A rigidity phenomenon for the Hardy–Littlewood maximal function

by

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Abstract. The Hardy–Littlewood maximal function $\mathcal{M}$ and the trigonometric function $\sin x$ are two central objects in harmonic analysis. We prove that $\mathcal{M}$ characterizes $\sin x$ in the following way: Let $f \in C^\alpha(\mathbb{R}, \mathbb{R})$ be a periodic function and $\alpha > 1/2$. If there exists a real number $0 < \gamma < \infty$ such that the averaging operator

$$(A_x f)(r) = \frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\, dz$$

has a critical point at $r = \gamma$ for every $x \in \mathbb{R}$, then

$$f(x) = a + b\sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

This statement can be used to derive a characterization of trigonometric functions as those nonconstant functions for which the computation of the maximal function $\mathcal{M}$ is as simple as possible. The proof uses the Lindemann–Weierstrass theorem from transcendental number theory.

1. Introduction and main results

1.1. Introduction. Maximal functions are a central object in harmonic analysis; conversely, harmonic analysis is built up from trigonometric functions. We were motivated by the simple question whether a maximal function is able to 'recognize' a trigonometric function in any particular way. We focus on the centered Hardy–Littlewood maximal function on the real line

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{2r} \int\limits_{x-r}^{x+r} |f(z)|\, dz.$$

Classical results are the embedding $\mathcal{M} : L^1 \to L^{1,\infty}$, where the sharp constant is known [12], as well as the embedding $\mathcal{M} : L^p \to L^p$ for $1 < p \leq \infty$.

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The wealth of theory developed around maximal functions can no longer be succinctly summarized: we refer to the book of Stein [16] for the classical theory and a survey of Wolff [19] on the Kakeya problem.

1.2. The interval length function r_f. Usual questions around maximal functions are concerned with their size: since $(\mathcal{M}f)(x) \geq |f(x)|$ at all Lebesgue points, it is of interest to understand mapping properties in L^p spaces. Our question goes in an orthogonal direction: how complicated is the dynamical behavior of the 'maximal' intervals? The question is not well-posed because there might be more than one interval centered at x over which the average value of the function coincides with the maximal function; in these cases, we opt for taking the smallest such interval. Formally, we define the length function $r_f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ associated to a periodic function f by

$$r_f(x) = \inf\left\{ r \geq 0 : \frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\,dz = \sup_{s>0} \frac{1}{2s} \int\limits_{x-s}^{x+s} f(z)\,dz \right\},$$

where the integral is to be understood as the point evaluation if $r = 0$. It is easy to see that r_f is well-defined and finite for periodic functions.

1.3. Main results. The purpose of this paper is to study the situation where for all $x \in \mathbb{R}$ both $r_f(x)$ and $r_{-f}(x)$ are either 0 or a fixed positive real number, and to show that this characterizes the trigonometric function. This theorem may be understood as a characterization of trigonometric functions by means of a dynamical aspect of the Hardy–Littlewood maximal function. It seems to have surprisingly little to do with traditional characterizations involving geometry, power series, differential equations or spectral theory. Indeed, we failed to find a slick reduction to any of the classical characterizations and ended up needing tools from transcendental number theory.

THEOREM 1. *Let $f \in C^\alpha(\mathbb{R}, \mathbb{R})$ be a periodic function and $\alpha > 1/2$. There exists a positive number $\gamma > 0$ such that the averaging operator*

$$(A_x f)(r) = \frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\,dz$$

has a critical point at $r = \gamma$ for every x if and only if

$$f(x) = a + b\sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

Theorem 1 is the strongest statement in this paper; it is relatively easy to deduce the following statement, which formulates everything in terms of the complexity of the maximal intervals.

THEOREM 2. *Let $f \in C^{\alpha}(\mathbb{R}, \mathbb{R})$ be a periodic function and $\alpha > 1/2$. Then*

$$\left| \bigcup_{x \in \mathbb{R}} \{r_f(x), r_{-f}(x)\} \right| \leq 2$$

if and only if

$$f(x) = a + b \sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

We emphasize that we do not even know whether the statement remains true if the constant 2 is replaced by a larger positive integer (but conjecture that it does). Another way of stating Theorem 2 is as follows: Suppose the periodic function f does not change sign and that both $\mathcal{M}f$ and $\mathcal{M}(-f)$ can be computed by checking the average over an interval of fixed length and comparing it with point evaluation, i.e. suppose there exists a fixed number $0 < \gamma < \infty$ such that f satisfies the equation

$$(\mathcal{M}f)(x) = \max\left(|f(x)|, \frac{1}{2\gamma} \int\limits_{x-\gamma}^{x+\gamma} |f(z)| \, dz \right) \quad \text{for all } x \in \mathbb{R}$$

and the same condition (with the same value γ) holds for $\mathcal{M}(-f)$. Then

$$f(x) = a + b \sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

1.4. A delay differential equation. Perhaps the most natural first step after seeing Theorem 1 would be to try a combination of differentiation and algebraic manipulations to obtain an ordinary differential equation (with the hope of it being $y'' + y = 0$). As it turns out, this does not work and produces much more interesting results instead. Differentiation in r implies that

$$0 = \partial_r \left(\frac{1}{2r} \int\limits_{x-r}^{x+r} f(z) \, dz \right) \bigg|_{r=\gamma} = -\frac{1}{2\gamma^2} \int\limits_{x-\gamma}^{x+\gamma} f(z) \, dz + \frac{1}{2\gamma}(f(x+\gamma) + f(x-\gamma)).$$

Assuming $f \in C^1$, this equation can now be differentiated in x and yields

$$f'(x+\gamma) - \frac{1}{\gamma}f(x+\gamma) = -f'(x-\gamma) - \frac{1}{\gamma}f(x-\gamma).$$

The qualitative theory of delay differential equations is a lot more complicated than the theory of ordinary differential equations because the space of solutions is *much* larger (uncountable): any C^1 function $g : [0, 2\gamma] \to \mathbb{R}$ with correct boundary conditions can always be extended to a solution of the delay differential equation. However, as an easy consequence of Theorem 2, we can show that there are few periodic solutions.

THEOREM 3. *Let $\gamma > 0$ be fixed and let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a solution of the delay differential equation*

$$f'(x+\gamma) - \frac{1}{\gamma}f(x+\gamma) = -f'(x-\gamma) - \frac{1}{\gamma}f(x-\gamma).$$

If f is periodic, then

$$f(x) = a + b\sin(cx + d) \quad \textit{for some } a, b, c, d \in \mathbb{R}.$$

Considering the large (uncountable) number of solutions, it is utterly remarkable that there are so few periodic solutions. We have not been able to locate any type of argument in the literature that would allow one to prove a result of this type.

1.5. Open questions. We believe that many of the assumptions can be weakened. Periodicity of the functions is necessary to allow the use of Fourier series on which our argument is based, but it seems natural to assume that the properties discussed could not hold for a nonperiodic function anyway. The assumption $f \in C^\alpha(\mathbb{R}, \mathbb{R})$ with $\alpha > 1/2$ is required at one point in the proof to enforce uniform convergence of the Fourier series; again, it seems to be an artefact of the method. We note that the condition $f \in C^\alpha(\mathbb{R}, \mathbb{R})$ with $\alpha > 1/2$ in our statements could always be replaced with the condition of f having an absolutely convergent Fourier series. The strongest statement we believe could be true is the following.

CONJECTURE. If $f \in L^\infty(\mathbb{R})$ is real-valued and r_f assumes only finitely many different values, then

$$f(x) = a + b\sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

A more daring conjecture would be that it suffices to assume that

$$\bigcup_{x \in \mathbb{R}} \{r_f(x)\} \subset \mathbb{R} \quad \text{is a Lebesgue-null set.}$$

If $r_f(x)$ is contained in a set of 'small' non-zero Lebesgue measure, what does that imply for the function? It seems to indicate, in some weak sense, that Fourier frequencies interact weakly (perhaps in the sense of a $\Lambda(p)$-property?). Furthermore, it seems that if f is given by a lacunary Fourier series, then

$$\bigcup_{x \in \mathbb{R}} \{r_f(x)\} \subset \mathbb{R} \quad \text{can have 'fractal' structure,}$$

again in a vague sense (small measure and a very large number of connected components); it could be of interest to try to understand quantitative versions of this basic intuition. One could also ask to which extent this persists in higher dimensions: in $\mathbb{R}^d$, if we consider the maximal function associated to axis-parallel rectangles and the natural analogue $r_f : \mathbb{R}^d \to \mathbb{R}^d$, then setting

$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} (a_i + b_i \sin x_i + c_i \cos x_i)$$

implies

$$\left| \bigcup_{x \in \mathbb{R}} \{r_f(x), r_{-f}(x)\} \right| \leq 2^d.$$

Already in two dimensions, there are many other natural maximal functions and it is not clear to us whether similar statements hold for any of them. We recall the Pompeiu conjecture [14]: if $K \subset \mathbb{R}^n$ is a simply connected Lipschitz domain and $f : \mathbb{R}^n \to \mathbb{R}$ is a nonzero continuous function such that the integral of f vanishes over every congruent copy of K—does this imply that K is a ball? Is there a connection between the Pompeiu conjecture and the maximal problem for disks?

The discrete setting. In light of the recent results [4, 6, 7] concerning the behavior of the maximal function on the lattice, this might be another interesting direction to investigate. For a function $f : \mathbb{Z} \to \mathbb{R}$, we define the maximal function as

$$(\mathcal{M}f)(n) = \sup_{r \in \mathbb{N}_{\geq 0}} \frac{1}{2r+1} \sum_{k=n-r}^{n+r} f(k).$$

The length function $r_f : \mathbb{Z} \to \mathbb{N}$ is defined as above. Numerical experiments show that the continuous case translates into the discrete setting: for generic parameters, functions of the form

$$f(n) = a + b\sin(cn + d), \qquad a, b, c, d \in \mathbb{R},$$

seem to give rise to two-valued r_f. We do not have a formal proof of this statement; it should be equivalent to a series of trigonometric inequalities that might actually be in the literature. The property is stable under small perturbations. However, there also exist completely different functions with
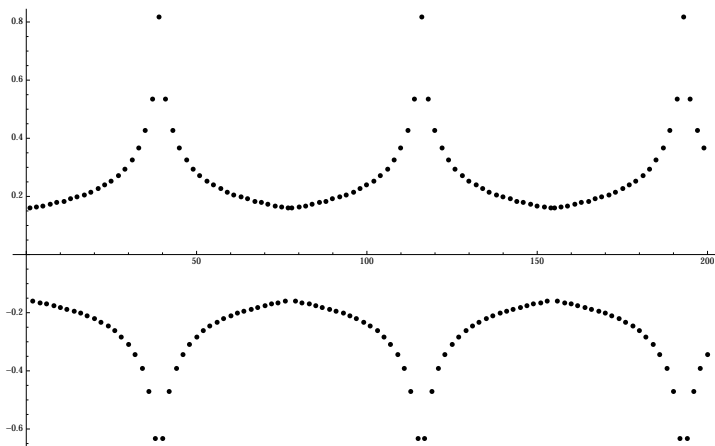

Fig. 1. A periodic function $f : \mathbb{Z} \to \mathbb{R}$ with $r_f(n) \in \{0, 2\}$

a two-valued r_f: the following example was found by Xiuyuan Cheng (personal communication). Taking

$$f(n) = \frac{(-1)^{n+1}}{(|n| + 1/2)^\alpha} \quad \text{for some } 0 < \alpha < 1/2,$$

introducing a cut-off and making it periodic can yield functions with r_f : $\mathbb{Z} \to \{0, 2\}$ (see Fig. 1 where $\alpha = 1/3$). This example is structurally totally different from the sine: having oscillations at scale 2 seems crucial. We do not know whether there are any solutions of other types and consider it to be a fascinating problem. A natural conjecture would be that if a periodic function $f : \mathbb{Z} \to \mathbb{R}$ satisfies

$$|f(n + 1) - f(n)| \leq \varepsilon \|f\|_{\ell^\infty}$$

and has a two-valued r_f, then

$$\inf_{a,b,c,d \in \mathbb{R}} \sup_{n \in \mathbb{Z}} |f(n) - (a + b\sin(cn + d))| \leq c(\varepsilon)\|f\|_{\ell^\infty}$$

for some function $c : \mathbb{R}_+ \to \mathbb{R}_+$ tending to 0 as ε tends to 0.

1.6. Related work. We are not aware of any related work in this direction. Our interest in the Hardy–Littlewood maximal function itself, however, stems from a series of recent interesting results on fine properties of $\mathcal{M}f$: since $\mathcal{M}f$ tries to maximize local averages, there is every reason to believe that it should decrease total variation—this turns out to be a surprisingly intricate problem. Motivated by a question of Kinnunen [10], Tanaka [17] showed for the *uncentered* Hardy–Littlewood maximal function $\mathcal{M}^*$ that

$$\|(\mathcal{M}^*f)'\| \leq 2\|f'\|_{L^1},$$

where the constant 2 was then improved to 1 by J. M. Aldaz & J. Pérez Lázaro [1]. Kurka [11] has recently proven the same inequality for the centered Hardy–Littlewood maximal function for a large universal constant. Carneiro & Svaiter [8] give corresponding results for the maximal heat flow and the maximal Poisson operator. The discrete question on the lattice $\mathbb{Z}$ has been investigated by Bober, Carneiro, Hughes & Pierce [4], and by Carneiro & Hughes [6] in higher dimensions. The result of Kurka in the discrete setting has been proven by Temur [18]. These results are well in line of what one would expect from a maximal function; however, it is quite interesting that all of them seem quite difficult to prove. Indeed, the sharp constant 1 for the centered maximal function on the real line is still merely conjectural.

Stokes wave. Equations of the type appearing in our proof seem to have previously surfaced in a completely different context: in a 1987 paper on the behavior of the Stokes wave of extreme form near its crest, Amick &

Fraenkel [2] encounter the equation

$$\sqrt{3}(1+z) = \tan\left(\frac{\pi}{2}z\right), \quad z \in \mathbb{C},$$

and require statements about the linear independence of the solutions of such an equation. All solutions $z_1, z_2, \ldots$ with $\mathrm{Re}(z) > -1$ are simple and the Amick–Fraenkel conjecture says that

$$\{1, z_1, z_2, \ldots\} \quad \text{is linearly independent over the rationals.}$$

Shargorodsky [15] showed that this is implied by the Schanuel conjecture. Our proof encounters similar issues but can be unconditionally resolved using the Lindemann–Weierstrass theorem.

2. Proofs

2.1. Outline. This section contains all the proofs. We first prove Theorem 1 and then show how it implies Theorems 2 and 3. The proof of Theorem 1 uses an expansion into Fourier series and the fact that averaging over intervals acts diagonally on the Fourier basis. This implies that a particular Fourier series has to vanish identically, which entails that all Fourier coefficients have to vanish identically—this can be reduced to a system of 'diophantine' equations (over $\mathbb{N} \times \mathbb{N} \times \mathbb{R}_+$). Using arguments from transcendental number theory, we can show that this system only has the trivial solution, which implies that the function has to be localized at one point in the frequency spectrum. The latter part of the argument is exploiting the arising structure in a very particular way and seems to only work in the very special case we are considering.

2.2. Proof of Theorem 1. Suppose that $f \in C^\alpha(\mathbb{R}, \mathbb{R})$ for some $\alpha > 1/2$ is periodic. Without loss of generality, we can use the symmetries of the statement to assume that the function has mean value 0 and smallest period 2π, and that it can be written as

$$f(x) = \sum_{k=1}^{\infty} (a_k \sin kx + b_k \cos kx).$$

We assume now that

$$\partial_r \left(\frac{1}{2r} \int_{x-r}^{x+r} f(z)\, dz \right) \bigg|_{r=\gamma} = 0 \quad \text{for all } x \in \mathbb{R}.$$

The trigonometric identities

$$\sin(x+r) - \sin(x-r) = 2 \sin r \cos x,$$
$$\cos(x+r) - \cos(x-r) = -2 \sin r \sin x$$

yield

$$\frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\, dz = \sum_{k=1}^{\infty} \frac{\sin rk}{rk} (a_k \sin kx + b_k \cos kx).$$

Here we invoke the classical theorem of Bernstein (see, e.g., [9]) stating that periodic functions in $C^{\alpha}(\mathbb{R}, \mathbb{R})$ for some $\alpha > 1/2$ have an absolutely convergent Fourier series. Furthermore, this allows us to interchange the sum and derivative with respect to r:

$$0 = \partial_r \left(\frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\, dz \right) \bigg|_{r=\gamma} = \sum_{k=1}^{\infty} \frac{\gamma k \cos \gamma k - \sin \gamma k}{\gamma^2 k} (a_k \sin kx + b_k \cos kx)$$

because

$$\left| \frac{\gamma k \cos \gamma k - \sin \gamma k}{\gamma^2 k} \right| \leq \frac{\gamma k + 1}{\gamma^2 k} \leq \frac{\gamma + 1}{\gamma^2}$$

and therefore

$$\sum_{k=1}^{\infty} \left| \frac{\gamma k \cos \gamma k - \sin \gamma k}{\gamma^2 k} (a_k \sin kx + b_k \cos kx) \right|$$

$$\leq \frac{\gamma + 1}{\gamma^2} \sum_{k=1}^{\infty} |a_k \sin kx + b_k \cos kx|.$$

The only way for a Fourier series to vanish everywhere is for all the coefficients to vanish. Note that $\gamma > 0$ and therefore

$$\frac{\gamma k \cos \gamma k - \sin \gamma k}{\gamma^2 k} = 0 \;\Leftrightarrow\; \gamma k = \tan \gamma k.$$

For any fixed k, it is certainly possible to choose γ in such a way that the equation is satisfied. It remains to show that no two such equations can be satisfied at the same time. Toward a contradiction, assume that

$$a_k^2 + b_k^2 > 0 \quad \text{for at least two different values of } k \in \mathbb{N}.$$

This would imply the existence of a solution $(\gamma, m, n) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N}$ of

$$\tan \gamma m = \gamma m,$$
$$\tan \gamma n = \gamma n,$$

with $\gamma > 0$ and $m \neq n$. If we could derive a contradiction from this assumption, it would imply that, independently of the value γ,

$$a_k^2 + b_k^2 > 0 \quad \text{can hold for at most one value of } k \in \mathbb{N},$$

from which the statement follows since then

$$f(x) = a_k \sin kx + b_k \cos kx.$$

It is not surprising that number theory enters here: one way of rephrasing the problem is that any two elements in the set

$$\{z \in \mathbb{R}_+ : \tan z = z\}$$

are linearly independent over $\mathbb{Q}$. The rest of the argument can be summarized as follows: the tangent has the powerful property of sending nonzero algebraic numbers to transcendental numbers. Any nonzero solution $\gamma \in \mathbb{R}$ of the equation $\tan(\gamma m) = \gamma m$ must therefore be transcendental, which means that it is never a root of a polynomial with rational coefficients. Using multiple angle formulas for the tangent, the assumption of any nontrivial solution γ satisfying two of these equations at the same time allows us to construct an explicit polynomial for which γ is a root—this contradiction will conclude the proof. We start with the cornerstone of the argument.

CLAIM ([13]). *If $x \neq 0$ is algebraic over $\mathbb{Q}$, then $\tan x$ is transcendental over $\mathbb{Q}$.*

Proof. Suppose $\tan x$ is algebraic over $\mathbb{Q}$. Then for some $n \in \mathbb{N}$ and some $r_k \in \mathbb{Q}$,

$$\sum_{k=0}^{n} r_k (\tan x)^k = 0.$$

We rewrite x using the exponential function

$$\tan x = \frac{1}{i} \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}.$$

Inserting this expression and multiplying by $(e^{ix} + e^{-ix})^n$ on both sides allows us to deduce that

$$\sum_{k=0}^{n} r_k \left(\frac{e^{ix} - e^{-ix}}{i} \right)^k (e^{ix} + e^{-ix})^{n-k} = 0.$$

Expanding all brackets, we may deduce that

$$\sum_{k=-n}^{n} r_k^* e^{ikx} = 0$$

for some $r_k^* \in \mathbb{Q}[i]$ not all of which are 0. We invoke the Lindemann–Weierstrass theorem in the formulation of Baker [3]: if $b_0, b_1, \ldots, b_m$ are nonzero algebraic numbers and $\beta_0, \beta_1, \ldots, \beta_m$ are distinct algebraic numbers, then

$$b_0 e^{\beta_0} + b_1 e^{\beta_1} + \cdots + b_m e^{\beta_m} \neq 0,$$

and this contradiction completes our proof. ∎

We now prove a little statement showing that integer multiples of fixed points $\tan x = x$ have a well-defined tangent. Equivalently, we want to guarantee that if $\tan \gamma = \gamma$, then $n\gamma = (m + 1/2)\pi$ has no solutions $(n, m) \in \mathbb{N}^2$.

CLAIM. *If $\gamma > 0$ satisfies $\tan \gamma n = \gamma n$ for some $n \in \mathbb{N}_{>0}$, then γ and π are linearly independent over $\mathbb{Q}$.*

Proof. Suppose that the statement fails and
$$\gamma n = (p/q)\pi.$$
Note that, by definition,
$$\tan(\tan(\pi p/q)) = \tan(\tan \gamma n) = \tan \gamma n = \tan(\pi p/q),$$
It is known that $\tan(\pi p/q)$ is an algebraic number (even the degree of the minimal polynomial is known, see [5]). This would be an instance of the tangent mapping the nonzero algebraic number $\tan(\pi p/q)$ to an algebraic number, contradicting the statement proven above. ∎

Suppose now that $(\gamma, m, n) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N}$ is a nontrivial solution of
$$\tan \gamma m = \gamma m,$$
$$\tan \gamma n = \gamma n.$$
Then γ has to be transcendental: if γ were algebraic, then γm would be algebraic, from which we could deduce that $\tan \gamma m$ is transcendental, which contradicts $\tan \gamma m = \gamma m$.

Now in order to derive a final contradiction exploiting the fact that γ is transcendental we use the addition theorem for the tangent:
$$\tan((n+1)x) = \tan(nx + x) = \frac{\tan nx + \tan x}{1 - \tan nx \tan x}.$$
Iterating this multiple-angle formula, we have
$$\tan nx = \frac{p_n(\tan(x))}{q_n(\tan(x))}$$
for two sequences of polynomials with integer coefficients satisfying the initial conditions $p_1(x) = x$ and $q_1(x) = 1$ and the recursion formulas
$$p_{n+1}(x) = p_n(x) + x q_n(x),$$
$$q_{n+1}(x) = q_n(x) - x p_n(x).$$
We know that $(\gamma, m, n) \in \mathbb{R}_{>0} \times \mathbb{N} \times \mathbb{N}$ solves
$$0 = n \tan \gamma m - m \tan \gamma n = n \frac{p_m(\tan \gamma)}{q_m(\tan \gamma)} - m \frac{p_n(\tan \gamma)}{q_n(\tan \gamma)},$$
and therefore
$$0 = n q_n(\tan \gamma) p_m(\tan \gamma) - m q_m(\tan \gamma) p_n(\tan \gamma).$$
It is easy to see that the polynomial on the right-hand side does not vanish identically by checking that
$$\frac{d^3}{dx^3}(n \tan xm - m \tan xn)\Big|_{x=0} = \frac{1}{3}(nm^3 - mn^3) \neq 0.$$

This implies that $\tan\gamma$ satisfies a polynomial equation with integer coefficients and thus $\tan\gamma$ is algebraic, which is a contradiction. ∎

2.3. Theorem 1 implies Theorem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of regularity C^α with $\alpha > 1/2$ such that

$$\left| \bigcup_{x\in\mathbb{R}} \{r_f(x), r_{-f}(x)\} \right| \le 2.$$

Using the symmetries of the maximal function, we may assume without loss of generality that f is periodic with period 2π and has vanishing mean value.

Let us first assume that

$$\left| \bigcup_{x\in\mathbb{R}} \{r_f(x), r_{-f}(x)\} \right| = 1.$$

Since f is periodic, it assumes a global maximum, from which it follows that if r_f were to be constant, it would have to be 0, which implies that f is constant and the statement holds. Thus we can focus on the remaining case of r_f and r_{-f} being two-valued (by the same reasoning, 0 always has to be one of the two values):

$$\bigcup_{x\in\mathbb{R}} \{r_f(x), r_{-f}(x)\} = \{0, \gamma\} \quad \text{for some real number } \gamma > 0.$$

We are now in the case where f is continuous, nonconstant and has vanishing mean; this allows us to partition $\mathbb{R}$ into three nonempty sets

$$
\begin{aligned}
I_1 &= \{x \in \mathbb{R} : f(x) < 0\}, \\
I_2 &= \{x \in \mathbb{R} : f(x) = 0\}, \\
I_3 &= \{x \in \mathbb{R} : f(x) > 0\}.
\end{aligned}
$$

We will now prove that

$$\partial_r \left(\frac{1}{2r} \int_{x-r}^{x+r} f(z)\,dz \right)\Bigg|_{r=\gamma} = 0 \quad \text{for all } x \in \mathbb{R}.$$

This is easy to see on I_1: If $x \in I_1$ and $r_f(x) = 0$, then the maximal function would have the value $f(x) < 0$. However, by taking the maximal interval of length 2π, we can at least get an average value of 0, which exceeds $f(x)$. This implies that $r_f(x) = \gamma$, which implies the statement. A similar argument works for $x \in I_3$, where the same reasoning implies $r_{-f}(x) = \gamma$, which implies

$$\partial_r \left(\frac{1}{2r} \int_{x-r}^{x+r} -f(z)\,dz \right)\Bigg|_{r=\gamma} = 0$$

and gives the statement after multiplication with -1. For $x \in I_2$ we have to

argue a bit differently. Suppose $r_f(x) = 0$. Then define $h : [0, \pi] \to \mathbb{R}$ by

$$h(r) = \frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\, dz.$$

By assumption, we have $h(0) = h(\pi) = 0$ and $h(r) \leq 0$.

If h vanishes identically, the derivative vanishes everywhere and in particular also in γ.

Suppose now that h does not vanish identically. Then it assumes a global minimum on that interval. By definition, this implies that $r_{-f}(x) > 0$ and thus by assumption $r_{-f}(x) = \gamma$, and this implies the statement as before, completing the reduction of Theorem 1 to Theorem 2. ∎

2.4. Theorem 1 implies Theorem 3. Let $\gamma > 0$ be fixed and let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a periodic solution of the delay differential equation

$$f'(x + \gamma) - \frac{1}{\gamma} f(x + \gamma) = -f'(x - \gamma) - \frac{1}{\gamma} f(x - \gamma).$$

This can be rephrased as

$$f'(x + \gamma) + f'(x - \gamma) = \frac{1}{\gamma}(f(x + \gamma) - f(x - \gamma)).$$

Integrating with respect to x yields

$$f(x + \gamma) + f(x - \gamma) = \frac{1}{\gamma} \int\limits_{x-\gamma}^{x+\gamma} f(z)\, dz + c,$$

where $c \in \mathbb{R}$ is some undetermined constant. Since f is periodic with some period P, we can deduce that the average value of the left-hand side is precisely

$$\lim_{y\to\infty} \frac{1}{y} \int\limits_{0}^{y} (f(x + \gamma) + f(x - \gamma))\, dx = \frac{2}{P} \int\limits_{0}^{P} f(x)\, dx.$$

On the other hand,

$$\lim_{y\to\infty} \frac{1}{y} \int\limits_{0}^{y} \left(\frac{1}{\gamma} \int\limits_{x-\gamma}^{x+\gamma} f(z)\, dz + c \right) dx = c + \frac{2}{P} \int\limits_{0}^{P} f(x)\, dx,$$

and thus $c = 0$. Thus, multiplying the equation with $(2\gamma)^{-1}$, we get

$$0 = \frac{1}{2\gamma}(f(x + \gamma) + f(x - \gamma)) - \frac{1}{2\gamma^2} \int\limits_{x-\gamma}^{x+\gamma} f(z)\, dz = \partial_r \left(\frac{1}{2r} \int\limits_{x-r}^{x+r} f(z)\, dz \right) \Bigg|_{r=\gamma}.$$

This is precisely the condition in Theorem 1 (with slightly higher regularity on f) and implies the result. ∎

3. Concluding remarks

3.1. A conjectured stronger statement. It seems reasonable to assume that for a periodic C^1 function $f : \mathbb{R} \to \mathbb{R}$ already

$$\left| \bigcup_{x \in \mathbb{R}} \{ r_f(x) \} \right| \leq 2$$

implies that f has to be a trigonometric function, i.e. that it suffices to demand that the computation of $\mathcal{M}f$ is 'simple' (in the sense described above) and not additionally that the computation of $\mathcal{M}(-f)$ be simple as well. After adding a suitable constant we can assume without loss of generality that f has vanishing mean and, by using the dilation symmetry, that $r_f(x) \in \{0, 1\}$. Then we know that $r_f(x) = 1$ whenever $f(x) < 0$ and that therefore

$$\partial_x \left(\partial_r \left(\frac{1}{2r} \int\limits_{x-r}^{x+r} f(y)\, dy \right) \bigg|_{r=1} \right) = 0 \quad \text{whenever } f(x) < 0.$$

The same explicit computation as before implies that one could derive the following statement.

CONJECTURE. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies

$$f'(x+1) - f(x+1) = -f'(x-1) - f(x-1) \quad \text{whenever } f(x) < 0.$$

Then

$$f(x) = a + b \sin(cx + d) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

This statement would be a quite curious strengthening of Theorem 3.

3.2. A Poincaré inequality. The purpose of this short section is to note a basic observation for the uncentered maximal function (which fails for the centered maximal function): if the uncentered maximal intervals are all rather short, then this should imply the presence of strong oscillation in the function. We give a very simple form of that statement. Let $f \in C^1([0,1])$ and consider the uncentered maximal function $\mathcal{M}^*$ defined via

$$(\mathcal{M}^* f)(x) = \sup_{J \ni x} \frac{1}{|J|} \int\limits_J f(x)\, dx,$$

where J ranges over all intervals $J \subset [0,1]$ containing x. We define analogously $r_f^*(x)$ as the length of the shortest interval necessary to achieve the maximal possible value.

It is clear from examples that, in a loose sense, strong oscillation implies that $r_f^*(x)$ is small: it will be optimal to choose the interval to be either a point evaluation or in such a way that one captures the one or two adjacent large amplitudes. An inverse result can be quantified as follows.
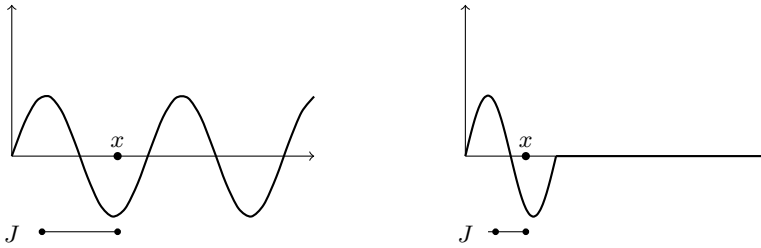
Fig. 2. Two examples where $r_f^*(x)$ is always small: both exhibit strong oscillation.

PROPOSITION. *Assume $f \in C^1([0,1])$ has mean value $\overline{f}$. Then we have the Poincaré inequality*

$$\int_0^1 |f(x) - \overline{f}| \, dx \leq 4\|r_f^*\|_{L^\infty(\mathbb{R})} \int_0^1 |f'(x)| \, dx.$$

If we were to replace $4\|r_f^*\|_{L^\infty}$ by the constant $1/2$, this would be the classical L^1-Poincaré inequality on $[0,1]$. Put differently, for a function $f : [0,1] \to \mathbb{R}$ with vanishing mean, we have an uncertainty relation between the total variation and $\|r_f^*\|_{L^\infty}$:

$$\mathrm{var}(f)\|r_f^*\|_{L^\infty} \geq \frac{1}{4}\int_0^1 |f(x)| \, dx.$$

It is easy to see that the statement has the sharp scaling: consider

$$f(x) = \sin N\pi x \quad \text{where} \quad \|r_f^*\|_{L^\infty} \sim N^{-1} \text{ and } \|f'\|_{L^1} \sim N.$$

Another example is given by taking a positive bump function $\phi \in C_c^\infty(0,1)$ and consider the rescaled function

$$f(x) = a\phi'(bx)$$

where

$$\int_0^1 |f(x) - \overline{f}| \, dx \sim ab^{-1}, \quad \|r^*\|_{L^\infty} \sim b^{-1} \quad \text{and} \quad \|f'\|_{L^1} \sim a.$$

We note that the result is not true for the centered maximal function: the function $f(x) = 1 - (x - 0.5)^2$ (or, more generally, any strictly concave function) satisfies $r_f \equiv 0$.

Proof of Proposition. We may suppose without loss of generality that f has vanishing mean, $\overline{f} = 0$. Now we write

$$\int_0^1 |f(x)| \, dx = 2\int_0^1 \chi_{f<0}(x)|f(x)| \, dx$$

and estimate the number on the right. It is easy to see that the set

$$\{x \in [0,1] : f(x) < 0\}$$

cannot contain an interval of length larger than $2\|r_f^*\|_{L^\infty}$ because $\overline{f} = 0$ implies that the maximal function is always nonnegative (one can always simply choose the entire interval). Clearly, for any $g \in C^1$,

$$\int\limits_s^t |g(x)|\, dx \leq (t-s)\int\limits_s^t |g'(x)|\, dx \quad \text{if } g(s) = 0.$$

We can now use that inequality on every connected component of $\{x \in [0,1] : f(x) < 0\}$; by the reasoning above, we will always have $t - s \leq 2\|r_f^*\|_{L^\infty}$, and therefore

$$\int\limits_0^1 \chi_{f<0}|f|\, dx \leq 2\|r_f^*\|_{L^\infty}\int\limits_0^1 |f'(x)|\, dx,$$

which implies the statement. ∎

It could be quite interesting to understand under which conditions and to which extent such improved Poincaré inequalities are true in higher dimensions and how they depend on the maximal function involved.

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