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NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURES

PAUL F. BAUM

Mathematics Department, McAllister Building, The Pennsylvania State University
University Park, PA 16802, USA;
Instytut Matematyczny Polskiej Akademii Nauk
ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail: baum@math.psu.edu

LUDWIK DĄBROWSKI

SISSA (Scuola Internazionale Superiore di Studi Avanzati)
Via Bonomea 265, 34136 Trieste, Italy
E-mail: dabrow@sissa.it

PIOTR M. HAJAC

Instytut Matematyczny Polskiej Akademii Nauk
ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail: pmh@impan.pl

Abstract. Within the framework of free actions of compact quantum groups on unital C*-algebras, we propose two conjectures. The first one states that, if $\delta\colon A \to A \otimes_{\min} H$ is a free coaction of the C*-algebra H of a non-trivial compact quantum group on a unital C*-algebra A, then there is no H-equivariant $*$-homomorphism from A to the equivariant join C*-algebra $A \circledast_\delta H$. For A being the C*-algebra of continuous functions on a sphere with the antipodal coaction of the C*-algebra of functions on $\mathbb{Z}/2\mathbb{Z}$, we recover the celebrated Borsuk–Ulam Theorem. The second conjecture states that there is no H-equivariant $*$-homomorphism from H to the equivariant join C*-algebra $A \circledast_\delta H$. We show how to prove the conjecture in the special case $A = C(SU_q(2)) = H$, which is tantamount to showing the non-trivializability of Pflaum's quantum instanton fibration built from $SU_q(2)$.

1. Introduction. The Borsuk–Ulam Theorem is a fundamental theorem of topology with an enormous amount of corollaries and generalizations [16]. Having a noncommutative generalization would provide further evidence for its fundamental nature.

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THEOREM 1.1 ([3]). *Let n be a positive natural number. If $f\colon S^n \to \mathbb{R}^n$ is continuous, then there exists a pair $(p,-p)$ of antipodal points on S^n such that $f(p) = f(-p)$.*

Assuming that both temperature and pressure are continuous functions, we can conclude that there are always two antipodal points on Earth with exactly the same pressure and temperature.

The logical negation of the theorem yields: There exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for all pairs $(p,-p)$ of antipodal points on S^n we have $f(p) \neq f(-p)$. For the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^n and $\mathbb{R}^n$, the latter statement is equivalent to: There exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.

Indeed, if $f\colon S^n \to \mathbb{R}^n$ is a continuous map with $f(p) \neq f(-p)$ for all $p \in S^n$, then the formula

$$\widetilde{f}(p) := \frac{f(p) - f(-p)}{\|f(p) - f(-p)\|} \tag{1.1}$$

defines a continuous $\mathbb{Z}/2\mathbb{Z}$-equivariant map from S^n to S^{n-1}. Also, composing any such map with the inclusion map $S^{n-1} \subset \mathbb{R}^n$ gives a nowhere vanishing continuous map $f\colon S^n \to \mathbb{R}^n$ with $f(-p) = -f(p) \neq f(p)$ for all $p \in S^n$. Consequently, the Borsuk–Ulam Theorem is equivalent to:

THEOREM 1.2 (equivariant formulation). *Let n be a positive natural number. There does* not *exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^n \to S^{n-1}$.*

Our next step is to reformulate the Borsuk–Ulam Theorem in terms of the join construction.

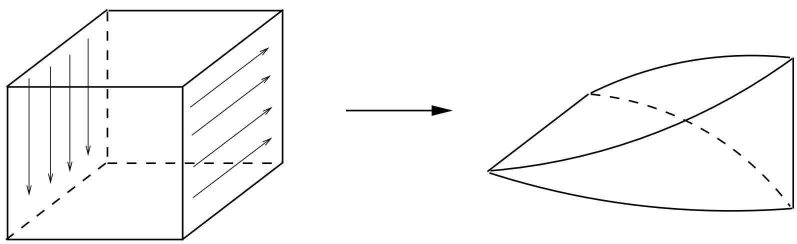
1.1. Classical equivariant join construction. Let $I = [0,1]$ be the closed unit interval and let X be a topological space. The *unreduced suspension* ΣX of X is the quotient of $I \times X$ by the equivalence relation R_S generated by

$$(0,x) \sim (0,x'), \qquad (1,x) \sim (1,x'). \tag{1.2}$$

Now take another topological space Y and, on the space $I \times X \times Y$, consider the equivalence relation R_J given by

$$(0,x,y) \sim (0,x',y), \qquad (1,x,y) \sim (1,x,y'). \tag{1.3}$$

The quotient space $X * Y := (I \times X \times Y)/R_J$ is called the *join* of X and Y. It resembles the unreduced suspension of $X \times Y$, but with only X collapsed at 0, and only Y collapsed at 1.



In particular, if Y is a one-point space, the join $X * Y$ is the *cone* CX of X. If Y is a two-point space with discrete topology, then the join $X * Y$ is the unreduced suspension ΣX of X.

If G is a topological group acting continuously on X and Y from the right, then the diagonal right G-action on $X \times Y$ induces a continuous action on the join $X * Y$. Indeed, the diagonal action of G on $I \times X \times Y$ factorizes to the quotient, so that the formula

$$\big([(t,x,y)],g\big) \longmapsto [(t,xg,yg)] \tag{1.4}$$

makes $X * Y$ a right G-space. It is immediate that this continuous action is free if the G-actions on X and Y are free.

If $Y = G$ with the right action assumed to be the group multiplication, we can construct the join G-space $X * Y$ in a different manner: at 0 we collapse $X \times G$ to G as before, and at 1 we collapse $X \times G$ to $(X \times G)/R_D$ instead of X. Here R_D is the equivalence relation generated by $(x,h) \sim (x',h')$, where $xh = x'h'$. More precisely, let R'_J be the equivalence relation on $I \times X \times G$ generated by

$$(0,x,h) \sim (0,x',h) \quad \text{and} \quad (1,x,h) \sim (1,x',h'), \text{ where } xh = x'h'. \tag{1.5}$$

The formula $[(t,x,h)]k := [(t,x,hk)]$ defines a continuous right G-action on $(I \times X \times G)/R'_J$. One can also easily check that the formula

$$X * G \ni [(t,x,h)] \longmapsto [(t,xh^{-1},h)] \in (I \times X \times G)/R'_J \tag{1.6}$$

yields a G-equivariant homeomorphism.

If we further specify also $X = G$ with the right action assumed to be the group multiplication, then the G-action on $X * Y = G * G$ is automatically free. Furthermore, since the action of G on $X * G$ is free whenever it is free on X, we conclude that the natural action on the iterated join of G with itself is also free. For instance, for $G = \mathbb{Z}/2\mathbb{Z}$ we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant identification $(\mathbb{Z}/2\mathbb{Z})^{*(n+1)} \cong S^n$, where S^n is the n-dimensional sphere with the antipodal action of $\mathbb{Z}/2\mathbb{Z}$.

1.2. Equivariant join of unital C*-algebras with free compact quantum group actions

Definition 1.3 (cf. Definition 4.1 in [5]). Let A_1 and A_2 be unital C*-algebras. We call the unital C*-algebra

$$A_1 \circledast A_2 := \big\{x \in C([0,1]) \otimes_{\min} A_1 \otimes_{\min} A_2 \;\big|\; (\mathrm{ev}_0 \otimes \mathrm{id})(x) \in A_2,\ (\mathrm{ev}_1 \otimes \mathrm{id})(x) \in A_1\big\}$$

the *join* C*-algebra of A_1 and A_2. Here $\otimes_{\min}$ stands for the spatial (minimal) tensor product, and $\mathrm{ev}_r\colon C([0,1]) \ni f \mapsto f(r) \in \mathbb{C}$ is the evaluation map at r on the C*-algebra of continuous functions on the unit interval.

Note that, due to the fact that minimal tensor products preserve injections (e.g., see [22, Proposition 4.22] or [24, Section 1.3]), for any unital C*-algebras A_1 and A_2 the natural maps

$$A_1 \ni a \longmapsto a \otimes 1 \in A_1 \otimes_{\min} A_2 \quad \text{and} \quad A_2 \ni a \longmapsto 1 \otimes a \in A_1 \otimes_{\min} A_2 \tag{1.7}$$

are injective. Observe also that, if $A_1 := C(X)$ and $A_2 := C(Y)$ are the C*-algebras of continuous functions on compact Hausdorff spaces X and Y respectively, then

$$A_1 \circledast A_2 = C(X * Y). \tag{1.8}$$

Next, let A be a unital C*-algebra, (H, Δ) a compact quantum group, and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a *coaction* of H on A (or an action of the compact quantum group (H, Δ) on A) iff

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \,|\, a \in A,\, h \in H\}^{\mathrm{cls}} = A \otimes_{\min} H$ (counitality).

Here "cls" stands for "closed linear span". A coaction δ is called *free* [9] iff

$$\big\{(x \otimes 1)\delta(y) \,|\, x, y \in A\big\}^{\mathrm{cls}} = A \otimes_{\min} H. \tag{1.9}$$

Since a diagonal coaction is not in general an algebra homomorphism, to obtain an equivariant version of our noncommutative join construction, we need to modify Definition 1.3 in the spirit of (1.5)–(1.6).

Definition 1.4 (cf. Definition 5.1 in [5]). Let (H, Δ) be a compact quantum group acting on a unital C*-algebra A via $\delta : A \to A \otimes_{\min} H$. We call the unital C*-algebra

$$A \circledast_\delta H := \big\{ f \in C([0,1], A \otimes_{\min} H) \;\big|\; f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \big\}$$

the *equivariant noncommutative join* of A and H.

Remark 1.5. The above definition overcomes the difficulty of constructing a join that is equivariant for the diagonal action of a quantum group by "gauging" the join construction of Definition 1.3. Another way of overcoming this difficulty is to use a braiding: see [7] and [18] for a bi-Galois and C*-algebraic approach respectively.

Theorem 1.6 ([1, Lemma 5.5 and Corollary 5.6]). *Let (H, Δ) be a compact quantum group, and A be a unital C*-algebra equipped with a free coaction $\delta\colon A \to A \otimes_{\min} H$. The $*$-homomorphism*

$$\mathrm{id} \otimes \Delta\colon C([0,1], A) \otimes_{\min} H \longrightarrow C([0,1], A) \otimes_{\min} H \otimes_{\min} H$$

restricts and corestricts to

$$\delta_\Delta\colon A \circledast_\delta H \longrightarrow (A \circledast_\delta H) \otimes_{\min} H.$$

Moreover, the thus defined action of the compact quantum group (H, Δ) on the join C-algebra $A \circledast_\delta H$ is* free.

Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations (see Woronowicz's Peter–Weyl theory of compact quantum groups [25]). Moreover, denoting by $\otimes$ the purely algebraic tensor product over the field $\mathbb{C}$ of complex numbers, we define the *Peter–Weyl subalgebra* of A [1] as

$$\mathcal{P}_H(A) := \big\{ a \in A \,|\, \delta(a) \in A \otimes \mathcal{O}(H) \big\}. \tag{1.10}$$

Using the coassociativity of δ, one can check that $\mathcal{P}_H(A)$ is a right $\mathcal{O}(H)$-comodule algebra. In particular, $\mathcal{P}_H(H) = \mathcal{O}(H)$. The assignment $A \mapsto \mathcal{P}_H(A)$ is functorial with respect to equivariant unital $*$-homomorphisms and comodule algebra maps. We call it the *Peter–Weyl functor*.

1.3. Pullback structure of equivariant joins. Let (H, Δ) be a compact quantum group, $\mathcal{O}(H)$ be its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations. Set

$$\begin{aligned} \mathcal{P}_1 &:= \left\{ f \in C([0, \tfrac{1}{2}], H) \otimes \mathcal{O}(H) \,|\, f(0) \in \mathbb{C} \otimes \mathcal{O}(H) \right\}, \\ \mathcal{P}_2 &:= \left\{ f \in C([\tfrac{1}{2}, 1], H) \otimes \mathcal{O}(H) \,|\, f(1) \in \Delta(\mathcal{O}(H)) \right\}, \\ \mathcal{P}_{12} &:= H \otimes \mathcal{O}(H). \end{aligned} \tag{1.11}$$

Here we identify elements of $C(I, H) \otimes \mathcal{O}(H)$ with functions $I \to H \otimes \mathcal{O}(H)$. The spaces just defined are $\mathcal{O}(H)$-comodule algebras with respect to the coaction $\mathrm{id} \otimes \Delta$. The coaction-invariant subalgebras $\mathcal{P}_i^{co\,\mathcal{O}(H)}$, $i = 1, 2$, can be identified respectively with the "left" and "right" cone of H:

$$\begin{aligned} C_1 H &:= \left\{ f \in C([0, \tfrac{1}{2}], H) \,|\, f(0) \in \mathbb{C} \right\}, \\ C_2 H &:= \left\{ f \in C([\tfrac{1}{2}, 1], H) \,|\, f(1) \in \mathbb{C} \right\}. \end{aligned} \tag{1.12}$$

We also see that $\mathcal{P}_{12}^{co\,\mathcal{O}(H)} = H$ and $\mathcal{P}_1 \cong C_1 H \otimes \mathcal{O}(H)$, $\mathcal{P}_2 \cong C_2 H \,\#\, \mathcal{O}(H)$. Here the smash product $\#$ is given by the adjoint action of $\mathcal{O}(H)$ on $C_2 H$:

$$(a \triangleright f)(t) = a_{(1)} f(t) S(a_{(2)}), \quad a \in H, \quad f \in C_2 H, \quad t \in [0, 1]. \tag{1.13}$$

Now let $\pi_i^P : \mathcal{P}_i \to \mathcal{P}_{12}$, $i = 1, 2$, be the evaluation maps at $\frac{1}{2}$. Then

$$\mathcal{P}_1 \underset{\mathcal{P}_{12}}{\times} \mathcal{P}_2 := \left\{ (p_1, p_2) \in \mathcal{P}_1 \times \mathcal{P}_2 \,|\, \pi_1^P(p_1) = \pi_2^P(p_2) \right\} \tag{1.14}$$

is the pullback comodule algebra for the diagram

$$\begin{array}{ccccc} & & \mathcal{P}_1 \underset{\mathcal{P}_{12}}{\times} \mathcal{P}_2 & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathcal{P}_1 & & & & \mathcal{P}_2\,. \\ & \searrow_{\pi_1^P} & & \swarrow_{\pi_2^P} & \\ & & \mathcal{P}_{12} & & \end{array} \tag{1.15}$$

Since the aforementioned evaluation maps are surjective, we conclude from [10] that all four comodule algebras in this diagram are principal. Furthermore, one can check [4] that the Peter–Weyl comodule algebra of the join C*-algebra $H \circledast_\Delta H$ coincides with the above pullback comodule algebra:

$$\mathcal{P}_H(H \circledast_\Delta H) \cong \mathcal{P}_1 \underset{\mathcal{P}_{12}}{\times} \mathcal{P}_2\,. \tag{1.16}$$

In diagram (1.15), the coaction-invariant subalgebras are C*-algebras. They assemble into the pullback diagram along the evaluation maps at $\frac{1}{2}$:

$$\begin{array}{ccccc} & & \Sigma H & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ C_1 H & & & & C_2 H\,, \\ & \searrow_{\pi_1} & & \swarrow_{\pi_2} & \\ & & H & & \end{array} \tag{1.17}$$

where ΣH is the unreduced suspension C*-algebra of H.

We end this section by unravelling the Milnor construction [17] for the specific case of the above pullback of unital C*-algebras. Given a finite-dimensional complex vector space V and an isomorphism of left H-modules $\chi : H \otimes V \to H \otimes V$, we construct the finitely generated projective left ΣH-module $M(C_1H \otimes V, C_2H \otimes V, \chi)$ (see [6]) as the pullback of the free left C_1H-module $C_1H \otimes V$ and the free left C_2H-module $C_2H \otimes V$:

$$\begin{array}{ccc} & M\,(C_1H \otimes V, C_2H \otimes V, \chi) & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ C_1H \otimes V & & C_2H \otimes V \\ \downarrow{\scriptstyle \mathrm{ev}_{1/2}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}_{1/2}\otimes\mathrm{id}} \\ H \otimes V & \xrightarrow[\chi]{} & H \otimes V\,. \end{array} \tag{1.18}$$

2. Noncommutative Borsuk–Ulam framework

2.1. Borsuk–Ulam Theorem for quantum spheres. Applying the Gelfand transform, Theorem 1.2 translates to:

$$\nexists\ \mathbb{Z}/2\mathbb{Z}\text{-equivariant } *\text{-homomorphism } C(S^n) \longrightarrow C(S^{n+1}). \tag{2.1}$$

Replacing the commutative C*-algebras of functions on spheres by noncommutative C*-algebras of q-deformed spheres, we obtain a noncommutative version of the Borsuk–Ulam Theorem.

In particular, we can consider it for the case of the equatorial Podleś quantum two-sphere [20]. We tensor the C*-algebra $C(S^2_{q\infty})$ of the equatorial Podleś quantum sphere with the algebra of continuous functions on the unit circle, act on the tensor product with the diagonal antipodal $\mathbb{Z}/2\mathbb{Z}$-action, and consider the invariant subalgebra. This gives a $U(1)$-C*-algebra A with the quantum real projective space C*-algebra $C(RP_q(2))$ (see [11]) as its $U(1)$-invariant part. Using the identity representation of $U(1)$, we associate with it a finitely generated projective module over $C(RP_q(2))$. One can prove that this module is not stably free [2], which implies that A cannot be a crossed product of $C(RP_q(2))$ and the integers. This proves the quantum Borsuk–Ulam Theorem for $n = 1$. For $n = 1$ and $q = 1$, this is a proof of the weather-on-Earth case of the Borsuk–Ulam Theorem (see the introduction).

For the arbitrary-dimension quantum spheres introduced in [23, 15], the quantum Borsuk–Ulam Theorem

$$\nexists\ C(\mathbb{Z}/2\mathbb{Z})\text{-equivariant } *\text{-homomorphism } C(S^n_q) \longrightarrow C(S^{n+1}_q) \circledast_\delta C(\mathbb{Z}/2\mathbb{Z}) \tag{2.2}$$

is proven in [26, Theorem 3]. Let us emphasize that our noncommutative generalization of (2.1) is different from the one proved in [26].

2.2. Noncommutative Borsuk–Ulam–type conjectures. The Borsuk–Ulam Theorem is equivalent to:

Theorem 2.1 (join formulation). *Let n be a positive natural number. There does* not *exist a $\mathbb{Z}/2\mathbb{Z}$-equivariant continuous map $\widetilde{f}\colon S^{n-1} * \mathbb{Z}/2\mathbb{Z} \to S^{n-1}$.*

This naturally leads to a classical Borsuk–Ulam–type conjecture:

CONJECTURE 2.2. *Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group G. Then, for the diagonal action of G on $X * G$, there does* not *exist a G-equivariant continuous map $f : X * G \to X$.*

Particular cases of the above statement going beyond the Borsuk–Ulam Theorem have already been studied in [8, 12, 13] (cf. [21] for weaker results for non-free $\mathbb{Z}/2\mathbb{Z}$-actions and maps from X to S^1).

Thus we have arrived at the main point of this paper:

CONJECTURE 2.3. *Let A be a unital C^*-algebra with a free action of a non-trivial compact quantum group (H, Δ). Also, let $A \circledast_\delta H$ be the equivariant noncommutative join C^*-algebra of A and H with the induced free action of (H, Δ) given by δ_Δ. Then*

TYPE 1 $\boxed{\nexists \ H\text{-equivariant } *\text{-homomorphism } A \longrightarrow A \circledast_\delta H}$,

TYPE 2 $\boxed{\nexists \ H\text{-equivariant } *\text{-homomorphism } H \longrightarrow A \circledast_\delta H}$.

Here the H-equivariance is defined with respect to coactions δ and δ_Δ (type 1), and with respect to Δ and δ_Δ (type 2).

3. Noncontractibility of compact quantum groups. In this section, we consider the special case $(A, \delta) = (H, \Delta)$ when the two types of Conjecture 2.3 coincide. In the classical setting, the thus restricted Conjecture 2.3 boils down to the non-contractibility of non-trivial compact Hausdorff groups, which is well known [14].

3.1. General setting. If X is a compact Hausdorff principal G-bundle, $A = C(X)$ and $H = C(G)$, then Conjecture 2.3 type 2 states that the principal G-bundle $X * G$ is not trivializable unless G is trivial. This is clearly true because otherwise $G * G$ would be trivializable, which is tantamount to G being contractible, and the only compact Hausdorff contractible group is trivial. On the other hand, we do not know whether Conjecture 2.3 type 1 holds in general in the commutative case (see Conjecture 2.2).

DEFINITION 3.1. Let P be a unital C*-algebra equipped with a free action of a compact quantum group (H, Δ) implemented by $\delta\colon P \to P \otimes_{\min} H$, and let B denote the fixed-point subalgebra for this action. We call such a triple (P, B, H) a *compact quantum principal bundle*. We say that a compact quantum principal bundle (P, B, H) is *trivializable* iff there exists an H-equivariant $*$-homomorphism $j : H \to P$, i.e. a $*$-homomorphism j such that $(j \otimes \mathrm{id}) \circ \Delta = \delta \circ j$.

PROPOSITION 3.2. *Let (H, Δ) be a compact quantum group and let $\mathcal{O}(H)$ be its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations. For any finite-dimensional corepresentation $\varrho : V \to \mathcal{O}(H) \otimes V$, if (P, B, H) is a trivializable compact quantum principal bundle, then the associated left B-module $\mathcal{P}_H(P) \Box_{\mathcal{O}(H)} V$ is free.*

Proof. Due to the H-colinearity of j, for any $h \in H$, we obtain $j(h_{(1)}) \otimes h_{(2)} = \delta(j(h))$. This shows that $j(\mathcal{O}(H)) \subseteq \mathcal{P}_H(P)$. Therefore, $\mathcal{P}_H(P)$ is a smash-product comodule algebra. Hence the associated left B-module $\mathcal{P}_H(P) \Box_{\mathcal{O}(H)} V$ is free. ■

THEOREM 3.3. *Let* (H,Δ) *be a compact quantum group,* $\mathcal{O}(H)$ *be its dense Hopf* $*$*-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations, and* $V \overset{\varrho}{\to} \mathcal{O}(H)\otimes V$ *be a finite-dimensional corepresentation of* $\mathcal{O}(H)$*. Next, let* $\mathcal{P}_H(H \circledast_\Delta H)$ *be the Peter–Weyl comodule algebra of the equivariant noncommutative join* $H \circledast_\Delta H$*, let* ΣH *be the unreduced suspension C*-algebra of* H *(the coaction-invariant subalgebra* $\mathcal{P}_H(H \circledast_\Delta H)^{co\,\mathcal{O}(H)}$*), and let* C_1H *and* C_2H *be respectively the left and right cones of* H*. Then*

$$\mathcal{P}_H(H \circledast_\Delta H)\Box_{\mathcal{O}(H)}V \cong M\big(C_1H\otimes V,\, C_2H\otimes V,\, (m\otimes\mathrm{id})\circ(\mathrm{id}\otimes(S\otimes\mathrm{id})\circ\varrho)\big) \tag{3.1}$$

as left modules over ΣH*. Here* m *and* S *are respectively the multiplication and the antipode of the Hopf algebra* $\mathcal{O}(H)$*, and the right-hand-side module is defined as the pullback module of* (1.18)*.*

Proof. Consider C_iH-module isomorphisms Λ_i, $i=1,2$, given by

$$\begin{gathered}\Lambda_i\colon C_iH\otimes V \overset{\mathrm{id}\otimes\varrho}{\longrightarrow} C_iH\otimes\mathcal{O}(H)\Box_{\mathcal{O}(H)}V \overset{J_i\otimes\mathrm{id}}{\longrightarrow} \mathcal{P}_i(H\circledast_\Delta H)\Box_{\mathcal{O}(H)}V,\\ J_i\colon C_iH\otimes\mathcal{O}(H)\ni c_i\otimes h\longmapsto c_ij_i(h)\in\mathcal{P}_i(H\circledast_\Delta H),\\ j_1\colon \mathcal{O}(H)\ni h\longmapsto 1\otimes h\in\mathcal{P}_1(H\circledast_\Delta H),\\ j_2\colon \mathcal{O}(H)\ni h\longmapsto (t\mapsto h_{(1)})\otimes h_{(2)}\in\mathcal{P}_2(H\circledast_\Delta H).\end{gathered} \tag{3.2}$$

Since the modules of (3.1) have the same pullback structure (see Section 1.3), they are isomorphic if the following equivalence holds:

$$\big((m\otimes\mathrm{id})\circ(\mathrm{id}\otimes(S\otimes\mathrm{id})\circ\varrho)\circ(\mathrm{ev}_{1/2}\otimes\mathrm{id})\big)(b_1) = \big(\mathrm{ev}_{1/2}\otimes\mathrm{id}\big)(b_2) \tag{3.3}$$

$$\Updownarrow$$

$$\big((\mathrm{ev}_{1/2}\otimes\mathrm{id})\circ\Lambda_1\big)(b_1) = \big((\mathrm{ev}_{1/2}\otimes\mathrm{id})\circ\Lambda_2\big)(b_2). \tag{3.4}$$

On simple tensors $b_i = c_i\otimes v_i$, the formula (3.3) boils down to

$$c_1(\tfrac12)S(v_{1(-1)})\otimes v_{1(0)} = c_2(\tfrac12)\otimes v_2\,, \tag{3.5}$$

and the formula (3.4) becomes

$$c_1(\tfrac12)v_{1(-1)}\otimes v_{1(0)} = c_2(\tfrac12)v_{2(-2)}v_{2(-1)}\otimes v_{1(0)}. \tag{3.6}$$

Applying twice the isomorphism $(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varrho)$ (its inverse is $(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes(S\otimes\mathrm{id})\circ\varrho)$) to both sides of (3.5) yields (3.6). ∎

3.2. Quantum instanton bundle. Assume now that H is the C*-algebra $C(SU_q(2))$ of the quantum group $SU_q(2)$. Then the equivariant noncommutative join

$$C(SU_q(2))\circledast_\Delta C(SU_q(2))$$

is Pflaum's quantum instanton fibration [19, 10]. Let α and β be the usual generators of $C(SU_q(2))$. Also, let

$$U := \begin{bmatrix}\alpha & -q\beta^*\\ \beta & \alpha^*\end{bmatrix}\in U_2(C(SU_q(2)))\subset M_2(C(SU_q(2))), \tag{3.7}$$

and let $V = \mathbb{C}^2$ be a left $\mathcal{O}(H)$-comodule via

$$\varrho_U : \mathbb{C}^2 \longrightarrow \mathcal{O}(H) \otimes \mathbb{C}^2, \qquad \varrho_U(e_i) := \sum_{j=1}^{2} U_{ij} \otimes e_j\,, \tag{3.8}$$

where $\{e_i\}_i$ is the standard basis of $\mathbb{C}^2$. (This corepresentation of $C(SU_q(2))$ is usually called the *fundamental representation* of $SU_q(2)$.) It follows from (3.1) that the left $\Sigma C(SU_q(2))$-module

$$E := \mathcal{P}_{C(SU_q(2))}\big(C(SU_q(2)) \circledast_\Delta C(SU_q(2))\big)\Box_{\mathcal{O}(C(SU_q(2)))}\mathbb{C}^2 \tag{3.9}$$

is isomorphic to the pullback module (1.18) for $\chi = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (S \otimes \mathrm{id}) \circ \varrho_U)$. Hence, by [6, Theorem 2.1], an idempotent p representing the K_0-class of E can be computed via [6, (2.7)] using the unitary matrix $a = U^*$. Moreover, as explained in [6, p. 77], there exists an even Fredholm module whose index pairing with E is equal to the index pairing of U^* with an appropriate odd Fredholm module. As the latter equals $-1 \neq 0$, we infer that the module E is not free. Therefore, Proposition 3.2 implies that Pflaum's quantum instanton fibration is not trivializable, i.e. Conjecture 2.3 holds. As explained at the beginning of Section 3.1, this means that $SU_q(2)$ is *not* contractible.

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