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Preperiodic dynatomic curves for $z \mapsto z^d + c$

by

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Abstract. The preperiodic dynatomic curve $\mathcal{X}_{n,p}$ is the closure in $\mathbb{C}^2$ of the set of (c, z) such that z is a preperiodic point of the polynomial $z \mapsto z^d + c$ with preperiod n and period p $(n, p \geq 1)$. We prove that each $\mathcal{X}_{n,p}$ has exactly $d - 1$ irreducible components, which are all smooth and have pairwise transverse intersections at the singular points of $\mathcal{X}_{n,p}$. We also compute the genus of each component and the Galois group of the defining polynomial of $\mathcal{X}_{n,p}$.

1. Introduction. Fix $d \geq 2$. For $c \in \mathbb{C}$, set $f_c(z) = z^d + c$. For $p \geq 1$, define

$$\check{\mathcal{X}}_{0,p} := \{(c, z) \in \mathbb{C}^2 \mid f_c^p(z) = z \text{ and for all } 0 < k < p,\ f_c^k(z) \neq z\},$$
$$\mathcal{X}_{0,p} := \text{the closure of } \check{\mathcal{X}}_{0,p} \text{ in } \mathbb{C}^2.$$

It is known that all $\mathcal{X}_{0,p}$ are affine algebraic curves, called the *periodic dynatomic curves*. These curves have been the subject of several studies in algebraic and holomorphic dynamical systems. The known results for these curves mainly concern smoothness (Douady–Hubbard [DH1], Milnor [Mil1], Buff–Tan [BT]); irreducibility (Bousch [B], Buff–Tan [BT], Morton [Mo], Lau–Schleicher [LS], Schleicher [S]); the genus (Bousch [B]) and the associated Galois groups (Bousch [B], Morton [Mo], Lau–Schleicher [LS], Schleicher [S]).

In the present work, we study some topological and algebraic properties of *preperiodic dynatomic curves*.

Definition 1.1. For $n \geq 0$ and $p \geq 1$, a point z is called a *p-periodic point* of f_c if $f_c^p(z) = z$ but $f_c^k(z) \neq z$ for $0 < k < p$, and an (n, p)*-preperiodic*

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point of f_c if $f_c^n(z)$ is a p-periodic point of f_c but $f_c^l(z)$ is not periodic for any $0 \le l < n$.

Now, for any $n, p \ge 1$, define

$$\check{\mathcal{X}}_{n,p} := \{(c,z) \in \mathbb{C}^2 \mid z \text{ is an } (n,p)\text{-preperiodic point of } f_c\},$$

$$\mathcal{X}_{n,p} := \text{the closure of } \check{\mathcal{X}}_{n,p} \text{ in } \mathbb{C}^2.$$

In fact, as we shall see below, all $\mathcal{X}_{n,p}$ are also affine algebraic curves, called the *preperiodic dynatomic curves*. Not much work has been done for this kind of curves. The special case $d = 2$ has been previously studied by Bousch [B], who established in this case that for any integers $n, p \ge 1$, the curve $\mathcal{X}_{n,p}$ is also smooth and irreducible (like the periodic dynatomic curves), and computed its associated Galois group.

The main purpose of this work is to extend these results to arbitrary $d \ge 2$. An obvious difference with the previous case is that, for $d > 2$, the curve $\mathcal{X}_{n,p}$ is no longer irreducible: it consists of $d-1$ irreducible components. We may understand this by a simple observation. Consider the curve $\mathcal{X}_{1,p}$ of $(1,p)$-preperiodic points, that is, the points z which are not p-periodic, but whose image $z_0 = f(z)$ is. The periodic point $z_{p-1} = f^{p-1}(z_0)$ is another preimage of z_0. Because $f_c(z) = z^d + c$, we have $z = \omega z_{p-1}$, where ω is a dth root of unity. According to the value of ω, we can partition the $(1,p)$-preperiodic points into $d-1$ classes, and this decomposition is of algebraic nature: it corresponds to a factorization of $f_c^{p+1}(z) - f_c(z)$.

We show that these $d-1$ components are smooth and irreducible. Our approach to smoothness is by using elementary calculations on quadratic differentials and Thurston's contraction principle, following the method of Buff–Tan [BT]. The approach to irreducibility is based on the connectedness of periodic dynatomic curves and then using induction on the preperiodic index n. Moreover, we study the features of the singular points of $\mathcal{X}_{n,p}$.

Following Bousch, we compute the genus of each irreducible component and the associated Galois group of the curve $\mathcal{X}_{n,p}$.

Here is a list of our main results. They should be compared with the results on periodic dynatomic curves.

Denote by $\{\nu_d(p)\}_{p\ge1}$ the unique sequence of positive integers satisfying the recursive relation

$$d^p = \sum_{k|p} \nu_d(k), \qquad d \ge 2 \text{ integer}, \tag{1.1}$$

and let $\varphi(m)$ be the Euler totient function (i.e., the number of positive integers less than m and coprime to m). For $n, p \ge 1$, define

$$M_{n,p} := \nu_d(p) d^{n-2}(d-1)\Big(n - 1 - \sum_{t=1}^{[(n-1)/p]} d^{-tp}\Big),$$

where $[x]$ denotes the maximal integer less than or equal to x, and

$$K_{n,p} := \nu_d(p)(d^{p-1}-1)d^{n-1-p}\Big(\sum_{t=1}^{[(n-1)/p]-1} d^{-t(p-1)} - \sum_{t=1}^{[(n-1)/p]-1} d^{-pt}\Big) + (d^{[(n-1)/p]}-1)\nu_d(p)d^{n-2-[(n-1)/p]p}$$

(see (5.3) and (5.4) for the computation of them). For $n, p \geq 1$, set

$$g_p(d) := 1 + \frac{dp-d-p-1}{2d}\nu_d(p) - \frac{d-1}{2d}\sum_{k|p,k<p}\varphi\Big(\frac{p}{k}\Big)k\cdot\nu_d(k),$$

$$g_{n,p} := 1 + \tfrac{1}{2}\nu_d(p)d^{n-2}(pd-d-p-1) + \tfrac{1}{2}(M_{n,p}+K_{n,p}) - \tfrac{1}{2}d^{n-2}(d-1)\sum_{k|p,k<p}\varphi\Big(\frac{p}{k}\Big)k\cdot\nu_d(k).$$

THEOREM 1.2. *For any $d \geq 2$ and $n, p \geq 1$, the preperiodic dynatomic curve $\mathcal{X}_{n,p}$ has the following properties:*

(1) *$\mathcal{X}_{n,p}$ is an affine algebraic curve. It has $d-1$ irreducible components and each one is smooth. Moreover, the components pairwise intersect at the singular points of $\mathcal{X}_{n,p}$. In particular, if $d = 2$, the curve $\mathcal{X}_{n,p}$ is smooth and irreducible.*

(2) *The genus of every irreducible component of $\mathcal{X}_{n,p}$ (in some kind of compactification) is $g_{n,p}(d)$, and all irreducible components are mutually homeomorphic.*

(3) *The Galois group associated with $\mathcal{X}_{n,p}$ is the same as that associated with $\mathcal{X}_{\leq n,p} := \bigcup_{l=0}^{n}\mathcal{X}_{l,p}$; it consists of all permutations of the roots of the defining polynomial of $\mathcal{X}_{\leq n,p}$ that commute with f_c and with the rotation of argument $1/d$.*

Here is a table comparing these various curves, where $\mathbf{S}_m$ denotes the group of permutations of $\{1,\dots,m\}$, and $G_{n,p}(d)$ is the Galois group of $\mathcal{X}_{n,p}$.

periodic $\mathcal{X}_{0,p}$	$d=2$	$d>2$
	irreducible	irreducible
	smooth	smooth
genus	$g_p(2)$	$g_p(d)$
Galois group	$\mathbf{S}_{\nu_2(p)/p} \ltimes \mathbb{Z}_p^{\nu_2(p)/p}$	$\mathbf{S}_{\nu_d(p)/p} \ltimes \mathbb{Z}_p^{\nu_d(p)/p}$

preperiodic $\mathcal{X}_{n,p}$, $n \geq 1$	$d=2$	$d>2$
	irreducible	$d-1$ irreducible components
	smooth	not smooth, but each component smooth
componentwise genus	$g_{n,p}(2)$	$g_{n,p}(d)$
Galois group	$G_{n,p}(2)$	$G_{n,p}(d)$
pairwise intersection	empty	$C_{n,p}$(singular) : singularity set of $\mathcal{X}_{n,p}$

This article is organized as follows:

In Section 2, we gather some preliminaries.

In Section 3, we prove that every $\mathcal{X}_{n,p}$ is an affine algebraic curve, and find its defining polynomial.

In Section 4, we give the irreducible factorization of $\mathcal{X}_{n,p}$, and prove that each irreducible factor is smooth and the irreducible components pairwise intersect at the singular points of $\mathcal{X}_{n,p}$.

In Section 5, we calculate the genus of each irreducible component.

In Section 6, we describe $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating its Galois group.

2. Preliminaries

1. Filled-in Julia set and Multibrot set. This material can be found in [DH1, DH2] and [Eb].

For $c \in \mathbb{C}$, we denote by K_c the *filled-in Julia set* of f_c, that is, the set of points $z \in \mathbb{C}$ whose orbit under f_c is bounded. We denote by M_d the *Multibrot set* in the parameter plane, that is, the set of $c \in \mathbb{C}$ for which the critical point 0 belongs to K_c. It is known that M_d is connected.

Assume $c \in M_d$. Then K_c is connected. There is a conformal isomorphism $\phi_c : \mathbb{C} \setminus \overline{K}_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying $\phi_c \circ f_c = (\phi_c)^d$ and $\phi_c'(\infty) = 1$ (i.e., $\phi_c(z)/z \to_{z\to\infty} 1$). The *dynamical ray* of angle $\theta \in \mathbb{T}$ is defined by

$$R_c(\theta) := \{z \in \mathbb{C} \setminus K_c \mid \arg(\phi_c(z)) = 2\pi\theta\}.$$

Assume $c \notin M_d$. Then K_c is a Cantor set and all periodic points of f_c are *repelling*, that is, $|(f^p)'(z)| > 1$ for $p \geq 1$ and all p-periodic points z. There is a conformal isomorphism $\phi_c : U_c \to V_c$ between neighborhoods of ∞ in $\mathbb{C}$ which satisfies $\phi_c \circ f_c = (\phi_c)^d$ on U_c. We may choose U_c so that U_c contains the critical value c, and V_c is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is a minimal $r_c(\theta) \geq 1$ such that ϕ_c^{-1} extends analytically along $R_0(\theta) \cap \{z \in \mathbb{C} \mid r_c(\theta) < |z|\}$. We denote by ψ_c this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\big(R_0(\theta) \cap \{z \in \mathbb{C} \mid r_c(\theta) < |z|\}\big).$$

As $|z| \searrow r_c(\theta)$, the point $\psi_c(re^{2\pi i\theta})$ converges to some $x \in \mathbb{C}$ [DH2, Prop. 8.3]. If $r_c(\theta) > 1$, then $x \in \mathbb{C} \setminus K_c$ is an iterated preimage of 0 and we say that $R_c(\theta)$ *bifurcates at* x. If $r_c(\theta) = 1$, then x belongs to K_c and we say that $R_c(\theta)$ *lands at* x.

There are three kinds of important parameters in M_d: superattracting, parabolic, and Misiurewicz parameters. Recall that a point z is said to be *p-periodic* if $f_c^p(z) = z$ but $f_c^k(z) \neq z$ for $0 < k < p$. We call a parametr $c \in \mathbb{C}$

- *p-superattracting* if 0 is p-periodic by f_c;
- *p-parabolic* if f_c has a p-periodic point z_0 with $(f^p)'(z_0) = 1$ or m-periodic point z_0 such that $m \mid p$ and $(f^m)'(z_0)$ is a (p/m)th root of unity;
- *(n,p)-Misiurewicz* if 0 is an (n,p)-preperiodic point of f_c.

A well-known result in complex dynamics says that any parabolic cycle of a rational map has a critical point in its basin, whose orbit eventually converges to but is disjoint from the cycle (see [Mil2, Thm. 10.15]). So for the family $\{f_c \mid c \in \mathbb{C}\}$ of unicritical polynomials, the three classes of parameters above are pairwise disjoint. We write this as a lemma, since it will be repeatedly used throughout the paper.

LEMMA 2.1. *If the critical point* 0 *is* (*pre*)*periodic for f_c, then c is not a parabolic parameter.*

2. Affine algebraic curve and singularity. This material can be found in [G].

A polynomial $f \in \mathbb{C}[x,y]$ is called *squarefree* if it is not divisible by $h(x,y)^2$ for any non-constant $h(x,y) \in \mathbb{C}[x,y]$. An *affine algebraic curve over* $\mathbb{C}$ is defined as

$$\mathcal{C} = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\},$$

where f is a non-constant squarefree polynomial in $\mathbb{C}[x,y]$, called the *defining polynomial* of $\mathcal{C}$. If $f = \prod_{i=1}^m f_i$, where f_i are the irreducible factors of f, we say that the affine curve defined by f_i is an *irreducible component* of $\mathcal{C}$.

Let $f \in \mathbb{C}[x,y]$. The *total degree* of $f(x,y)$ as a multivariate polynomial is the highest degree of its terms, denoted by $\mathrm{Deg}(f)$. Correspondingly, we denote by $\deg_x(f)$ and $\deg_y(f)$ the degrees of f when considered as a polynomial in x and y respectively. The following lemma is repeatedly used in this paper.

LEMMA 2.2.

(1) *If $f = f_1 f_2$ with $f_1, f_2 \in \mathbb{C}[x,y]$, then* $\mathrm{Deg}(f) = \mathrm{Deg}(f_1) + \mathrm{Deg}(f_2)$, $\deg_x(f) = \deg_x(f_1) + \deg_x(f_2)$ *and* $\deg_y(f) = \deg_y(f_1) + \deg_y(f_2)$.

(2) *For $f_1, f_2 \in \mathbb{C}[x,y]$, if $f(x,y) = f_1(x, f_2(x,y))$, then*

$$\deg_y(f) = \deg_y(f_1) \cdot \deg_y(f_2).$$

(3) *For $f_1, f_2 \in \mathbb{C}[x,y]$, if $f(x,y) = f_1(x, f_2(x,y))$ and* $\mathrm{Deg}(f_1) = \deg_y(f_1) \geq 1$, $\mathrm{Deg}(f_2) > 1$, *then* $\mathrm{Deg}(f) = \mathrm{Deg}(f_1) \cdot \mathrm{Deg}(f_2)$.

Proof. (1) Refer to [F, Section 1.1].

(2) This is straightforward by a simple computation.

(3) Set $d_1 := \mathrm{Deg}(f_1)$ and $d_2 := \mathrm{Deg}(f_2)$. By assumption, $\deg_y(f_1) = d_1 \geq 1$ and $d_2 > 1$. On the one hand, since $\mathrm{Deg}(f_1) = \deg_y(f_1) = d_1$, there

is a unique term in f_1 of the form $a_1 y^{d_1}$, where a_1 is a non-zero constant. So, from (1) and $d_1 \geq 1$, it follows that $\mathrm{Deg}(a_1 f_2^{d_1}) = d_1 d_2$. On the other hand, any other term of f_1 has the form $a x^s y^t$, where a is a non-zero constant and either $s + t < d_1$, or $s + t = d_1$ and $s \geq 1$. From (1) and $d_2 > 1$,

$$\mathrm{Deg}(x^s f_2^t(x, y)) = s + t d_2 < d_1 d_2.$$

So we get $\mathrm{Deg}(f) = d_1 d_2$. ∎

Let $\mathcal{C}$ be an affine algebraic curve for $\mathbb{C}$ defined by $f \in \mathbb{C}[x, y]$, and let $P = (a, b) \in \mathcal{C}$. The *multiplicity* of $\mathcal{C}$ at P, denoted by $\mathrm{mult}_P(\mathcal{C})$, is defined as the order s of the first non-vanishing term in the Taylor expansion of f at P, i.e.,

$$f(x, y) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{t=0}^{s} \binom{s}{t} (x - a)^t (y - b)^{s-t} \frac{\partial^s f}{\partial x^t \partial y^{s-t}}(a, b).$$

If $\mathrm{mult}_P(\mathcal{C}) = 1$, the point P is called a *smooth point* of $\mathcal{C}$. If $\mathrm{mult}_P(\mathcal{C}) = r > 1$, then we say that P is a *singular point of multiplicity* r. We say that mult$C$ or f is *smooth* if any point of $\mathcal{C}$ is smooth. Note that the first non-vanishing term is a homogeneous polynomial of $x - a$ and $y - b$, so all its irreducible factors are linear and they are called the *tangents* of $\mathcal{C}$ at P.

A singular point P of multiplicity r on an affine plane curve $\mathcal{C}$ is called *ordinary* if the r tangents to $\mathcal{C}$ at P are distinct.

The following result provides a topological interpretation of the irreducibility of polynomials.

LEMMA 2.3. *A squarefree polynomial $f \in \mathbb{C}[x, y]$ is irreducible if and only if the set of smooth points of f is connected.*

3. Periodic dynatomic curves. In this paper, some of the proofs and statements rely on the work concerning the periodic curves $\mathcal{X}_{0,p}$. We list the related results in the following lemma. Their proofs can be found in [B], [BT], [Eb], [GO], [LS], [Mil1], [S].

By abuse of notation, we will identify polynomials in $\mathbb{C}[c, z]$ as polynomials in $\mathbf{C}[z]$ with $\mathbf{C} = \mathbb{C}[c]$. Denote by $\mathbf{K}$ a fixed algebraically closed field containing $\mathbf{C}$.

Let $f \in \mathbb{C}[c, z]$. By the *zeros* of $f \in \mathbb{C}[c, z]$, we mean the points $(c, z) \in \mathbb{C}^2$ with $f(c, z) = 0$. By the *roots* of $f \in \mathbf{C}[z]$, we mean the roots of f in $\mathbf{K}$ when f is considered as a polynomial in $\mathbf{C}[z]$.

Recall that $\{\nu_d(p)\}_{p \geq 1}$ is a unique sequence of positive integers satisfying the recursive relation $d^p = \sum_{k|p} \nu_d(k)$, $\mathrm{Deg}(f)$ denotes the total degree of f, and $\deg_z(f)$ denotes the degree of f as a polynomial in $\mathbf{C}[z]$.

LEMMA 2.4. *Let $\mathcal{X}_{0,p}$ be a periodic dynatomic curve. Then:*

(i) ([B, BT]) *There exists a unique sequence $\{Q_{0,p} \in \mathbf{C}[z]\}_{p\geq 1}$ of monic polynomials such that for all $p \geq 1$,*

$$\Phi_{0,p}(c,z) := f_c^{\circ p}(z) - z = \prod_{k|p} Q_{0,k}(c,z).$$

Moreover, $\mathrm{Deg}(Q_{0,p}) = \deg_z(Q_{0,p}) = \nu_d(p)$.

(ii) ([BT]) *Let $c_0 \in \mathbb{C}$. Then a point z_0 is a root of $Q_{0,p}(c_0, z) \in \mathbb{C}[z]$ if and only if one of the following three mutually exclusive conditions is satisfied:*

(1) *z_0 is a p-periodic point of f_{c_0} and $[f_{c_0}^{\circ p}]'(z_0) \neq 1$,*
(2) *z_0 is a p-periodic point of f_{c_0} and $[f_{c_0}^{\circ p}]'(z_0) = 1$,*
(3) *z_0 is an m-periodic point of f_{c_0}, where m is a proper factor of p, and $[f_{c_0}^{\circ m}]'(z_0)$ is a primitive (p/m)th root of unity.*

(iii) ([B, BT, GO, LS, S]) *The polynomial $Q_{0,p}$ is smooth and irreducible for all $p \geq 1$ and*

$$\mathcal{X}_{0,p} = \{(c,z) \in \mathbb{C} \mid Q_{0,p}(c,z) = 0\}.$$

(iv) ([B, BT, GO]) *The projection $\pi_{0,p} : \mathcal{X}_{0,p} \to \mathbb{C}$ defined by $\pi_{0,p}(c,z) = c$ is a degree $\nu_d(p)$ (given in (1.1)) branched covering with two kinds of critical points:*

(1) $C_{0,p}$(primitive) $= \{(c,z) \in \mathcal{X}_{0,p} \mid (c,z)$ *satisfies* (ii)(2)$\}$. *In this case, (c,z) is a simple critical point.*
(2) $C_{0,p}$(satellite) $= \{(c,z) \in \mathcal{X}_{0,p} \mid (c,z)$ *satisfies* (ii)(3)$\}$. *In this case, the multiplicity of the critical point (c,z) is $p/m - 1$.*

The critical value set of $\pi_{0,p}$ consists of the parabolic parameters of period p.

(v) ([Eb, Mil1]) *The projection $\varpi_{0,p} : \mathcal{X}_{0,p} \to \mathbb{C}$ defined by $\varpi_{0,p}(c,z) = z$ is a degree $\nu_d(p)/d$ branched covering, which is injective near each point $(c_0, 0) \in \mathcal{X}_{0,p}$.*

(vi) ([B]) *The Galois group $G_{0,p}$ for the polynomial $Q_{0,p} \in \mathbf{C}[z]$ consists of the permutations of roots of $Q_{0,p} \in \mathbf{C}[z]$ that commute with f_c.*

3. The defining polynomial for $\mathcal{X}_{n,p}$. The objective of this section is to show that $\mathcal{X}_{n,p}$ is an affine algebraic curve, and find its defining polynomial.

Recall that $\mathbf{C}$ denotes the ring $\mathbb{C}[c]$. For $n \geq 0$ and $p \geq 1$, set $\Phi_{n,p}(c,z) = f_c^{\circ(n+p)}(z) - f_c^{\circ n}(z)$.

LEMMA 3.1. *The polynomial $\Phi_{n,p} \in \mathbf{C}[z]$ has no multiple roots. Consequently, it is squarefree.*

Proof. It is enough to show that there exists $c_0 \in \mathbb{C}$ such that all roots of $\Phi_{n,p}(c_0, z)$ are simple. In fact, given $c_0 \in \mathbb{C} \setminus M_d$, a point z_0 is a root of $\Phi_{n,p}(c_0, z) \in \mathbb{C}[z]$ if and only if z_0 is an (l,k)-preperiodic point of f_{c_0}, where $0 \le l \le n$ and $k \,|\, p$. For such a c_0, the critical point 0 goes to infinity and all periodic points of f_{c_0} are repelling. It follows that

$$(\partial \Phi_{n,p}/\partial z)(c_0, z_0) = [f_{c_0}^{\circ n}]'(z_0)([f_{c_0}^{\circ p}]'(z_0) - 1) \neq 0,$$

which completes the proof. ∎

Lemma 3.2. *There exists a unique doubly indexed sequence* $\{Q_{n,p} \in \mathbf{C}[z]\}_{n,p\ge 1}$ *of squarefree monic polynomials such that for all* $n, p \ge 1$,

$$\Phi_{n,p}(c,z) = \Phi_{n-1,p}(c,z) \prod_{k|p} Q_{n,k}(c,z). \tag{3.1}$$

Moreover, $\mathrm{Deg}(Q_{n,p}) = \deg_z(Q_{n,p}) = \nu_d(p)(d-1)d^{n-1}$.

Proof. The definition of $\{Q_{n,p}\}_{n,p\ge1}$ is based on the polynomials $\{Q_{0,p}\}_{p\ge 1}$ of Lemma 2.4(i). We first show that $Q_{0,p}(c,z)$ divides $Q_{0,p}(c, f_c(z))$ for any $p \ge 1$. Since the polynomials $Q_{0,p}(c, f_c(z)) \in \mathbf{C}[z]$ are monic, we may perform Euclidean division to find a monic quotient $Q \in \mathbf{C}[z]$ and a remainder $R \in \mathbf{C}[z]$ with $\deg(R) < \deg(Q_{0,p})$ such that $Q_{0,p}(c, f_c(z)) = Q_{0,p}Q + R$. We need to show that $R = 0$, which will enable us to set $Q_{1,p}(c,z) := Q$.

By Lemmas 3.1 and 2.4(i), the polynomial $Q_{0,p} \in \mathbf{C}[z]$ has no repeated factors. So its discriminant $\Delta_{0,p} \in \mathbb{C}[c]$ does not identically vanish, and hence $\Delta_{0,p}(c) \neq 0$ outside a finite set. Fix $c_0 \in \mathbb{C}$ such that $\Delta_{0,p}(c_0) \neq 0$. Then any root z_0 of $Q_{0,p}(c_0, z)$ is simple. By Lemma 2.4(ii), the point z_0 is also a root of $Q_{0,p}(c_0, f_{c_0}(z))$. As a consequence, $R(c_0, z) = 0$ for all $z \in \mathbb{C}$. Since this is true for every c_0 outside a finite set, we have $R = 0$ as required.

For $n, p \ge 1$, we define $Q_{n,p}(c,z) := Q_{1,p}(c, f_c^{n-1}(z))$. It is clear that each $Q_{n,p} \in \mathbf{C}[z]$ is monic. Note that $\Phi_{n,p}(c,z) = \Phi_{0,p}(c, f_c^n(z))$ for any $n, p \ge 1$, and so

$$\begin{aligned}
\Phi_{n,p}(c,z) &= \Phi_{0,p}(c, f_c^n(z)) \overset{\text{Lem. 2.4}}{=} \prod_{k|p} Q_{0,k}(c, f_c^n(z)) \\
&= \prod_{k|p} Q_{0,k}(c, f_c^{n-1}(z)) Q_{1,k}(c, f_c^{n-1}(z)) \\
&= \prod_{k|p} Q_{0,k}(c, f_c^{n-1}(z)) \prod_{k|p} Q_{1,k}(c, f_c^{n-1}(z)) \\
&= \Phi_{0,p}(c, f_c^{n-1}(z)) \prod_{k|p} Q_{n,k}(c,z) = \Phi_{n-1,p}(c,z) \prod_{k|p} Q_{n,k}(c,z).
\end{aligned}$$

Since each $\Phi_{n,p}$ is squarefree (Lemma 3.1), so is each $Q_{n,p}$.

Repeatedly applying Lemma 2.2(2) & (3), we find $\mathrm{Deg}(f_c^k(z)) = \deg_z(f_c^k(z)) = d^k$ for $k \ge 1$. It follows that $\mathrm{Deg}(\Phi_{n,p}) = \deg_z(\Phi_{n,p}) = d^{n+p}$

for $n \geq 0$ and $p \geq 1$. Then by the recursive formulas (3.1), (1.1) and Lemma 2.2(1), the degree conclusion in the lemma holds. ■

By the definition of $Q_{n,p}$, we get the inductive formulas

$$
(3.2)\qquad \begin{aligned} Q_{n-1,p}(c, f_c(z)) &= Q_{n,p}(c, z), \quad n \geq 2, \\ Q_{0,p}(c, f_c(z)) &= Q_{0,p}(c, z) Q_{1,p}(c, z), \end{aligned}
$$

for each $p \geq 1$. This implies that we can obtain the properties of $Q_{n,p}$ by induction on n.

In fact, $Q_{n,p}(c, z)$ is the defining polynomial of $\mathcal{X}_{n,p}$. To see this, we will now study the properties of the roots of $Q_{n,p}(c_0, z) \in \mathbb{C}[z]$ for any $c_0 \in \mathbb{C}$.

PROPOSITION 3.3. *Let $n, p \geq 1$ be integers and $c_0 \in \mathbb{C}$. Then $z_0 \in \mathbb{C}$ is a root of $Q_{n,p}(c_0, z)$ if and only if one of the following mutually exclusive conditions holds:*

1. *z_0 is an (n, p)-preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) \neq 1$.*
2. *z_0 is an (n, p)-preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) = 1$.*
3. *z_0 is an (n, m)-preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and m is a proper factor of p with $[f_{c_0}^m]'(f_{c_0}^n(z_0))$ a primitive (p/m)th root of unity.*
4. *z_0 is an (n, p)-preperiodic point of f_{c_0} such that $f_{c_0}^l(z_0) = 0$ for some $0 \leq l < n$.*
5. *$f_{c_0}^{(n-1)}(z_0) = 0$ and 0 is a p-periodic point of f_{c_0}.*

We remark that in case (4), the case of $l = n - 1$ never occurs.

Proof. Fix $c_0 \in \mathbb{C}$. The proof goes by induction on n. If $n = 1$, then $Q_{0,p}(c, f_c(z)) = Q_{0,p}(c, z) \cdot Q_{1,p}(c, z)$. We claim that z_0 is a common root of $Q_{0,p}(c_0, z)$ and $Q_{1,p}(c_0, z)$ if and only if $z_0 = 0$ is a p-periodic point of f_{c_0}.

For sufficiency, we only need to note that, in this case, 0 is a multiple root of $Q_{0,p}(c_0, f_{c_0}(z))$, but a simple root of $Q_{0,p}(c_0, z)$ by Lemma 2.4(ii). For necessity, z_0 must be a multiple root of $Q_{0,p}(c_0, f_{c_0}(z))$. It follows that either $f_{c_0}(z_0)$ is a multiple root of $Q_{0,p}(c_0, z)$, or z_0 is a critical point of f_{c_0}. In the former case, by Lemma 2.4(iv), c_0 is a parabolic parameter and $f_{c_0}(z_0)$ is a parabolic periodic point. This means that $Q_{0,p}(c_0, f_{c_0}(z))$ and $Q_{0,p}(c_0, z)$ have the same zero multiplicity at z_0. Thus $Q_{1,p}(c_0, z_0) \neq 0$. In the latter case, we have $z_0 = 0$, and by Lemma 2.4(ii), 0 is a p-periodic point of f_{c_0}.

Such c_0, z_0 correspond to condition (5). In any other case, z_0 is a root of $Q_{1,p}(c_0, z)$ if and only if $f_{c_0}(z_0)$ is a root of $Q_{0,p}(c_0, z)$ but z_0 is not periodic. In fact, if it were, it would have the same period and multiplier as its first image. By Lemma 2.4(ii), $Q_{0,p}(c_0, z_0)$ would vanish, a contradiction.

Then Theorem 2.4(2) implies that z_0 satisfies one of conditions (1)–(4) in Proposition 3.3.

Assume that the proposition is established for $1 \le l < n$. Then $Q_{n,p}(c,z) = Q_{n-1,p}(c, f_c(z))$. So for any $c_0 \in \mathbb{C}$, z_0 is a root of $Q_{n,p}(c_0, z)$ if and only if $f_{c_0}(z_0)$ is a root of $Q_{n-1,p}(c_0, z)$. By Lemma 2.1, if $f_{c_0}(z_0)$ has property (2) or (3), then the orbit of z_0 does not contain 0. Therefore by the inductive assumption, z_0 satisfies one of the five listed conditions. ∎

In Proposition 3.3, the zeros of $Q_{n,p}(c,z)$ are divided into five classes. We give some notation for sets consisting of zeros of various classes:

The set	The points in the set
$C_{n,p}$(primitive)	(c,z) satisfies condition (2) in Proposition 3.3
$C_{n,p}$(satellite)	(c,z) satisfies condition (3) in Proposition 3.3
$C_{n,p}$(Misiurewicz)	(c,z) satisfies condition (4) in Proposition 3.3
$C_{n,p}$(singular)	(c,z) satisfies condition (5) in Proposition 3.3

Recall that for any $n, p \ge 1$, the sets $\check{\mathcal{X}}_{n,p}$ and $\mathcal{X}_{n,p}$ are defined by

$$\check{\mathcal{X}}_{n,p} = \{(c,z) \in \mathbb{C}^2 \mid z \text{ is an } (n,p)\text{-preperiodic point of } f_c\},$$
$$\mathcal{X}_{n,p} = \text{the closure of } \check{\mathcal{X}}_{n,p} \text{ in } \mathbb{C}^2.$$

PROPOSITION 3.4. *For $n, p \ge 1$, we have*

$$\mathcal{X}_{n,p} = \{(c,z) \mid Q_{n,p}(c,z) = 0\}, \quad \mathcal{X}_{n,p} \setminus \check{\mathcal{X}}_{n,p} = C_{n,p}(\text{satellite}) \cup C_{n,p}(\text{singular}).$$

Proof. Set $X := \{(c,z) \mid Q_{n,p}(c,z) = 0\}$. Then X is a closed, perfect set. By the definition of $\check{\mathcal{X}}_{n,p}$ and Proposition 3.3, we have

$$X \setminus \big(C_{n,p}(\text{satellite}) \cup C_{n,p}(\text{singular})\big) = \check{\mathcal{X}}_{n,p} \subset X. \tag{3.3}$$

We claim that the sets $C_{n,p}$(satellite) and $C_{n,p}$(singular) are both finite. If so, we get

$$X = \overline{X \setminus (C_{n,p}(\text{satellite}) \cup C_{n,p}(\text{singular}))} = \overline{\check{\mathcal{X}}_{n,p}} = \mathcal{X}_{n,p} \subset X.$$

Hence it remains to check the claim.

If $(c_0, z_0) \in C_{n,p}$(satellite), then $f_{c_0}^{n+p}(z_0) - f_{c_0}^n(z_0) = 0$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) = 1$. Hence c_0 is a root of the resultant $R \in \mathbb{C}[c]$ of the equations $f_c^{n+p}(z) - f_c^n(z) = 0$ and $(f_c^p)'(f_c^n(z)) = 1$. For c outside the Multibrot set, all the periodic points of f_c are repelling, so $f_c^{n+p}(z) - f_c^n(z)$ and $(f_c^p)'(f_c^n(z)) - 1$ do not have a common root. It follows that R is not identically zero, and hence its roots form a finite set. If $(c_0, z_0) \in C_{n,p}$(singular), then $Q_{0,p}(c_0, 0) = 0$ by Proposition 3.3(5) and Lemma 2.4(ii), whereas the roots of $Q_{0,p}(c,0)$ form a finite set. ∎

4. The irreducible factorization of $Q_{n,p}$. In this section, we will show that the curve $\mathcal{X}_{n,p}$, $n \geq 1$, has $d-1$ smooth irreducible components, and analyze the properties of its singular points. We always assume $n \geq 1$ without explicit mention.

4.1. Factorization of $Q_{n,p}$ and the features of its singular points. Recall that for $f \in \mathbb{C}[c,z]$, $\mathrm{Deg}(f)$ the total degree of f and $\deg_z(f)$ is the degree of f in z.

LEMMA 4.1. (Algebraic version) *There exists a unique sequence* $\{q^j_{n,p} \in \mathbf{C}[z]\}_{1\leq j\leq d-1}$ *of monic polynomials such that*

$$Q_{n,p}(c,z) = \prod_{j=1}^{d-1} q^j_{n,p}(c,z).$$

All points in $C_{n,p}(\text{singular})$ *are zeros of* $q^j_{n,p} \in \mathbb{C}[c,z]$*, and there are no other common zeros for* $q^i_{n,p}$ *and* $q^j_{n,p}$ *with* $i \neq j$*. Moreover,* $\mathrm{Deg}(q^j_{n,p}) = \deg_z(q^j_{n,p}) = \nu_d(p)d^{n-1}$.

(Topological version) *Define* $\mathcal{V}^j_{n,p} = \{(c,z) \in \mathbb{C}^2 \mid q^j_{n,p}(c,z) = 0\}$ $(1 \leq j \leq d-1)$*. Then* $C_{n,p}(\text{singular}) \subset \mathcal{V}^j_{n,p}$ *for each* j*, and the sets* $\{\mathcal{V}^j_{n,p} \setminus C_{n,p}(\text{singular})\}_{1\leq j\leq d-1}$ *are pairwise disjoint.*

Proof. Recall that $\mathbf{C} = \mathbb{C}[c]$ and $\mathbf{K}$ is a fixed algebraically closed field containing $\mathbf{C}$.

Let Δ be a root of $Q_{0,p} \in \mathbf{C}[z]$. Then by Lemma 2.4(i),

$$\Phi_{0,p}(c,\Delta) = f_c^p(\Delta) - \Delta = 0.$$

We see that Δ is periodic under f_c and $\Delta, \ldots, f_c^{p-1}(\Delta)$ are roots of $\Phi_{0,p}$. Note that $\Phi_{0,p}(c,0) = f_c^p(0)$ is a polynomial in c of degree d^{p-1}, so $\Delta \neq 0$. Consequently, $\omega\Delta, \ldots, \omega^{d-1}\Delta$ are not roots of $Q_{0,p}$, where $\omega = e^{2\pi i/d}$, because they are not periodic under f_c. Then by the equation $Q_{0,p}(c, f_c(z)) = Q_{0,p}(c,z)Q_{1,p}(c,z)$ (see (3.2)), we see that $\omega\Delta, \ldots, \omega^{d-1}\Delta$ are roots of $Q_{1,p} \in \mathbf{C}[z]$.

Let us factorize $Q_{0,p}$ in $\mathbf{K}$ by

$$Q_{0,p}(c,z) = \prod_{i=1}^{\nu_d(p)} (z - \Delta_i)$$

($\Delta_{s_1} \neq \Delta_{s_2}$ for $s_1 \neq s_2$, because all roots of $\Phi_{0,p} \in \mathbf{C}[z]$ are simple by Lemma 3.1, and so are $Q_{0,p}$ by Lemma 2.4(i)). Then $Q_{1,p}$ can be expressed as

$$Q_{1,p} = \prod_{i=1}^{\nu_d(p)} (z - \omega\Delta_i)\cdots(z - \omega^{d-1}\Delta_i) = \prod_{j=1}^{d-1}\prod_{i=1}^{\nu_d(p)} (z - \omega^j\Delta_i). \tag{4.1}$$

To see this, we first note that for any $s, t \in [1, d-1]$ and $i_1 \neq i_2 \in [1, \nu_d(p)]$, $\omega^s \Delta_{i_1} \neq \omega^t \Delta_{i_2}$. But this is impossible because both Δ_{i_1} and Δ_{i_2} are periodic. Thus $\{\omega\Delta_i, \ldots, \omega^{d-1}\Delta_i\}_{i=1}^{\nu_d(p)}$ are pairwise distinct roots of $Q_{1,p} \in \mathbf{C}[z]$ by the discussion above, so $\prod_{i=1}^{\nu_d(p)}(z-\omega\Delta_i)\cdots(z-\omega^{d-1}\Delta_i)$ is a divisor of $Q_{1,p}$. As its degree is $(d-1)\nu_d(p)$, equal to the degree of $Q_{1,p}$, and $Q_{1,p}$ is monic, we get (4.1). For $j \in [1, d-1]$, set

$$\begin{aligned} q^j_{1,p}(c,z) &= \prod_{i=1}^{\nu_d(p)} (z - \omega^j \Delta_i) = (\omega^j)^{\nu_d(p)} \prod_{i=1}^{\nu_d(p)} (\omega^{-j} z - \Delta_i) \\ &= (\omega^j)^{\nu_d(p)} Q_{0,p}(c, \omega^{-j} z). \end{aligned} \tag{4.2}$$

Note that $d \mid \nu_d(p)$, so $(\omega^j)^{\nu_d(p)} = 1$. Then $q^j_{1,p}(c,z)$ is a monic polynomial in $\mathbf{C}[z]$ satisfying

$$Q_{1,p}(c,z) = \prod_{j=1}^{d-1} q^j_{1,p}(c,z). \tag{4.3}$$

This gives a factorization of $Q_{1,p}$ in $\mathbf{C}[z]$. By (4.2) and the degree conclusion in Lemma 2.4(i), $\operatorname{Deg}(q^j_{1,p})$ and $\deg_z(q^j_{1,p})$ are both $\nu_d(p)$.

For $n \geq 2$, we can define $q^j_{n,p}(c,z)$ inductively by the formula $q^j_{n,p}(c,z) = q^j_{n-1,p}(c, f_c(z))$. Using induction, the degree conclusion in the lemma follows directly from Lemma 2.2(2)&(3). As $Q_{n,p}(c,z) = Q_{n-1,p}(c, f_c(z))$, we have

$$Q_{n,p}(c,z) = \prod_{j=1}^{d-1} q^j_{n,p}(c,z). \tag{4.4}$$

This is a factorization of $Q_{n,p}(c,z)$ in $\mathbf{C}[z]$.

It remains to prove that each $q^j_{n,p}(c,z)$ has the remaining properties announced in the lemma. For $n = 1$, since $q^j_{1,p}(c,z) = Q_{0,p}(c, \omega^{-j}z)$, we set that: (c_0, z_0) is a common root of $q^i_{1,p}(c,z)$ and $q^j_{1,p}(c,z)$ for some $1 \leq i \neq j \leq d-1 \Leftrightarrow$ both $(c_0, \omega^{-i}z_0)$ and $(c_0, \omega^{-j}z_0)$ are zeros of $Q_{0,p}(c,z)$. It follows that $\omega^{-i}z_0$ and $\omega^{-j}z_0$ are both periodic points of f_{c_0}, hence $z_0 = 0$. Note that, in case (3) of Lemma 2.4(ii), the critical point 0 is never periodic (Lemma 2.1), so 0 has period p. It follows that $(c_0, z_0) \in C_{1,p}(\text{singular})$. On the other hand, if $(c_0, z_0) \in C_{1,p}(\text{singular})$, then $(c_0, \omega^{-i}z_0) = (c_0, \omega^{-j}z_0) = (c_0, 0)$ is a zero of $Q_{0,p}(c,z)$. For $n \geq 2$, the conclusion can be deduced from the case of $n = 1$ and the definition of $q^j_{n,p}(c,z)$. ■

For convenience, we summarize the definitions of $q^j_{1,p}$ in terms of $Q_{0,p}$ and the inductive definitions of $q^j_{n,p}$ $(n \geq 2)$ in terms of $q^j_{n-1,p}$ as a corollary.

COROLLARY 4.2. *For any $p \geq 1$, $1 \leq j \leq d-1$, and $\omega = e^{2\pi i/d}$, we have*

$$\begin{cases} q^j_{1,p}(c,z) = Q_{0,p}(c, \omega^{-j} z), \\ q^j_{n,p}(c,z) = q^j_{n-1,p}(c, f_c(z)), \quad n \geq 2. \end{cases}$$

EXAMPLE 4.3. Here are some examples of $Q_{n,p}$ and their decompositions. Let $d = 3$. Suppose $p = 1$; then $Q_{0,1}(c,z) = z^3 + c - z$,

$$\begin{aligned} Q_{1,1}(c,z) &= c^2 + cz + z^2 + 2cz^3 + z^4 + z^6 \\ &= (z^3 + c - e^{-\frac{2}{3}\pi i} z)(z^3 + c - e^{-\frac{4}{3}\pi i} z) \\ &= q^1_{1,1}(c,z) \cdot q^2_{1,1}(c,z) \end{aligned}$$

and

$$\begin{aligned} Q_{2,1}(c,z) &= 3c^2 + 3c^4 + (c^6 + 3c + 10c^3 + 6c^5)z^3 + (1 + 12c^2 + 15c^4)z^6 \\ &\quad + (6c + 20c^3)z^9 + (1 + 15c^2)z^{12} + 6cz^{15} + z^{18} \\ &= \big((1 - e^{-\frac{2}{3}\pi i})c + c^3 + (3c^2 - e^{-\frac{2}{3}\pi i})z^3 + 3cz^6 + z^9\big) \\ &\quad \times \big((1 - e^{-\frac{4}{3}\pi i})c + c^3 + (3c^2 - e^{-\frac{4}{3}\pi i})z^3 + 3cz^6 + z^9\big) \\ &= q^1_{2,1}(c,z) \cdot q^2_{2,1}(c,z). \end{aligned}$$

Suppose $p = 2$; then $Q_{0,2}(c,z) = 1 + c^2 + cz + z^2 + 2cz^3 + z^4 + z^6$ and

$$\begin{aligned} Q_{1,2}(c,z) &= 1 + 2c^2 + c^4 - (c + c^3) - z^2 + (3c + 4c^3)z^3 - 3c^2 z^4 \\ &\quad + (1 + 6c^2)z^6 - 3cz^7 + 4cz^9 - z^{10} + z^{12} \\ &= (1 + c^2 + e^{-\frac{2}{3}\pi i} z + e^{-\frac{4}{3}\pi i} z^2 + 2cz^3 + e^{-\frac{2}{3}\pi i} z^4 + z^6) \\ &\quad \times (1 + c^2 + e^{-\frac{4}{3}\pi i} z + e^{-\frac{2}{3}\pi i} z^2 + 2cz^3 + e^{-\frac{4}{3}\pi i} z^4 + z^6) \\ &= q^1_{1,2}(c,z) \cdot q^2_{1,2}(c,z). \end{aligned}$$

From Lemma 4.1, we see that in the case $d \geq 3$, the polynomial $Q_{n,p}$ is both reducible and non-smooth, because $C_{n,p}$(singular), which is non-empty, is contained in the set of singular points of $Q_{n,p}$.

We now turn to the study of the components $q^j_{n,p}(c,z)$. The following theorem is the core of this section.

THEOREM 4.4. *Given $d \geq 2$, for any $n, p \geq 1$ and $j \in [1, d-1]$, the polynomial $q^j_{n,p}(c,z)$ is smooth and irreducible.*

The proof of this theorem is postponed to §4.2.

By this theorem, all components $\mathcal{V}^j_{n,p}$ are Riemann surfaces. Together with Lemma 4.1, this implies that the singularity set of $\mathcal{X}_{n,p}$ is equal to $C_{n,p}$(singular). The next proposition characterizes the features of these singularities.

PROPOSITION 4.5. *Given $d \geq 2$, for $n, p \geq 1$, each singularity (c_0, z_0) of $\mathcal{X}_{n,p}$ has multiplicity $d-1$. Furthermore, if $f_{c_0}^l(z_0) = 0$ for some $0 \leq l \leq n-2$, then $\mathcal{X}_{n,p}$ has one tangent of multiplicity $d-1$ at (c_0, z_0); otherwise, the singularity (c_0, z_0) is ordinary.*

Proof. Let (c_0, z_0) be a singular point of $\mathcal{X}_{n,p}$. Since each component of $\mathcal{X}_{n,p}$ is smooth and they pairwise intersect at (c_0, z_0), the first non-vanishing term of $Q_{n,p}(c, z)$ at (c_0, z_0) is $d-1$. Hence the multiplicity of the singularity (c_0, z_0) is $d-1$.

If $n = 1$, by Proposition 3.3(5), the fact that $(c_0, z_0) \in C_{1,p}$(singular) implies that $z_0 = 0$ and $(c_0, 0) \in \mathcal{X}_{0,p}$. According to Lemma 2.1, c_0 is not a parabolic parameter. Then Lemma 2.4(iv) shows that $(c_0, 0)$ is not a critical point of $\pi_{0,p}$, and hence $(\partial Q_{0,p}/\partial z)(c_0, 0) \neq 0$. Meanwhile, according to Lemma 2.4(v), $(\partial Q_{0,p}/\partial c)(c_0, 0) \neq 0$. Thus $Q_{0,p}(c, z)$ has a local expression

$$Q_{0,p}(c, z) = a_{0,p}(c - c_0) + b_{0,p}z + \text{higher order terms}$$

around $(c_0, 0)$ with $a_{0,p}, b_{0,p} \neq 0$. It follows that

$$q_{1,p}^j(c, z) = Q_{0,p}(c, \omega^{-j}z) = a_{0,p}(c - c_0) + b_{0,p}\omega^{-j}z + \text{higher order terms}.$$

Therefore the tangents of $\mathcal{V}_{1,p}^j$ $(1 \leq j \leq d-1)$ at $(c_0, 0)$ are pairwise distinct.

For $n \geq 2$, we denote by $a_{n,p}^j(c - c_0) + b_{n,p}^j(z - z_0)$ the equation of the tangent of $\mathcal{V}_{n,p}^j$ at (c_0, z_0). By the formula $q_{n,p}^j(c, z) = q_{1,p}(c, f_c^{n-1}(z))$ (Corollary 4.2), we have

$$\begin{cases} a_{n,p}^j = \dfrac{\partial q_{n,p}^j}{\partial c}(c_0, z_0) = \dfrac{\partial q_{1,p}^j}{\partial c}(c_0, 0) + \dfrac{\partial q_{1,p}^j}{\partial z}(c_0, 0)\dfrac{\partial f_c^{n-1}}{\partial c}(c_0, z_0) \\ \qquad\;\; = a_{0,p} + b_{0,p}\omega^{-j}\dfrac{\partial f_c^{n-1}}{\partial c}(c_0, z_0), \\ b_{n,p}^j = \dfrac{\partial q_{n,p}^j}{\partial z}(c_0, z_0) = \dfrac{\partial q_{1,p}^j}{\partial z}(c_0, 0)(f_{c_0}^{n-1})'(z_0) = b_{0,p}\omega^{-j}(f_{c_0}^{n-1})'(z_0). \end{cases}$$

If there exists $0 \leq l \leq n-2$ such that $f_{c_0}^l(z_0) = 0$, then $(f_{c_0}^{n-1})'(z_0) = 0$, and hence $b_{n,p}^j = 0$. It follows that the first non-vanishing term of $Q_{n,p}$ at (c_0, z_0) is $a(c-c_0)^{d-1}$ where a is a non-zero constant, i.e., $\mathcal{X}_{n,p}$ has the tangent $c = c_0$ of multiplicity $d-1$ at (c_0, z_0). In the other cases, we get $(f_{c_0}^{n-1})'(z_0) \neq 0$. Combining this with the fact that $a_{0,p}, b_{0,p} \neq 0$, it is not difficult to check that the pairs $(a_{n,p}^j, b_{n,p}^j)$ $(1 \leq j \leq d-1)$ are pairwise non-colinear. Hence the tangents of $\mathcal{V}_{n,p}^j$ $(1 \leq j \leq d-1)$ at (c_0, z_0) are pairwise distinct, that is, (c_0, z_0) is ordinary. ∎

4.2. Proof of the smoothness and irreducibility of $q_{n,p}^j$. The objective here is to prove Theorem 4.4.

The approach to proving the smoothness is similar to that in [BT]. The idea is to prove that some partial derivative of $q^j_{n,p}$ is non-vanishing. Following A. Epstein [E], we will express this derivative as the coefficient of a quadratic differential of the form $(f_c)_\star\mathcal{Q} - \mathcal{Q}$. Thurston's contraction principle gives $(f_c)_\star\mathcal{Q} - \mathcal{Q} \neq 0$, whence our partial derivative is non-zero.

The approach to the irreducibility is based on the connectedness of the periodic curve $\mathcal{X}_{0,p}$. Then we will show the connectedness of the $\mathcal{V}^j_{n,p}$ using a branched covering, by induction on n.

Here we list some definitions and results about quadratic differentials and Thurston's contraction principle. The proofs can be found in [BT] and [Le].

We use $\mathcal{Q}(\mathbb{C})$ to denote the set of meromorphic quadratic differentials on $\mathbb{C}$ whose poles (if any) are all simple. If $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ and U is a bounded open subset of $\mathbb{C}$, the norm

$$\|\mathcal{Q}\|_U := \iint_U |q|$$

is well-defined and finite.

For $f : \mathbb{C} \to \mathbb{C}$ a non-constant polynomial and $\mathcal{Q} = q\,dz^2$ a meromorphic quadratic differential on $\mathbb{C}$, the pushforward $f_*\mathcal{Q}$ is defined to be the quadratic differential

$$f_*\mathcal{Q} := Tq\,dz^2 \quad \text{with} \quad Tq(z) := \sum_{f(w)=z} \frac{q(w)}{f'(w)^2}.$$

If $Q \in \mathcal{Q}(\mathbb{C})$, then also $f_*\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$. The following lemma is a weak version of Thurston's contraction principle.

Lemma 4.6. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \neq \mathcal{Q}$.*

The formulas below appeared in [Le, Chapter 2]; we write them together as a lemma.

Lemma 4.7 (Levin). *For $f = f_c$, we have*

$$\begin{cases} f_*\left(\dfrac{dz^2}{z}\right) = 0, \\ f_*\left(\dfrac{dz^2}{z-a}\right) = \dfrac{1}{f'(a)}\left(\dfrac{dz^2}{z-f(a)} - \dfrac{dz^2}{z-c}\right) \quad \text{if } a \neq 0. \end{cases} \tag{4.5}$$

To prove the irreducibility of $q^j_{n,p}$, we need the following lemma.

Lemma 4.8. *For each $n, p \geq 1$ and $1 \leq j \leq d-1$, the polynomial $q^j_{n,p}(c, 0)$ (in the variable c) has degree $\nu_d(p)d^{n-2}$.*

Proof. For $n = 1$, we see that $q^j_{1,p}(c, 0) = Q_{0,p}(c, 0)$ from Corollary 4.2. Then the result follows directly from Lemma 2.4(v).

For $n \geq 2$, we have $q^j_{n,p}(c,0) = q^j_{1,p}(c, f^{n-1}(0))$. Since $\mathrm{Deg}(q^j_{1,p}) = \deg_z(q^j_{1,p}) = \nu_d(p)$ (see Lemma 4.1) and $\mathrm{Deg}(f_c^{n-1}(0)) = d^{n-2}$ (which is easily checked), we have

$$\mathrm{Deg}(q^j_{n,p}(c,0)) = \mathrm{Deg}(q^j_{1,p}(c,z)) \cdot \mathrm{Deg}(f_c^{n-1}(0)) = \nu_d(p) d^{n-2}$$

by Lemma 2.2(2) & (3). ∎

Proof of Theorem 4.4. The proof goes by induction on n.

For $n = 1$, as $q^j_{1,p}(c,z) = Q_{0,p}(c, \omega^{-j} z)$ and $Q_{0,p}(c,z)$ is smooth and irreducible, we know that $q^j_{1,p}(c,z)$ are smooth and irreducible. Assume that for $1 \leq l < n$, the polynomials $q^j_{l,p}(c,z)$ are smooth and irreducible. We will show that $q^j_{n,p}(c,z)$ are then smooth and irreducible. Fix any $j_0 \in [1, d-1]$.

Smoothness of $q^{j_0}_{n,p}$. As $q^{j_0}_{n,p}(c,z) = q^{j_0}_{n-1,p}(c, f_c(z))$, for any zero (c_0, z_0) of $q^{j_0}_{n,p}(c,z)$ we have

$$\tag{4.6} \begin{cases} \dfrac{\partial q^{j_0}_{n,p}}{\partial c}(c_0,z_0) = \dfrac{\partial q^{j_0}_{n-1,p}}{\partial c}(c_0,w_0) + \dfrac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0,w_0), \\ \dfrac{\partial q^{j_0}_{n,p}}{\partial z}(c_0,z_0) = \dfrac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0,w_0) \cdot f'_{c_0}(z_0), \end{cases}$$

where $w_0 = f_{c_0}(z_0)$. Hence if $z_0 \neq 0$, by the smoothness of $\mathcal{V}^{j_0}_{n,p}$ (inductive assumption), $[\partial q^{j_0}_{n,p}/\partial c](c_0,z_0)$ and $[\partial q^{j_0}_{n,p}/\partial z](c_0,z_0)$ cannot be 0 simultaneously; it follows that $q^{j_0}_{n,p}(c,z)$ is smooth at (c_0,z_0). So it remains to prove that $q^{j_0}_{n,p}(c,z)$ is smooth at $(c_0,0) \in \mathcal{V}^{j_0}_{n,p}$. In this situation, c_0 is either a p-periodic superattracting parameter or an (n,p)-Misiurewicz parameter, and $[\partial q^{j_0}_{n,p}/\partial z](c_0,0) = 0$. So we have to show $[\partial q^{j_0}_{n,p}/\partial c](c_0,0) \neq 0$.

In the former case $f^{n-1}_{c_0}(0) = 0$, so $p \mid n-1$. As $q^{j_0}_{n,p}(c,z) = q^{j_0}_{1,p}(c, f_c^{n-1}(z))$, we have

$$\tag{4.7} \frac{\partial q^{j_0}_{n,p}}{\partial c}(c_0,0) = \frac{\partial q^{j_0}_{1,p}}{\partial c}(c_0,0) + \frac{\partial q^{j_0}_{1,p}}{\partial z}(c_0,0) \frac{\partial f_c^{n-1}}{\partial c}(c_0,0).$$

As $Q_{0,p}(c_0,0) = 0$ and $p \mid n-1$, differentiating both sides of the equation

$$f_c^{n-1}(z) - z = \prod_{k \mid n-1} Q_{0,k}(c,z),$$

which appears in Lemma 2.4(i), with respect to c and z respectively at $(c_0,0)$, we obtain

$$\tag{4.8} \begin{cases} \dfrac{\partial f_c^{n-1}}{\partial c}(c_0,0) = \dfrac{\partial Q_{0,p}}{\partial c}(c_0,0) \displaystyle\prod_{\substack{k \mid n-1 \\ k \neq p}} Q_{0,k}(c_0,0), \\ \qquad\qquad -1 = \dfrac{\partial Q_{0,p}}{\partial z}(c_0,0) \displaystyle\prod_{\substack{k \mid n-1 \\ k \neq p}} Q_{0,k}(c_0,0). \end{cases}$$

Since $q_{1,p}^{j_0}(c,z) = Q_{0,p}(c, \omega^{-j_0} z)$, we have

$$\frac{\partial q_{1,p}^{j_0}}{\partial c}(c_0,0) = \frac{\partial Q_{0,p}}{\partial c}(c_0,0), \qquad \frac{\partial q_{1,p}^{j_0}}{\partial z}(c_0,0) = \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0).$$

By substituting these two formulas into (4.7) and applying (4.8), we find

(4.9)

$$\begin{aligned}\frac{\partial q_{n,p}^{j_0}}{\partial c}(c_0,0) &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0) + \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0)\frac{\partial Q_{0,p}}{\partial c}(c_0,0)\prod_{\substack{k\mid n-1\\ k\neq p}} Q_{0,k}(c_0,0)\\ &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0)\Bigg(1 + \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0)\prod_{\substack{k\mid n-1\\ k\neq p}} Q_{0,k}(c_0,0)\Bigg)\\ &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0)(1-\omega^{-j_0}).\end{aligned}$$

By Lemma 2.4(v), $[\partial Q_{0,p}/\partial c](c_0,0)$ is non-zero, hence so is $[\partial q_{n,p}^{j_0}/\partial c](c_0,0)$.

In the latter (Misiurewicz) case, since

$$\frac{\partial Q_{n,p}}{\partial c}(c_0,0) = \prod_{1\le j\neq j_0\le d-1} q_{n,p}^{j}(c_0,0)\cdot \frac{\partial q_{n,p}^{j_0}}{\partial c}(c_0,0)$$

and $(c_0,0)$ is not a zero of $\prod_{j\neq j_0} q_{n,p}^{j}(c,z)$ by Lemma 4.1, we have only to show $[\partial Q_{n,p}/\partial c](c_0,0)\neq 0$. Furthermore, since

$$\frac{\partial \Phi_{n,p}}{\partial c}(c_0,0) = \Phi_{n-1,p}(c_0,0)\cdot \prod_{k\mid p,k<p} Q_{n,k}(c_0,0)\cdot \frac{\partial Q_{n,p}}{\partial c}(c_0,0)$$

and $\Phi_{n-1,p}(c_0,0)\cdot\prod_{k\mid p,k<p} Q_{n,k}(c_0,0) \neq 0$, we shall equivalently show $[\partial\Phi_{n,p}/\partial c](c_0,0)\neq 0$. We shall find a meromorphic quadratic differential $\mathcal{Q}$ with simple poles such that

$$(f_{c_0})_*\mathcal{Q} = \mathcal{Q} + \frac{\partial \Phi_{n,p}}{\partial c}(c_0,0)\cdot\frac{dz^2}{z-c_0}.$$

Then by Lemma 4.6, we obtain $[\partial\Phi_{n,p}/\partial c](c_0,0)\neq 0$.

We shall use the following notation:

$$\begin{aligned} z_k &:= f_{c_0}^{\circ n+k}(0), & \delta_k &:= f'_{c_0}(z_k) = d z_k^{d-1}, & 0\le k\le p-1,\\ y_l &:= f_{c_0}^{l}(0), & \varepsilon_l &:= f'_{c_0}(y_l) = d y_l^{d-1}, & 1\le l\le n-1.\end{aligned}$$

With this notation and some calculation, we get

$$\begin{aligned}\frac{\partial\Phi_{n,p}}{\partial c}(c_0,0) &= \frac{\partial f_c^{\circ(n+p)}}{\partial c}(c_0,0) - \frac{\partial f_c^{\circ n}}{\partial c}(c_0,0)\\ &= (\delta_0\cdots\delta_{p-1}-1)(\varepsilon_{n-1}\cdots\varepsilon_1 + \cdots + \varepsilon_{n-1}\varepsilon_{n-2} + \varepsilon_{n-1} + 1)\\ &\quad + \delta_{p-1}\cdots\delta_1 + \cdots + \delta_{p-1} + 1.\end{aligned}$$

Denote

$$\alpha = (\delta_0 \cdots \delta_{p-1} - 1)(\varepsilon_{n-1} \cdots \varepsilon_1 + \cdots + \varepsilon_{n-1}\varepsilon_{n-2} + \varepsilon_{n-1} + 1).$$

Let

$$\mathcal{Q} = \sum_{k=0}^{p-1} \frac{\rho_k}{z - z_k} dz^2 + \sum_{l=1}^{n-1} \frac{\lambda_l}{z - y_l} dz^2$$

be a quadratic differential in $\mathcal{Q}(\mathbb{C})$. Here ρ_k $(0 \leq k \leq p-1)$, λ_l $(1 \leq l \leq n-1)$ are undetermined coefficients (note that $y_1 = c_0$). Applying Lemma 4.7 and writing f for f_{c_0}, we have

$$\begin{aligned} f_*\mathcal{Q} &= \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \left(\frac{dz^2}{z - z_{k+1}} - \frac{dz^2}{z - c_0} \right) + \sum_{l=1}^{n-2} \frac{\lambda_l}{\varepsilon_l} \left(\frac{dz^2}{z - y_{l+1}} - \frac{dz^2}{z - c_0} \right) \\ &\quad + \frac{\lambda_{n-1}}{\varepsilon_{n-1}} \left(\frac{dz^2}{z - z_0} - \frac{dz^2}{z - c_0} \right) \\ &= \left(\frac{\rho_{p-1}}{\delta_{p-1}} + \frac{\lambda_{n-1}}{\varepsilon_{n-1}} \right) \frac{dz^2}{z - z_0} + \frac{\rho_0}{\delta_0} \frac{dz^2}{z - z_1} + \cdots + \frac{\rho_{p-2}}{\delta_{p-2}} \frac{dz^2}{z - z_{p-1}} \\ &\quad + \left(\alpha - \sum_{l=1}^{n-1} \frac{\lambda_l}{\varepsilon_l} \right) \frac{dz^2}{z - y_1} + \frac{\lambda_1}{\varepsilon_1} \frac{dz^2}{z - y_2} + \cdots + \frac{\lambda_{n-2}}{\varepsilon_{n-2}} \frac{dz^2}{z - y_{n-1}} \\ &\quad - \left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0}. \end{aligned}$$

We want to choose $\mathcal{Q}$ so that

$$f_*\mathcal{Q} - \mathcal{Q} = -\left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k} \right) \frac{dz^2}{z - c_0}.$$

This amounts to solving the following linear system for the unknown coefficient vector $(\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})$:

$$\begin{pmatrix} \frac{1}{\delta_0} & -1 & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \frac{1}{\delta_{p-2}} & -1 & & & & & & \\ -1 & & & & \frac{1}{\delta_{p-1}} & & & & & & \frac{1}{\varepsilon_{n-1}} \\ & & & & & 1 + \frac{1}{\varepsilon_1} & \frac{1}{\varepsilon_2} & \frac{1}{\varepsilon_3} & \cdots & \frac{1}{\varepsilon_{n-2}} & \frac{1}{\varepsilon_{n-1}} \\ & & & & & \frac{1}{\varepsilon_1} & -1 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \frac{1}{\varepsilon_{n-2}} & -1 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{p-2} \\ \rho_{p-1} \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Denoting by A the coefficient matrix, we have

$$\det(A) = \frac{(-1)^{n-1}\alpha}{\delta_0 \cdots \delta_{p-1} \cdot \varepsilon_1 \cdots \varepsilon_{n-1}}.$$

Thus whether $\alpha = 0$ or not, this linear system has non-zero solutions, and one of its solutions is

$$\begin{aligned}
\rho_0 &= \delta_0 \cdots \delta_{p-1}, \\
\rho_1 &= \delta_1 \cdots \delta_{p-1}, \\
&\vdots \\
\rho_{p-1} &= \delta_{p-1}, \\
\lambda_1 &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \cdots \varepsilon_1, \\
&\vdots \\
\lambda_{n-2} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1}\varepsilon_{n-2}, \\
\lambda_{n-1} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1}.
\end{aligned} \tag{4.10}$$

Therefore, for $(\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})$ satisfying (4.10), we have

$$f_*\mathcal{Q} - \mathcal{Q} = -\left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k}\right) \frac{dz^2}{z - c_0} = -\frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) \cdot \frac{dz^2}{z - c_0}.$$

As a consequence $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$.

Irreducibility of $q_{n,p}^{j_0}$. For $n \geq 2$, $q_{n,p}^j(c,z)$ is defined by $q_{n,p}^j(c,z) = q_{n-1,p}^j(c, f_c(z))$. Interpreting these equations from a topological point of view, we obtain a sequence of maps

$$\{\wp_{n,p}^j : \mathcal{V}_{n,p}^j \to \mathcal{V}_{n-1,p}^j,\ (c,z) \mapsto (c, f_c(z)) \mid n \geq 2,\ p \geq 1,\ 1 \leq j \leq d-1\}.$$

Note that for $n = 1$, we can also define a map $\wp_{1,p}^j : \mathcal{V}_{1,p}^j \to \mathcal{X}_{0,p}$ by $\wp_{1,p}^j(c,z) = (c, f_c(z))$. By the smoothness of $\mathcal{V}_{n,p}^j$, we can check the following results.

- The map $\wp_{1,p}^j : \mathcal{V}_{1,p}^j \to \mathcal{X}_{0,p}$ is a homeomorphism. To see this, notice that $q_{1,p}^j(c,z) = Q_{0,p}(c, \omega^{-j} z)$ (Corollary 4.2), so we can define a map $\phi_{1,p}^j$ from $\mathcal{X}_{0,p}$ to $\mathcal{V}_{1,p}^j$ by sending $(c_0, w_0) \in \mathcal{X}_{0,p}$ to $(c_0, \omega^j z_0) \in \mathcal{V}_{1,p}^j$, where z_0 is the point in the orbit of w_0 under f_{c_0} with $f_{c_0}(z_0) = w_0$. By a simple computation, we can see that $\phi_{1,p}^j \circ \wp_{1,p}^j = \mathrm{id}_{\mathcal{V}_{1,p}^j}$ and $\wp_{1,p}^j \circ \phi_{1,p}^j = \mathrm{id}_{\mathcal{X}_{0,p}}$. Hence $\wp_{1,p}^j$ is a homeomorphism.
- For $n \geq 2$, the map $\wp_{n,p}^j : \mathcal{V}_{n,p}^j \to \mathcal{V}_{n-1,p}^j$ is a degree d branched covering with critical set

$$D_{n,p}^j = \{(c, 0) \mid q_{n,p}^j(c, 0) = 0\},$$

and each critical point has multiplicity $d - 1$.

In fact, $(c_0, w_0) \in \mathcal{V}^j_{n-1,p} \setminus \wp(D^j_{n,p})$ has d preimages $(c_0, z_1), \ldots, (c_0, z_d)$ under $\wp^j_{n,p}$, where $z_1, \ldots, z_d$ are the preimages of w_0 under f_{c_0}. Fix $i \in [1, d]$. If $[\partial q^j_{n,p}/\partial z](c_0, z_i) \neq 0$, then by (4.6), $[\partial q^j_{n-1,p}/\partial z](c_0, w_0) \neq 0$. This implies that some neighborhoods of (c_0, z_i) and (c_0, w_0) can each be parameterized by c. In such two local coordinates, the map $\wp^j_{n,p}$ has a local expression $c \mapsto c$ near (c_0, z_i), which means that $\wp^j_{n,p}$ is a local homeomorphism near (c_0, z_i). If $[\partial q^j_{n,p}/\partial z](c_0, z_i) = 0$, then by (4.6), the fact that $z_i \neq 0$ and the smoothness of $q^j_{n,p}$, we find that $[\partial q^j_{n-1,p}/\partial z](c_0, w_0) = 0$ and

$$\frac{\partial q^j_{n,p}}{\partial c}(c_0, z_i) = \frac{\partial q^j_{n-1,p}}{\partial c}(c_0, w_0) \neq 0.$$

This implies that some neighborhoods of (c_0, z_i) and (c_0, w_0) can each be parameterized by z, and $c'(z_i) = 0$. In such two local coordinates, the map $\wp^j_{n,p}$ has a local expression $z \mapsto f_{c(z)}(z)$ near (c_0, z_i). Since $z_i \neq 0$, we have $\frac{df_{c(z)}(z)}{dz}\big|_{z=z_i} = dz_i \neq 0$, which still means that $\wp^j_{n,p}$ is a local homeomorphism near (c_0, z_i).

By the discussion above, we can see that

$$\wp^j_{n,p} : \mathcal{V}^j_{n,p} \setminus (\wp^j_{n,p})^{-1}(\wp(D^j_{n,p})) \to \mathcal{V}^j_{n-1,p} \setminus \wp(D^j_{n,p})$$

is a degree d covering. On the other hand, any point in $\wp^j_{n,p}(D^j_{n,p})$ has only one preimage, which belongs to $D^j_{n,p}$. Hence $\wp : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{n-1,p}$ is a degree d branched covering (because $(\wp^j_{n,p})^{-1}(\wp(D^j_{n,p})) = D^j_{n,p}$ and $D^j_{n,p}$ is finite), and the local degree of $\wp^j_{n,p}$ at each point of $D^j_{n,p}$ is d.

By the smoothness of $q^{j_0}_{n,p}(c, z)$ and the inductive irreducibility assumption, we know that $\mathcal{V}^{j_0}_{n-1,p}$ and each connected component of $\mathcal{V}^{j_0}_{n,p}$ are Riemann surfaces. Then the restriction of $\wp^{j_0}_{n,p}$ to any connected component of $\mathcal{V}^{j_0}_{n,p}$ is also a branched covering. Lemma 4.8 implies that the critical set $D^{j_0}_{n,p}$ of $\wp^{j_0}_{n,p}$ is non-empty. Since each critical point has multiplicity $d - 1$, the set $\mathcal{V}^{j_0}_{n,p}$ must be connected. By Lemma 2.3 and the smoothness of $q^{j_0}_{n,p}$, we conclude that $q^{j_0}_{n,p}(c, z)$ is irreducible in $\mathbb{C}[c, z]$. ∎

5. Genus of the compactification of $\mathcal{V}^j_{n,p}$. In the previous section, we have seen that $\mathcal{X}_{n,p}$ consists of $d - 1$ Riemann surfaces $\mathcal{V}^j_{n,p}$ pairwise intersecting at the singular points of $\mathcal{X}_{n,p}$. In order to give a complete topological description of $\mathcal{X}_{n,p}$, we also need the topological characterization of each $\mathcal{V}^j_{n,p}$.

In fact, by adding an ideal boundary point at each end of $\mathcal{V}^j_{n,p}$, we obtain a compactification of $\mathcal{V}^j_{n,p}$, denoted by $\widehat{\mathcal{V}}^j_{n,p}$, such that $\widehat{\mathcal{V}}^j_{n,p}$ is a compact Riemann surface (see §5.1). The genus of $\widehat{\mathcal{V}}^j_{n,p}$ is calculated in §5.2. Topologically, $\mathcal{X}_{n,p}$ is completely determined by the number of its singular points, the genus of $\widehat{\mathcal{V}}^j_{n,p}$ and the number of ideal boundary points added to $\mathcal{V}^j_{n,p}$ (or the number of ends of $\mathcal{V}^j_{n,p}$).

5.1. Compactification of $\mathcal{V}^j_{n,p}$. Denote by $\pi^j_{n,p}\colon \mathcal{V}^j_{n,p} \to \mathbb{C}$ the projection from $\mathcal{V}^j_{n,p}$ to the parameter plane, i.e., $\pi^j_{n,p}(c,z) = c$. It is easy to see that

$$\pi^j_{n,p} = \pi_{0,p} \circ \wp^j_{1,p} \circ \cdots \circ \wp^j_{n-1,p} \circ \wp^j_{n,p} \tag{5.1}$$

where $\pi_{0,p}$ is the projection from $\mathcal{X}_{0,p}$ to the parameter plane and $\wp^j_{n,p}$ is defined in the proof of irreducibility. It follows that $\pi^j_{n,p}$ is a degree $\nu_d(p)d^{n-1}$ branched covering. To study the critical points of $\pi^j_{n,p}$, we define a subset $C^{\mathrm{crit}}_{n,p}(\text{singular})$ of $C_{n,p}(\text{singular})$ by

$$\begin{aligned} &C^{\mathrm{crit}}_{n,p}(\text{singular}) \\ &\quad = \{(c,z) \in C_{n,p}(\text{singular}) \mid f^l_c(z) = 0 \text{ for some } 0 \le l \le n-2\}. \end{aligned} \tag{5.2}$$

Lemma 5.1. *For any $l,p \ge 1$, the critical set of $\pi^j_{l,p}$ is the union of $C^j_{l,p}(\text{primitive})$, $C^j_{l,p}(\text{satellite})$, $C^j_{l,p}(\text{Misiurewicz})$ and $C^{\mathrm{crit}}_{l,p}(\text{singular})$, where $C^j_{l,p}(\mathrm{M}) := C_{l,p}(M) \cap \mathcal{V}^j_{l,p}$ and M indicates different properties.*

Proof. We first note that (c_0,z_0) is a critical point of $\pi^j_{l,p}$ if and only if $[\partial q^j_{l,p}/\partial z](c_0,z_0) = 0$. By Lemma 2.4(iv) and the fact that $\wp^j_{1,p}$ is a homeomorphism (shown in the proof of irreducibility of $q^j_{l,p}$), the critical set of $\pi^j_{1,p}$ is $C^j_{1,p}(\text{primitive}) \cup C^j_{1,p}(\text{satellite})$. In the case $l = 1$, $C_{l,p}(\text{Misiurewicz})$ and $C^{\mathrm{crit}}_{l,p}(\text{singular})$ are empty.

For $l \ge 2$, by Corollary 4.2, we have $q^j_{l,p}(c,z) = q^j_{1,p}(c, f^{l-1}_c(z))$. Then a point (c_0,z_0) is critical for $\pi^j_{l,p}$ if and only if

$$\frac{\partial q^j_{l,p}}{\partial z}(c_0,z_0) = \frac{\partial q^j_{1,p}}{\partial z}(c_0, f^{l-1}_{c_0}(z_0)) \cdot (f^{l-1}_{c_0})'(z_0) = 0.$$

Equivalently, either $(c_0, f^{l-1}_{c_0}(z_0))$ is a critical point of $\wp^j_{1,p}$, or $f^l_{c_0}(z_0) = 0$ for some $0 \le q \le n-2$. By Proposition 3.3, the former happens if and only if $(c_0,z_0) \in C^j_{l,p}(\text{primitive}) \cup C^j_{l,p}(\text{satellite})$, and the latter if and only if $(c_0,z_0) \in C^j_{l,p}(\text{Misiurewicz}) \cup C^{\mathrm{crit}}_{l,p}(\text{singular})$. ∎

From this lemma, we see that the critical value set of $\pi^j_{n,p}$ is contained in the union of the sets of parabolic, superattracting and Misiurewicz parameters. Hence $\mathbb{C} \setminus M_d$ contains no critical values. It follows that each connected component of $(\wp^j_{n,p})^{-1}(\mathbb{C} \setminus M_d)$, called an *end* of $\mathcal{V}^j_{n,p}$, is conformal to $\mathbb{C} \setminus \overline{\mathbb{D}}$.

By adding an ideal boundary point at the infinitely far boundary, each end of $\mathcal{V}^j_{n,p}$ is conformal to the unit disk, and thus $\mathcal{V}^j_{n,p}$ becomes a compact Riemann surface. This gives a kind of compactification of $\mathcal{V}^j_{n,p}$, and in the next subsection we will calculate the genus of this compact Riemann surface.

More precisely, let $\{\mathcal{E}^j_{n,p,i} \mid 1 \le i \le m^j_{n,p}\}$ be the ends of $\mathcal{V}^j_{n,p}$. Denote by $\infty^j_{n,p,i}$ the point added at the infinitely far boundary of $\mathcal{E}^j_{n,p,i}$. Then the surface $\widehat{\mathcal{V}}^j_{n,p} := \mathcal{V}^j_{n,p} \cup \{\infty^j_{n,p,i}\}_{i=1}^{m^j_{n,p}}$ is a compactification of $\mathcal{V}^j_{n,p}$, and $\widehat{\mathcal{E}}^j_{n,p,i} := \mathcal{E}^j_{n,p,i} \cup \{\infty^j_{n,p,i}\}$ is called an *end* of $\widehat{\mathcal{V}}^j_{n,p}$. In this case, the map $\pi^j_{n,p}$ can be extended to

$$\widehat{\pi}^j_{n,p} : \widehat{\mathcal{V}}^j_{n,p} \to \widehat{\mathbb{C}}$$

by setting $\widehat{\pi}^j_{n,p}(\infty^j_{n,p,i}) = \infty$.

5.2. Calculation of the genus of $\widehat{\mathcal{V}}^j_{n,p}$. Now, for any $n, p \ge 1$ and $j \in [1, d-1]$, we have obtained a branched covering $\widehat{\pi}^j_{n,p} \colon \widehat{\mathcal{V}}^j_{n,p} \to \widehat{\mathbb{C}}$ of degree $\nu_d(p)d^{n-1}$ between two compact Riemann surfaces. By the Riemann–Hurwitz formula, we have

$$2 - 2g^j_{n,p} + \text{total number of critical points of } \widehat{\pi}^j_{n,p} = 2\nu_d(p)d^{n-1},$$

where $g^j_{n,p}$ denotes the genus of $\widehat{\mathcal{V}}^j_{n,p}$. So in order to calculate the genus of $\widehat{\mathcal{V}}^j_{n,p}$, we only need to count the number of critical points of $\widehat{\pi}^j_{n,p}$ with multiplicity. By Lemma 5.1, we know that the set of critical points of $\widehat{\pi}^j_{n,p}$ consists of $C^j_{n,p}$(primitive), $C^j_{n,p}$(satellite), $C^j_{n,p}$(Misiurewicz), $C^{\mathrm{crit}}_{n,p}$(singular) and maybe some added ideal boundary points. So we will count them class by class.

Counting the points of $C^j_{n,p}$(primitive) *and* $C^j_{n,p}$(satellite). Bousch [B] counts the number of critical points in $C_{0,p}$(primitive) and $C_{0,p}$(satellite). His argument can be directly extended to our case (see also [Sil, Thm. 4.17]), so we only give the result. The numbers of critical points (counted with multiplicity) of $\widehat{\pi}^j_{n,p}$ in $C^j_{n,p}$(primitive) and $C^j_{n,p}$(satellite) are

$$d^{n-1}p\Big[(d-1)\nu_d(p)/d - \sum_{k|p,\, k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)\Big]$$

and

$$d^{n-1} \sum_{k|p,\, k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(p/k-1).$$

Counting the points of $C^j_{n,p}$(Misiurewicz). Recall that $D^j_{s,p} = \{(c,0) \in \mathbb{C}^2 \mid q^j_{s,p}(c,0) = 0\}$, $s \ge 2$, is the set of critical points of $\wp^j_{s,p}$. By Proposition 3.3, if $(c,0) \in D^j_{s,p}$, then c is either an (s,p)-Misiurewicz parameter or a p-superattracting parameter. So we divide $D^j_{s,p}$ into

$$D^j_{s,p}(\text{Misiurewicz}) = \{(c,0) \in D^j_{s,p} \mid c \text{ is a Misiurewicz parameter}\},$$
$$D^j_{s,p}(\text{period}) = \{(c,0) \in D^j_{s,p} \mid c \text{ is a superattracting parameter}\}.$$

By the definition of $C^j_{n,p}(\text{Misiurewicz})$, we have

$$C^j_{n,p}(\text{Misiurewicz}) = \bigcup_{s=2}^{n} (h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz})),$$

where $h^j_{n,s,p} := \wp^j_{s+1,p} \circ \cdots \circ \wp^j_{n,p} : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{s,p}$.

Fix any $s \in [2,n]$. Since the degree of $q^j_{s,p}(c,0)$ is $\nu_d(p)d^{s-2}$ (Lemma 4.8) and $[\partial q^j_{s,p}/\partial c](c,0) \neq 0$ at each $(c,0) \in D^j_{s,p}$ (shown in the proof of smoothness of $q^j_{s,p}(c,z)$), we get $\#D^j_{s,p} = \nu_d(p)d^{s-2}$. By Proposition 3.3(v), $D^j_{s,p}(\text{period})$ is non-empty if and only if $p \mid s-1$. In this case, we also see that $D^j_{s,p}(\text{period}) = \{(c,0) \mid Q_{0,p}(c,0) = 0\}$, so $\#D^j_{s,p}(\text{period})$ equals $\nu_d(p)/d$ if $p \mid s-1$, and 0 otherwise. It follows that

$$\#D^j_{s,p}(\text{Misiurewicz}) = \begin{cases} \nu_d(p)d^{s-2} & \text{if } p \nmid s-1, \\ \nu_d(p)d^{s-2} - \nu_d(p)/d & \text{if } p \mid s-1. \end{cases}$$

Note that the critical value set of $h^j_{n,s,p}$ is disjoint from $D^j_{s,p}(\text{Misiurewicz})$, so

$$\#(h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz})) = \#D^j_{s,p}(\text{Misiurewicz}) \cdot d^{n-s}$$

and each point in $(h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz}))$ is a critical point of $\widehat{\pi}^j_{n,p}$ with multiplicity $d-1$. Therefore the number of critical points (counted with multiplicity) of $\widehat{\pi}^j_{n,p}$ in $C^j_{n,p}(\text{Misiurewicz})$, denoted by $M_{n,p}$, is

$$M_{n,p} = \sum_{s=2}^{n} \#D^j_{s,p}(\text{Misiurewicz}) \cdot d^{n-s} \cdot (d-1) \tag{5.3}$$
$$= \nu_d(p)d^{n-2}(d-1)\Big(n-1-\sum_{t=1}^{[(n-1)/p]} d^{-tp}\Big).$$

Counting the points of $C^{\text{crit}}_{n,p}(\text{singular})$. Recall that $C^{\text{crit}}_{n,p}(\text{singular})$ consists of all points of the form (c,z) with $f^{n-1}_c(z) = 0$ and such that there exists $l \in [0,n-2]$ with $f^l(z) = 0$ and such that 0 is p-periodic.

We divide $C_{n,p}(\text{singular})$ into subsets $C^t_{n,p}(\text{singular})$ which consist of points $(c,z) \in C_{n,p}(\text{singular})$ such that

$$n-1-tp = \min\{l \mid f^l_c(z) = 0\}.$$

Here t can take the values $0,\dots,[(n-1)/p]$, where $[x]$ denotes the integer part of x, and the sets $C^t_{n,p}$ are pairwise disjoint and form a partition of $C_{n,p}(\text{singular})$. From (5.2), we see that $C^{\text{crit}}_{n,p}(\text{singular})$ is the union of

$C^t_{n,p}$(singular), $t \geq 1$. Hence $\#C^{\mathrm{crit}}_{n,p}$(singular) $= 0$ if $n-1 < p$. So in the following discussion, we only treat the case of $n-1 \geq p$, i.e., $[(n-1)/p] \geq 1$.

Let $t \geq 1$. We have $(c,z) \in C^t_{n,p}$(singular) if and only if

$$(c,0) \in D^j_{tp+1,p}(\text{period}), \quad f_c^{n-1-tp}(z) = 0 \quad \text{and} \quad (f_c^{n-1-tp})'(z) \neq 0.$$

Hence

$$\begin{aligned} &C^t_{n,p}(\text{singular}) \\ &\quad = (h^j_{n,tp+1,p})^{-1}(D^j_{tp+1,p}(\text{period})) \setminus (h^j_{n,(t+1)p+1,p})^{-1}(D^j_{(t+1)p+1,p}(\text{period})) \end{aligned}$$

if $(t+1)p+1 \leq n$, and $C^t_{n,p}(\text{singular}) = (h^j_{n,tp+1,p})^{-1}(D^j_{tp+1,p}(\text{period}))$ otherwise. So

$$\begin{aligned} &\#C^t_{n,p}(\text{singular}) \\ &\quad = \begin{cases} d^{n-1-tp} \cdot \nu_d(p)/d & \text{if } t = [(n-1)/p], \\ d^{n-1-tp} \cdot \nu_d(p)/d - d \cdot d^{n-1-(t+1)p} \cdot \nu_d(p)/d & \text{if } 1 \leq t < [(n-1)/p]. \end{cases} \end{aligned}$$

On the other hand, $h^j_{n,tp+1,p} : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{tp+1,p}$ is injective in a neighborhood of any point $(c,z) \in C^t_{n,p}$(singular), and $\pi^j_{kp+1,p} : \mathcal{V}^j_{tp+1} \to \mathbb{C}$ has local degree d^t at $(c,0)$, so the number of critical points counted with multiplicity in $C^t_{n,p}$(singular) is $(d^t-1)\#C^t_{n,p}$(singular). Thus the total number of critical points counted with multiplicity in $C_{n,p}$(singular), in the case of $[(n-1)/p] \geq 1$, is

$$\begin{aligned} (5.4) \qquad K_{n,p} &:= \sum_{t=1}^{[(n-1)/p]} (d^t - 1)\#C^t_{n,p}(\text{singular}) \\ &= \nu_d(p)(d^{p-1}-1)d^{n-1-p}(\xi_{n,p} - \zeta_{n,p}) + (d^{[(n-1)/p]} - 1)\nu_d(p)d^{n-2-[(n-1)/p]p} \end{aligned}$$

where $\xi_{n,p} := \sum_{t=1}^{[(n-1)/p]-1} d^{-t(p-1)}$ and $\zeta_{n,p} := \sum_{t=1}^{[(n-1)/p]-1} d^{-pt}$.

Note that when $[(n-1)/p] = 0$, the number computed by formula (5.4) is 0, which is still equal to the cardinality of $C^{\mathrm{crit}}_{n,p}$(singular). So the number $K_{n,p}$, defined by (5.4), is equal to the number of critical points counted with multiplicity in $C^{\mathrm{crit}}_{n,p}$(singular) in all cases.

Counting the ideal boundary points. Bousch [B] and Milnor [Mil1] show that the local degree of $\pi_{0,p}$ at each ideal boundary point is 2 (in the case of $d = 2$) by analysing the asymptotic behavior of $f_c(z)$ as (c,z) goes to an ideal boundary point on $\mathcal{X}_{0,p}$. Their argument can be easily generalized to our case with degree $d \geq 2$. Just to be self-contained we give an alternative proof using the monodromy action (Lemma 5.3 below). By Lemma 5.3, the local degree of $\widehat{\mathcal{V}}^j_{n,p}$ at each ideal boundary point is d. It follows that the number of ideal boundary points is $\nu_d(p)d^{n-2}$ because $\widehat{\pi}^j_{n,p}$ is a degree $\nu_d(p)d^{p-1}$

branched covering. So the number of critical points counted with multiplicity is $\nu_d(p)d^{n-2}(d-1)$.

By the Riemann–Hurwitz formula, we have

$$g^j_{n,p} = 1 + \tfrac{1}{2}\nu_d(p)d^{n-2}(pd - p - 1 - d) + \tfrac{1}{2}(M_{n,p} + K_{n,p}) - \tfrac{1}{2}d^{n-2}(d-1)\sum_{k|p,\, k<p} k\nu_d(k)\varphi(p/k).$$

Here are the genera of some examples.

d	n	p	$\nu_d(p)$	$M_{n,p}$	$K_{n,p}$	$g_{n,p}$
3	1	1	3	0	0	0
3	2	1	3	4	2	1
2	2	2	2	2	0	0
2	3	2	2	7	1	1
2	2	3	6	6	0	2

COROLLARY 5.2. *For fixed $n, p \geq 1$, the surfaces $\mathcal{V}^j_{n,p}$, $1 \leq j \leq d-1$, are pairwise homeomorphic.*

Proof. Topologically, the surface $\mathcal{V}^j_{n,p}$ is completely determined by the genus and the number of ideal boundary points of $\widehat{\mathcal{V}}^j_{n,p}$, and these two numbers are independent of j. ∎

LEMMA 5.3. *All ideal boundary points are critical points of $\widehat{\pi}^j_{n,p}$ with multiplicity $d-1$.*

Proof. We first give a symbolic description of the dynamics on the filled-in Julia set for a parameter outside the Multibrot set.

If $c \in \mathbb{C} \setminus M_d$, the Julia set of f_c is a Cantor set. If $c \in R_{M_d}(\theta)$ with $\theta \neq 0$ not necessarily periodic, the dynamical rays $R_c(\theta/d), \dots, R_c((\theta + d - 1)/d)$ bifurcate at the critical point. The set $R_c(\theta/d) \cup \dots \cup R_c((\theta+d-1)/d) \cup \{0\}$ decomposes the complex plane into d connected components. We denote by U_0 the component containing $R_c(0)$ and by $U_1, \dots, U_{d-1}$ the other components in counterclockwise order.

The orbit of a point $x \in K_c$ has an itinerary with respect to this partition. In other words, to each $x \in K_c$, we can associate a sequence in $\mathbb{Z}_d^{\mathbb{N}}$ whose jth entry is k if $f_c^{\circ j-1}(x) \in U_k$. This gives a map $\iota_c : K_c \to \mathbb{Z}_d^{\mathbb{N}}$, which is bijective for any $c \in \mathbb{C} \setminus M_d$. Moreover, the dynamic of f_c on K_c is conjugate to shift on $\mathbb{Z}_d^{\mathbb{N}}$ via the map ι_c.

Now let $\pi := \pi^j_{n,p}\big|_{\mathcal{E}^j_{n,p,i}}$. The map $\pi : \mathcal{E}^j_{n,p,i} \to \mathbb{C} \setminus M_d$ is a covering of degree $d^j_{n,p,i}$. Fix $c_0 \in \mathbb{C} \setminus (M_d \cup R_{M_d}(0))$ with $d^j_{n,p,i} = \#(\pi^{-1}(c_0))$. Since $\mathcal{E}^j_{n,p,i}$ is connected, the monodromy group induced by π, denoted by $\mathrm{Mon}(\pi)$, acts on $\pi^{-1}(c_0)$ transitively. Thus for any fixed $(c_0, z_0) \in \pi^{-1}(c_0)$, the set $\pi^{-1}(c_0)$ is exactly the orbit of (c_0, z_0) under $\mathrm{Mon}(\pi)$.

Let $c_t : [0,1] \to \mathbb{C} \setminus M_d$ be an oriented simple closed curve based at c_0 that intersects $R_{M_d}(0)$ only at c_{t_0}. Let z_t be the (n,p)-preperiodic point of f_{c_t} obtained from the analytic continuation of z_0 along c_t. Note that as c varies in $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, the (n,p)-preperiodic points of f_c and the dynamical rays $R_c(0)$ and $R_c((\theta_c + s)/d)$ $(s \in \mathbb{Z}_d)$ vary continuously. Consequently,

$$\begin{cases} \iota_{c_t}(z_t) = \iota_{c_0}(z_0) & \text{for } t \in [0, t_0), \\ \iota_{c_t}(z_t) = \iota_{c_0}(z_1) & \text{for } t \in (t_0, 1]. \end{cases}$$

Furthermore, on the one hand, z_t and $R_{c_t}(0)$ vary continuously for $t \in [0,1]$. On the other hand, when c_t passes through $R_{M_d}(0)$, the dynamical rays $R_{c_t}((\theta_t + s)/d)$ $(s \in \mathbb{Z}_d)$ jump from $R_{c_{t_-}}((\theta_{t_-} + s)/d)$ to $R_{c_{t_+}}((\theta_{t_+} + s + 1)/d)$, $t_- < t_0 < t_+$. So if $\iota_{c_0}(z_0) = \beta_n \cdots \beta_1 \overline{\epsilon_1 \cdots \epsilon_p}$, then

$$\iota_{c_0}(z_1) = (\beta_n + 1) \cdots (\beta_1 + 1)\overline{(\epsilon_1 + 1) \cdots (\epsilon_p + 1)}. \tag{5.5}$$

Hence σ_{c_t}, the element of $\mathrm{Mon}(\pi)$ induced by c_t maps (c_0, z_0) to (c_0, z_1) with z_1 satisfying (5.5). Since $\pi_1(\mathbb{C} \setminus M_d, c_0) = \langle c_t \rangle$, we have

$$\begin{aligned}(\pi^j_{n,p}|_{\mathcal{E}^j_{n,p,i}})^{-1}(c_0) \\ = \{(c_0, z) \mid \iota_{c_0}(z) = (\beta_n + s) \cdots (\beta_1 + s)\overline{(\epsilon_1 + s) \cdots (\epsilon_p + s)},\ s \in \mathbb{Z}_d\}.\end{aligned}$$

As a consequence, $d^j_{n,p,i} = d$. ∎

6. The Galois group of $Q_{n,p}(c, z)$. The objective here is to study $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating its associated Galois group.

Recall that $\mathbf{C}$ denotes the ring $\mathbb{C}[c]$, and $\mathbf{K}$ is a fixed algebraically closed field containing $\mathbf{C}$. Since the characteristic of $\mathbb{C}(c)$ is 0, any polynomial $f \in \mathbf{C}[z]$ induces a finite Galois extension $\mathbb{C}(c)(f)$ over $\mathbb{C}(c)$ (see [W, Thms. 3.2.6, 2.7.14]), where $\mathbb{C}(c)(f)$ is the splitting field of f, and hence a Galois group $G(f) := \mathrm{Gal}(\mathbb{C}(c)(f)/\mathbb{C}(c))$. In particular, we denote the Galois group of $Q_{n,p}$ by $G_{n,p}$.

For all $n \geq 0$ and $p \geq 1$, denote by $\mathfrak{R}_{n,p}$ the set of roots of $Q_{n,p} \in \mathbf{C}[z]$. By (3.2), we have $f_c(\mathfrak{R}_{n,p}) = \mathfrak{R}_{n-1,p}$ if $n \geq 1$ and $f_c(\mathfrak{R}_{0,p}) = \mathfrak{R}_{0,p}$. Set

$$\mathfrak{R}_{\leq n,p} := \bigcup_{0 \leq l \leq n} \mathfrak{R}_{l,p}.$$

Then $f_c(\mathfrak{R}_{\leq n,p}) \subset \mathfrak{R}_{\leq n,p}$ and the action of f_c induces a directed graph structure consisting of a certain number of disjoint cycles of order p, to each vertex of which is attached a tree of height n. More precisely, for each $0 \leq l \leq n$, we consider the roots in $\mathfrak{R}_{l,p}$ as the vertices of level l, and two vertices $\Delta_1, \Delta_2 \in \mathfrak{R}_{\leq n,p}$ are connected by an oriented edge from Δ_1 to Δ_2 if

$f_c(\Delta_1) = \Delta_2$. Thus $\mathfrak{R}_{\leq n,p}$ has a graph structure, and we denote this graph by $\mathfrak{R}^T_{\leq n,p}$ (see Figure 1).

EXAMPLE. For $d = 3$, $p = 4$, $n = 2$, the directed graph $\mathfrak{R}^T_{\leq 2,4}$ has 18 pairwise isomorphic connected components. We draw one below.

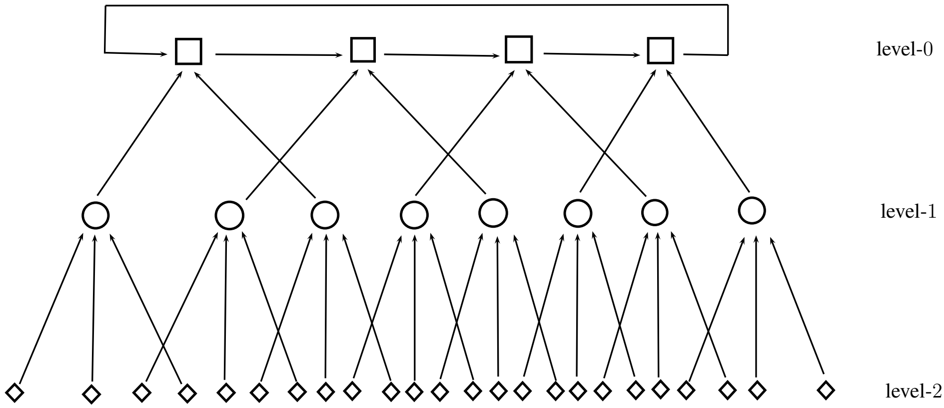

Fig. 1. A connected component of $\mathfrak{R}^T_{\leq 2,4}$

On the other hand, note that $\mathfrak{R}_{\leq n,p}$ is the set of roots of

$$Q_{\leq n,p} := \prod_{l=0}^{n} Q_{l,p} \in \mathbf{C}[z].$$

So, correspondingly, we consider the Galois group $G_{\leq n,p}$ of $Q_{\leq n,p}$. Firstly, we have the following simple result.

PROPOSITION 6.1. *For all $n \geq 0$ and $p \geq 1$, we have $G_{n,p} = G_{\leq n,p}$.*

Proof. By (3.2), any root of $Q_{l,p} \in \mathbf{C}[z]$ $(0 \leq l \leq n)$ can be written as a polynomial with coefficients in $\mathbf{C}$ of roots of $Q_{n,p}$. It follows that the splitting field of $Q_{\leq n,p} = \prod_{l=0}^{n} Q_{l,p}$ over $\mathbb{C}(c)$ is the same as that of $Q_{n,p}$ over $\mathbb{C}(c)$. Hence $G_{\leq n,p} = G_{n,p}$. ∎

By this proposition, computing the Galois group $G_{n,p}$ is equivalent to computing $G_{\leq n,p}$. Let $\sigma \in G_{\leq n,p}$. Since it fixes the base field $\mathbb{C}(c)$ pointwise, we have $\sigma(\mathfrak{R}_{l,p}) = \mathfrak{R}_{l,p}$ and $f_c \circ \sigma = \sigma \circ f_c$. Hence σ induces an *automorphism* of the graph $\mathfrak{R}^T_{\leq n,p}$, i.e., σ is a permutation of the vertices of $\mathfrak{R}^T_{\leq n,p}$ of each fixed level, and $\Delta_1, \Delta_2 \in \mathfrak{R}_{\leq n,p}$ are connected by an edge from Δ_1 to Δ_2 if and only if $\sigma(\Delta_1), \sigma(\Delta_2)$ are connected by an edge from $\sigma(\Delta_1)$ to $\sigma(\Delta_2)$. Clearly, different elements of $G_{\leq n,p}$ induce different automorphisms of $\mathfrak{R}^T_{\leq n,p}$. So $G_{\leq n,p}$ can be seen as a subgroup of $\mathrm{Aut}(\mathfrak{R}^T_{\leq n,p})$, the automorphism group of the graph $\mathfrak{R}^T_{\leq n,p}$.

In the case $d = 2$, Bousch [B] proved that

$$G_{\le n,p} \simeq \mathrm{Aut}(\mathfrak{R}^T_{\le n,p}) \simeq H_{\le n,p}(f_c),$$

where $H_{\le n,p}(f_c)$ denotes the set of all permutations of $\mathfrak{R}_{\le n,p}$ that commute with f_c. In the general case, the result is similar but needs a small modification. We will exhibit this point in the following.

Let $\sigma \in G_{\le n,p}$. As σ fixes the field $\mathbb{C}(c)$ pointwise, it must satisfy the following two conditions:

(P1) σ commutes with f_c, i.e., $\sigma \circ f_c = f_c \circ \sigma$.
(P2) σ commutes with the rotation of argument $1/d$. That is, if $\sigma(\Delta) = \widetilde{\Delta}$ for $\Delta, \widetilde{\Delta} \in \mathfrak{R}_{\le n,p}$, then $\sigma(\omega^j \Delta) = \omega^j \widetilde{\Delta}$, where $\omega = e^{2\pi i/d}$ and $1 \le j \le d-1$.

Therefore, if a permutation of $\mathfrak{R}_{\le n,p}$ is to be a candidate for being an element of $G_{\le n,p}$, it should satisfy conditions (P1) and (P2).

In fact, in the case of $d = 2$, condition (P1) implies (P2). To see this, let Δ_{n-1} be a root of $Q_{n-1,p}$ ($n \ge 1$) and $\Delta_n, -\Delta_n$ be its preimages under f_c. Let $\sigma \in G_{\le n,p}$ and set $\widetilde{\Delta}_n := \sigma(\Delta_n)$. By (P1), σ must map $-\Delta_n$ to $-\widetilde{\Delta}_n$, so (P2) holds. Therefore, (P1) alone may be sufficient for a permutation of $\mathfrak{R}_{\le n,p}$ to be an element of $G_{\le n,p}$, and Bousch [B] proved this point.

However, the situation is a little different in the case of $d \ge 3$. With the notations Δ_{n-1}, Δ_n, $\widetilde{\Delta}_n$ and σ as above, Δ_{n-1} has now at least three preimages, which are $\Delta_n, \omega\Delta_n, \ldots, \omega^{d-1}\Delta_n$. From condition (P1), we only know that σ maps $\{\omega\Delta_n, \ldots, \omega^{d-1}\Delta_n\}$ bijectively to $\{\omega\widetilde{\Delta}_n, \ldots, \omega^{d-1}\widetilde{\Delta}_n\}$, but cannot get $\sigma(\omega^j\Delta_n) = \omega^j(\widetilde{\Delta}_n)$ for $1 \le j \le d-1$. So, in case $d \ge 3$, condition (P2) cannot be omitted.

We wish to prove that, except (P1) and (P2), no other restrictions are imposed on $G_{\le n,p}$. The proof is similar to that of [B, Chapter III, Theorem 4].

Theorem 6.2. *The Galois group $G_{\le n,p}$ consists of all permutations on $\mathfrak{R}_{\le n,p}$ which commute with f_c and with the rotation of argument $1/d$.*

Proof. We denote by r_d the rotation of argument $1/d$, and by $H_{\le n,p}(f_c, r_d)$ the set of permutations of $\mathfrak{R}_{\le n,p}$ which commute with f_c and r_d. By the definition, it is not difficult to check that $H_{\le n,p}(f_c, r_d)$ leaves each $\mathfrak{R}_{l,p}$, and hence $\mathfrak{R}_{\le l,p}$, invariant for $0 \le l \le n$.

Define a group homomorphism

$$\phi_n : G_{\le n,p} \to H_{\le n,p}(f_c, r_d)$$

by letting $\phi_n(\sigma)$ be the restriction of σ to $\mathfrak{R}_{\le n,p}$. According to the discussion above, we just need to prove the surjectivity of ϕ_n.

Note first that the result is true for $n = 0$ by Lemma 2.4(vi).

For $n = 1$, since $H_{\le 1,p}(f_c, r_d)$ leaves $\mathfrak{R}_{0,p}$ invariant, there is a natural homomorphism from $H_{\le 1,p}(f_c, r_d)$ to $H_{\le 0,p}(f_c, r_d)$ with $\widetilde{\tau} \mapsto \widetilde{\tau}|_{\mathfrak{R}_{0,p}}$. It has an inverse which maps $\tau \in H_{\le 0,p}(f_c, r_d)$ to $\widetilde{\tau} \in H_{\le 1,p}(f_c, r_d)$ such that $\widetilde{\tau}|_{\mathfrak{R}_{0,p}} = \tau$ and $\widetilde{\tau}(\omega^j \Delta) = \omega^j \tau(\Delta)$ for each $\Delta \in \mathfrak{R}_{0,p}, j \in [1, d-1]$. Thus $H_{\le 1,p}(f_c, r_d) \simeq H_{\le 0,p}(f_c, r_d)$. Note that $G_{1,p} = G_{0,p}$ (because the splitting fields of $Q_{0,p}$ and $Q_{1,p}$ over $\mathbb{C}(c)$ coincide), so the result is true for $n = 1$.

Now we argue by induction on n. Assume $\phi_{n-1} : G_{\le n-1,p} \to H_{\le n-1,p}(f_c, r_d)$ is surjective ($n \ge 2$).

Let $\tau \in H_{\le n,p}(f_c, r_d)$. As τ commutes with f_c, it leaves $\mathfrak{R}_{\le n-1,p}$ invariant. Then $\tau|_{n-1}$, the restriction of τ to $\mathfrak{R}_{\le n-1,p}$, belongs to $H_{\le n-1,p}(f_c, r_d)$. By the inductive assumption, there is a $\sigma_{n-1} \in G_{\le n-1,p}$ with $\phi_{n-1}(\sigma_{n-1}) = \tau|_{n-1}$. From Galois theory (see [W, Thm. 2.88]), we can find $\sigma \in G_{\le n,p}$ whose restriction to the splitting field of $Q_{\le n-1,p}$ over $\mathbb{C}(c)$ coincides with σ_{n-1}. Set $\tau' := \tau \cdot \phi_n(\sigma)^{-1}$. Then $\tau' \in H_{\le n,p}(f_c, r_d)$ and τ' fixes $\mathfrak{R}_{\le n-1,p}$ pointwise. Now it remains to prove that $G_{\le n,p}$ contains τ', i.e., there exists $\sigma' \in G_{\le n,p}$ with $\phi_n(\sigma') = \tau'$, because if so, then $\tau = \phi_n(\sigma')\phi_n(\sigma) = \phi_n(\sigma'\sigma)$, which implies ϕ_n is surjective.

Set $\kappa_l := \nu_d(p)(d-1)d^{l-1}$ for each $l \ge 1$ (which is the number of roots of $Q_{l,p}$), and denote

$$\mathfrak{R}_{n,p} = \{\Delta_n^i, \omega\Delta_n^i, \dots, \omega^{d-1}\Delta_n^i\}_{i=1}^{\kappa_{n-1}}.$$

Since τ' fixes $\mathfrak{R}_{\le n-1,p}$ pointwise and commutes with both f_c and r_d, it can be expressed as a product

$$\tau' = \prod_{i=1}^{\kappa_{n-1}} (s_i, s_i + 1, \dots, d-1, 0, \dots, s_i - 1),$$

where $(s_i, s_i + 1, \dots, d-1, 0, \dots, s_i - 1)$ is the cyclic permutation of $(\Delta_n^i, \dots, \omega^{d-1}\Delta_n^i)$ mapping Δ_n^i to $\omega^{s_i}\Delta_n^i$ and so on. Notice that all these cyclic permutations $(s_i, \dots, s_i - 1)$ pairwise commute.

The argument in this section is a classical correspondence between Galois theory and covering theory (see [Z] for the details). Let $V_{n,p}$ be the set of singular values of the projection $\pi : \mathcal{X}_{n,p} \to \mathbb{C}$. Then $\pi_{n,p}$ restricts to a cover from the complement of the preimage of $V_{n,p}$ in $\mathcal{X}_{n,p}$ to the complement of $V_{n,p}$ in $\mathbb{C}$. For all c_0 not in $V_{n,p}$, there is thus an action of $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ on the roots

$$Z_{n,p} = \{z_n^i, \dots, \omega^{d-1} z_n^i\}_{i=1}^{\kappa_{n-1}}$$

of $Q_{n,p}(c_0, z)$ seen as a polynomial in z with complex coefficients. By the correspondence between Galois theory and covering theory (see [Z, Thm. 1]), there is a (non-unique) choice of bijection between the roots of $Q_{n,p} \in \mathbf{C}[z]$ and the roots of $Q_{n,p}(c_0, z) \in \mathbb{C}[z]$ such that the set of permutations induced by the Galois group on $\mathfrak{R}_{\le n,p}$ is conjugated by this bijection to the set of permutations of $Z_{n,p}$ induced by $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$. Thus we get a surjective (non-injective, usually) morphism from $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ to the Galois group.

Moreover, this bijection is such that any polynomial relation between the Δ_n^i with coefficients in $\mathbb{C}(c)$ will give a relation between the z_n^i, with c_0 being substituted for c. This implies that the action of $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ on $Z_{n,p}$ preserves commutation with multiplication by ω.

Therefore, by the discussion above, to obtain the required permutation τ', we only need to find a path in the basic group $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ whose monodromy action on $\{(z_n^i, \ldots, \omega^{d-1} z_n^i)\}_{i=1}^{\kappa_{n-1}}$ induces the same permutation as τ'. We now show the following result, which is sufficient: for any $i \in [1, \kappa_{n-1}]$, there exists a path in $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ whose monodromy action induces the permutation $(s_i, s_i + 1, \ldots, s_i - 1)$.

Fix any $i \in [1, \kappa_{n-1}]$. Suppose that $\{(c_0, z_n^i), (c_0, \omega z_n^i), \ldots, (c_0, \omega^{d-1} z_n^i)\}$ belong to $\mathcal{V}_{n,p}^t$. Let $\hat{c}$ be an (n,p)-Misiurewicz parameter with $(\hat{c}, 0) \in \mathcal{V}_{n,p}^t$. Such a $\hat{c}$ exists because the set $D_{n,p}^t$(Misiurewicz) is non-empty (see Section 5.2). By (3.2), we have $(\hat{c}, \hat{c}) \in \mathcal{X}_{n-1,p}$. Since $\hat{c}$ is a Misiurewicz parameter and the orbit of $\hat{c}$ does not contain 0, the point $(\hat{c}, \hat{c})$ belongs to no sets in Lemma 5.1 in the case $l = n - 1$. Hence $w = \hat{c}$ is a simple root of the equation $Q_{n-1,p}(\hat{c}, w) = 0$ (in w). So by the Implicit Function Theorem, the equation $Q_{n-1,p}(c, w) = 0$ has a unique solution $w = w(c)$ close to $\hat{c}$ fullfilling $w(\hat{c}) = \hat{c}$. Thus, a neighborhood of $(\hat{c}, 0)$ in $\mathcal{X}_{n,p}$ can be written as

$$\{(c, z_c) \cup (c, \omega z_c) \cup \cdots \cup (c, \omega^{d-1} z_c) \mid |c - \hat{c}| < \epsilon\},$$

where z_c is one of the preimages of $w(c)$ under f_c near 0.

When c makes a small turn around $\hat{c}$, the set $\{z_c, \omega z_c, \ldots, \omega^{d-1} z_c\}$ gets a cyclic permutation with $\omega^j z_c$ mapped to $\omega^{j+1} z_c$, because $\pi_{n,p}$ is a degree d covering in a punctured neighborhood of $(\hat{c}, 0)$ (which is shown in §5.2), and the other (n,p)-preperiodic points of f_c remain fixed, since $\pi_{n,p}$ is injective around each point $(\hat{c}, \xi)$ with ξ a non-zero (n,p)-preperiodic point of $f_{\hat{c}}$. So if we choose a path $\gamma \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ homotopic to $\hat{c}$, the permutation induced by γ's monodromy action gives the cyclic permutation $(2, \ldots, d, 1)$ of $(z_n^*, \omega z_n^*, \ldots, \omega^{d-1} z_n^*)$ for an (n,p)-preperiodic point z_n^* of f_{c_0} fullfilling $(c_0, z_n^*) \in \mathcal{V}_{n,p}^t$, and keeps the other (n,p)-preperiodic points of f_{c_0} fixed. Now we join (c_0, z_n^i) and (c_0, z_n^*) by a curve from (c_0, z_n^i) to (c_0, z_n^*) in $\mathcal{V}_{n,p}^t \setminus \pi_{n,p}^{-1}(V_{n,p})$, and denote its projection under $\pi_{n,p}$ by β. Then $\beta \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ and the path $\beta \cdot \gamma^{s_i} \cdot \beta^{-1}$ is what we expect. ∎

Applying this theorem, we can also characterize the Galois group $G_{\leq n,p}$ by the automorphisms of the directed graph $\mathfrak{R}_{\leq n,p}^T$, as in the $d = 2$ case. For $d \geq 3$, denote by $\mathrm{Aut}(\mathfrak{R}_{n,p}^T, r_d)$ the set of automorphisms of $\mathfrak{R}_{\leq n,p}^T$ that commute with the rotation of argument $1/d$, and by $H_{\leq n,p}(f_c, r_d)$ the set of permutations on $\mathfrak{R}_{\leq n,p}$ that commute with f_c and the rotation of argument $1/d$.

Corollary 6.3. *For $n \geq 0$ and $p \geq 1$,*

$$G_{\leq n,p} \simeq \mathrm{Aut}(\mathfrak{R}^T_{n,p}, r_d) \simeq H_{\leq n,p}(f_c, r_d).$$

Following Bousch [B, Chap. 3, III] and Silverman [Sil, §3.9], we express the Galois group $G_{n,p}$ ($n \geq 2$) as a wreath product.

Definition 6.4. Let G be a group and Σ be a subgroup of $\mathbf{S}_m$, where $\mathbf{S}_m$ denotes the set of permutations of $\{1, \ldots, m\}$. Denote by $\Sigma \ltimes G^m$ the wreath product of G and Σ. As a set, it consists of $g = \sigma(g_1, \ldots, g_m)$ where $g_i \in G$ and $\sigma \in \Sigma$. The multiplication is defined by

$$g \cdot h = \sigma_g(g_1, \ldots, g_m) \cdot \sigma_h(h_1, \ldots, h_m) = \sigma_g \circ \sigma_h(g_{\sigma_h(1)} \cdot h_1, \ldots, g_{\sigma_h(m)} \cdot h_m).$$

Under this multiplication law, $\Sigma \ltimes G^m$ is a group with

$$g^{-1} = \sigma_g^{-1}(g^{-1}_{\sigma_g^{-1}(1)}, \ldots, g^{-1}_{\sigma_g^{-1}(1)})$$

and unit element $(1, \ldots, 1)$.

Bousch [B] showed that $G_{0,p}$ is isomorphic to $\mathbf{S}_{\nu_d(p)/p} \ltimes (\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}$ (see also [Sil, §3.9]). From the proof of Theorem 6.2, we have seen that $G_{1,p} = G_{0,p}$, so

$$G_{1,p} \simeq \mathbf{S}_{\nu_d(p)/p} \ltimes (\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}.$$

For $n \geq 2$, we can give inductively an isomorphic model of $G_{n,p}$ by a wreath product. Recall that $\kappa_n = \nu_d(p)(d-1)d^{n-1}$ ($n \geq 1$) is the number of roots of $Q_{n,p}$.

Proposition 6.5. *For $n \geq 2$, we have $G_{n,p} \cong G_{n-1,p} \ltimes (\mathbb{Z}/d\mathbb{Z})^{\kappa_{n-1}}$, where the action of $G_{n-1,p}$ on $(1, \ldots, \kappa_{n-1})$ comes from the action of $G_{n-1,p}$ on the roots of $Q_{n-1,p}$, of which there are exactly κ_{n-1}.*

Proof. For $n \geq 2$, we denote by $(\varDelta^i_{n-1})_{i=1}^{\kappa_{n-1}}$ the roots of $Q_{n-1,p} \in \mathbf{C}[z]$, and denote by

$$(\{\varDelta^i_n, \omega\varDelta^i_n, \ldots, \omega^{d-1}\varDelta^i_n\})_{i=1}^{\kappa_{n-1}}$$

the roots of $Q_{n,p}$ such that $f_c(\varDelta^i_n) = \varDelta^i_{n-1}$.

$\varDelta^1_{n-1}$	$\varDelta^2_{n-1}$	$\ldots$	$\varDelta^{\kappa_{n-1}}_{n-1}$
$\varDelta^1_n$	$\varDelta^2_n$	$\ldots$	$\varDelta^{\kappa_{n-1}}_n$
$\omega\varDelta^1_n$	$\omega\varDelta^2_n$	$\ldots$	$\omega\varDelta^{\kappa_{n-1}}_n$
$\vdots$	$\vdots$		$\vdots$
$\omega^{d-1}\varDelta^1_n$	$\omega^{d-1}\varDelta^2_n$	$\ldots$	$\omega^{d-1}\varDelta^{\kappa_{n-1}}_n$

We define a group homomorphism

$$W : G_{n,p} \to G_{n-1,p} \ltimes (\mathbb{Z}/d\mathbb{Z})^{\kappa_{n-1}}$$

by $W(\sigma)=\sigma|_{n-1}(s_1,\dots,s_{\kappa_{n-1}})$, where $\sigma|_{n-1}$ is the restriction of σ to the splitting field of $Q_{n-1,p}$ over $\mathbb{C}(c)$, and the ith digit in $(s_1,\dots,s_{\kappa_{n-1}})$ is s_i if and only if once $\sigma(\varDelta_{n-1}^i)=\varDelta_{n-1}^t$ for some $1\le t\le\kappa_{n-1}$, then $\sigma(\varDelta_n^i)=\omega^{s_i}\varDelta_n^t$. The injectivity of W is straightforward by the action of $G_{n,p}$ on $\mathfrak{R}_{\le n,p}$, and the surjectivity of W is due to Theorem 6.2. ∎

To end this section, we compute $G_{n,p}$ for some small n,p. Note that although $G_{1,p}$ is isomorphic to a subgroup $\mathbf{S}_{\nu_d(p)/p}\ltimes(\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}$ of $\mathbf{S}_{\nu_d(p)}$, it is indeed a subgroup of $\mathbf{S}_{\nu_d(p)(d-1)}$. So mimicking the action of $G_{1,p}$ on

$$\{\omega\varDelta_1^1,\dots,\omega\varDelta_1^{\nu_d(p)};\dots;\omega^{d-1}\varDelta_1^1,\dots,\omega^{d-1}\varDelta_1^{\nu_d(p)}\},$$

we define a subgroup $\mathbf{P}_{\nu_d(p)(d-1),d}$ of $\mathbf{S}_{\nu_d(p)(d-1)}$ such that $\tau\in\mathbf{P}_{\nu_d(p)(d-1),d}$ if and only if

$$\begin{aligned}&\tau\big(1,\dots,\nu_d(p);\dots;(d-2)\nu_d(p)+1,\dots,(d-2)\nu_d(p)+\nu_d(p)\big)\\&\quad=\big(\sigma(1),\dots,\sigma(\nu_d(p));\dots;(d-2)\nu_d(p)+\sigma(1),\dots,(d-2)\nu_d(p)+\sigma(\nu_d(p))\big)\end{aligned}$$

for a $\sigma\in\mathbf{S}_{\nu_d(p)}$. Then $\mathbf{P}_{\nu_d(p)(d-1),d}\simeq\mathbf{S}_{\nu_d(p)}\simeq G_{1,p}$, and $\mathbf{P}_{\nu_d(p)(d-1),d}=\mathbf{S}_{\nu_d(p)}$ in the case $d=2$. The results of computation are listed in the following table.

d	n	p	$\nu_d(p)$	κ_{n-1}	$G_{n,p}(d)$
3	1	1	3	–	$\mathbf{S}_3\simeq\mathbf{P}_{6,3}$
3	2	1	3	6	$\mathbf{P}_{6,3}\ltimes(\mathbb{Z}/3\mathbb{Z})^6$
3	3	1	3	18	$(\mathbf{P}_{6,3}\ltimes(\mathbb{Z}/3\mathbb{Z})^6)\ltimes(\mathbb{Z}/3\mathbb{Z})^{18}$
2	3	2	2	4	$((\mathbb{Z}/2\mathbb{Z})\ltimes(\mathbb{Z}/2\mathbb{Z})^2)\ltimes(\mathbb{Z}/2\mathbb{Z})^4$
2	2	3	6	6	$((\mathbb{Z}/2\mathbb{Z})\ltimes(\mathbb{Z}/3\mathbb{Z})^2)\ltimes(\mathbb{Z}/2\mathbb{Z})^6$

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