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QUANTUM HOMOGENEOUS SUPERSPACES AND QUANTUM DUALITY PRINCIPLE

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Abstract. We define the concept of quantum section of a line bundle of a homogeneous superspace and we employ it to define the concept of quantum homogeneous projective superspace. We also suggest a generalization of the QDP to the quantum supersetting.

1. Introduction. The purpose of this paper is to define the notion of quantum homogeneous projective and affine supervarieties via the idea of *quantum section*; then we proceed to describe the quantum duality principle in this setting. We also want to reinterpret within this framework the construction of the quantum chiral conformal and Minkowski superspaces appeared in [6, 7, 19], which are of considerable importance in physics.

This goal is achieved with a careful rewriting of the definitions and results appearing in [9] in the supersetting. Such a rewriting is complicated by the fact that, in general, the quotient of a simple algebraic supergroup G by a parabolic subgroup P is not a projective supervariety and many of the results in this framework will not hold, due to the rigidity of the projective superspace. We are then forced to assume the existence of an ample line bundle corresponding to a projective embedding of G/P. This is clearly a restrictive hypothesis, which however has very interesting physical applications, since the examples that we are going to discuss, namely the quantum conformal and Minkowski superspaces, indeed satisfy such requirement.

Through the examples of the quantum conformal and Minkowski superspaces we then take a glimpse of the Quantum Duality Principle (QDP), originally stated by Drinfeld in [11] and later on described more in details by Gavarini in [22]. The more general theory of QDP in the supercontext is very interesting, but not available yet, we plan to explore it in a forthcoming paper.

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2. Supergeometry preliminaries. In this section we recall few facts about projective supergeometry. We give an approach which is most suited for the quantization that we introduce in the next sections.

This material is generally known, though maybe not in this form. For more details see [25, 26] and [3], [19].

Let $\Bbbk$ be our ground field, $\mathrm{char}(\Bbbk) = 0$, $\Bbbk$ algebraically closed.

Definition 2.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ endowed with a sheaf of superalgebras $\mathcal{O}_S$ such that the stalk at a point $x \in |S|$ denoted by $\mathcal{O}_{S,x}$ is a local superalgebra.

Definition 2.2. We say that a superalgebra A is *affine* if A_1 is finitely generated as A_0 module and A/J is an ordinary affine algebra (i.e. it is finitely generated and with no nilpotents), where J is the ideal generated by the odd elements.

We define the *affine supervariety* X associated with A, the superspace $X = (|X|, \mathcal{O}_X)$, where $|X|$ is the topological space of the ordinary affine variety defined by A/J, while $\mathcal{O}_X$ is the sheaf of superalgebras defined as follows: it is the unique sheaf on $|X|$ with $\mathcal{O}_X(U_f) = A_f$, for $f \in A_0$ and $U_f = \big\{x \in |X| \bigm| f(x) \neq 0\big\}$ and with global sections A. We shall denote with X_r, the ordinary algebraic variety corresponding to the affine algebra A/J.

A *morphism* of affine supervarieties is a morphism of the underlying superspaces, though one can readily see that it corresponds (contravariantly) to a morphism of the corresponding coordinate superalgebras:

$$\{\text{morphisms } X \longrightarrow Y\ \} \qquad \longleftrightarrow \qquad \{\text{morphisms } \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)\ \}$$

We define *algebraic supervariety* a superspace which is locally isomorphic to an affine supervariety.

We are interested in algebraic supervarieties which admit an embedding into the projective superspace $\mathbb{P}^{m|n}$ (see [3], Ch. 10 for its definition and more details on the embeddings).

A morphism of a supervariety X into the projective superspace $\mathbb{P}^{m|n}$ is associated with a $1|0$-line bundle, as it happens for the classical setting.

Definition 2.3. We say that a line bundle $\mathcal{L}$ on X is *very ample*, if it corresponds to a projective embedding of the supervariety X; we say that $\mathcal{L}$ is *ample* if $\mathcal{L}^n$ is very ample.

We are now restricting our attention to the supervarieties of the form G/P, for G an algebraic supergroup, G_r reductive, P a closed algebraic subsupergroup of G, P_r a parabolic subgroup of G_r.

Let $\chi : P \longrightarrow \Bbbk^\times$ be a character of P. We can associate to χ a line bundle $\mathcal{L}$ on G/P whose global sections are (see [5]):

$$H^0(G/P, \mathcal{L}) = \big\{ f : G \longrightarrow \Bbbk^{1|1} \,|\, f(gh) = \chi^{-1}(h) f(g) \big\}$$

where we employ the functor of points notation, that we are unable to explain here (see [3], Ch. 3 for more details). In the forthcoming paper [5], discussing the theory of supersymmetric spaces, there will be a thorough discussion on several equivalent ways to define vector bundles on the quotient G/P.

From now on we assume $\mathcal{L}$ to be very ample.

DEFINITION 2.4. Let G be an algebraic supergroup, P a subsupergroup of G such that G/P is an algebraic supervariety. Let $\chi : P \longrightarrow \Bbbk^\times$ be a character of P and $\mathcal{L}$ the corresponding line bundle as above. Define:

$$\mathcal{R}(G/P, \mathcal{L}) := \sum H^0(G/P, \mathcal{L}^n),$$
$$\text{where} \quad H^0(G/P, \mathcal{L}^n) := \big\{ f : G \longrightarrow \Bbbk^{1|1} \,|\, f(gh) = \chi^{-n}(h) f(g) \big\}.$$

We call $\mathcal{R}(G/P, \mathcal{L})$ the *coordinate superalgebra of the supervariety* G/P with respect to the projective embedding given by $\mathcal{L}$.

We may also denote $\mathcal{R}(G/P, \mathcal{L})$ with $\mathcal{O}(G/P)$ (though one should not interpret this as the global sections). Similarly we denote $H^0(G/P, \mathcal{L}^n)$ also with $\mathcal{O}(G/P)_n$.

The next proposition turns out to be crucial for our treatment, since it translates the geometric definitions into Hopf algebraic terms, so to make them suitable for the quantum deformations we discuss in the next section.

NOTATION. Throughout the paper, if A is a Hopf (super)algebra, Δ_A, ϵ_A, S_A will denote its comultiplication, counit and antipode. Whenever there is no danger of confusion we will drop the index A. We also will use the Sweedler notation, that is we express the comultiplication of an element formally as:

$$\Delta(f) = \sum f^{(1)} \otimes f^{(2)}.$$

PROPOSITION 2.5. *Let the notation be as above. Let the supervariety G/P be embedded into some projective superspace via the line bundle $\mathcal{L}$. Let $\pi : \mathcal{O}(G) \longrightarrow \mathcal{O}(P) = \mathcal{O}(G)/I(P)$. Then there exists a $t \in \mathcal{O}(G)$ such that*

$$\big((\mathrm{id} \otimes \pi) \circ \Delta\big)(t) = t \otimes \pi(t), \quad \pi(t^m) \neq \pi(t^n) \quad \forall m \neq n \in \mathbb{N}$$
$$\mathcal{O}\big(G/P\big)_n = \big\{ f \in \mathcal{O}(G) \,\big|\, (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes \pi(t^n) \big\}$$
$$\mathcal{O}\big(G/P\big) = \bigoplus\nolimits_{n \in \mathbb{N}} \mathcal{O}\big(G/P\big)_n$$

and $\mathcal{O}\big(G/P\big)$ is generated in degree 1, namely by $\mathcal{O}\big(G/P\big)_1$.

Proof. If $f \in \mathcal{O}\big(G/P\big)_n$, then

$$\pi(f) = \pi\Big(\sum_{(f)} \epsilon(f_{(1)}) f_{(2)}\Big) = (\epsilon \otimes \pi)(\Delta(f)) = (\epsilon \otimes \mathrm{id})\big((\mathrm{id} \otimes \pi)\Delta(f)\big) = \epsilon(f)\lambda^n$$

where $\lambda = S(\chi)$, S the antipode in $\mathcal{O}(P)$.

Now, by assumption there exists a non-zero global section of the line bundle on G/P, i.e. a regular function $t \in \mathcal{O}\big(G/P\big)_1 \setminus \{0\}$ on G and it is not restrictive to assume $\epsilon(t) \neq 0$. By the above (for $n = 1$), up to dividing out by $\epsilon(t)$, we can assume that $\pi(t) = \lambda$, which corresponds to $S(\chi)$. The result follows immediately. ∎

3. Quantum homogeneous supervarieties. Let $\mathbb{k}$ be our algebraically closed ground field of characteristic zero.

Let G be an algebraic Poisson supergroup, $\mathcal{O}(G)$ its function algebra. In other words, $\mathcal{O}(G)$ is a *Poisson* Hopf algebra, so we have a Poisson bracket $\{\ ,\ \} : \mathcal{O}(G) \otimes \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$ which is compatible with the Hopf algebra structure. Moreover, $\mathfrak{g}$ is a Lie bialgebra and the same holds for its dual space $\mathfrak{g}^*$, these two Lie bialgebras structures being dual to each other.

Definition 3.1. By *quantization* of $\mathcal{O}(G)$, we mean a Hopf algebra $\mathcal{O}_q(G)$ over the ground ring $\mathbb{k}_q := \mathbb{k}\left[q, q^{-1}\right]$, where q is an indeterminate, such that:

1. the specialization of $\mathcal{O}_q(G)$ at $q = 1$, that is $\mathcal{O}_q(G)/(q-1)\mathcal{O}_q(G)$, is isomorphic to $\mathcal{O}(G)$ as a Poisson Hopf algebra;
2. $\mathcal{O}_q(G)$ is torsion-free, as a $\mathbb{k}_q$-module;
3. if $I_G := (q-1)\mathcal{O}_q(G) + \mathrm{Ker}\left(\epsilon_{\mathcal{O}_q(G)}\right)$, then $\bigcap_{n\geq 0} I_G^n = \bigcap_{n\geq 0} (q-1)^n \mathcal{O}_q(G)$.

We also call $\mathcal{O}_q(G)$ a *quantum deformation* of G, or for short *quantum group*.

Similarly, we say that a $\mathbb{k}_q$-algebra $\mathcal{O}_q(X)$ is a *quantization* of the commutative $\mathbb{k}$-superalgebra $\mathcal{O}(X)$ if it is torsion-free and $\mathcal{O}_q(X)/(q-1)\mathcal{O}_q(X) \cong \mathcal{O}(X)$.

For more details on quantization and its significance, we invite the reader to consult [8] and all the references therein.

Let X be an affine supervariety with an action of the supergroup G that makes X an affine homogeneous superspace. Then $\mathcal{O}(X)$, the affine superalgebra of X, admits a natural coaction of $\mathcal{O}(G)$.

If X is a projective supervariety, then we may associate to it the $\mathbb{Z}$-graded superalgebra $\mathcal{O}(X)$ obtained through its embedding into the projective superspace $\mathbb{P}^{m|n}$. If X is a homogenous space for G, then $\mathcal{O}(X)$ admits a coaction of $\mathcal{O}(G)$.

These observations prompt the following definition, which will be crucial for our development.

Definition 3.2. Let X be a projective supervariety, which is a homogeneous space for a supergroup G, and let $\mathcal{O}(X)$ be its superalgebra with respect to a given projective embedding. Let $\mathcal{O}_q(X)$ be a quantization of $\mathcal{O}(X)$. We say that $\mathcal{O}_q(X)$ is a *quantum homogeneous supervariety*, if $\mathcal{O}_q(X)$ admits a coaction of the quantum supergroup $\mathcal{O}_q(G)$, reducing to the coaction of $\mathcal{O}(G)$ on $\mathcal{O}(X)$ when $q = 1$.

One of course may give the same definition also for the affine case.

We now restrict our attention to the case of $X = G/P$ for P a closed subsupergroup of G. The theory of quotients of algebraic supergroups is far from being completely understood, however we consider only the case in which $X = G/P$ is a projective supervariety. Notice that there are quotients of a simple supergroup G by a closed affine subsupergroup, which are neither affine nor projective: for example the Grassmannian of $1|1$ spaces into $\mathbb{C}^{2|2}$ as Manin points out in [26], Ch. 4. We shall not pursue this example in the present work, see [25] for more details.

We now want to develop the theory of quantum superprojective varieties in analogy with Sec. 3 in [9].

Let us assume we have the projective supervariety $X = G/P$, where P is a parabolic subsupergroup of G, that is, its Lie superalgebra contains a Borel subalgebra of Lie(G). Assume X is embedded into some $\mathbb{P}^{m|n}$ via some super line bundle $\mathcal{L}$. Let t be a (non-zero) global section of $\mathcal{L}$. As we already have remarked in Section 2, t, $\epsilon(t) \neq 0$, contains all of the information to reconstruct the line bundle $\mathcal{L}$ (see Proposition 2.5 and its proof).

We want to translate to the quantum setting the notion of super line bundle and super projective embedding.

Definition 3.3. We define *quantum section* of the line bundle $\mathcal{L}$ on G/P given by t, any $d \in \mathcal{O}_q(G)$ such that

(1) $\Delta_\pi(d) = d \otimes \pi(d)$, i.e. $\Delta(d) \in \big(d \otimes d + \mathcal{O}_q(G) \otimes I_q(P)\big)$,
(2) $d \bmod (q-1)\mathcal{O}_q(G) = t \quad \big(\in \mathcal{O}(G)\big)$

where $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(P) := \mathcal{O}_q(G)/I_q(P)$, $I_q(P)$ is a quantization of the ideal $I_q(P)$ defining P and $\Delta_\pi = (id \otimes \pi) \circ \Delta$.

Since $d \in \mathcal{O}_q(G)$ reduces to t, when we specialize $q = 1$, and t contains all of the information to reconstruct the line bundle $\mathcal{L}$, we may think of d as a quantum deformation of the line bundle $\mathcal{L}$. Since $\mathcal{L}$ corresponds in the ordinary setting to an embedding of G/P into a projective superspace, there is a $\mathbb{Z}$-graded superalgebra $\mathcal{O}(G/P)$ associated with it (see Definition 2.4). We now want to define a quantization of such superalgebra: it will be our model for the quantum homogeneous supervariety corresponding to G/P with respect to the projective embedding given by $\mathcal{L}$.

Definition 3.4. Let d be a quantum section of $\mathcal{L}$. Define:

$$\mathcal{O}_q(G/P) := \sum \mathcal{O}_q(G/P)_n,$$
$$\text{where} \quad \mathcal{O}_q(G/P)_n := \{f \in \mathcal{O}_q(G) \,|\, (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes \pi(d^n)\}.$$

Remark 3.5. In [9] an element $d \in \mathcal{O}_q(G)$ satisfying the conditions (1) and (2) of Definition 3.3 is called a *pre-quantum section*, while the terminology "quantum section" is reserved for a d satisfying an extra condition, namely $\mathcal{O}_q\big(G/P\big)_r \cdot \mathcal{O}_q\big(G/P\big)_s \subseteq \mathcal{O}_q\big(G/P\big)_{r+s}$, for all $r, s \in \mathbb{N}$. However, when P is an algebraic supergroup, this extra condition is automatically satisfied and it is one of the main points of the next theorem.

The next theorem shows the importance of quantum sections.

Theorem 3.6. *Let d be a quantum section on G/P. Then*

1. *We have*:
$$\mathcal{O}_q\big(G/P\big)_r \cdot \mathcal{O}_q\big(G/P\big)_s \subseteq \mathcal{O}_q\big(G/P\big)_{r+s},$$
for all $r, s \in \mathbb{N}$. Hence, $\mathcal{O}_q\big(G/P\big)$ is a graded subalgebra *of $\mathcal{O}_q(G)$, its n-th graded summand ($n \in \mathbb{N}$) being $\mathcal{O}_q\big(G/P\big)_n$.*
Furthermore, we have:
$$\mathcal{O}_q\big(G/P\big) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q\big(G/P\big)_n \subset \mathcal{O}_q(G).$$

2. *The grading in* (1) *is compatible with all other structures of* $\mathcal{O}_q\big(G/P\big)$, *so that* $\mathcal{O}_q\big(G/P\big)$ *is a graded* $\mathcal{O}_q(G)$*-comodule algebra, via the restriction of the comultiplication* Δ *in* $\mathcal{O}_q(G)$, *where we take on* $\mathcal{O}_q(G)$ *the trivial grading*:
$$\Delta|_{\mathcal{O}_q(G/P)} : \mathcal{O}_q\big(G/P\big) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q\big(G/P\big).$$
3. *For every* $c \in \Bbbk_q$, *we have* $\mathcal{O}_q\big(G/P\big) \cap c\mathcal{O}_q(G) = c\mathcal{O}_q\big(G/P\big)$. *In particular,*
$$\mathcal{O}_q\big(G/P\big) \cap (q-1)\mathcal{O}_q(G) = (q-1)\mathcal{O}_q\big(G/P\big).$$

Hence $\mathcal{O}_q\big(G/P\big)$ *is a projective homogeneous quantum supervariety for the coaction of the quantum supergroup* $\mathcal{O}_q(G)$.

Proof. This proof is very similar to the non-supersetting that one finds in [9]. However, since in there the setting is slightly more general (still non-super) and the result we need to prove now is scattered through several pages, we prefer to give a direct proof in our setting, given the importance of this result in our present treatment.

(1) To ease the notation, let $\Delta_\pi = (\mathrm{id} \otimes \pi) \circ \Delta$ and let us define $\overline{u} = \pi(u)$, where $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(P)$.

Let $f \in \mathcal{O}_q\big(G/P\big)_r$, $g \in \mathcal{O}_q\big(G/P\big)_s$. By definition we have:
$$\Delta(f) = f \otimes d^r + f_1 \otimes \phi_1\,, \qquad f_1 \in \mathcal{O}_q(G),\ \phi_1 \in I_q(P),$$
$$\Delta(g) = g \otimes d^s + g_1 \otimes \gamma_1\,, \qquad g_1 \in \mathcal{O}_q(G),\ \gamma_1 \in I_q(P).$$

We have
$$\begin{aligned}\Delta(fg) = \Delta(f)\Delta(g) &= \big(f \otimes d^r + f_1 \otimes \phi_1\big)\,\big(g \otimes d^s + g_1 \otimes \gamma_1\big)\\ &= fg \otimes d^{r+s} + fg_1 \otimes d^r\gamma_1 + f_1 g \otimes \phi_1 d^s + f_1 g_1 \otimes \phi_1\gamma_1.\end{aligned}$$

Now, $d^r\gamma_1, \phi_1\gamma_1, \phi_1 d^s \in I_q(P)$ because $I_q(P)$ is a two sided ideal ($\mathcal{O}_q(P) = \mathcal{O}_q(G)/I_q(P)$ is a quantum algebraic subsupergroup of $\mathcal{O}_q(G)$).

We now need to prove that the sum $\sum_{n\in\mathbb{N}} \mathcal{O}_q\big(G/P\big)_n$ is *direct*, so that
$$\mathcal{O}_q\big(G/P\big) := \sum_{n\in\mathbb{N}} \mathcal{O}_q\big(G/P\big)_n = \bigoplus_{n\in\mathbb{N}} \mathcal{O}_q\big(G/P\big)_n\,.$$

Indeed, let $\sum_n c_n f_n = 0$ be a linear dependence relation, with $f_n \in \mathcal{O}_q\big(G/P\big)_n$ and $c_n \in \Bbbk_q$. Applying Δ_π to this relation we get $\sum_n c_n f_n \otimes \overline{d^n} = 0$. But the $\overline{d^n}$'s are linearly independent (being the $\overline{t}^n$ linearly independent); thus $c_n = 0$ for all n.

(2) We need to show that if $f \in \mathcal{O}_q\big(G/P\big)_r$, then $\Delta(f) \in \mathcal{O}_q\big(G/P\big)_r \otimes \mathcal{O}_q(G)$. This is a direct calculation:
$$\begin{aligned}(\mathrm{id} \otimes \pi \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(f) &= (\mathrm{id} \otimes \pi \otimes \mathrm{id})(\Delta \otimes \mathrm{id})[f \otimes d^r + a \otimes h]\\ &= (\mathrm{id} \otimes \pi \otimes \mathrm{id})[\Delta(f) \otimes d^r + \Delta(a) \otimes h]\\ &= f \otimes \overline{d}^r \otimes d^r + (\mathrm{id} \otimes \pi)\Delta(a) \otimes h.\end{aligned}$$

with $a \in \mathcal{O}_q(G)$, $h \in I_q(P)$.

(3) is a direct calculation. ∎

The importance of this theorem is clear: in order to have a quantization of a projective homogeneous supervariety it is enough to verify the existence of a quantum section d.

We shall take this point of view in the next section where we construct explicit examples of such quantum supervarieties.

We now give a remark, which will turn out to be especially important for the applications. For a complete discussion in the ordinary quantum setting see [9], Section 4.

Remark 3.7. Let us consider the comultiplication $\Delta : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$ in $\mathcal{O}_q(G)$. We can extend Δ uniquely to a coassociative morphism of topological algebras

$$\widetilde{\Delta} : \widetilde{\mathcal{O}_q}(G) \longrightarrow \widetilde{\mathcal{O}_q}(G) \,\widetilde{\otimes}\, \widetilde{\mathcal{O}_q}(G).$$

$\widetilde{\mathcal{O}_q}(G)$ is the I_G-adic completion of $\mathcal{O}_q(G)$, where I_G the augmentation ideal in $\mathcal{O}_q(G)$. Most interestingly $\widetilde{\Delta}$ actually restricts to a coassociative algebra morphism:

$$\widehat{\Delta} : \mathcal{O}_q(G)\big[d^{-1}\big] \longrightarrow \mathcal{O}_q(G)\big[d^{-1}\big] \,\widetilde{\otimes}\, \mathcal{O}_q(G)\big[d^{-1}\big].$$

In fact, as d is a quantum section we have

$$\Delta(d) = d \otimes d + \sum_i h_i \otimes k_i \,, \quad \text{for some} \quad h_i \in \mathcal{O}_q(G),\ k_i \in I_q(P).$$

Since d is Ore (see [8], p. 232), we can re-write $\tilde{\Delta}(d) = \Delta(d)\big(d^{-1} \otimes d^{-1}\big)(d \otimes d)$, which reads

$$\tilde{\Delta}(d) = \Big(1 \otimes 1 + \sum_i h_i\, d^{-1} \otimes k_i d^{-1}\Big)(d \otimes d).$$

This in turn implies

$$\begin{aligned}\tilde{\Delta}(d^{-1}) = \tilde{\Delta}(d)^{-1} &= (d \otimes d)^{-1}\Big(1 \otimes 1 + \sum_i h_i\, d^{-1} \otimes k_i\, d^{-1}\Big)^{-1} \\ &= \big(d^{-1} \otimes d^{-1}\big) \sum_{n=0}^{+\infty} (-1)^n \Big(\sum_i h_i\, d^{-1} \otimes k_i\, d^{-1}\Big)^n\end{aligned}$$

where the bottom term does belong to $\mathcal{O}_q(G)[d^{-1}] \,\widetilde{\otimes}\, \mathcal{O}_q(G)[d^{-1}]$, as expected, because $k_i \in I_q(P) \subset I_G$ (for every i), hence the last formal series above is convergent in the $I_{G\otimes}$-adic topology.

We conclude this section with a couple of remarks on some important issues we are unable to properly discuss in the present work, but represent possible directions for future research.

Remark 3.8.

1. We have constructed a projective embedding of the quantum supervariety G/P, in other words we have provided a quantization of the $\mathbb{Z}$-graded superalgebra through a quantum section corresponding to a projective embedding of G/P given by a line bundle. We however have not given conditions on when such embedding exists in the classical setting or when one can actually quantize it.

2. In our construction the $\mathbb{Z}_2$-graded nature of our objects plays actually only a motivational role, but, since our approach is mainly algebraic, there is in principle no obstruction to extend what we are doing for example to $\mathbb{Z}$-graded geometry.

4. The quantum conformal and Minkowski chiral superspaces. In this section we want to apply the theory developed in the previous section to some examples which have physical relevance: the conformal and Minkowski superspaces.

We start with a brief description of the classical conformal and Minkowski superspaces from a mathematical point of view. For the physical interpretation, as well for a thorough mathematical treatment of the classical (i.e. unquantized) part, we send the reader to [25, 19].

Let our ground field be $\mathbb{k} = \mathbb{C}$[1].

Definition 4.1. Let $G = \mathrm{SL}_{4|1}$ and P the parabolic subsupergroup given through the functor of points notation by:

$$P(A) = \left\{ \begin{pmatrix} a & b & \beta \\ 0 & c & \gamma \\ 0 & \delta & d \end{pmatrix} \right\} \subset G(A)$$

where a, b, c are 2×2 matrices with entries in A_0, β, γ, δ^t are 1×2 matrices with entries in A_1 and $d \in A_0$, for A a superalgebra. We define the *conformal chiral superspace* as the quotient G/P. This is identified with the Grassmannian supervariety of $2|0$ spaces in the superspace $\mathbb{C}^{4|1}$. It is a homogeneous supervariety (see [3], Ch. 8 for more details). The supergroup $G = \mathrm{SL}_{4|1}$ is called the *conformal supergroup*. We define the *Minkowski chiral superspace* as the big cell inside G/P corresponding in the given coordinates to the invertibility of the 2×2 principal diagonal block determinant.

The affine supergroups $\mathrm{SL}_{4|1}$ and P are best described for our purposes through their associated superalgebras. We have:

$$\mathcal{O}(\mathrm{SL}_{4|1}) = \mathbb{k}[a_{ij}]/(\mathrm{Ber} - 1), \quad \mathcal{O}(P) = \mathbb{k}[a_{ij}]/(a_{kl}, \mathrm{Ber} - 1), \quad i, j = 1, \dots, 5,$$

for $3 \leq k \leq 5$, $l = 1, 2$. The generators a_{ij} have parity $i + j$, with $p(i) = p(j) = 0$ for $1 \leq i, j \leq 4$, $p(5) = 0$.

The quotient G/P admits a projective embedding via the super line bundle $\mathcal{L}$ corresponding to the character $\chi : P \longrightarrow \mathbb{k}^\times$, $\chi_A(h) = \det(a_{ij})$, $i, j = 1, 2$ (see [24] for more details).

We have the following proposition.

Proposition 4.2. *Let the notation be as above.*

(1) *The homogeneous superalgebra corresponding to the embedding of G/P into $\mathbb{P}^{7|4}$ via the super line bundle $\mathcal{L}$ is*

$$\mathcal{O}(G/P) = \mathbb{k}[d_{ij}, d_{55}] \subset \mathcal{O}(G)$$

where $d_{ij} = a_{i1}a_{j2} - a_{i2}a_{j1} \in \mathcal{O}(G)$, $d_{55} = a_{51}a_{52}$, $i < j = 1, \dots, 5$.

[1] In physics the real forms of the conformal and Minkowski superspace are very important however they are obtained through involutions of the quantum superalgebras we are going to construct. We do not pursue this important and delicate point here (refer to [18] for a thorough treatment).

(2) *Furthermore* $\mathcal{O}(G/P)$ *has the following presentation in terms of generators and relations*:

$$\mathcal{O}(G/P) = \Bbbk[q_{ij}, q_{55}]/I_P$$

where I_P *is the ideal of the super Plücker relations:*

$$\begin{aligned} & q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0 && (\textit{classical Plücker relation}),\\ & q_{ij}q_{k5} - q_{ik}q_{j5} + q_{jk}q_{i5} = 0, && 1 \leq i < j < k \leq 4,\\ & q_{i5}q_{j5} = q_{55}q_{ij}, && 1 \leq i < j \leq 4\,,\\ & q_{i5}q_{55} = 0. \end{aligned}$$

Proof.

(1) According to Definition 2.5 we need to verify that

$$(id \otimes \pi)\Delta(d_{i_1 j_1} \dots d_{i_r j_r}) = d_{i_1 j_1} \dots d_{i_r j_r} \otimes \pi(d_{12}^r)$$

This is a calculation based on the formula

$$\Delta(d_{ij}) = \sum d_{kl}^{ij} \otimes d_{ij}^{kl}$$

and the fact that $\pi(d_{ij}^{kl}) = \delta_{ij,kl}$, where $d_{ij}^{kl} = a_{ik}a_{jl} - a_{il}a_{jk}$. We then need to show that the linear combinations of the $d_{i_1 j_1} \dots d_{i_r j_r}$'s are the only elements in $\mathcal{O}(G/P)$. This is again a direct calculation that is based on the same statement being true in the ordinary setting (i.e. the non-super one). One may find this calculation in [19], though the context there is quite different.

(2) As for the presentation of the superalgebra $\mathcal{O}(G/P)$, we have a morphism:

$$\Bbbk[q_{ij}, q_{55}]/I_P \longrightarrow \mathcal{O}(G/P), \quad q_{ij} \mapsto d_{ij},\ i \leq j, \quad q_{55} \mapsto d_{55},$$

as one can check with a direct calculation.

To prove such a morphism is injective, one replicates a standard argument on the superversion of the Young tableaux and straightening algorithm. The question is to show that there are no other relations among the d_{ij} besides the ones in I_P. We give a brief sketch on how this is done, sending the reader to [18], Ch. 4, Sec. 4. One first shows that the monomials $d_{i_1 j_1} \dots d_{i_r j_r}$ in which the indices $(i_1, \dots, i_r)$ $(j_1, \dots, j_r)$ form a superstandard tableau generate the superalgebra $\mathcal{O}(G/P)$. We call such monomials *standard*. This is obtained by using the straightening algorithm, which is really the same as the classical setting (see [20], p. 110). We then prove they form a basis for $\mathcal{O}(G/P)$ (see [20], p. 110). Suppose there is a relation R: we can write each of its terms using the standard monomial and then we will either have an identically zero relation, hence in the ideal I_P, or we reach a contradiction, because the standard monomials are linearly independent. ∎

We now turn to the quantum setting.

In [27] Manin has given a quantum deformation of the supergroup $\mathrm{SL}_{4|1}$.

Definition 4.3. We define *quantum conformal supergroup*

$$\mathcal{O}_q(\mathrm{SL}_{4|1}) \underset{\mathrm{def}}{=} \Bbbk_q\langle a_{ij}, d_1, d_2\rangle/I_M + (\mathrm{Ber}_q - 1, d_1 \det\nolimits_{q,1} - 1, d_2 \det\nolimits_{q,2} - 1)$$

where I_M is the two sided homogeneous ideal generated by the relations:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \quad j < l,$$
$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \quad i < k,$$
$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \quad i < k,\ j > l \quad \textit{or} \quad i > k,\ j < l,$$
$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kj})}(q^{-1} - q)a_{kj}a_{il}, \quad i < k,\ j < l.$$

where $\pi(a_{ij}) = p(i) + p(j)$ denotes the parity of a_{ij}. Ber_q is the *quantum berezinian* and is defined as

$$\mathrm{Ber}_q := \det_q(a_{ij}) \det_{q^{-1}}(S(a_{kl}))$$

$\det_q(a_{ij})$ is the quantum determinant of the quantum matrix $(a_{ij})_{1\leq i,j\leq 4}$, while $\det_{q^{-1}}(S(a_{kl}))$ is the quantum determinant of the quantum matrix a_{55}, that is, the element a_{55} (for more details see [32], [29]).

We now consider the quantum supergroup $\mathcal{O}_q(P)$ obtained by quotienting $\mathcal{O}_q(G)$ by the two sided (Hopf) ideal $(a_{i1}, a_{j2},\ i, j = 3, 4, 5)$.

We want to give a quantum section d according to Definition 3.3 corresponding to the super line bundle $\mathcal{L}$ defined in Proposition 4.2. This will give us directly a quantization of G/P as quantum projective homogeneous superspace.

It is very natural to start with the element $d = \det_q(a_{ij})$, for $i, j = 1, 2$. This is the quantum determinant of the 2×2 block diagonal matrix of the first two rows and columns (of indeterminates).

Lemma 4.4. *$d := a_{11}a_{22} - q^{-1}a_{12}a_{21}$ is a quantum section of the line bundle $\mathcal{L}$ (see Proposition 4.2) on G/P.*

Proof. We need to verify $(\mathrm{id}\otimes\pi)\Delta(d) = d\otimes\pi(d)$ and that $d \equiv a_{11}a_{22}-a_{12}a_{21}$, $\mathrm{mod}(q-1)$. The second statement is clear. As for the first one, we have the formula (see [18], Ch. 4, Sec. 4):

$$\Delta(D^{kl}_{ij}) = \sum_{r,s} D^{rs}_{ij} \otimes D^{kl}_{rs}$$

where $D^{kl}_{ij} := a_{ik}a_{jl} - q^{-1}a_{il}a_{jk}$, hence in this notation $d = D^{12}_{12}$. Hence we have

$$\big((\mathrm{id}\otimes\pi)\circ\Delta\big)\big(D^{12}_{12}\big) = (\mathrm{id}\otimes\pi)\Big(\sum_{r,s} D^{rs}_{12}\otimes D^{12}_{rs}\Big) = \sum_{r,s} D^{rs}_{12}\otimes\pi(D^{12}_{rs}) = D^{12}_{12}\otimes\pi(D^{12}_{12})$$

because $\pi(D^{12}_{rs}) = \delta_{12,rs}$. ∎

Using d, we can perform the construction of the algebra $\mathcal{O}_q\big(G/P\big)$. Note: from now on we shall remove the column indices when equal to $(1, 2)$.

We have this important corollary, which is immediate after our discussion.

Corollary 4.5. *The superalgebra $\mathcal{O}_q\big(G/P\big)$ is a quantum projective supervariety and it is a quantization of $\mathcal{O}\big(G/P\big)$, in the sense of Definition 3.2. In particular we have a well defined coaction of $\mathcal{O}_q(G)$ on $\mathcal{O}_q\big(G/P\big)$, coming from the restriction of the comultiplication.*

This coaction has a special significance in physics, since it describes the quantization of the conformal supergroup $\mathcal{O}_q(G)$ acting on the conformal superspace $\mathcal{O}_q\big(G/P\big)$.

We now consider the projective localization $\mathcal{O}_q(G/P)[d^{-1}]$ (see [18] for more details of this technique). Physically it is the quantum deformation of the affine superalgebra representing the Minkowski superspace. We have the following proposition, the proof is a direct calculation.

PROPOSITION 4.6. *The quantum Minkowski space $\mathcal{O}_q(G/P)[d^{-1}]$ is a quantum homogeneous supervariety with respect to the restriction of the coaction $\widetilde{\Delta}$ for the supergroup $\mathcal{O}_q(P) = \mathcal{O}_q(G)/(a_{ij}, i = 1, 2)$.*

This proposition establishes a quantization of the coaction of the quantum super-Poincaré $\mathcal{O}_q(P)$ on the quantum Minkowski superspace $\mathcal{O}_q(G/P)[d^{-1}]$. The fact that this coaction is well defined is a consequence of the Remark 3.7. One can also be very explicit and compute the expression for such a coaction, we send the reader to [18], Ch. 5, Sec. 5 for more details.

5. The quantum duality principle. The quantum duality principle is not available in full generality in the supercategory, however we shall show in the example of the quantum conformal superspace, how one may immediately obtain it, with a few caveat at some key points. We now proceed and sketch the main constructions; we plan to explore in full these questions in a forthcoming paper.

In the ordinary (i.e. non-super) setting, we have the Drinfeld functor:

$$\vee : (\mathrm{qfg}) \longrightarrow (\mathrm{que})$$

from the category of quantum functions algebras of quantum groups (qfg) to the category of quantum universal enveloping algebras (que). We cannot explain here the construction and the properties of such functor; we send the reader to [22]. This functor can be effectively understood as a machinery producing from a given quantum group, in the sense of Definition 3.1, a quantization of the corresponding universal enveloping algebra. The functor can also be reversed (see [22]). In [10], the functor was extended to go from the category of quantum functions algebras (qfa) of quantum homogeneous spaces to the category of (que) quantized universal enveloping algebras. Later on (see [17, 9]) the recipe was further generalized to comprehend the case of quantizations of homogeneous projective varieties and this is the case we want to consider for our supergeneralization.

Let $G = \mathrm{SL}_{4|1}$. As in the ordinary setting, given the quantum group $\mathcal{O}_q(G)$, we obtain immediately a super-Poisson structure on G by

$$\big\{\bar{a}, \bar{b}\big\} := (q-1)^{-1}(ab - ba)\big|_{q=1} \qquad \forall\, \bar{a}, \bar{b} \in \mathcal{O}(G).$$

Such a structure is not immediately available as its ordinary counterpart, since the quantum $\mathcal{O}_q(G)$ is defined in a more involved way. We shall nevertheless proceed and define the QDP functor.

In analogy with what was done in [17] we give the following definition.

Definition 5.1. Let G be a supergroup, $\mathcal{O}_q(G)$ a quantum supergroup.

Let $J_G := \mathrm{Ker}(\epsilon : \mathcal{O}_q(G) \longrightarrow \Bbbk)$ be the augmentation ideal of $\mathcal{O}_q(G)$. Also, let $I_G := J_G + (q-1)\mathcal{O}_q(G)$. We define

$$\mathcal{O}_q(G)^\vee := \sum_{n\geq 0} (q-1)^{-n} I_G^n = \sum_{n\geq 0} \big((q-1)^{-1} I_G\big)^n = \bigcup_{n\geq 0} \big((q-1)^{-1} I_G\big)^n.$$

This is a well defined $\Bbbk_q$-subalgebra of $\Bbbk(q) \otimes_{\Bbbk_q} \mathcal{O}_q(G)$. Notice also that

$$\mathcal{O}_q(G)^\vee = \sum_{n\geq 0} (q-1)^{-n} J_G^n = \sum_{n\geq 0} \big((q-1)^{-1} J_G\big)^n.$$

Let G/P be a quotient of supergroups, which has a projective supervariety structure and let $\mathcal{O}_q(G/P)$ be a quantum projective supervariety. We define

$$\mathcal{O}_q(G/P)^\vee := \sum_{n\geq 0} (q-1)^{-n} \big(J_{G/P}^{\mathrm{loc}}\big)^n = \sum_{n\geq 0} \big((q-1)^{-1} J_{G/P}^{\mathrm{loc}}\big)^n,$$

the unital $\Bbbk_q$-subalgebra of $\Bbbk(q) \otimes_{\Bbbk_q} \mathcal{O}_q^{\mathrm{loc}}\big(G/P\big)$ generated by $(q-1)^{-1} J_{G/P}^{\mathrm{loc}}$, or — which amounts to be the same — by $(q-1)^{-1} I_{G/P}^{\mathrm{loc}}$, where by definition we set $I_{G/P}^{\mathrm{loc}} := J_{G/P}^{\mathrm{loc}} + (q-1)\mathcal{O}_q^{\mathrm{loc}}\big(G/P\big)$.

The suffix "loc" means that we are adjoining to our ring d^{-1} the inverse of d. This involves some subtleties, which however are the same as in the ordinary setting, see [9].

We are not going to fully develop the theory and obtain the QDP recipe in general, however in the example of the quantum conformal supergroup and superspace we have the following theorem, whose proof is a simple reworking of the ordinary result, found in [17, 9]. We are planning to explore the general case and to write the details of the discussed example in a forthcoming paper.

Theorem 5.2. *Let the notation and hypothesis be as above. Let* $\mathfrak{g} = \mathrm{Lie}(G)$ *and* $\mathfrak{h} = \mathrm{Lie}(P)$. *Then*

1. $\mathcal{O}_q(G)^\vee$ *is a quantum deformation of* $\mathcal{U}(\mathfrak{g}^*)$.
2. $\mathcal{O}_q(G/P)^\vee$ *is a quantum deformation of* $\mathcal{U}(\mathfrak{h}^\perp)$, $\mathfrak{h} = \mathrm{Lie}(P)$.

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