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HOPF–GALOIS EXTENSIONS FOR MONOIDAL HOM-HOPF ALGEBRAS

BY

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Abstract. Hopf–Galois extensions for monoidal Hom-Hopf algebras are investigated. As the main result, Schneider's affineness theorem in the case of monoidal Hom-Hopf algebras is shown in terms of total integrals and Hopf–Galois extensions. In addition, we obtain an affineness criterion for relative Hom-Hopf modules which is associated with faithfully flat Hopf–Galois extensions of monoidal Hom-Hopf algebras.

1. Introduction. The study of nonassociative algebras was originally motivated by certain problems in physics and in other branches of mathematics. Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov [17, 19, 20] as part of a study of deformations of Witt algebras and Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, resulting in the Hom-Jacobi identity

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,$$

where α is a Lie algebra endomorphism. Because of the close relation to discrete and deformed vector fields and differential calculus, Hom-Lie algebras have been widely studied recently (see [1, 2, 5, 6, 18, 24, 26, 29, 31]).

Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by Makhlouf and Silvestrov [22]. Hom-associative algebras and related structures have recently become rather popular, due to the prospect of having a general framework in which one can produce many types of natural deformations of algebras, including Hom-coassociative coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infini-

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tesimal Hom-bialgebras, Hom-power associative algebras, quasi-triangular Hom-bialgebras (see [3, 7, 8, 9, 11, 12, 16, 21, 24, 30, 32]), and so on.

Makhlouf and Silvestrov further investigated Hom-associative algebras and Hom-coassociative coalgebras in [23, 24]. Here the associativity of algebras and the coassociativity of coalgebras were twisted by endomorphisms. Hom-bialgebras are both Hom-associative algebras and Hom-coassociative coalgebras such that the comultiplication and counit are morphisms of algebras. These objects are slightly different from the ones studied in this paper (see Section 1).

The theory of Hopf–Galois extensions, which has its roots in the Galois theory for groups acting on commutative rings, plays an important role in the theory of Hopf algebras. There are two important applications of Hopf–Galois extensions: Kreimer–Takeuchi type theorems and Schneider's affineness theorems.

In fact, this paper is a follow-up of the Ph.D. thesis [4] of the first named author under the supervision of the second author. The fundamental theorem and Maschke's theorem were considered in the category of relative Hom-Hopf modules in [10]. The main purpose of this paper is to study the theory of Hopf–Galois extensions for monoidal Hom-Hopf algebras and the affineness criterion for relative Hom-Hopf modules. The paper is organized as follows. In Section 2, relative Hom-Hopf modules are introduced. In Section 3, we prove the affineness criterion for relative Hom-Hopf modules associated with faithfully flat Hopf–Galois extensions. In Section 4, we consider Schneider's affineness theorem in the case of monoidal Hom-Hopf algebras in terms of total integrals and Hom-Hopf Galois extensions.

2. Preliminaries. Throughout this paper, let k be a fixed field. Unless otherwise specified, linearity, modules and $\otimes$ are all meant over k. And we freely use the Hopf algebras terminology introduced in [14], [25], [27] and [28]. For a coalgebra C, we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$; for a right C-comodule M, we denote its coaction by $\rho : m \mapsto m_{(0)} \otimes m_{(1)}$, for any $m \in M$, where we omit the summation symbols for convenience.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k-modules. There is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$ whose objects are couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in \mathsf{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_k$ such that $\nu f = f \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \quad \text{and} \quad (k, \mathsf{id}).$$

Briefly speaking, all Hom-objects are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathsf{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [3], where the associator

$\widetilde{a}$ is given by the formula

$$\widetilde{a}_{M,N,L} = a_{M,N,L}((\mu \otimes \mathsf{id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathsf{id} \otimes \varsigma^{-1}))a_{M,N,L} \tag{2.1}$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$, and the unitors $\widetilde{l}$ and $\widetilde{r}$ are

$$\widetilde{l}_M = \mu l_M = l_M(\mathsf{id} \otimes \mu), \quad \widetilde{r}_M = \mu r_M = r_M(\mu \otimes \mathsf{id}).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the *Hom-category* associated to the monoidal category $\mathcal{M}_k$. A k-submodule $N \subseteq M$ is called a *subobject* of (M, μ) if μ restricts to an automorphism of N, that is, $(N, \mu|_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Since the category $\mathcal{M}_k$ has left duality, so does the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Now let M^* be the left dual of $M \in \mathcal{M}_k$, and let $b_M : k \to M \otimes M^*$ and $d_M : M^* \otimes M \to k$ be the coevaluation and evaluation maps. Then the left dual of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ is $(M^*, (\mu^*)^{-1})$, and the coevaluation and evaluation maps are given by

$$\widetilde{b}_M = (\mu \otimes \mu^*)^{-1} b_M, \quad \widetilde{d}_M = d_M(\mu^* \otimes \mu).$$

We now recall from [3] some information about Hom-structures.

Definition 2.1. A *unital monoidal Hom-associative algebra* is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \to A, \, a \otimes b \mapsto ab, \quad \alpha \in \mathsf{Aut}(A)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \qquad a1_A = 1_A a = \alpha(a), \tag{2.2}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \tag{2.3}$$

for all $a, b, c \in A$.

Note that the first part of (2.2) can be rewritten as $a(b\alpha^{-1}(c)) = (\alpha^{-1}(a)b)c$. In the language of Hopf algebras, m is called *Hom-multiplication*, α is the *twisting automorphism* and 1_A is the *unit*. Henceforth, the terminology in Definition 1.1 will in general be slightly abused for simplicity by dropping the words "unital" and "Hom-associative". And we denote the monoidal Hom-algebra by (A, α).

Our definition of monoidal Hom-algebras is different from those in [23, 24] in the following sense. The same twisted associativity condition (2.2) holds in both cases. However, the unitality condition in those papers is the usual untwisted one: $a1_A = 1_A a = a$ for any $a \in A$, and the twisting map α need not be monoidal (that is, (2.3) is not required).

Let (A, α) and (A', α') be two monoidal Hom-algebras. A *Hom-algebra map* $f : (A, \alpha) \to (A', \alpha')$ is a linear map such that $f\alpha = \alpha' f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Definition 2.2. A *counital monoidal Hom-coassociative coalgebra* is an object (C, γ) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \to$

$C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \to k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2, \tag{2.4}$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad \varepsilon(\gamma(c)) = \varepsilon(c), \tag{2.5}$$

for all $c \in C$.

Note that the first part of (2.4) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$. Analogously to monoidal Hom-algebras, "monoidal Hom-coalgebra" will be short for "counital monoidal Hom-coassociative coalgebra". The definition of monoidal Hom-coalgebras here is somewhat different from that of counital Hom-coassociative coalgebras in [23, 24]. Their coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of an automorphism, and Hom-comultiplication is not comultiplicative. The superiority of our definition is that our objects admit duality. Hence more results in Hopf algebras can be extended to the monoidal-Hom case.

Let (C, γ) and (C', γ') be two monoidal Hom-coalgebras. A *Hom-coalgebra map* $f : (C, \gamma) \to (C', \gamma')$ is a linear map such that $f\gamma = \gamma' f$, $\Delta f = (f \otimes f)\Delta$ and $\varepsilon' f = \varepsilon$.

Definition 2.3. A *monoidal Hom-bialgebra* $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra, and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1_k.$$

For any bialgebra $(H, m, \eta, \Delta, \varepsilon)$, and any bialgebra endomorphism α of H, the authors of [23] showed that $(H, \alpha, \alpha m, \eta, \Delta\alpha, \varepsilon)$ is a Hom-bialgebra in their terminology. In our case, there is a monoidal Hom-bialgebra $(H, \alpha, \alpha m, \eta, \Delta\alpha^{-1}, \varepsilon)$, provided that $\alpha : H \to H$ is a bialgebra automorphism.

Definition 2.4. A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called *antipode*) $S : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S\alpha = \alpha S$) such that for any $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1 S(h_2). \tag{2.6}$$

In fact, a monoidal Hom-Hopf algebra is a Hopf algebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Further, the antipodes of monoidal Hom-Hopf algebras have similar properties to those of Hopf algebras: they are morphisms of Hom-anti-(co)algebras. Since α is bijective and commutes with the antipode S, we have $S\alpha^{-1} = \alpha^{-1}S$. For a finite-dimensional monoidal Hom-Hopf algebra $(H, \alpha, m, \eta, \Delta, \varepsilon, S)$, the dual $(H^*, (\alpha^*)^{-1})$ is also a monoidal Hom-Hopf algebra with the following structure maps: for all $g, h \in H$ and $h^*, g^* \in H^*$,

$$\langle h^*g^*, h\rangle = \langle h^*, h_1\rangle\langle g^*, h_2\rangle, \quad 1_{H^*} = \varepsilon,$$
$$\langle \Delta(h^*), g \otimes h\rangle = \langle h^*, gh\rangle, \quad \varepsilon_{H^*} = \eta,$$
$$(\alpha^*)^{-1}(h^*) = h^*\alpha^{-1}, \quad S^*(h^*) = h^*S^{-1}.$$

Now we recall the actions and coactions over monoidal Hom-algebras and monoidal Hom-coalgebras respectively.

Definition 2.5. Let (A, α) be a monoidal Hom-algebra. A *right* (A, α)-*Hom-module* consists of (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \otimes a) = m \cdot a$, such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \quad m \cdot 1_A = \mu(m),$$
$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a),$$

for all $a, b \in A$ and $m \in M$.

Similarly, we can define left (A, α)-Hom-modules. A monoidal Hom-algebra (A, α) can be considered as a Hom-module over itself by Hom-multiplication. Let $(M, \mu), (N, \nu)$ be two left (A, α)-Hom-modules. A morphism $f : M \to N$ is called *left* (A, α)-*linear* (or a *left* (A, α)-*Hom-module map*) if $f(a \cdot m) = a \cdot f(m)$ for any $a \in A$ and $m \in M$, and $f\mu = \nu f$. We denote the category of left (A, α)-Hom-modules by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}({}_H\mathcal{M})$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}({}_H\mathcal{M})$ via the left (H, α)-action

$$h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \tag{2.7}$$

where (H, α) is a monoidal Hom-bialgebra.

Definition 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A *right* (C, γ)-*Hom-comodule* is an object (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that

$$\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = m_{(0)(0)} \otimes (m_{(0)(1)} \otimes \gamma^{-1}(m_{(1)})),$$
$$m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m), \quad \rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}),$$

for all $m \in M$.

(C, γ) is a Hom-comodule over itself via Hom-comultiplication. Let $(M, \mu), (N, \nu)$ be two right (C, γ)-Hom-comodules. A morphism $g : M \to N$ is called *right* (C, γ)-*colinear* (or a *right* (C, γ)-*Hom-comodule map*) if $g\mu = \nu g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$ for any $m \in M$. The category of right (C, γ)-Hom-comodules is denoted by $\widetilde{\mathcal{H}}(\mathcal{M}^C)$. We also denote the set of morphisms in $\widetilde{\mathcal{H}}(\mathcal{M}^H)$ from M to N by $\widetilde{\mathcal{H}}(\mathrm{Com}_H(M, N))$. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^H)$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^H)$ with Hom-comodule structure

$$\rho(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}. \tag{2.8}$$

In the following, we introduce the invariants and coinvariants on Hom-modules and Hom-comodules respectively.

Definition 2.7. Let (H,α) be a monoidal Hom-Hopf algebra.

(1) If (M,μ) is a left (H,α)-Hom-module, then the *invariant of* (H,α) *on* (M,μ) is the set

$$M^H = \{m \in M \mid h \cdot m = \varepsilon(h)\mu(m)\}.$$

(2) If (N,ν) is a right (H,α)-Hom-comodule with comodule structure ρ, then the *coinvariant of* (H,α) *on* (N,ν) is the set

$$N^{\mathrm{co}\,H} = \{n \in N \mid \rho(n) = \nu^{-1}(n) \otimes 1_H\}.$$

If H is finite-dimensional, then a right (H,α)-Hom-comodule (N,ν) can be considered as a left $(H^*,(\alpha^*)^{-1})$-Hom-module with the action $h^* \cdot n = \langle h^*, n_{(1)}\rangle \nu^2(n_{(0)})$. Then we have

$$\begin{aligned} N^{\mathrm{co}\,H} &= \{n \in N \mid \rho(n) = \nu^{-1}(n) \otimes 1_H\} \\ &= \{n \in N \mid h^* \cdot n = \langle h^*, 1_H\rangle \nu(n)\} = N^{H^*}. \end{aligned} \tag{2.9}$$

Definition 2.8. Let (H,α) be a monoidal Hom-Hopf algebra. A *right* (H,α)*-Hom-Hopf module* (M,μ) is defined as a right (H,α)-Hom-module and a right (H,α)-Hom-comodule as well, obeying the following compatibility condition:

$$\rho(m \cdot h) = m_{(0)} \cdot h_1 \otimes m_{(1)} h_2 \tag{2.10}$$

for all $m \in M$ and $h \in H$.

Morphisms of right (H,α)-Hom-Hopf modules are both right (H,α)-linear and right (H,α)-colinear. We denote by $\widetilde{\mathcal{H}}(\mathcal{M}_H^H)$ the category of right (H,α)-Hom-Hopf modules.

If (M,μ) is a right (H,α)-Hom-Hopf module, then so is $(M^{\mathrm{co}\,H} \otimes H, \mu|_{M^{\mathrm{co}\,H}} \otimes \alpha)$, with the following action and coaction:

$$(m \otimes h) \cdot g = \mu(m) \otimes hg, \qquad \rho(m \otimes h) = (\mu^{-1}(m) \otimes h_1) \otimes h_2,$$

for all $m \in M$ and $h, g \in H$.

3. Relative Hom-Hopf modules. In this section, we study relative Hom-Hopf modules and the adjoint functors in the category of relative Hom-Hopf modules.

Definition 3.1. Let (H,α) be a monoidal Hom-Hopf algebra. A *right* (H,α)*-Hom-comodule algebra* is both a monoidal Hom-algebra and a right (H,α)-Hom-comodule (A,β) with the coaction $\rho_A : A \to A \otimes H$ such that

ρ_A is a morphism of Hom-algebras, that is, for any $a, b \in A$,

$$\begin{aligned} \rho_A(ab) &= \rho_A(a)\rho_A(b), \\ \rho_A(1_A) &= 1_A \otimes 1_H, \\ \rho_A\beta &= (\beta \otimes \alpha)\rho_A. \end{aligned} \tag{3.1}$$

We always assume (A, β) is a right (H, α)-Hom-comodule algebra.

Let $(H, m_H, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra and (A, m_A, ρ) a right H-comodule algebra. If $\alpha : H \to H$ is a Hopf algebra automorphism, then there is a monoidal Hom-Hopf algebra $H_\alpha = (H, m_\alpha = \alpha m_H, \eta, \ \Delta_\alpha = \Delta\alpha^{-1}, \varepsilon, S, \alpha)$ by [3, Proposition 1.14]. Let $\beta \in \mathsf{Aut}(A)$ be an algebra automorphism such that $\rho\beta = (\beta \otimes \alpha)\rho$. Then it is easy to show by direct computation that $A_\beta = (A, m_\beta = \beta m_A, \rho_\beta = \rho\beta^{-1}, \beta)$ is a right (H_α, α)-Hom-comodule algebra. And the compatibility condition (3.1) for ρ_β and m_β is just a consequence of the compatibility $\rho(ab) = \rho(a)\rho(b)$ in the comodule algebra (A, m_A, ρ).

Definition 3.2. Let (A, β, ρ_A) be a right (H, α)-Hom-comodule algebra. (M, μ) is called a *right* (H, A)*-Hom-Hopf module* if (M, μ) is both in $\widetilde{\mathcal{H}}(\mathcal{M}_A)$ and in $\widetilde{\mathcal{H}}(\mathcal{M}^H)$, and the following diagram commutes:

$$\begin{array}{ccccc}
M \otimes A & \xrightarrow{\psi_M} & M & \xrightarrow{\rho_M} & M \otimes H \\
\downarrow{\scriptstyle \rho_M \otimes \rho_A} & & & & \uparrow{\scriptstyle \psi_M \otimes m_H} \\
(M \otimes H) \otimes (A \otimes H) & & & & (M \otimes A) \otimes (H \otimes H) \\
\downarrow{\scriptstyle \widetilde{a}} & & & & \uparrow{\scriptstyle \widetilde{a}^{-1}} \\
M \otimes (H \otimes (A \otimes H)) & & & & M \otimes (A \otimes (H \otimes H)) \\
\downarrow{\scriptstyle \mathsf{id} \otimes \widetilde{a}^{-1}} & & & & \uparrow{\scriptstyle \mathsf{id} \otimes \widetilde{a}} \\
M \otimes ((H \otimes A) \otimes H) & & \xrightarrow{\mathsf{id} \otimes (\tau \otimes \mathsf{id})} & & M \otimes ((A \otimes H) \otimes H)
\end{array}$$

where ψ_M is the right (A, β)-Hom-module structure on (M, μ), ρ_M is the right (H, α)-Hom-comodule structure on (M, μ), m_H is the multiplication of H, and τ is the flip map.

The diagram expresses the compatibility condition for (H, A)-Hom-Hopf module, which can be rewritten as

$$\rho_M(m \cdot a) = m_{(0)} \cdot a_{(0)} \otimes m_{(1)} a_{(1)} \tag{3.2}$$

for all $m \in M$ and $a \in A$. A morphism of right (H, A)-Hom-Hopf modules is both a right (A, β)-Hom-module map and a right (H, α)-Hom-comodule map. We denote by $\widetilde{\mathcal{H}}(M_A^H)$ the category of right (H, A)-Hom-Hopf modules. Similarly, we can define the category $\widetilde{\mathcal{H}}({}_A M^H)$ of left-right (H, A)-Hom-Hopf modules.

In fact, the right (H,α)-Hom-comodule algebra (A, m_A, ρ_A, β) is a right (H,A)-Hom-Hopf module over itself via the Hom-comodule structure ρ_A and the Hom-multiplication m_A, since the compatibility condition of (H,A)-Hom-Hopf modules is just the equality (3.1).

Example 3.3. (1) We can induce a relative Hom-Hopf module from a relative Hopf module (M,ψ,ρ), which is similar to inducing a Hom-comodule algebra from a comodule algebra. We just need to twist the action ψ and coaction ρ into $\psi_\mu = \mu\psi$ and $\rho_\mu = \rho\mu^{-1}$ respectively, where $\mu : M \to M$ is an automorphism such that $\mu\psi = \psi(\mu\otimes\beta)$ and $\rho\mu = (\mu\otimes\alpha)\rho$.

(2) Let (A,β) be a right (H,α)-Hom-comodule algebra, and (M,μ) be a right (A,β)-Hom-module. Then $(M\otimes H, \mu\otimes\alpha)$ is a right (H,A)-Hom-Hopf module, with the right (A,β)-Hom-module structure $\psi : (M\otimes H)\otimes A \to M\otimes H$, $(m\otimes h)\otimes a \mapsto (m\otimes h)\cdot a = m\cdot a_{(0)} \otimes ha_{(1)}$, and the right (H,α)-Hom-comodule structure $\rho : M\otimes H \to (M\otimes H)\otimes H$, $m\otimes h \mapsto (\mu^{-1}(m)\otimes h_1)\otimes\alpha(h_2)$. Here we just check the compatibility condition (3.2): for any $m\in M$, $h\in H$ and $a\in A$,

$$\begin{aligned}(m\otimes h)_{(0)}\cdot a_{(0)} \otimes (m\otimes h)_{(1)}a_{(1)} &= (\mu^{-1}(m)\otimes h_1)\cdot a_{(0)} \otimes \alpha(h_2)a_{(1)}\\ &= (\mu^{-1}(m)\cdot a_{(0)(0)} \otimes h_1 a_{(0)(1)})\otimes\alpha(h_2)a_{(1)}\\ &= (\mu^{-1}(m)\cdot\beta^{-1}(a_{(0)})\otimes h_1 a_{(1)1})\otimes\alpha(h_2)\alpha(a_{(1)2})\\ &= (\mu^{-1}(m\cdot a_{(0)})\otimes h_1 a_{(1)1})\otimes\alpha(h_2 a_{(1)2})\\ &= \rho(m\cdot a_{(0)}\otimes ha_{(1)}) = \rho((m\otimes h)\cdot a).\end{aligned}$$

In particular, $(A\otimes H, \beta\otimes\alpha)\in\widetilde{\mathcal{H}}(M_A^H)$.

Let (M,μ) be a right (H,α)-Hom-module and (N,ν) a left (H,α)-Hom-module. The *tensor product over* (H,α) of (M,μ) and (N,ν) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is defined as

$$(M\otimes_H N, \mu\otimes\nu) = \{m\otimes n \in M\otimes N \mid m\cdot h\otimes\nu(n) = \mu(m)\otimes h\cdot n\}.$$

And dually, let (M,μ) be a right (H,α)-Hom-comodule and (N,ν) a left (H,α)-Hom-comodule. The *co-tensor product space* $(M\,\square_H\, N, \mu\otimes\nu)$ in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is defined as the following set:

$$\{m\otimes n\in M\otimes N \mid (m_{(0)}\otimes m_{(1)})\otimes\nu^{-1}(n) = (\mu^{-1}(m)\otimes n_{(-1)})\otimes n_{(0)}\}.$$

Let (A,β,ρ_A) be a right (H,α)-Hom-comodule algebra. We denote $B = A^{\mathrm{co}\,H}$. If (N,ν) is a right (H,A)-Hom-Hopf module, then $(N^{\mathrm{co}\,H}, \nu|_{N^{\mathrm{co}\,H}})$ is a right $(B,\beta|_B)$-Hom-submodule of (N,ν). Obviously, $N\,\square_H\, k\cong N^{\mathrm{co}\,H}$, where (k,id) is a trivial (H,α)-Hom-comodule.

For any right (A,β)-Hom-module (M,μ), $(M\otimes_B A, \mu\otimes\alpha)$ is a right (H,A)-Hom-Hopf module with the action $(m\otimes a)\otimes b\mapsto \mu(m)\otimes a\beta^{-1}(b)$ and coaction $m\otimes a\mapsto(\mu^{-1}(m)\otimes a_{(0)})\otimes\alpha(a_{(1)})$. This defines the induction functor $F:\widetilde{\mathcal{H}}(\mathcal{M}_B)\to\widetilde{\mathcal{H}}(M_A^H)$, $M\mapsto M\otimes_B A$. In fact, F is a left adjoint to the functor of coinvariants $G:\widetilde{\mathcal{H}}(M_A^H)\to\widetilde{\mathcal{H}}(\mathcal{M}_B), N\mapsto N^{\mathrm{co}\,H}$ (see the following result).

Proposition 3.4. *(F,G) is a pair of adjoint functors with the unit*

$$\eta_{(M,\mu)}:M\to(M\otimes_B A)^{\mathrm{co}\,H},\qquad m\mapsto\mu^{-1}(m)\otimes 1_A,$$

and counit

$$\epsilon_{(N,\nu)}:N^{\mathrm{co}\,H}\otimes_B A\to N,\qquad n\otimes a\mapsto n\cdot a,$$

where $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_B)$ and $(N,\nu)\in\widetilde{\mathcal{H}}(M_A^H)$.

Proof. First $\eta_{(M,\mu)}$ and $\epsilon_{(N,\nu)}$ are well-defined. In fact, for any $m\in M$, obviously $\mu^{-1}(m)\otimes 1_A\in(M\otimes_B A)^{\mathrm{co}\,H}$, and $\epsilon_{(N,\nu)}(n\otimes ba)=n\cdot(ba)=\epsilon_{(N,\nu)}(\nu^{-1}(n)\cdot b\otimes\beta(a))$ for any $n\in N$ and $a,b\in A$. Hence we only need to check the triangular identity:

$$\epsilon_{F(M,\mu)}F\eta_{(M,\mu)}(m\otimes a)=(\mu^{-1}(m)\otimes 1_A)\cdot a=m\otimes a,$$
$$G\epsilon_{(N,\nu)}\eta_{G(N,\nu)}(n)=\nu^{-1}(n)\cdot 1_A=n.\ \blacksquare$$

In the same way, the induction functor $F:\widetilde{\mathcal{H}}({}_B\mathcal{M})\to\widetilde{\mathcal{H}}({}_AM^H)$, $M\mapsto A\otimes_B M$, is a left adjoint to $N\mapsto N^{\mathrm{co}\,H}$.

Similarly, for the left-right (H,A)-Hom-Hopf module category $\widetilde{\mathcal{H}}(M_A^H)$, there is another pair of adjoint functors

$$F'=A\otimes_B - :\widetilde{\mathcal{H}}({}_B\mathcal{M})\to\widetilde{\mathcal{H}}({}_AM^H),$$
$$G'=(-)^{\mathrm{co}\,H}:\widetilde{\mathcal{H}}({}_AM^H)\to\widetilde{\mathcal{H}}({}_B\mathcal{M}),$$

where $\widetilde{\mathcal{H}}({}_B\mathcal{M})$ is the category of left (B,β)-Hom-modules.

4. Hopf–Galois extensions. In this section, we give some affineness theorems, providing additional sufficient conditions for (F,G) and (F',G') to be pairs of inverse equivalences. We always assume that (H,α) is a monoidal Hom-Hopf algebra with antipode S, and (A,β) is a right (H,α)-Hom-comodule algebra.

Definition 4.1. An (H,α)-Hom-module (M,μ) is called *flat* over k if the tensor product preserves exact sequences in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, and *faithfully flat* if taking the tensor product with a sequence produces an exact sequence if and only if the original sequence is exact in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

For the concepts of flat and faithfully flat see [13]. Since $(A\otimes H,\beta\otimes\alpha)$ is in $\widetilde{\mathcal{H}}(M_A^H)$ and A is k-flat, we have

$$(A\otimes H)^{\mathrm{co}\,H}\cong A\otimes H^{\mathrm{co}\,H}\cong A\otimes k\cong A.$$

So the counit map in Proposition 3.4 is $\epsilon_{A\otimes H} : (A\otimes H)^{\mathrm{co}\,H}\otimes_B A \to A\otimes H$, which can be translated to the following map:

$$\mathsf{can} : A\otimes_B A \to A\otimes H.$$

We find easily that

$$\mathsf{can}(a\otimes b) = (\beta^{-1}(a)\otimes 1_H)\cdot b = \beta^{-1}(a)b_{(0)}\otimes\alpha(b_{(1)})$$

for all $a, b\in A$.

Similarly, $(A\otimes H, \beta\otimes\alpha)\in\widetilde{\mathcal{H}}({}_AM^H)$, and the corresponding adjunction map $\epsilon'_{A\otimes H}$ now defines another map

$$\mathsf{can}' : A\otimes_B A \to A\otimes H$$

given by

$$\mathsf{can}'(a\otimes b) = a\cdot(\beta^{-1}(b)\otimes 1_H) = a_{(0)}\beta^{-1}(b)\otimes\alpha(a_{(1)}).$$

PROPOSITION 4.2. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode S, and (A,β) a right (H,α)-Hom-comodule algebra. The map $f : A\otimes H\to A\otimes H$ given by*

$$a\otimes h\mapsto\beta(a_{(0)})\otimes a_{(1)}S\alpha^{-1}(h)$$

is an isomorphism. Furthermore, $\mathsf{can}' = f\circ\mathsf{can}$*, so* can *is an isomorphism if and only if* can' *is.*

Proof. For any $a\in A$ and $h\in H$, it is easy to check that the inverse of f is

$$f^{-1}(a\otimes h) = \beta(a_{(0)})\otimes S^{-1}\alpha^{-1}(h)a_{(1)},$$

by the Hom-coassociativity of the Hom-comodule algebra (A,β), the Hom-associativity of (H,α) and the property of the antipode; and for any $a, b\in A$,

$$f\circ\mathsf{can}(a\otimes b) = \mathsf{can}'(a\otimes b).\ \blacksquare$$

DEFINITION 4.3. Consider a right (H,α)-Hom-comodule algebra (A,β) and its coinvariance $(B,\beta|_B)$. Then (A,β) is called a *Hopf–Galois extension* of $(B,\beta|_B)$ if can or can' is an isomorphism.

If the functors (F,G) or (F',G') is a pair of inverse equivalence of categories, then clearly can and can' are isomorphisms.

Now we consider equivalent conditions for (F,G) to be a pair of inverse equivalences. Explicitly, (F,G) is a pair of inverse equivalences if and only if (A,β) is a faithfully flat Hopf–Galois extension of $(B,\beta|_B)$. This comes from the results of Doi and Takeuchi [15].

THEOREM 4.4. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode S, and (A,β) a right (H,α)-Hom-comodule algebra. Then the following are equivalent:*

(1) (A,β) *is faithfully flat as a left* $(B,\beta|_B)$*-Hom-module, and* (A,β) *is a Hopf–Galois extension of* $(B,\beta|_B)$*;*
(2) (F,G) *is a pair of inverse equivalences between* $\widetilde{\mathcal{H}}({}_B\mathcal{M})$ *and* $\widetilde{\mathcal{H}}(M_A^H)$*.*

Proof. (2)⇒(1). We have already seen that (A,β) is a Hopf–Galois extension of B. Let $(M,\mu)\to(M',\mu')$ be an injective map of right (B,β)-Hom-modules. The equivalence of $\widetilde{\mathcal{H}}({}_B\mathcal{M})$ and $\widetilde{\mathcal{H}}(M_A^H)$ implies that $M\otimes_B A\to M'\otimes_B A$ is monic in $\widetilde{\mathcal{H}}(M_A^H)$, and of course monic in $\widetilde{\mathcal{H}}(M_A)$. Thus (A,β) is left $(B,\beta|_B)$-flat. Faithful flatness also follows from the equivalence of $\widetilde{\mathcal{H}}({}_B\mathcal{M})$ and $\widetilde{\mathcal{H}}(M_A^H)$ in a similar way: assume that we have a sequence

$$0\to M'\to M\to M''\to 0$$

such that

$$0\to M'\otimes_B A\to M\otimes_B A\to M''\otimes_B A\to 0$$

is exact in $\widetilde{\mathcal{H}}(M_A)$. The three right (A,β)-Hom-modules have the structure of right (H,A)-Hom-Hopf modules, and the sequence is also exact in $\widetilde{\mathcal{H}}(M_A^H)$. It stays exact after applying G to it, by the equivalence of $\widetilde{\mathcal{H}}(M_A^H)$ and $\widetilde{\mathcal{H}}({}_B\mathcal{M})$. Thus the original sequence

$$0\to M'\to M\to M''\to 0$$

is also exact in $\widetilde{\mathcal{H}}({}_B\mathcal{M})$.

(1)⇒(2). For any $(N,\rho_N,\nu)\in\widetilde{\mathcal{H}}(M_A^H)$, we will prove that the counit ϵ_N is an isomorphism. If (X,ι) is a right (A,β)-Hom-module, then the map can_X is defined as the composition

$$X\otimes_B A\xrightarrow{\cong}X\otimes_A(A\otimes_B A)\xrightarrow{X\otimes\mathsf{can}}X\otimes_A(A\otimes H)\xrightarrow{\cong}X\otimes H,$$

given by

$$\mathsf{can}_X(x\otimes a)=xa_{(0)}\otimes\alpha(a_{(1)}).$$

Since can is an isomorphism, so is can_X. Now we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & N^{\mathrm{co}\,H}\otimes_B A & \longrightarrow & N\otimes_B A & \underset{(\nu^{-1}\otimes\eta_H)\otimes A}{\overset{\rho_N\otimes_B A}{\rightrightarrows}} & (N\otimes H)\otimes_B A\\
 & & \downarrow{\scriptstyle\epsilon_N} & & \downarrow{\scriptstyle\mathsf{can}_N} & & \downarrow{\scriptstyle\mathsf{can}_{N\otimes H}}\\
0 & \longrightarrow & N & \xrightarrow[\rho_N]{} & N\otimes H & \underset{\widetilde{a}^{-1}(N\otimes\Delta_H)}{\overset{\rho_N\otimes H}{\rightrightarrows}} & (N\otimes H)\otimes H
\end{array}$$

where $\eta_H:k\to H$ is the structure map of (H,α). The top row is exact, since $N^{\mathrm{co}\,H}$ is the equalizer of ρ_N and $\nu^{-1}\otimes\eta_H$, and A is flat as a left (B,β)-Hom-module. Meanwhile, the equalizer of $\rho_N\otimes H$ and $\widetilde{a}^{-1}(N\otimes\Delta_H)$ is $N\,\square_H\,H\cong N$. So the bottom row is also exact. Since can_N and $\mathsf{can}_{N\otimes H}$ are isomorphisms, so is ϵ_N by the five lemma.

In addition, the unit $\eta_M : M \to (M \otimes_B A)^{\mathrm{co}\, H}$ is also an isomorphism. Indeed, define $i_1, i_2 : M \otimes_B A \to M \otimes_B (A \otimes_B A)$ by

$$i_1(m \otimes a) = m \otimes (1_A \otimes \beta^{-1}(a)), \qquad i_2(m \otimes a) = m \otimes (\beta^{-1}(a) \otimes 1_A),$$

for all $m \in M$ and $a \in A$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \xrightarrow{M \otimes \eta_A} & M \otimes_B A & \overset{i_1}{\underset{i_2}{\rightrightarrows}} & M \otimes_B (A \otimes_B A) \\
 & & \Big\downarrow{\scriptstyle \eta_M} & & \Big\| & & \Big\downarrow{\scriptstyle M \otimes \mathsf{can}} \\
0 & \longrightarrow & (M \otimes_B A)^{\mathrm{co}\, H} & \xrightarrow[\subseteq]{} & M \otimes_B A & \overset{M \otimes \rho_A}{\underset{M \otimes (\beta^{-1} \otimes \eta_H)}{\rightrightarrows}} & M \otimes_B (A \otimes H)
\end{array}$$

since

$$(M \otimes \mathsf{can}) i_1(m \otimes a) = m \otimes (a_{(0)} \otimes a_{(1)}) = (M \otimes \rho_A)(m \otimes a)$$

and

$$(M \otimes \mathsf{can}) i_2(m \otimes a) = m \otimes (\beta^{-1}(a) \otimes 1_H) = \big(M \otimes (\beta^{-1} \otimes \eta_H)\big)(m \otimes a).$$

The top row is exact because A is faithfully flat as a left (B, β)-Hom-module. The bottom row is also exact by the definition of coinvariants. Since can is an isomorphism, so is the adjunction unit η_M, again by the five lemma. ∎

5. Total integrals. In this section we consider Schneider's affineness theorems under the assumption that there exists a total integral.

Definition 5.1. Let (A, β, ρ_A) be a right (H, α)-Hom-comodule algebra. A morphism $\varphi : (H, \alpha) \to (A, \beta)$ is called a *total integral* for (A, β) if φ is a right (H, α)-Hom-comodule map such that $\varphi(1_H) = 1_A$.

For the main result of this section we need some lemmas.

Lemma 5.2 (see [10, Theorem 2.5]). *Let (A, β) be a right (H, α)-Hom-comodule algebra. Then the following are equivalent:*

(1) *there is a total integral,*
(2) *(A, β) is an injective (H, α)-Hom-comodule,*
(3) *all right (H, A)-Hom-Hopf modules are injective as (H, α)-Hom-comodules,*
(4) *there is a right (H, α)-colinear map $\varphi : (H, \alpha) \to (A, \beta)$ with $\varphi(1_H)$ invertible in A.*

Let $(M, \mu) \in \widetilde{\mathcal{H}}({}^H M)$. Then $(M, \mu) \in \widetilde{\mathcal{H}}(M^H)$ via $m \mapsto m_{(0)} \otimes S(m_{(-1)})$, for any $m \in M$. Applying the induction functor $A \otimes - : \widetilde{\mathcal{H}}(M^H) \to \widetilde{\mathcal{H}}({}_A M^H)$

we find that $A \otimes M \in \widetilde{\mathcal{H}}({}_AM^H)$ with structure maps

$$\psi^l : a \otimes (b \otimes m) \mapsto (\beta^{-1}(a)b) \otimes \mu(m),$$
$$\rho^r : a \otimes m \mapsto (a_{(0)} \otimes m_{(0)}) \otimes a_{(1)}S(m_{(-1)}).$$

LEMMA 5.3. *With notation as above, we have*

$$(A \otimes M)^{\mathrm{co}\,H} = A \,\square_H\, M$$

for any $M \in \widetilde{\mathcal{H}}({}^HM)$.

Proof. This is straightforward. ∎

LEMMA 5.4. *If* $(N, \nu) \in \widetilde{\mathcal{H}}(M_A^H)$, *then we have well-defined maps*

$$i : N^{\mathrm{co}\,H} \to A \,\square_H\, N, \qquad n \mapsto 1_A \otimes \nu^{-1}(n),$$

and

$$p : A \,\square_H\, N \to N^{\mathrm{co}\,H}, \qquad a \otimes n \mapsto n \cdot a,$$

such that $pi = N^{\mathrm{co}\,H}$, *where the left* (H, α)*-Hom-comodule structure on* (N, ν) *is given by* $n \mapsto S(n_{(1)}) \otimes n_{(0)}$.

Proof. First, the counitality and Hom-coassociativity of (N, ν) imply that (N, ν) is also a left (H, α)-Hom-comodule via $n \mapsto S(n_{(1)}) \otimes n_{(0)}$.

Next, i is well-defined, since taking $n \in N^{\mathrm{co}\,H}$, we obviously have $i(n) = 1_A \otimes \nu^{-1}(n) \in A \,\square_H\, N$ by the left action on N and the definition of coinvariants.

Also, p is well-defined. Taking $a \otimes n \in A \,\square_H\, N$, we have

$$(a_{(0)} \otimes a_{(1)}) \otimes \nu^{-1}(n) = (\beta^{-1}(a) \otimes S(n_{(1)})) \otimes n_{(0)}.$$

Applying ρ_N to the last fact and using the Hom-coassociativity of (N, ν), we obtain

$$(a_{(0)} \otimes a_{(1)}) \otimes (n_{(0)} \otimes n_{(1)}) = (\beta^{-1}(a) \otimes S\alpha(n_{(1)2})) \otimes (n_{(0)} \otimes \alpha(n_{(1)1})). \tag{5.1}$$

Hence,

$$\begin{aligned}
\rho_N(n \cdot a) &= n_{(0)} \cdot a_{(0)} \otimes n_{(1)}a_{(1)} \\
&\overset{(5.1)}{=} n_{(0)} \cdot \beta^{-1}(a) \otimes \alpha(n_{(1)1})S\alpha(n_{(1)2}) \\
&= n_{(0)} \cdot \beta^{-1}(a) \otimes \varepsilon(n_{(1)})1_H \\
&= \nu^{-1}(n) \cdot \beta^{-1}(a) \otimes 1_H \\
&= \nu^{-1}(n \cdot a) \otimes 1_H.
\end{aligned}$$

That is, $p(a \otimes n) \in N^{\mathrm{co}\,H}$, as required.

Finally, $pi(n) = p(1_A \otimes \nu^{-1}(n)) = \nu^{-1}(n) \cdot 1_A = n$ for all $n \in N$. ∎

Proposition 5.5. *Let (A, β) be a right (H, α)-Hom-comodule algebra. Then the following are equivalent:*

(1) *(A, β) is right (H, α)-coflat,*
(2) *$G = (-)^{\mathrm{co}\,H} : \widetilde{\mathcal{H}}(M_A^H) \to \widetilde{\mathcal{H}}(M_B)$ is an exact functor,*
(3) *$G' = (-)^{\mathrm{co}\,H} : \widetilde{\mathcal{H}}({}_AM^H) \to \widetilde{\mathcal{H}}({}_BM)$ is an exact functor.*

Proof. (1)⇒(2). It is clear that G is left exact. Assume that $f : (N, \nu) \to (N', \nu')$ is surjective in $\widetilde{\mathcal{H}}(M_A^H)$. Then $A \,\square_H\, f$ is surjective because (A, β) is right (H, α)-coflat. Since f is a morphism in $\widetilde{\mathcal{H}}(M_k)$, there is a commutative diagram

$$\begin{array}{ccc} A \,\square_H\, N & \xrightarrow{A\square_H f} & A \,\square_H\, N' \\ {\scriptstyle i}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle i}\uparrow\downarrow{\scriptstyle p} \\ N^{\mathrm{co}\,H} & \xrightarrow{f} & N'^{\,\mathrm{co}\,H} \end{array}$$

which implies that $f : N^{\mathrm{co}\,H} \to N'^{\,\mathrm{co}\,H}$ is surjective, where p, i are the maps defined in Lemma 5.4.

(3)⇒(1). From Lemma 5.3, we know that $(A \otimes M)^{\mathrm{co}\,H} = A \,\square_H\, M$. Then $A \,\square_H\, (-)$ is the composition

$$\widetilde{\mathcal{H}}({}^HM) \xrightarrow{A\otimes(-)} \widetilde{\mathcal{H}}({}_AM^H) \xrightarrow{G'} \widetilde{\mathcal{H}}({}_BM).$$

$A \otimes (-)$ is exact since (A, β) is k-flat, and G' is also exact by assumption. It follows that $A \,\square_H\, (-)$ is exact. Hence (A, β) is right (H, α)-coflat.

(1)⇒(3). We can apply (1)⇒(2) to A^{op} as an H^{op}-Hom-comodule algebra. Therefore,

$$N \to N^{\mathrm{co}\,H}, \qquad N \in \widetilde{\mathcal{H}}({}_AM^H) = \widetilde{\mathcal{H}}(M_{A^{\mathrm{op}}}^{H^{\mathrm{op}}}),$$

is exact.

(2)⇒(1). Similar: apply (3)⇒(1) to A^{op}. ∎

Lemma 5.6. *Assume that (A, β) is a right (H, α)-Hom-comodule algebra, and $\varphi : (H, \alpha) \to (A, \beta)$ is a total integral. For any $(M, \mu) \in \widetilde{\mathcal{H}}(M_B)$, the adjunction unit $\eta_M : M \to (M \otimes_B A)^{\mathrm{co}\,H}$ is an isomorphism.*

Proof. Define a map in $\widetilde{\mathcal{H}}(M_k)$ as

$$t : (A, \beta) \to (B, \beta|_B), \qquad a \mapsto a_{(0)}\varphi(S(a_{(1)})).$$

It is easy to prove that t is well-defined. That is, $t(a) \in A^{\mathrm{co}\,H} = B$.

Now define

$$\phi_M : (M \otimes_B A)^{\mathrm{co}\,H} \to M, \qquad m \otimes a \mapsto m \cdot t(a).$$

Then ϕ_M is the inverse of η_M. ∎

Theorem 5.7. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode S, and (A, β) a right (H, α)-Hom-comodule algebra. If* can *is surjective, and if there exists a total integral $\varphi : (H,\alpha) \to (A,\beta)$, then the adjoint pair (F,G) is a pair of inverse equivalences between $\widetilde{\mathcal{H}}(M_A^H)$ and $\widetilde{\mathcal{H}}(M_B)$.*

Proof. In Lemma 5.6 we have shown that the adjunction unit is an isomorphism. We need to show that so is the counit ϵ_N for all $N \in \widetilde{\mathcal{H}}(M_A^H)$.

First, we prove this for $N = V \otimes A$, where (V,ν) is an arbitrary object in $\widetilde{\mathcal{H}}(M_k)$, and the (H,A)-Hom-Hopf module structure on $(V \otimes A, \nu \otimes \beta)$ is induced by the structure on (A,β), that is,

$$(v \otimes a) \cdot b = \nu(v) \otimes a\beta^{-1}(b), \qquad \rho_{V\otimes A}(v \otimes a) = (\nu^{-1}(v) \otimes a_{(0)}) \otimes \alpha(a_{(1)}),$$

for any $v \in V$ and $a, b \in A$. By Lemma 5.6,

$$(V \otimes A)^{\mathrm{co}\,H} \cong (V \otimes (B \otimes_B A))^{\mathrm{co}\,H} \cong ((V \otimes B) \otimes_B A)^{\mathrm{co}\,H} \cong V \otimes B.$$

Then we have a commutative diagram

$$\begin{array}{ccc} (V \otimes B) \otimes_B A & \xrightarrow{\ \cong\ } & V \otimes (B \otimes_B A) \\ \cong \downarrow & & \downarrow \cong \\ (V \otimes A)^{\mathrm{co}\,H} \otimes_B A & \xrightarrow{\ \epsilon_{V\otimes A}\ } & V \otimes A \end{array}$$

It follows that $\epsilon_{V\otimes A}$ is an isomorphism.

By Lemma 5.2, we know that the coaction $\rho_A : A \to A \otimes H$ has a section $\lambda_A : A \otimes H \to A$. And λ_A is a right (H,α)-Hom-comodule map with explicit form

$$\lambda_A(a \otimes h) = \beta(a_{(0)})\varphi\big(S(a_{(1)}\alpha^{-1}(h))\big)$$

for any $a \in A$ and $h \in H$.

It is not difficult to check that $N \otimes (A \otimes H) \in \widetilde{\mathcal{H}}(M_A^H)$ with the structure maps

$$(n \otimes (a \otimes h)) \cdot b = \nu^{-1}(n) \otimes \big(a\beta^{-1}(b_{(0)}) \otimes h\alpha^{-1}(b_{(1)})\big),$$
$$\rho_{N\otimes(A\otimes H)}(n \otimes (a \otimes h)) = \big(\nu^{-1}(n) \otimes (\beta^{-1}(a) \otimes h_1)\big) \otimes \alpha^2(h_2),$$

for all $a \in A$, $h \in H$ and $n \in N$. Define a map in $\widetilde{\mathcal{H}}(M_k)$ by

$$f : N \otimes (A \otimes H) \to N, \qquad n \otimes (a \otimes h) \mapsto \nu(n_{(0)}) \cdot \lambda_A\big(a \otimes S\alpha^{-1}(n_{(1)})\alpha^{-1}(h)\big).$$

First, we note that f is surjective, because for any $n \in N$,

$$\begin{aligned} f\big(n_{(0)} \otimes (1_A \otimes \alpha^{-1}(n_{(1)}))\big) &= \nu(n_{(0)(0)}) \cdot \lambda_A\big(1_A \otimes S\alpha^{-1}(n_{(0)(1)})\alpha^{-2}(n_{(1)})\big) \\ &= n_{(0)} \cdot \lambda_A\big(1_A \otimes S\alpha^{-1}(n_{(1)1})\alpha^{-1}(n_{(1)2})\big) \\ &= n_{(0)} \cdot \lambda_A(1_A \otimes \varepsilon(n_{(1)})1_H) = \nu^{-1}(n) \cdot 1_A = n. \end{aligned}$$

Next, f is right (H,α)-colinear by the following computation: for all $h \in H$,

$a \in A$ and $n \in N$,

$$
\begin{aligned}
\rho_A f(n\otimes(a\otimes h)) &= \rho_A(\nu(n_{(0)})\cdot\lambda_A(a\otimes S\alpha^{-1}(n_{(1)})\alpha^{-1}(h)))\\
&= \nu(n_{(0)(0)})\cdot\lambda_A(a\otimes S\alpha^{-1}(n_{(1)})\alpha^{-1}(h))_{(0)}\otimes\alpha(n_{(0)(1)})\lambda_A(a\otimes S\alpha^{-1}(n_{(1)})\alpha^{-1}(h))_{(1)}\\
&= \nu(n_{(0)(0)})\cdot\lambda_A\left(\beta^{-1}(a)\otimes(S\alpha^{-1}(n_{(1)})\alpha^{-1}(h))_1\right)\otimes\alpha(n_{(0)(1)})\alpha((S\alpha^{-1}(n_{(1)})\alpha^{-1}(h))_2)\\
&= \nu(n_{(0)(0)})\cdot\lambda_A\left(\beta^{-1}(a)\otimes S\alpha^{-1}(n_{(1)2})\alpha^{-1}(h_1)\right)\otimes\alpha(n_{(0)(1)})(S(n_{(1)1})h_2)\\
&= n_{(0)}\cdot\lambda_A\left(\beta^{-1}(a)\otimes S(n_{(1)22})\alpha^{-1}(h_1)\right)\otimes\alpha(n_{(1)1})(S\alpha(n_{(1)21})h_2)\\
&= n_{(0)}\cdot\lambda_A\left(\beta^{-1}(a)\otimes S\alpha^{-1}(n_{(1)2})\alpha^{-1}(h_1)\right)\otimes\alpha^2(n_{(1)11})(S\alpha(n_{(1)12})h_2)\\
&= n_{(0)}\cdot\lambda_A\left(\beta^{-1}(a)\otimes S\alpha^{-1}(n_{(1)2})\alpha^{-1}(h_1)\right)\otimes(\alpha(n_{(1)11})S\alpha(n_{(1)12}))\alpha(h_2)\\
&= n_{(0)}\cdot\lambda_A\left(\beta^{-1}(a)\otimes S\alpha^{-1}(n_{(1)2})\alpha^{-1}(h_1)\right)\otimes\varepsilon(n_{(1)1})1_H\alpha(h_2)\\
&= n_{(0)}\cdot\lambda_A\left(\beta^{-1}(a)\otimes S\alpha^{-2}(n_{(1)})\alpha^{-1}(h_1)\right)\otimes\alpha^2(h_2)\\
&= (f\otimes H)\left((\nu^{-1}(n)\otimes(\beta^{-1}(a)\otimes h_1))\otimes\alpha^2(h_2)\right)\\
&= (f\otimes H)\rho_{N\otimes(A\otimes H)}(n\otimes(a\otimes h)),
\end{aligned}
$$

where the second step follows by the compatibility (3.2), and the third step holds since λ_A is right (H,α)-colinear and the coaction on $(A\otimes H,\beta\otimes\alpha)$ is given by $\rho(a\otimes h)=(\beta^{-1}(a)\otimes h_1)\otimes\alpha(h_2)$. Finally, we conclude that f is a split epimorphism in $\widetilde{\mathcal{H}}(M^H)$.

Since H is projective as a k-module, $A\otimes H$ is projective as a left (A,β)-Hom-module. The map $\mathsf{can}: A\otimes A\to A\otimes H$ is a left (A,β)-linear epimorphism because

$$
\begin{aligned}
\mathsf{can}(c\cdot(a\otimes b)) = \mathsf{can}(\beta^{-1}(c)a\otimes\beta(b)) &= \left(\beta^{-2}(c)\beta^{-1}(a)\right)\beta(b_{(0)})\otimes\alpha^2(b_{(1)})\\
&= \beta^{-1}(c)(\beta^{-1}(a)b_{(0)})\otimes\alpha^2(b_{(1)}) = c\cdot\mathsf{can}(a\otimes b)
\end{aligned}
$$

for any $a,b,c\in A$. Thus can has an (A,β)-linear splitting, and a fortiori has a splitting in $\widetilde{\mathcal{H}}(M_k)$.

It is easy to check that $N\otimes(A\otimes A)\in\widetilde{\mathcal{H}}(M_A^H)$ with structure maps

$$
\begin{aligned}
(n\otimes(a\otimes a'))\cdot b &= \nu(n)\otimes(\alpha(a)\otimes a'\beta^{-2}(b)),\\
\rho_{N\otimes(A\otimes A)}(n\otimes(a\otimes b)) &= \left(\nu^{-1}(n)\otimes(\beta^{-1}(a)\otimes b_{(0)})\right)\otimes\alpha^2(b_{(1)}).
\end{aligned}
$$

Then

$$N\otimes\mathsf{can}: N\otimes(A\otimes A)\to N\otimes(A\otimes H)$$

is a morphism in $\widetilde{\mathcal{H}}(M_A^H)$, which is surjective and split in $\widetilde{\mathcal{H}}(M_k)$. Therefore,

$$g=f(N\otimes\mathsf{can}): N\otimes(A\otimes A)\to N$$

is surjective and split in $\widetilde{\mathcal{H}}(M_k)$.

Set $N'=\ker(g)$. Then there is an exact sequence

$$0\to N'\to N\otimes(A\otimes A)\xrightarrow{g} N\to 0 \tag{5.2}$$

in $\widetilde{\mathcal{H}}(M_A^H)$ which is split as a sequence in $\widetilde{\mathcal{H}}(M_k)$. Indeed, (5.2) is also a split exact sequence of (H,α)-Hom-comodules by Lemma 5.2.

Repeating the resolution with N' instead of N, we obtain another exact sequence in $\widetilde{\mathcal{H}}(M_A^H)$:

$$0 \to N'' \to N' \otimes (A \otimes A) \xrightarrow{g'} N' \to 0, \tag{5.3}$$

which is split in $\widetilde{\mathcal{H}}(M^H)$. Now set $N_1 = N \otimes (A \otimes A)$ and $N_2 = N' \otimes (A \otimes A)$. Combining (5.2) and (5.3), we obtain the exact sequence

$$N_2 \xrightarrow{g'} N_1 \xrightarrow{g} N \to 0$$

in $\widetilde{\mathcal{H}}(M^H)$. Since (5.2) and (5.3) are both split exact in $\widetilde{\mathcal{H}}(M^H)$, they stay exact after taking (H, α)-coinvariants and combining them. Thus we have an exact sequence in $\widetilde{\mathcal{H}}(M_B)$

$$N_2^{\mathrm{co}\,H} \to N_1^{\mathrm{co}\,H} \to N^{\mathrm{co}\,H} \to 0.$$

Tensor functors are right exact, so finally we obtain an exact sequence

$$N_2^{\mathrm{co}\,H} \otimes_B A \to N_1^{\mathrm{co}\,H} \otimes_B A \to N^{\mathrm{co}\,H} \otimes_B A \to 0$$

in $\widetilde{\mathcal{H}}(M_A^H)$. Thus, there is a commutative diagram

$$\begin{array}{ccccccc}
N_2^{\mathrm{co}\,H} \otimes_B A & \longrightarrow & N_1^{\mathrm{co}\,H} \otimes_B A & \longrightarrow & N^{\mathrm{co}\,H} \otimes_B A & \longrightarrow & 0 \\
\downarrow{\scriptstyle \epsilon_{N_2}} & & \downarrow{\scriptstyle \epsilon_{N_1}} & & \downarrow{\scriptstyle \epsilon_{N}} & & \\
N_2 & \xrightarrow{g'} & N_1 & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

where both the bottom and the top lines are exact sequences in $\widetilde{\mathcal{H}}(M_A^H)$. Since $N_1 = N \otimes (A \otimes A) \cong (N \otimes A) \otimes A$ and $N_2 = N' \otimes (A \otimes A) \cong (N' \otimes A) \otimes A$ are (H, A)-Hom-Hopf modules of the form $V \otimes A$, where V is an object in $\widetilde{\mathcal{H}}(M_k)$, we see that ϵ_{N_1} and ϵ_{N_2} are isomorphisms. Hence so is ϵ_N. ∎

LEMMA 5.8. *Let (H, α) be a monoidal Hom-Hopf algebra. Assume that there is an isomorphism $M \,\square_H\, Q \cong \widetilde{\mathcal{H}}(\mathrm{Com}_H(Q^*, M))$ for any right (H, α)-Hom-comodule (M, μ) and any finite-dimensional left (H, α)-Hom-comodule (Q, κ). Then (M, μ) is right (H, α)-coflat if and only if it is an injective object in $\widetilde{\mathcal{H}}(M^H)$.*

Proof. If (M, μ) is injective in $\widetilde{\mathcal{H}}(M^H)$, then there is an (H, α)-Hom-colinear map

$$\lambda_M : M \otimes H \to M$$

splitting ρ_M, that is, $\lambda_M \rho_M = \mathsf{id}_M$. Let $f : (N, \nu) \to (W, \omega)$ be surjective in $\widetilde{\mathcal{H}}(M^H)$ and take $m \otimes w \in M \,\square_H\, W$. Since f is surjective, we can find an $n \in N$ such that $f(n) = w$. To show (M, μ) is right (H, α)-coflat, we only need to show $M \,\square_H\, f : M \,\square_H\, N \to M \,\square_H\, W$ is surjective. The proof is left to the reader.

Conversely, if (N,μ,ρ_N) is a finite-dimensional right (H,α)-Hom-comodule, then the natural morphism

$$\theta : H \otimes N^* \to \mathrm{Hom}(N,H), \qquad \theta(h \otimes n^*)(n) = n^*(\nu(n))\alpha(h),$$

is a linear isomorphism. $(N^*,(\nu^*)^{-1})$ is a left (H,α)-Hom-comodule via the coaction

$$\rho_{N^*} : N^* \to H \otimes N^*, \qquad \rho_{N^*}(n^*) = \theta^{-1}((n^* \otimes H)\rho_N).$$

Then

$$M \,\square_H\, N^* \cong \widetilde{\mathcal{H}}(\mathrm{Com}_H(N^{**},M)) \cong \widetilde{\mathcal{H}}(\mathrm{Com}_H(N,M)),$$

by assumption. Since (M,μ) is coflat we deduce that (M,μ) is (N,ν)-injective, which means that for any Hom-subcomodule (N',ν) of (N,ν), and any $f \in \widetilde{\mathcal{H}}(\mathrm{Com}_H(N',M))$, there exists $g \in \widetilde{\mathcal{H}}(\mathrm{Com}_H(N,M))$ such that $g|_{(N',\nu)} = f$. Then (M,μ) is also an injective object in $\widetilde{\mathcal{H}}(M^H)$, and the proof is similar to the non-Hom-case in [14, Theorem 2.4.17]. ∎

THEOREM 5.9. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode, and (A,β) a right (H,α)-Hom-comodule algebra. Assume that there is an isomorphism $M \,\square_H\, Q \cong \widetilde{\mathcal{H}}(\mathrm{Com}_H(Q^*,M))$ for any right (H,α)-Hom-comodule (M,μ) and any finite-dimensional left (H,α)-Hom-comodule (Q,κ). Then the following assertions are equivalent:*

(1) *there exists a total integral $\varphi : (H,\alpha) \to (A,\beta)$, and the map* can *is surjective,*
(2) *F and G are mutually inverse equivalences between $\widetilde{\mathcal{H}}(M_A^H)$ and $\widetilde{\mathcal{H}}(M_B)$,*
(3) *F' and G' are mutually inverse equivalences between $\widetilde{\mathcal{H}}({}_AM^H)$ and $\widetilde{\mathcal{H}}({}_BM)$,*
(4) *A is a Hopf–Galois extension of B, and is faithfully flat as a left (B,β)-Hom-module,*
(5) *A is a Hopf–Galois extension of B, and is faithfully flat as a right (B,β)-Hom-module.*

Proof. (1)⇒(2) follows by Theorem 5.7, and (2)⇔(4) follows by Theorem 4.4. Now we only need to show (4)⇒(1). Suppose that A is a Hopf–Galois extension of B, and is faithfully flat as a left (B,β)-Hom-module. In order to show that there is a total integral, by Lemma 5.2 we only need to show that (A,β) is an injective object in $\widetilde{\mathcal{H}}(M^H)$. Equivalently, by Lemma 5.8 we have to show that (A,β) is right (H,α)-coflat.

For any $(V,\nu) \in \widetilde{\mathcal{H}}({}^HM)$, $A \,\square_H\, V$ is a right $(B,\beta|_B)$-Hom-module via $(a \otimes v)\cdot b = a\beta^{-1}(b) \otimes \nu(v)$. Define a map

$$\varpi : (A \square_H V) \otimes_B A \to (A \otimes_B A) \square_H V,$$
$$(a \otimes v) \otimes a' \mapsto (a \otimes \beta^{-1}(a')) \otimes \nu(v),$$

where the right (H, α)-Hom-comodule structure on $A \otimes_B A$ is given by $\rho_{A \otimes_B A}(a \otimes a') = (a_{(0)} \otimes \beta^{-1}(a')) \otimes \alpha(a_{(1)})$. Since A is flat as a left B-Hom-module, ϖ is an isomorphism as a left B-Hom-module. Since can is bijective, can' is an isomorphism. Thus we have the following sequence of left B-Hom-module isomorphisms:

$$(A \square_H V) \otimes_B A \cong (A \otimes_B A) \square_H V \cong (A \otimes H) \square_H V$$
$$\cong A \otimes (H \square_H V) \cong A \otimes V.$$

For any exact sequence

$$0 \to U \to V \to W \to 0$$

in $\widetilde{\mathcal{H}}({}^H M)$, the sequence

$$0 \to A \otimes U \to A \otimes V \to A \otimes W \to 0$$

is also exact in $\widetilde{\mathcal{H}}({}_k M)$, since k is a field. Hence, we have the exact sequence

$$0 \to (A \square_H U) \otimes_B A \to (A \square_H V) \otimes_B A \to (A \square_H W) \otimes_B A \to 0.$$

Since A is faithfully flat as a left B-Hom-module, we finally obtain the exact sequence

$$0 \to A \square_H U \to A \square_H V \to A \square_H W \to 0,$$

which implies that A is right (H, α)-coflat. ∎

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