)^2$, thus completing the proof of (3.5) by induction. ∎

3.2. Part II: solution of (2.1). We now use (3.2) to determine $\mathbb{A}_t$.

Since $\mathbb{E} - \mathbb{F}\mathbb{D} = (1, 0, 0, \dots)$, from Remark 1.6 it follows that $\mathbb{A}_t\mathbb{F}\mathbb{D} = \mathbb{A}_t$. Therefore, multiplying (2.1) by $\mathbb{D}$ from the right we get

$$\mathbb{A}_t = \mathbb{F}\mathbb{A}_t\mathbb{D} + \mathbb{H}_t\mathbb{D}.$$

Iterating this, we get

$$\mathbb{A}_t = \sum_{k=0}^{\infty}\mathbb{F}^k\mathbb{H}_t\mathbb{D}^{k+1}, \tag{3.8}$$

which is well defined as the series consists of finite sums elementwise. Since $\mathbb{H}_t$ is given by (3.2), we get

$$\begin{aligned}\mathbb{A}_t &= \frac{1}{1+\sigma t}\sum_{k=0}^{\infty}\mathbb{F}^k(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\varphi_t(\mathbb{D})\mathbb{D}\mathbb{D}^{k+1} \\ &= \frac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\sum_{k=0}^{\infty}\mathbb{F}^k\mathbb{D}^{k+1}\varphi_t(\mathbb{D})\mathbb{D}.\end{aligned} \tag{3.9}$$

We now note that

$$\sum_{k=0}^{\infty}\mathbb{F}^k\mathbb{D}^{k+1} = \mathbb{D}_1. \tag{3.10}$$

(This can be seen either by examining each element of the sequence, or by solving the equation $\mathbb{D}_1\mathbb{F} - \mathbb{F}\mathbb{D}_1 = \mathbb{E}$, which is just a product formula for the derivative, by the previous technique. The latter equation is of course of the same form as (2.1).)

Replacing the series in (3.9) by the right hand side of (3.10) we get (2.7). This ends the proof of Theorem 2.5.

4. Integral representation. In this section it is more convenient to use linear operators on $\mathcal{P}$ instead of sequences of polynomials. We assume that the polynomial process $\{\mathsf{P}_{s,t} : 0 \leq s \leq t\}$ corresponds to a quadratic harness from Definition 1.4, and as before we use the parameters η, θ, σ, τ and γ to describe the quadratic harness. Infinitesimal generators of several quadratic harnesses, all different from those in Theorem 2.5, have been studied in this language by several authors.

For quadratic harnesses with parameters $\eta = \sigma = 0$ and $\gamma = q \in (-1, 1)$, according to [13], the infinitesimal generator A_t acting on a polynomial f is

$$\mathsf{A}_t(f)(x) = \int_{\mathbb{R}} \frac{\partial}{\partial x}\left(\frac{f(y) - f(x)}{y - x}\right) \nu_{x,t}(dy), \tag{4.1}$$

where $\nu_{x,t}(dy)$ is a uniquely determined probability measure. By inspecting the recurrences for the orthogonal polynomials $\{Q_n\}$ and $\{W_n\}$, from [13, Theorem 1.1(ii)] one can read off that for $q^2 t \geq (1+q)\tau$ the probability measure $\nu_{x,t}$ can be expressed in terms of the transition probabilities $P_{s,t}(x, dy)$ of the Markov process by the formula $\nu_{x,t}(dy) = P_{tq^2-(1+q)\tau,t}(\theta + qx, dy)$. In this form, the formula coincides with that of Anshelevich [1, Corollary 22] who considered the case $\eta = \theta = \tau = \sigma = 0$ and $\gamma = q \in [-1, 1]$. (However, the domain of the generator in [1] is much larger than the polynomials.) Earlier results in [5, p. 392], [6, Example 4.9], and [7] dealt with infinitesimal generators for quadratic harnesses such that $\sigma = \eta = \gamma = 0$.

The following result gives an explicit formula for the infinitesimal generator of the evolution corresponding to the "free quadratic harness" in the integral form similar to (4.1). The main new feature is the presence of an extra quadratic factor in front of the integral in expression (4.3) below for the infinitesimal generator. Denote

$$\alpha = \frac{\eta + \theta\sigma}{1 - \sigma\tau}, \qquad \beta = \frac{\eta\tau + \theta}{1 - \sigma\tau}. \tag{4.2}$$

THEOREM 4.1. *Fix $\sigma, \tau \geq 0$ such that $\sigma\tau < 1$ and $\eta, \theta \in \mathbb{R}$ such that $1 + \alpha\beta > 0$. Let $\gamma = -\sigma\tau$ and let φ_t be a continuous solution of* (2.8) *with $\varphi_t(0) = 1$. Then φ_t is a moment generating function of a unique probability measure ν_t, and the generator of the quadratic harness with the above parameters acts on $p \in \mathcal{P}$ as*

$$\mathsf{A}_t(p)(x) = \frac{1 + \eta x + \sigma x^2}{1 + \sigma t} \int \frac{\partial}{\partial x}\left(\frac{p(y) - p(x)}{y - x}\right) \nu_t(dy), \qquad t > 0. \tag{4.3}$$

Proof. Since $\sigma\tau < 1$, the solution of the quadratic equation (2.8) with $\varphi_t(0) = 1$ is

$$(4.4)\qquad \varphi_t(z) = \frac{z(t\eta - \theta) + 2t\sigma + \sigma\tau + 1}{2(t+\tau)(z^2 + z\eta + \sigma)} - \frac{\sqrt{(z(t\eta - \theta) + 2t\sigma + \sigma\tau + 1)^2 - 4(z^2 + z\eta + \sigma)(1 + t\sigma)(t+\tau)}}{2(t+\tau)(z^2 + z\eta + \sigma)}.$$

(We omit the rewrite for the case $\sigma = \eta = 0$.)

We will identify ν_t through its Cauchy–Stieltjes transform

$$G_{\nu_t}(z) = \int \frac{1}{z - x}\, \nu_t(dx)$$

which in our case will be well defined for all real z large enough.

To this end we compute

$$(4.5)\qquad \varphi_t(1/z)/z = \frac{(1 + \sigma\tau + 2\sigma t)z + t\eta - \theta}{2(t+\tau)(\sigma z^2 + \eta z + 1)} - \frac{\sqrt{[(1 - \sigma\tau)z - (\alpha + \sigma\beta)t - \beta - \alpha\tau]^2 - 4(1 + \sigma t)(t+\tau)(1 + \alpha\beta)}}{2(t+\tau)(\sigma z^2 + \eta z + 1)}.$$

Under our assumptions, the above expression is well defined for large enough $z \in \mathbb{R}$.

Expression (4.5) coincides with the Cauchy–Stieltjes transform in [21, Proposition 2.3], with their parameters

$$c_{\mathrm{SY}} = \frac{1 - \sigma\tau}{1 + \sigma t}, \quad \alpha_{\mathrm{SY}} = \frac{\eta\tau + \theta}{1 - \sigma\tau}, \quad a_{\mathrm{SY}} = \frac{2\eta\tau + \theta\sigma\tau + \theta + t(\eta\sigma\tau + \eta + 2\theta\sigma)}{(\sigma\tau - 1)^2}$$

and

$$b_{\mathrm{SY}} = \frac{(\sigma t + 1)(t+\tau)\big(\eta^2\tau + \eta\theta(\sigma\tau + 1) + \theta^2\sigma + (1 - \sigma\tau)^2\big)}{(1 - \sigma\tau)^3}.$$

(We added subscript "SY" to avoid confusion with our use of α in (4.2).)

This shows that $\varphi_t(1/z)/z$ is the Cauchy–Stieltjes transform of a unique compactly supported probability measure ν_t. For a more detailed description of ν_t and explicit formulas for its discrete and absolutely continuous components we refer to [21, Theorem 2.1]; see also Remark 4.2 below.

It is well known that a Cauchy–Stieltjes transform is an analytic function in the upper complex plane, determines the measure uniquely, and if it extends to real z with $|z|$ large enough then the corresponding moment generating function is well defined for all $|z|$ small enough and is given by $G_{\nu_t}(1/z)/z = \varphi_t(z)$. This shows that $\varphi_t(z)$ is the moment generating function of the probability measure ν_t.

Next we observe that (3.3) in operator notation reads

$$\mathsf{H}_t(x^n) = \frac{1+\eta x+\sigma x^2}{1+\sigma t}\sum_{k=1}^{n} c_k(t)x^{n-k}.$$

Writing $c_k(t) = \int y^{k-1}\,\nu_t(dy)$, we therefore get

$$\mathsf{H}_t(f)(x) = \frac{1+\eta x+\sigma x^2}{1+\sigma t}\int \frac{f(y)-f(x)}{y-x}\,\nu_t(dy). \tag{4.6}$$

Since the operator version of relation (2.1) is $\mathsf{A}_t(x^{n+1}) = \mathsf{H}_t(x^n) + x\mathsf{A}(x^n)$, we derive (4.3) from (4.6) by induction on n; for a similar reasoning see [13, Lemma 2.4]. ∎

Remark 4.2. Denote by $\pi_{t,\eta,\theta,\sigma,\tau}(dx)$ the univariate law of X_t for the free quadratic harness (X_t) with parameters η, θ, σ, τ as in [9, Section 3]. Then ν_t is given by

$$\nu_t(dx) = \frac{1}{t(t+\tau)}(t^2+\theta tx+\tau x^2)\pi_{t,\eta,\theta,\sigma,\tau}(dx)\,. \tag{4.7}$$

We read off this answer from [9, (3.4)] using the following elementary relation between Cauchy–Stieltjes transforms:

If $\nu(dx) = (ax^2+bx+c)\pi(dx)$ and $m = \int x\,\pi(dx)$ then the Cauchy–Stieltjes transforms of π and ν are related by the formula

$$G_\nu(z) = (az^2+bz+c)G_\pi(z) - am - az - b. \tag{4.8}$$

In our setting, $m = 0$, $a = \frac{\tau}{t(t+\tau)}$, $b = \theta/(t+\tau)$, $c = t/(t+\tau)$, and [9, (3.4)] gives

$$\begin{aligned} G_\pi(z) = {} & \frac{\tau z+\theta t}{\tau z^2+\theta tz+t^2} + \frac{t[(1+\sigma\tau+2\sigma t)z+t\eta-\theta]}{2(\sigma z^2+\eta z+1)(\tau z^2+\theta tz+t^2)} \\ & - \frac{t\sqrt{[(1-\sigma\tau)z-(\alpha+\sigma\beta)t-\beta-\alpha\tau]^2-4(1+\sigma t)(t+\tau)(1+\alpha\beta)}}{2(\sigma z^2+\eta z+1)(\tau z^2+\theta tz+t^2)}. \end{aligned} \tag{4.9}$$

Inserting this expression into the right hand side of (4.8) we get the right hand side of (4.5). Uniqueness of Cauchy–Stieltjes transform implies (4.7).

5. Some other cases. Several other cases can be worked out by a similar technique based on (2.2). Recasting [18, Section IV] in our notation, for $m = 0, 1, \ldots$ we have

$$\mathbb{D}_1^{m+1}\mathbb{F} - \mathbb{F}\mathbb{D}_1^{m+1} = (m+1)\mathbb{D}_1^m, \tag{5.1}$$

and if

$$\mathbb{H} = \sum_{k=1}^{\infty}\frac{c_k}{k!}\,\mathbb{D}_1^k$$

then

$$\mathbb{A} = \sum_{m=1}^{\infty} \frac{c_m}{(m+1)!} \mathbb{D}_1^{m+1}. \tag{5.2}$$

The generator for a Lévy process (ξ_t) with exponential moments acts on polynomials in x via $\kappa(\partial/\partial x)$, where $t\kappa(\theta) = \log \mathrm{E}(\exp(\theta\xi_t))$; this well-known formula appears e.g. in [1, Section 3].

5.1. Centered Poisson process. For example, the quadratic harness with $\gamma = 1$ and $\eta = \sigma = \tau = 0$ which corresponds to the Poisson process can also be analyzed by the algebraic technique. Equation (2.4) takes the form

$$\mathbb{H}\mathbb{F} - \mathbb{F}\mathbb{H} = \mathbb{E} + \theta\mathbb{H}. \tag{5.3}$$

We get

$$\mathbb{H} = \frac{1}{\theta}(e^{\theta\mathbb{D}_1} - \mathbb{E}).$$

It follows from (5.2) that

$$\mathbb{A} = \frac{1}{\theta^2}(e^{\theta\mathbb{D}_1} - \mathbb{E} - \theta\mathbb{D}_1).$$

5.2. A non-Lévy example. Here we consider a quadratic harness with parameters $\gamma = 1$ and $\tau = \sigma = 0$. This quadratic harness appeared under the name of the *quantum Bessel process* in [4] (see also [20] for a multidimensional version), and as the *classical bi-Poisson* process in [11].

Then equation (2.4) takes the form

$$\mathbb{H}\mathbb{F} - \mathbb{F}\mathbb{H} = \mathbb{E} + \theta\mathbb{H} + \eta(\mathbb{F} - t\mathbb{H}) = \mathbb{E} + \eta\mathbb{F} + (\theta - t\eta)\mathbb{H}. \tag{5.4}$$

For $\mathbb{H} = (\mathbb{E} + \eta\mathbb{F})\tilde{\mathbb{H}}$ we obtain

$$(\mathbb{E} + \eta\mathbb{F})(\tilde{\mathbb{H}}\mathbb{F} - \mathbb{F}\tilde{\mathbb{H}}) = (\mathbb{E} + \eta\mathbb{F})(\mathbb{E} + (\theta - t\eta)\tilde{\mathbb{H}}).$$

Comparing this with (5.3) we conclude that by uniqueness the solution of (5.4) is

$$\mathbb{H} = \frac{1}{\theta - t\eta}(\mathbb{E} + \eta\mathbb{F})(e^{(\theta - t\eta)\mathbb{D}_1} - \mathbb{E}).$$

From (3.8) it follows that

$$\mathbb{A} =: \mathbb{A}(\mathbb{H}) = (\mathbb{E} + \eta\mathbb{F})\, \mathbb{A}(\tilde{\mathbb{H}}).$$

Therefore we obtain

$$\mathbb{A} = \frac{1}{(\theta - t\eta)^2}(\mathbb{E} + \eta\mathbb{F})\big(e^{(\theta - t\eta)\mathbb{D}_1} - \mathbb{E} - (\theta - t\eta)\mathbb{D}_1\big).$$

Acknowledgements. Research of JW was supported in part by NCN grant 2012/05/B/ST1/00554. Research of WB was supported in part by the Taft Research Center at the University of Cincinnati.

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