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Less than one implies zero

by

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Abstract. In this paper we show that from an estimate of the form $\sup_{t\geq 0} \|C(t) - \cos(at)I\| < 1$, we can conclude that $C(t)$ equals $\cos(at)I$. Here $(C(t))_{t\geq 0}$ is a strongly continuous cosine family on a Banach space.

1. Introduction. Let $(T(t))_{t\geq 0}$ denote a strongly continuous semigroup on the Banach space X with infinitesimal generator A. It is well-known that

$$\limsup_{t\to 0^+} \|T(t) - I\| < 1 \tag{1.1}$$

implies that A is a bounded operator (see e.g. [14, Remark 3.1.4]). That the stronger assumption of having

$$\sup_{t\geq 0} \|T(t) - I\| < 1 \tag{1.2}$$

implies that $T(t) = I$ for all $t \geq 0$ seems not to be equally well-known among researchers working in the area of strongly continuous semigroups. The result was proved in the sixties (see e.g. Wallen [15] and Hirschfeld [10]). We refer the reader to [3, Lemma 10] for a more detailed listing of related references.

In this paper we investigate a similar question for cosine families $(C(t))_{t\geq 0}$. Recently, Bobrowski and Chojnacki showed in [3, Theorem 4] that

$$\sup_{t\geq 0} \|C(t) - \cos(at)I\| < 1/2 \tag{1.3}$$

implies $C(t) = \cos(at)I$ for all $t \geq 0$. They used this to conclude that scalar cosine families are isolated points within the space of bounded strongly continuous cosine families acting on a fixed Banach space, equipped with the supremum norm.

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The purpose of this note is to extend the result of [3] by showing that the half in (1.3) may be replaced by one. More precisely, we prove the following.

THEOREM 1.1. *Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family on the Banach space X and let $a \geq 0$. If*

$$\sup_{t\geq 0} \|C(t) - \cos(at)I\| < 1, \tag{1.4}$$

then $C(t) = \cos(at)I$ for all $t \geq 0$.

Between the first draft (arXiv:1310.6202, version 1) and this version of the manuscript, Chojnacki showed in [6] that Theorem 1.1 even holds for cosine families on normed algebras indexed by general abelian groups and without assuming strong continuity. Furthermore, Bobrowski, Chojnacki and Gregosiewicz [4] and independently Esterle [7] extended Theorem 1.1 to

$$\sup_{t\geq 0} \|C(t) - \cos(at)I\| < \frac{8}{3\sqrt{3}} \approx 1.54 \;\Rightarrow\; C(t) = \cos(at)I \;\forall t \geq 0.$$

This is optimal as $\sup_{t\geq 0} |\cos(3t) - \cos(t)| = 8/(3\sqrt{3})$. Again, their results do not require the strong continuity assumption and hold for cosine families on general normed algebras with a unity element.

Let us remark that the case $a = 0$ is special. In a *three-line-proof* [1], Arendt showed that $\sup_{t\geq 0} \|C(t) - I\| < 3/2$ still implies that $C(t) = I$ for all $t \geq 0$. In [13], we proved that for $(C(t))_{t\geq 0}$ strongly continuous,

$$\sup_{t\geq 0} \|C(t) - I\| < 2 \;\Rightarrow\; C(t) = I \;\forall t \geq 0. \tag{1.5}$$

Moreover, we were able to show the following *zero-two law*:

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2 \;\Rightarrow\; \lim_{t\to 0^+} \|C(t) - I\| = 0, \tag{1.6}$$

which can be seen as the cosine families version of (1.1). Recently, Chojnacki [5] and Esterle [8] also extended (1.5) and (1.6), allowing for, not necessarily strongly continuous, cosine familes on general Banach algebras with a unity element (in [5], even general normed algebras with a unity are considered). In the next section we prove Theorem 1.1 for $a \neq 0$ using elementary techniques, which seem to be less involved than the technique used in [3]. As mentioned, the case $a = 0$ can be found in [13] (see also [3, 4, 5, 6]).

2. Proof of Theorem 1.1. Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family on the Banach space X with infinitesimal generator A with domain $D(A)$ and spectrum $\sigma(A)$. For $\lambda \in \mathbb{C}$ that lies in the resolvent set $\rho(A)$, we define $R(\lambda, A) = (\lambda I - A)^{-1}$. For an introduction to cosine families we refer to e.g. [2, 9].

Let us assume that for some $r > 0$,

$$\sup_{t\geq 0} \|C(t) - \cos(at)I\| = r. \tag{2.1}$$

If $a > 0$ we may apply scaling on t. Hence in that situation, without loss of generality we can take $a = 1$, thus

$$\sup_{t\geq 0} \|C(t) - \cos(t)I\| = r. \tag{2.2}$$

The following lemma is essential in proving Theorem 1.1.

LEMMA 2.1. *Let $(C(t))_{t\geq 0}$ be a cosine family such that* (2.1) *holds for $r < 1$ and $a \geq 0$. Then the spectrum of its generator A satisfies $\sigma(A) \subseteq \{-a^2\}$.*

Proof. The case $r = 0$ is trivial, thus let $r > 0$. From (2.1) it follows in particular that the cosine family $(C(t))_{t\geq 0}$ is bounded. Using [9, Lemma 5.4] we conclude that for every $s \in \mathbb{C}$ with positive real part, s^2 lies in the resolvent set of A. Thus the spectrum of A lies on the non-positive real axis.

To determine the spectrum, we use the following identity (see [11, Lemma 4]). For $\lambda \in \mathbb{C}$, $s \in \mathbb{R}$ and $x \in D(A)$ we have

$$\frac{1}{\lambda}\int_0^s \sinh(t-s)C(t)(\lambda^2 I - A)x\,dt = (\cosh(\lambda s)I - C(s))x.$$

By this and the definition of the approximate point spectrum,

$$\sigma_{\mathrm{ap}}(A) = \Big\{\lambda \in \mathbb{C} \;\Big|\; \exists (x_n)_{n\in\mathbb{N}} \subset D(A),\ \|x_n\| = 1,\ \lim_{n\to\infty} \|(A - \lambda I)x_n\| = 0\Big\},$$

it follows that if $\lambda^2 \in \sigma_{\mathrm{ap}}(A)$, then $\cosh(\lambda s) \in \sigma_{\mathrm{ap}}(C(s))$. Hence,

$$\cosh(s\sqrt{\sigma_{\mathrm{ap}}(A)}) \subset \sigma_{\mathrm{ap}}(C(s)), \qquad \forall s \in \mathbb{R}. \tag{2.3}$$

Since $\sigma(A) \subset \mathbb{R}_0^-$, the boundary of the spectrum equals the spectrum. Combining this with the fact that the boundary of the spectrum is contained in the approximate point spectrum, we see that $\sigma(A) = \sigma_{\mathrm{ap}}(A)$. Let $-\lambda^2 \in \sigma(A)$ for $\lambda \geq 0$. Then, by (2.3),

$$\cosh(\pm si\lambda) = \cos(s\lambda) \in \sigma_{\mathrm{ap}}(C(s)), \qquad \forall s \in \mathbb{R}.$$

If $\lambda \neq a$, we can find $\tilde{s} > 0$ such that $|\cos(\tilde{s}\lambda) - \cos(a\tilde{s})| \geq 1$ (see Lemma 2.2). Since $\cos(\tilde{s}\lambda) \in \sigma_{\mathrm{ap}}(C(\tilde{s}))$, we find a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(C(\tilde{s}) - \cos(\tilde{s}\lambda)I)x_n\| = 0$. Since

$$\|(C(\tilde{s}) - \cos(a\tilde{s})I)x_n\| \geq |\cos(\tilde{s}\lambda) - \cos(a\tilde{s})| - \|(C(\tilde{s}) - \cos(\tilde{s}\lambda)I)x_n\|,$$

we conclude that $\|C(\tilde{s}) - \cos(a\tilde{s})I\| \geq 1$. This contradicts assumption (2.1) as $r < 1$. ■

LEMMA 2.2. *If $a, b \geq 0$ and $a \neq b$, then $\sup_{t\geq 0} |\cos(at) - \cos(bt)| > 1$.*

Proof. If $a = 0$, the assertion is clear as $\cos(\pi) = -1$. Hence, let $a, b > 0$. By scaling, it suffices to prove that

$$\forall a \in (0,1)\ \exists s \geq 0: \quad |\cos(as) - \cos(s)| > 1.$$

Since $\cos(2k\pi) = 1$ for $k \in \mathbb{Z}$ and $\cos(as) < 0$ for $t \in \frac{\pi}{a}\big(\frac{1}{2} + 2m, \frac{3}{2} + 2m\big)$, $m \in \mathbb{Z}$, we are done if we find $(k, m) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$k \in \frac{1}{a}\bigg(\frac{1}{4} + m, \frac{3}{4} + m\bigg).$$

This is equivalent to $ka - m \in (1/4, 3/4)$. It is easy to check that for $a \in (2^{-n-1}, 2^{-n}] \cup [1 - 2^{-n-1}, 1 - 2^{-n})$ we can choose $k = 2^{n-1}$ and $m = \lfloor ka \rfloor$. ∎

As mentioned before, we may assume that $a = 1$, and thus we consider equation (2.2) and assume that $r < 1$. Hence we know that the norm of the difference $e(t) = C(t) - \cos(t)I$ is uniformly below one, and we want to show that it equals zero. The idea is to use the following inequality:

$$\bigg\| \int_0^\infty h_n(q,t) e(t)\, dt \bigg\| \leq r \int_0^\infty |h_n(q,t)|\, dt, \tag{2.4}$$

with $h_n(q,t) = e^{-qt} \cos(t)^{2n+1}$, $n \in \mathbb{N}$, where $q > 0$ is an auxiliary variable to be dealt with later.

Since $(C(t))_{t \geq 0}$ is bounded, it is well-known (see e.g. [9, Lemma 5.4]) that for s with $\Re(s) > 0$, $s^2 \in \rho(A)$ and we can define $E(s)$ as the Laplace transform of $e(t)$,

$$E(s) := \int_0^\infty e^{-st} e(t)\, dt = s(s^2 I - A)^{-1} - \frac{s}{s^2+1} I. \tag{2.5}$$

To calculate the left-hand side of (2.4) we need the following two results. We omit the proof of the first as it can easily be checked by the reader.

LEMMA 2.3. *Let $n \in \mathbb{N}$. Then, for all $t \in \mathbb{R}$,*

$$\cos(t)^{2n+1} = \sum_{k=0}^{n} a_{2k+1,2n+1} \cos((2k+1)t),$$

where $a_{2k+1,2n+1} = 2^{-2n}\binom{2n+1}{n-k}$.

PROPOSITION 2.4. *For $h_n(q,t) = -2e^{-qt}\cos(t)^{2n+1}$ and $q > 0$ we have*

$$\int_0^\infty h_n(q,t) e(t)\, dt = a_{1,2n+1} \frac{g(q)}{q} I + a_{1,2n+1} B(A,q) + G(A,q),$$

where a_n is as in Lemma 2.3, $g(q) = \frac{2q^2+4}{q^2+4}$,

$$B(A,q) = R((q+i)^2, A) 2q[A - (q^2+1)I] R((q-i)^2, A),$$

and $G(A,q)$ is such that $\lim_{q \to 0^+} q \cdot G(A,q) = 0$ in the operator norm.

Proof. By Lemma 2.3, we have

$$\int_0^\infty h_n(q,t)e(t)\,dt = -\sum_{k=0}^{n} a_{2k+1,2n+1} 2\int_0^\infty e^{-qt}\cos((2k+1)t)\,e(t)\,dt$$
$$= -\sum_{k=0}^{n} a_{2k+1,2n+1}[E(q+(2k+1)i)+E(q-(2k+1)i)].$$

Let us first consider the term in the sum corresponding to $k=0$. By (2.5),

$$E(q\pm i) = \frac{q\pm i}{q(q\pm 2i)}[q(q\pm 2i)R((q\pm i)^2)-I], \tag{2.6}$$

where $R(\lambda)$ abbreviates $R(\lambda, A)$. Hence,

$$\begin{aligned} E(q+i)+E(q-i) &= -\frac{g(q)}{q} + (q+i)R((q+i)^2)+(q-i)R((q-i)^2)\\ &= -\frac{g(q)}{q} + R((q+i)^2)[(q+i)((q-i)^2-A)\\ &\qquad + ((q-i)^2-A)(q-i)]R((q-i)^2)\\ &= -\frac{g(q)}{q} + R((q+i)^2)2q[q^2I+I-A]R((q-i)^2)\\ &= -\frac{g(q)}{q} - B(A,q). \end{aligned}$$

Thus, it remains to show that $qG(A,q)$ with

$$G(A,q) := -\sum_{k=1}^{n} a_{2k+1,2n+1}[E(q+(2k+1)i)-E(q-(2k+1)i)]$$

goes to 0 as $q\to 0^+$. Let $d_k=(2k+1)i$. By (2.5),

$$E(q\pm(2k+1)i) = (q\pm d_k)R((q\pm d_k)^2) - \frac{q\pm d_k}{(q\pm d_k)^2+1}I.$$

Thus, since $d_k^2\in\rho(A)$ for $k\neq 0$ by Lemma 2.1,

$$\lim_{q\to 0^+} E(q\pm(2k+1)i) = \pm d_kR(d_k^2)\pm\frac{d_k}{d_k^2+1},$$

for $k\neq 0$, hence, $\lim_{q\to 0^+} q\cdot G(A,q)=0$. Therefore, the assertion follows. ∎

LEMMA 2.5. *For any $n\in\mathbb{N}$ and $a_{1,2n+1}$ chosen as in Lemma 2.3 we have:*

- $b_n := \lim_{q\to 0^+} q\cdot\int_0^\infty e^{-qt}|\cos(t)^n|\,dt$ *exists and* $b_n\geq b_{n+1}$*,*
- $a_{1,2n+1} = 2b_{2n+2}$*,*
- $\lim_{n\to\infty}\frac{a_{1,2n+1}}{2b_{2n+1}} = 1$*.*

Proof. Because $t \mapsto |\cos(t)^n|$ is π-periodic,

$$q\int_0^\infty e^{-qt}|\cos(t)^n|\,dt = \frac{q\int_0^\pi e^{-qt}|\cos(t)^n|\,dt}{1-e^{-q\pi}},$$

which goes to $\frac{1}{\pi}\int_0^\pi |\cos(t)^n|\,dt$ as $q \to 0^+$. Furthermore,

$$2_{b_{2n+2}} = \frac{2}{\pi}\int_0^\pi |\cos(t)^{2n+2}|\,dt = \frac{1}{\pi}\int_0^{2\pi} \cos(t)^{2n+1}\cos(t)\,dt$$

equals $a_{1,2n+1}$ by the Fourier series of $\cos(t)^{2n+1}$ (see Lemma 2.3). By the same lemma we find that for $n \geq 1$,

$$\frac{a_{1,2n-1}}{a_{1,2n+1}} = \frac{2^{-2n+2}\binom{2n-1}{n}}{2^{-2n}\binom{2n+1}{n}} = \frac{(2n+1)2n}{4(n+1)n},$$

which goes to 1 as $n \to \infty$. This implies that $\frac{a_{1,2n+1}}{2b_{2n+1}}$ goes to 1 because

$$a_{1,2n+1} = 2b_{2n+2} \leq 2b_{2n+1} \leq 2b_{2n} = a_{1,2n-1}, \qquad n \in \mathbb{N}. \blacksquare$$

Proof of Theorem 1.1. Let $r = 1 - 2\varepsilon$ for some $\varepsilon > 0$. By Lemma 2.5 we can choose $n \in \mathbb{N}$ such that

$$r\frac{2b_{2n+1}}{a_{1,2n+1}} < 1 - \varepsilon. \tag{2.7}$$

Let us abbreviate $a_{1,2n+1}$ by a_{2n+1}. By (2.4) and Proposition 2.4, for $q > 0$,

$$\|a_{2n+1}[g(q)I + qB(A,q)] + G(A,q)\| \leq 2rq\int_0^\infty e^{-qt}|\cos(t)^{2n+1}|\,dt,$$

hence,

$$\left\|I + \frac{q}{g(q)}\left(B(A,q) + \frac{1}{a_{2n+1}}G(A,q)\right)\right\| \leq \frac{2rq}{g(q)a_{2n+1}}\int_0^\infty e^{-qt}|\cos(t)^{2n+1}|\,dt.$$

For $q \to 0^+$, we have $g(q) \to 1^+$, $qG(A,q) \to 0$ by Proposition 2.4, and Lemma 2.5 yields $q\int_0^\infty e^{-qt}|\cos(t)^{2n+1}|\,dt \to b_{2n+1}$. Thus, there exists $q_0 > 0$ (depending only on ε and n) such that

$$\left\|I + \frac{q}{g(q)}B(A,q)\right\| \leq r\frac{2b_{2n+1}}{a_{2n+1}} + \varepsilon =: \delta, \qquad \forall q \in (0,q_0),$$

Since $\delta < 1$ by (2.7), $B(A,q)$ is invertible for $q \in (0,q_0)$. Moreover,

$$\|B(A,q)^{-1}\| \leq \frac{q}{g(q)}\cdot\frac{1}{1-\delta}.$$

Since for $x \in D(A)$,

$$B(A,q)^{-1}x = \frac{1}{2}((q-i)^2 - A)q^{-1}[A - (q^2+1)I]^{-1}((q+i)^2 - A)x,$$

we conclude that

$$\|((q-i)^2 - A)R(q^2+1, A)((q+i)^2 - A)x\| \le \frac{q^2}{g(q)} \cdot \frac{2}{1-\delta}\|x\|.$$

As $q \to 0^+$, the right-hand side goes to 0, whereas the left-hand side tends to $\|(I+A)[A-I]^{-1}(I+A)x\|$ as $1 \in \rho(A)$. Since $-1 \in \rho(A)$, we derive $(I+A)x = 0$. Therefore, $A = -I$, since $D(A)$ is dense in X. ∎

Remark 2.6. A related question is if condition (1.4) can be replaced by

$$\limsup_{t\to\infty} \|C(t) - \cos(at)\| = r,$$

for some $r < 1$ (or even some $r \ge 1$) such that Theorem 1.1 still holds. For $a = 0$, an affirmative answer was given in [12] for $r = 2$. There, the techniques used rely on the results obtained by Esterle [8]. For $a > 0$, it seems that a similar approach might work. This is the subject of ongoing work.

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