

MULTIPLIER HOPF ALGEBROIDS ARISING FROM WEAK MULTIPLIER HOPF ALGEBRAS

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Abstract. It is well-known that any weak Hopf algebra gives rise to a Hopf algebroid. Moreover it is possible to characterize those Hopf algebroids that arise in this way.

Recently, the notion of a weak Hopf algebra has been extended to the case of algebras without identity. This led to the theory of weak multiplier Hopf algebras. Similarly also the theory of Hopf algebroids was recently developed for algebras without identity. They are called multiplier Hopf algebroids. Then it is quite natural to investigate the expected link between weak multiplier Hopf algebras and multiplier Hopf algebroids. This relation has been considered already in the original paper on multiplier Hopf algebroids. In this note, we investigate the connection further.

First we show that any regular weak multiplier Hopf algebra gives rise, in a natural way, to a regular multiplier Hopf algebroid. Secondly we give a characterization, mainly in terms of the base algebra, for a regular multiplier Hopf algebroid to have an underlying weak multiplier Hopf algebra. We illustrate this result with some examples. In particular, we give examples of multiplier Hopf algebroids that do not arise from a weak multiplier Hopf algebra.

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1. Introduction. Let G be a *finite groupoid*. Consider the algebra $K(G)$ of all complex functions on G with point-wise operations. It will be denoted by A . Define a coproduct $\Delta : A \rightarrow A \otimes A$ by

$$\Delta(f)(p, q) = \begin{cases} f(pq) & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where $f \in A$ and $p, q \in G$. We identify $A \otimes A$ with the algebra $K(G \times G)$ of all complex functions on the Cartesian product $G \times G$ of G with itself. It is not hard to see that Δ is a homomorphism. It is also coassociative in the sense that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ where ι denotes the identity map. The algebra A clearly has an identity. However, the coproduct Δ need not be unital. This will be the case only when the product pq is defined for all pairs $(p, q) \in G \times G$, i.e. when G is a group. The pair (A, Δ) is a *weak Hopf algebra* as introduced in [B-N-S].

If G is *no longer* assumed to be *finite*, we take for A the algebra $K(G)$ of complex functions with *finite support* instead of all complex functions. This is now an algebra without identity. The product is non-degenerate and we can consider the multiplier algebra $M(A)$. It is naturally identified with the algebra $C(G)$ of all complex functions on G . We can still define a coproduct Δ as in the formula (1.1) above but now Δ maps A to $M(A \otimes A)$, the multiplier algebra of $A \otimes A$. Again $A \otimes A$ is identified with $K(G \times G)$, the algebra of complex functions with finite support in $G \times G$, while $M(A \otimes A)$ is identified with $C(G \times G)$, the algebra of all complex functions on $G \times G$. In this case, the pair (A, Δ) is a *weak multiplier Hopf algebra* as introduced and studied in [VD-W1].

Next consider any *weak Hopf algebra* (A, Δ) . It has a unique antipode S . Denote by ε_s and ε_t respectively the source and target maps, defined from A to itself by

$$\varepsilon_s(a) = \sum S(a_{(1)})a_{(2)} \quad \text{and} \quad \varepsilon_t(a) = \sum a_{(1)}S(a_{(2)})$$

for $a \in A$. We use the Sweedler notation $\sum a_{(1)} \otimes a_{(2)}$ for $\Delta(a)$. The ranges of ε_s and ε_t will be denoted by B and C respectively. Then B and C are two commuting subalgebras of A . If we use S_B and S_C for the restrictions of the antipode S to these subalgebras B and C respectively, it turns out that S_B is an anti-isomorphism from B to C and that S_C is an anti-isomorphism from C to B . Observe that in general, they are not inverses of each other.

By assumption, here the algebra A is an algebra with an identity 1. We have that $\Delta(1) \in B \otimes C$ and that B and C are the smallest subspaces of A with this property. In other words, B is the left leg of $\Delta(1)$ and C is the right leg of $\Delta(1)$. Recall that $\Delta(1)$ is not necessarily equal to $1 \otimes 1$. If that is the case, that is when (A, Δ) is actually a Hopf algebra, we have that B and C are just the scalar multiples of the identity and the source and target maps are essentially the counit of the Hopf algebra (A, Δ) .

Consider now two *balanced tensor products*. The first one, denoted as ${}_B A \otimes A^B$, is characterized by the requirement that $xa \otimes b = a \otimes S_B(x)b$ for all $a, b \in A$ and $x \in B$. In the second one, denoted as ${}^C A \otimes A_C$, we have $a \otimes by = aS_C(y) \otimes b$ for all $a, b \in A$ and $y \in C$. Denote by π_B and π_C the canonical quotient maps from $A \otimes A$ to ${}_B A \otimes A^B$ and from $A \otimes A$ to ${}^C A \otimes A_C$ respectively. Define

$$\Delta_B(a) = \pi_B(\Delta(a)) \quad \text{and} \quad \Delta_C(a) = \pi_C(\Delta(a))$$

for $a \in A$. The maps

$$\Delta_B : A \rightarrow {}_B A \otimes A^B \quad \text{and} \quad \Delta_C : A \rightarrow {}^C A \otimes A_C$$

are a left and a right coproduct that turn the above data (A, B, C, S_B, S_C) into a *Hopf algebroid*.

In the course of the construction, further balanced tensor products will appear. The notation for such tensor products is explained in Section 3.2.

There is much more to explain about the construction above. We will do this further in the paper where in fact, we describe the construction in the more general case of a regular weak multiplier Hopf algebra.

This is the main topic of this work and it takes us to the content of the paper.

Content of the paper. *Section 2* is a preliminary section. It mainly contains a review of the notion of a *regular weak multiplier Hopf algebra*, one of the basic ingredients of this paper and some of its basic properties. Special attention is given to the source and target maps and the source and target algebras. We do not recall the notion of a *multiplier Hopf algebroid* in this preliminary section. As it turns out, it will be easier to do it *on the way*, in Section 3, where we pass from a regular weak multiplier Hopf algebra to a multiplier Hopf algebroid.

Section 3 is devoted to the configuration of the algebras A, B, C and the anti-isomorphisms S_B and S_C mentioned above, and the relation with separability idempotents as studied in [VD2]. This is the basic material, commonly needed in both Section 4 and Section 5. If a Hopf algebroid is coming from a weak Hopf algebra as explained above, then (A, B, C, S_B, S_C) form a *quantum graph* with unit, and the separability idempotent in this case will be nothing else but $\Delta(1)$. In this section we consider the generalizations of these objects to the framework of weak multiplier Hopf algebras. We will not yet consider the coproducts. That will only be done in the next two sections.

In *Section 4* we show how we can associate a multiplier Hopf algebroid to any regular weak multiplier Hopf algebra. The procedure is quite standard and there are no surprises. It is indeed very similar to the same construction in the case of a weak Hopf algebra, with the typical problems due to the fact that the algebras have no identity and that the coproducts do not map into the tensor product, but into a bigger space.

During the process, we recall the various notions that we encounter in the definition of a multiplier Hopf algebroid. We finish the section by looking at the concrete expressions for the counit, as considered in the theory of multiplier Hopf algebroids, in terms of the counital maps, considered in the theory of weak multiplier Hopf algebras. We also illustrate some formulas involving the antipode in the two settings.

In *Section 5* we consider the opposite direction. We start with a regular multiplier Hopf algebroid and we discuss the necessary and sufficient conditions for it to come from a weak multiplier Hopf algebra as in the previous section.

The main condition is that the base algebra B is separable Frobenius. This means that there is a separability idempotent $E \in M(B \otimes C)$ in the sense of [VD2]. However, this is not sufficient. The anti-isomorphisms $S_B : B \rightarrow C$ and $S_C : C \rightarrow B$, that are given for the multiplier Hopf algebroid have to coincide with the anti-isomorphisms induced by E . In other words, we have to require that

$$E(x \otimes 1) = E(1 \otimes S_B(x)) \quad \text{and} \quad (1 \otimes y)E = (S_C(y) \otimes 1)E$$

for all $x \in B$ and $y \in C$ (where S_B and S_C are the original anti-isomorphisms that come with the definition of the multiplier Hopf algebroid).

Still, this will not be sufficient. We also need the equality of the counits ε and ε' on A defined by

$$\varepsilon = \varphi_B \circ \varepsilon_B \quad \text{and} \quad \varepsilon' = \varphi_C \circ \varepsilon_C.$$

Here φ_B and φ_C are the unique linear functionals on B and C respectively characterized by

$$(\varphi_B \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi_C)E = 1$$

and ε_B and ε_C are the counital maps that are obtained within the theory for the original multiplier Hopf algebroid.

In this section, we further discuss these conditions and compare them with the conditions found for a Hopf algebroid to arise from a weak Hopf algebra as studied in [B]. For example, we shall see that equality of the counits ε and ε' implies that the functionals φ_B and φ_C admit *modular automorphisms*, given by $S_B^{-1}S_C^{-1}$ and $S_B S_C$, respectively.

In *Section 6* we discuss some examples and special cases.

In the last section, *Section 7*, we draw some conclusions and discuss possible future research related with the results obtained in this paper.

Conventions and basic references. We only consider algebras over the field of complex numbers (although we believe that most of our results are still valid for more general fields). We do not require our algebras to have a unit, but we do need that the product is non-degenerate (as a bilinear form). Our algebras are all idempotent, either by assumption, or by a result. In fact, they even will have local units and this implies that the product is non-degenerate and that the algebra is idempotent.

For algebras with a non-degenerate product, it is possible to define the multiplier algebra $M(A)$. It can be characterized as the largest algebra with identity containing A as an essential two-sided ideal. We use 1 for the unit in the multiplier algebras. Of course $M(A) = A$ if and only if A already has an identity. If A is non-degenerate, then so is the tensor product $A \otimes A$ and we consider also its multiplier algebra $M(A \otimes A)$. We have natural imbeddings

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

In general, for a non-unital algebra, these two inclusions are strict.

The opposite algebra A^{op} is the algebra A , but endowed with the opposite product.

We use ι for the identity map. We use ζ to denote the flip map on $A \otimes A$ and for its extension to the multiplier algebra $M(A \otimes A)$. A coproduct is a map from A to $M(A \otimes A)$. The composition of Δ with ζ is denoted by Δ^{cop} .

We will sometimes use the Sweedler notation for coproduct. So we write $\sum_{(a)} a_{(1)} \otimes a_{(2)}$ for $\Delta(a)$ when Δ is a coproduct on the algebra A and $a \in A$. Some useful information about the use of the Sweedler notation in the case of non-unital algebras can be found e.g. in [VD1], [VD2] and [VD4].

For the theory of weak Hopf algebras we refer to [B-N-S] and for the relation with Hopf algebroids to [B].

For the theory of weak multiplier Hopf algebras, we refer to [VD-W1] and [VD-W2]. In particular, we find in [VD-W2] the necessary results about the source and target maps and source and target algebras that are very important for the described procedure. See also [VD-W0] for a motivational paper. The theory of multiplier Hopf algebroids has been developed recently in [T-VD]. An important role is played by separability idempotents. For these objects, we refer to [VD2]. Whenever in this paper we consider separability idempotents, it will be in the sense of [VD2], the first version. In the more recent version these are called *regular* separability idempotents because there a more general notion is also studied. So, using the new terminology, we are only considering here regular separability idempotents.

The conventions used in the original papers on weak multiplier Hopf algebras [VD-W1] and [VD-W2] are not always the same as in the newer paper on multiplier Hopf algebroids [T-VD]. We will mention such a difference whenever it occurs.

2. Preliminaries. In this preliminary section, we mainly recall the notion of a *regular weak multiplier Hopf algebra* as well as some of the basic properties. We also give the necessary definitions and results about the source and target maps and the source and target algebras.

We could have chosen to do the same here for the *multiplier Hopf algebroids*. However, it turned out to be more convenient to recall the notions and some of the results during the process in the next two sections, where we describe the passage from a regular weak multiplier Hopf algebra to a multiplier Hopf algebroid.

For details about the first objects, we refer to [VD-W1], [VD-W2] and [VD-W0]. The main reference for multiplier Hopf algebroids is [T-VD]. Further information can be found in the references given in the introduction.

2.1. Weak multiplier Hopf algebras. The basic ingredient is a *non-degenerate* and *idempotent* algebra A over the field of complex numbers and a *regular* and *full coproduct* Δ on A , admitting a *counit*.

DEFINITION 2.1. A *regular coproduct* on an algebra A is a homomorphism

$$\Delta : A \rightarrow M(A \otimes A)$$

so that elements of the form

$$\Delta(a)(1 \otimes b) \quad (1 \otimes b)\Delta(a) \quad \Delta(a)(b \otimes 1) \quad (b \otimes 1)\Delta(a)$$

all belong to $A \otimes A$ for all $a, b \in A$. It is assumed to satisfy *coassociativity*. This can be formulated in different ways, e.g. we assume

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b, c \in A$. In any case, the first requirement is necessary to be able to formulate coassociativity in any of such forms. A coproduct is called *full* if the smallest subspaces V, W of A so that

$$\Delta(a)(1 \otimes b) \in V \otimes A \quad \text{and} \quad (a \otimes 1)\Delta(b) \in A \otimes W$$

for all $a, b \in A$ are actually all of A . A *counit* for the pair (A, Δ) is a linear map $\varepsilon : A \rightarrow \mathbb{C}$ satisfying

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab \quad \text{and} \quad (\iota \otimes \varepsilon)((a \otimes 1)\Delta(b)) = ab$$

for all a, b .

To formulate fullness of the coproduct, it is not needed that it is regular. The same is true for the characterizing properties of the counit. However we assume regularity of the coproduct from the very beginning. This is the only case that will be treated in this paper. For the more general notion, we refer to [VD-W1].

If the algebra has an identity, so that $\Delta : A \rightarrow A \otimes A$, fullness of the coproduct follows from the existence of the counit. Also the uniqueness of the counit follows immediately from the definition in that case. However, as discussed in [VD-W0], this is not expected in general. On the other hand, for a full coproduct, it can be shown that the counit is unique.

Also remark that, because the coproduct is assumed to be regular, the defining property of the counit can also be formulated with the factors on the other side. In fact, if A is replaced by A^{op} , nothing above changes and Δ remains a regular and full coproduct and ε is still the counit. Similarly if we replace Δ by Δ^{cop} , obtained from Δ by composing it with the flip map ζ on $A \otimes A$, extended to $M(A \otimes A)$, nothing changes here.

A *regular weak multiplier Hopf algebra* is a pair (A, Δ) of a non-degenerate idempotent algebra with a regular and full coproduct, such that there is a counit and with some further assumptions, formulated in the terms of the ranges and the kernels of the *canonical maps* T_1, T_2, T_3 and T_4 given by

$$\begin{aligned} T_1(a \otimes b) &= \Delta(a)(1 \otimes b) & \text{and} & & T_2(a \otimes b) &= (a \otimes 1)\Delta(b) \\ T_3(a \otimes b) &= (1 \otimes b)\Delta(a) & \text{and} & & T_4(a \otimes b) &= \Delta(b)(a \otimes 1) \end{aligned}$$

when $a, b \in A$. Remark that the maps T_3 and T_4 are the maps T_1 and T_2 for the opposite algebra A^{op} (with the original coproduct).

For the *precise conditions* we refer to the original papers mentioned before.

We now collect the *main properties* that we will be using further in this paper.

In any weak multiplier Hopf algebra (regular or not), the following holds.

PROPOSITION 2.2. *Let (A, Δ) be a weak multiplier Hopf algebra. By assumption, there is a unique idempotent $E \in M(A \otimes A)$ such that*

$$E(A \otimes A) = \Delta(A)(A \otimes A) \quad \text{and} \quad (A \otimes A)E = (A \otimes A)\Delta(A).$$

It is the smallest idempotent element in $M(A \otimes A)$ such that $\Delta(a) = E\Delta(a) = \Delta(a)E$ for all $a \in A$. The coproduct Δ can be extended uniquely to a homomorphism $\Delta : M(A) \rightarrow M(A \otimes A)$ with the property that still $\Delta(m) = E\Delta(m) = \Delta(m)E$ for all $m \in M(A)$. Similarly we have extensions of the homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ to $M(A \otimes A)$ and these are unique if we require that

$$\begin{aligned} (\Delta \otimes \iota)(m) &= (E \otimes 1)((\Delta \otimes \iota)(m)) = ((\Delta \otimes \iota)(m))(E \otimes 1) \\ (\iota \otimes \Delta)(m) &= (1 \otimes E)((\iota \otimes \Delta)(m)) = ((\iota \otimes \Delta)(m))(1 \otimes E) \end{aligned}$$

for all $m \in M(A \otimes A)$. For these extensions we have $\Delta(1) = E$ and also $(\Delta \otimes \iota)E = (\iota \otimes \Delta)E$. Finally also

$$(\Delta \otimes \iota)E = (E \otimes 1)(1 \otimes E). \tag{2.1}$$

We are using the same symbols for the extensions of the homomorphisms involved. Also remark that the last property (2.1) is very natural. Indeed, by the construction of the extensions, we will have that $(\Delta \otimes \iota)E$ is an idempotent, smaller than $E \otimes 1$ and $1 \otimes E$. So the last property is saying that $(\Delta \otimes \iota)E$ is as big as possible. Property (2.1) is sometimes referred to as the weak comultiplicativity of the unit. This makes sense once we know that $\Delta(1) = E$ for the extension of Δ . See [K-VD] for the use of this terminology and [VD-W0] for a motivation of the equation (2.1).

One of the main results in the theory is the existence of a unique *antipode*. Recall that we assume *regularity* of the weak multiplier Hopf algebra. Then we have the following properties.

PROPOSITION 2.3. *There is a bijective linear map $S : A \rightarrow A$ such that the maps R_1 and R_2 , defined on $A \otimes A$ by*

$$\begin{aligned} R_1(a \otimes b) &= \sum_{(a)} a_{(1)} \otimes S(a_{(2)})b \\ R_2(a \otimes b) &= \sum_{(b)} aS(b_{(1)}) \otimes b_{(2)}, \end{aligned}$$

are well-defined maps from $A \otimes A$ to itself and so that R_1 and R_2 are generalized inverses of the canonical maps T_1 and T_2 , respectively.

We are using the Sweedler notation here. The equations can be covered by multiplying with an element of A from the left in the first factor, for the first equation, and with an element of A from the right in the second factor, for the second equation. It is not obvious that the right hand sides are in $A \otimes A$ but that is actually the case.

That R_1 is a generalized inverse of T_1 means that

$$T_1 R_1 T_1 = T_1 \qquad R_1 T_1 R_1 = R_1$$

and similarly for T_2 and R_2 .

In general, such generalized inverses are not unique. They are determined by a choice of the projection on the kernel and on the range of the corresponding maps. Such a choice is part of the axioms of a weak multiplier Hopf algebra and it depends only on the canonical multiplier E . It follows that also the map S , as in the above proposition, is

uniquely determined. It is called the antipode of the weak multiplier Hopf algebra (A, Δ) . As expected, it is an anti-isomorphism of the algebra A and it also flips the coproduct Δ .

If we replace A by A^{op} , the antipode S is replaced by the inverse S^{-1} . Then it is easy to find formulas for the appropriate generalized inverses R_3 and R_4 of T_3 and T_4 respectively.

From these equations, we can conclude the following result about the antipode.

PROPOSITION 2.4. *The antipode satisfies*

$$\sum_{(a)} a_{(1)} S(a_{(2)}) a_{(3)} = a \quad \text{and} \quad \sum_{(a)} S(a_{(1)}) a_{(2)} S(a_{(3)}) = S(a)$$

for all $a \in A$.

If we multiply the left hand side of the first equation with an element of A , left or right, it turns out that the use of the Sweedler notation is justified. Similarly for the second equation. So one first has to consider these equations in $M(A)$. As we get elements in A on the right, we see that actually the equations will hold in A , but that is not trivial. We get the same formulas with the inverse S^{-1} of the antipode, provided that we either flip the product or the coproduct.

Because R_1 is a generalized inverse of T_1 , it follows that $T_1 R_1$ projects onto the range of T_1 . Similarly for $T_2 R_2$ and for the two other maps $T_3 R_3$ and $T_4 R_4$. It then follows from the choices mentioned earlier that the following holds.

PROPOSITION 2.5. *We have*

$$\begin{aligned} T_1 R_1(a \otimes b) &= E(a \otimes b) & \text{and} & & T_2 R_2(a \otimes b) &= (a \otimes b) E \\ T_3 R_3(a \otimes b) &= (a \otimes b) E & \text{and} & & T_4 R_4(a \otimes b) &= E(a \otimes b) \end{aligned}$$

for all $a, b \in A$.

The second set of formulas is obtained from the first one by replacing A by A^{op} and taking into account that E remains the same.

These formulas describe the ranges of these four canonical maps. This means that we have

$$\begin{aligned} \Delta(A)(1 \otimes A) &= E(A \otimes A) & \text{and} & & (A \otimes 1)\Delta(A) &= (A \otimes A)E \\ (1 \otimes A)\Delta(A) &= (A \otimes A)E & \text{and} & & \Delta(A)(A \otimes 1) &= E(A \otimes A). \end{aligned}$$

In fact, these equations are part of the basic assumptions in the definition of a regular weak multiplier Hopf algebra.

We have similar formulas for the *kernels*.

PROPOSITION 2.6. *There exists idempotents F_1, F_2, F_3 and F_4 in $M(A \otimes A^{\text{op}})$ given by*

$$\begin{aligned} F_1 &= (\iota \otimes S)E & \text{and} & & F_3 &= (\iota \otimes S^{-1})E \\ F_2 &= (S \otimes \iota)E & \text{and} & & F_4 &= (S^{-1} \otimes \iota)E. \end{aligned}$$

They give the kernels of the four canonical maps in the sense that

$$\begin{aligned} R_1 T_1(a \otimes b) &= (a \otimes 1)F_1(1 \otimes b) & \text{and} & & R_2 T_2(a \otimes b) &= (a \otimes 1)F_2(1 \otimes b) \\ R_3 T_3(a \otimes b) &= (1 \otimes b)F_3(a \otimes 1) & \text{and} & & R_4 T_4(a \otimes b) &= (1 \otimes b)F_4(a \otimes 1) \end{aligned}$$

for all a, b in A .

These results follow from the formulas in the previous proposition, provided we use that S is an anti-isomorphism of the algebra A . Indeed, we have e.g.

$$R_1(\iota \otimes S) = (\iota \otimes S)T_3 \quad \text{and} \quad T_1(\iota \otimes S) = (\iota \otimes S)R_3$$

and this will give the formula for R_1T_1 . Similarly for the other cases.

From these formulas, we also find

$$\begin{aligned} \text{Ker}(T_1) &= (A \otimes 1)(1 - F_1)(1 \otimes A) & \text{and} & & \text{Ker}(T_2) &= (A \otimes 1)(1 - F_2)(1 \otimes A) \\ \text{Ker}(T_3) &= (1 \otimes A)(1 - F_3)(A \otimes 1) & \text{and} & & \text{Ker}(T_4) &= (1 \otimes A)(1 - F_4)(A \otimes 1). \end{aligned}$$

We refer to [VD-W1] for these results. In that paper, it is also shown that the multipliers F_i are completely determined by E itself and the antipode is not needed to characterize them.

2.2. The source and target maps. Having formulated the basic results about regular weak multiplier Hopf algebras, we now recall the *source* and *target maps*, the *source and target algebras* and their main properties needed further in this paper.

DEFINITION 2.7. The source and target maps ε_s and ε_t from A to $M(A)$ are defined by

$$\varepsilon_s(a) = \sum_{(a)} S(a_{(1)})a_{(2)} \quad \text{and} \quad \varepsilon_t(a) = \sum_{(a)} a_{(1)}S(a_{(2)}).$$

Recall that by assumption, $\varepsilon_s(a)$ as defined above, is a left multiplier. It is also a right multiplier because of the result in Proposition 2.3. Similarly $\varepsilon_t(a)$ is a right multiplier by the assumptions on the coproduct and a left multiplier because of Proposition 2.3.

In what follows, we denote $\varepsilon_s(A)$ by B and $\varepsilon_t(A)$ by C .

For any $a \in A$, $x \in B$ and $y \in C$ we have

$$\varepsilon_s(ax) = \varepsilon_s(a)x \quad \text{and} \quad \varepsilon_s(ay) = S(y)\varepsilon_s(a) \tag{2.2}$$

$$\varepsilon_t(ya) = y\varepsilon_t(a) \quad \text{and} \quad \varepsilon_t(xa) = \varepsilon_t(a)S(x). \tag{2.3}$$

PROPOSITION 2.8. *The images B and C of the source and target maps $\varepsilon_s(A)$ and $\varepsilon_t(A)$ are non-degenerate idempotent algebras. They sit nicely in the multiplier algebra $M(A)$ in the sense that*

$$BA = AB = A \quad \text{and} \quad CA = AC = A.$$

As a consequence, the imbeddings extend to imbeddings of their multiplier algebras $M(B)$ and $M(C)$ in $M(A)$. These multiplier algebras are denoted by A_s and A_t , respectively. They are called the source and target algebras and they are characterized in $M(A)$ by

$$A_s = \{x \in M(A) \mid \Delta(x) = E(1 \otimes x)\}$$

$$A_t = \{y \in M(A) \mid \Delta(y) = (y \otimes 1)E\}.$$

The formulas in (2.2) and (2.3) also hold for $x \in A_s$ and $y \in A_t$.

The ranges of the source and target maps are, in a sense that can be made precise, the left and the right leg of E respectively. Consequently the algebras A_s and A_t commute. This implies that the formulas in the proposition above still hold when the product in A is reversed.

Finally we have the following important property.

PROPOSITION 2.9. *We have $E \in M(B \otimes C)$ and E is a separability idempotent in the sense of [VD2]. The associated antipodal maps from B to C and from C to B are precisely the restriction of the antipode (or rather of its extension to the multiplier algebra $M(A)$). This means that*

$$E(x \otimes 1) = E(1 \otimes S(x)) \quad \text{and} \quad (1 \otimes y)E = (S(y) \otimes 1)E$$

for all $x \in B$ and $y \in C$.

Remark that $E(1 \otimes y) \in B \otimes C$ for all $y \in C$ and if we use a Sweedler type notation $E_1 \otimes E_2$ for E we have $S_B(E_1)E_2y = y$ when $y \in C$. Similarly, $(x \otimes 1)E \in B \otimes C$ and $xE_1S_C(E_2) = x$ for all $x \in B$. We write these properties as

$$S_B(E_1)E_2 = 1 \quad \text{and} \quad E_1S_C(E_2) = 1.$$

See Proposition 1.8 in [VD2].

We finish this section with the remark that the algebras B and C have local units (see Proposition 1.9 in [VD2]). The same is true for the original algebra A (see Proposition 4.9 in [VD-W1]).

3. Quantum graphs and separability idempotents. In this section, we introduce the notion of a *quantum graph with local units* and investigate the relation between such quantum graphs and separability idempotents. These quantum graphs are the basic ingredients for a regular multiplier Hopf algebroid whose underlying algebra has local units. On the other side, *separability idempotents*, as studied in [VD2], play a basic role in the theory of weak multiplier Hopf algebras. And since this paper is devoted to the relation between these two objects, it is quite natural to start with the study of the relation between these basic ingredients.

Observe that in this section, *we do not yet consider the coproducts*. This will be done in the next two sections.

In a first item of this section, we will collect properties about separability idempotents (as introduced in [VD2]) and the concept of a separable Frobenius algebra.

3.1. Separable Frobenius algebras. We consider an algebra B with local units. In particular, the product is non-degenerate and B is idempotent.

A linear functional φ on B is called *faithful* if the maps

$$x \mapsto \varphi(\cdot x) \quad \text{and} \quad x \mapsto \varphi(x \cdot)$$

from B to the linear dual B' are both injective. If these maps also have the same range, there exists a unique automorphism σ of B so that $\varphi(x_1x_2) = \varphi(x_2\sigma(x_1))$ for all x_1, x_2 in B . Because B is assumed to be idempotent, this implies also that φ is σ -invariant.

We call σ the *modular automorphism* of φ .

Observe that the existence of a faithful linear functional on the algebra B automatically implies that the product is non-degenerate.

If B is a finite-dimensional algebra with identity, the existence of a faithful linear functional is equivalent with B being Frobenius. Moreover, in this case, a faithful linear functional will automatically have a modular automorphism. In the literature, the inverse of σ is called the Nakayama automorphism. We refer to [M] for the notion of the

Nakayama automorphism in the case of (finite-dimensional) Frobenius algebras. However, in the case of an infinite-dimensional algebra, possibly without identity, this result is not true. There can be a linear functional that is faithful but without a modular automorphism. We refer to [VD2] for an example.

We now recall the following concept from [VD2].

DEFINITION 3.1. Let φ be a linear functional on the algebra B . Assume that it is faithful and admits a modular automorphism as above. Let C be the algebra B^{op} and use S for the identity map from B to C . It is an anti-isomorphism by definition. Assume that there is an idempotent $E \in M(B \otimes C)$ satisfying

$$(\varphi(\cdot x) \otimes \iota)E = S(x) \tag{3.1}$$

for all x . Then we call φ a *separating functional* on B . If the algebra B admits such a separating functional, we call B *separable Frobenius*.

We need to give a couple of remarks here.

REMARKS 3.2.

(i) If B is a finite-dimensional algebra with identity and φ a faithful linear functional, there is always an element $E \in B \otimes C$ determined by the equality (3.1). However, it can happen that E is not an idempotent. In fact one can easily construct examples where $E^2 = 0$. See [VD2] for an example. Given φ the idempotent E is unique if it exists. This follows from the faithfulness of φ . This means that we really define a property of the linear functional φ on the algebra B .

(ii) It can be shown that, if a separating functional on the algebra exists, then any faithful linear functional will have a modular automorphism and it will be separating as well.

In what follows, we assume that B is separable Frobenius and that φ is a given separating functional on B as in Definition 3.1. The modular automorphism (that exists by assumption) will be denoted by σ_B . Finally, the given anti-isomorphism from B to C will now be denoted by S_B .

Then we have the following result.

PROPOSITION 3.3. *We have an anti-isomorphism $S_C : C \rightarrow B$ defined by $S_C = \sigma_B^{-1} S_B^{-1}$. Define a linear functional φ_C on C by $\varphi_C(y) = \varphi_B(S_B^{-1}(y))$ when $y \in C$. It is a faithful functional with modular automorphism σ_C given by $\sigma_C = S_B S_C$. We not only have $(\varphi(\cdot x) \otimes \iota)E = S_B(x)$ for all $x \in B$ but now also*

$$(\iota \otimes \varphi_C(y \cdot))E = S_C(y)$$

for all $y \in C$. The element E is a separability idempotent in $M(B \otimes C)$ with associated anti-isomorphisms S_B and S_C . This means that

$$E(x \otimes 1) = E(1 \otimes S_B(x)) \quad \text{and} \quad (1 \otimes y)E = (S_C(y) \otimes 1)E$$

for all $x \in B$ and $y \in C$. The linear functionals φ_B and φ_C are the distinguished linear functionals.

The proof is straightforward by using techniques from [VD2].

So we see that given a separable Frobenius algebra B with a separating linear functional φ , the unique associated idempotent E is a separability idempotent in $M(B \otimes C)$ where $C = B^{\text{op}}$. The identity map from B to C is the antipodal map $S_B : B \rightarrow C$, characterized by the equation $E(x \otimes 1) = E(1 \otimes S_B(x))$ for all $x \in B$. Remark that it is not essential that C is identified with B^{op} . Obviously, any algebra C with a given anti-isomorphism $S_B : B \rightarrow C$ as a starting point will work as well.

In fact, also if we have two faithful functionals φ_B and φ'_B , related by an automorphism of B , we will still get essentially the same separability idempotent.

We also have the following converse result.

PROPOSITION 3.4. *Suppose that B and C are non-degenerate algebras and that E is a separability idempotent in $M(B \otimes C)$. Then B is separable Frobenius. The unique linear functional φ_B on B satisfying $(\varphi_B \otimes \iota)E = 1$ in $M(C)$ is a separating functional on B .*

In Section 5, where we characterize those multiplier Hopf algebroids that arise from a weak multiplier Hopf algebra, we will start with a multiplier Hopf algebroid whose base algebra B is separable Frobenius as defined in Definition 3.1 above. The separability idempotent E that we find in Proposition 3.3 will eventually be the canonical idempotent of the associated weak multiplier Hopf algebra. In Section 4, we start from a weak multiplier Hopf algebra and construct a multiplier Hopf algebroid from it. In this case we have given the canonical idempotent E . It is a separability idempotent.

Let us note that separable Frobenius algebras as defined in Definition 3.1 are closely related to coseparable coalgebras as defined, for example, in Section 26 of [B-W]. The precise relation of between these two notions is discussed in the second version of [VD2].

Next we consider the *second basic ingredient*, used in the theory of multiplier Hopf algebroids. First we look at it independently from the previous item. Later in this section, we will relate the two concepts.

3.2. Quantum graphs with local units. The starting point for the definition of a regular multiplier Hopf algebroid in [T-VD] is an algebra A with commuting subalgebras $B, C \subseteq M(A)$ and anti-isomorphisms between B and C . These algebras are not assumed to be unital but to satisfy several regularity conditions. These conditions hold automatically if A has local units, and since we will only be interested in this case, the following terminology will be convenient.

DEFINITION 3.5. A *quantum graph with local units* consists of

- (i) an algebra A that has local units and therefore is non-degenerate and idempotent,
- (ii) two commuting subalgebras B and C of $M(A)$ satisfying

$$BA = AB = A \quad \text{and} \quad CA = AC = A, \quad (3.2)$$

- (iii) two anti-isomorphisms

$$S_B : B \rightarrow C \quad \text{and} \quad S_C : C \rightarrow B.$$

Note that the anti-isomorphisms S_B and S_C are in general not expected to be each other's inverses.

The following example will be the starting point in the next section where we pass from a regular weak multiplier Hopf algebra to a regular multiplier Hopf algebra.

EXAMPLE 3.6. A regular weak multiplier Hopf algebra (A, Δ) gives rise in a natural way to a quantum graph with local units, where B and C are the images of the source and target maps, as sitting in $M(A)$, while S_B and S_C are the restrictions of the antipode (after it has been extended to the multiplier algebra). The fact that A has local units is proven Proposition 4.9 in [VD-W1].

In Section 5, we will start from a quantum graph with local units as above.

Following [T-VD], we shall use the following notation.

NOTATION 3.7. Suppose that (A, B, C, S_B, S_C) is a quantum graph with local units.

We denote elements of B by x, x', \dots and elements of C by y, y', \dots , and reserve a, b, c for elements of A .

We regard A as a *left* or *right module* over the algebra B with respect to the module multiplications given by

$$x \cdot a := xa, \quad a \cdot x := ax, \quad a \cdot x := S_B(x)a \quad \text{or} \quad x \cdot a := aS_B(x),$$

and denote the corresponding left or right B -module by

$${}_B A, \quad A_B, \quad A^B \quad \text{and} \quad {}^B A,$$

respectively. Similarly, we write ${}_C A, A_C, A^C$ or ${}^C A$ when we regard A as a left or right module over C .

Regarding left or right B -modules as right or left B^{op} -modules, respectively, we can form *various balanced tensor products* over the algebras B or B^{op} , for example,

$${}_B A \otimes A_B, \quad {}_B A \otimes A^B, \quad A^B \otimes {}^B A,$$

and similarly we can form various balanced tensor products of A with itself over the algebras C or C^{op} .

3.3. Quantum graphs with a separable Frobenius base algebra. Suppose that (A, B, C, S_B, S_C) is a quantum graph with local units, that is, A is an algebra with local units, B and C are commuting subalgebras sitting nicely in $M(A)$, and $S_B : B \rightarrow C$ and $S_C : C \rightarrow B$ are anti-isomorphisms.

Now we *add equivalent additional assumptions* which make a link between the material studied in the first item and the one discussed in the second item of this section.

LEMMA 3.8. *The following conditions are equivalent:*

- (i) B is a separable Frobenius algebra and there exists a separating linear functional φ_B whose modular automorphism σ_B coincides with the inverse of the given automorphism $S_C S_B$ of B .
- (ii) C is a separable Frobenius algebra and there exists a separating linear functional φ_C whose modular automorphism σ_C coincides with the given automorphism $S_B S_C$ of C .

(iii) *There exists a (regular) separability idempotent $E \in M(B \otimes C)$ such that*

$$E(x \otimes 1) = E(1 \otimes S_B(x)) \quad \text{and} \quad (1 \otimes y)E = (S_C(y) \otimes 1)E$$

for all $x \in B$ and $y \in C$.

If these conditions hold, then the separating linear functionals φ_B and φ_C in conditions (i) and (ii) and the separability idempotent E in condition (iii) satisfy

$$(\varphi_B \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi_C)E = 1.$$

Proof. Assume first the existence of a separating linear functional φ_B as in condition (i). Denote by C' the opposite algebra of B , by $S'_B : B \rightarrow C'$ the canonical anti-isomorphism and let $S'_{C'} = \sigma_B^{-1} S'_B^{-1} : C' \rightarrow B$. Then there exists a separability idempotent $E' \in M(B \otimes C')$ such that $E'(x \otimes 1) = E'(1 \otimes S'_B(x))$ and $(1 \otimes y')E' = (S'_{C'}(y') \otimes 1)E'$ for all $x \in B$ and $y' \in C'$. The multiplier

$$E := (\iota \otimes S_B(S'_B)^{-1})(E') \in M(B \otimes C)$$

is an idempotent and satisfies $E(x \otimes 1) = E(1 \otimes S_B(x))$ and $(1 \otimes S_B(S'_B)^{-1}(y'))E = (S'_{C'}(y') \otimes 1)E$ for all $x \in B$ and $y' \in C'$. Since $S'_{C'} S'_B S_B^{-1} = \sigma_B^{-1} S_B^{-1} = S_C$ by assumption, the last equation implies $(1 \otimes y)E = (S_C(y) \otimes 1)E$ for all $y \in C$. Therefore, condition (iii) holds.

A similar argument shows that condition (ii) implies condition (iii).

Finally, condition (iii) implies (i) and (ii) by Proposition 3.4. See also Propositions 2.1 and 2.3 in [VD2]. ■

In Section 5, we shall introduce another condition (equality of two counits or invariance of the counit under the antipode) which will imply the conditions on the modular automorphisms σ_B and σ_C above.

In Section 4 we arrive at the situation considered above when we start from a regular weak multiplier Hopf algebra. In Section 5, it will be part of the assumptions.

Till the end of this section, we assume that the equivalent conditions (i)–(iii) in Lemma 3.8 hold.

Using the separability idempotent E , we obtain *concrete realizations* of the balanced tensor products introduced in Notation 3.7 as subspaces of $A \otimes A$. We begin with the tensor products ${}_B A \otimes A^B$ and ${}^C A \otimes A_C$.

PROPOSITION 3.9.

(i) *Denote by*

$$\pi_B : A \otimes A \rightarrow {}_B A \otimes A^B$$

the canonical quotient map. There is a well-defined map

$$\theta_B : {}_B A \otimes A^B \rightarrow A \otimes A, \quad a \otimes b \mapsto E(a \otimes b),$$

and $\pi_B \circ \theta_B$ is the identity map on ${}_B A \otimes A^B$.

(ii) *Denote by*

$$\pi_C : A \otimes A \rightarrow {}^C A \otimes A_C$$

the canonical quotient map. There is a well-defined map

$$\theta_C : {}^C A \otimes A_C \rightarrow A \otimes A, \quad a \otimes b \mapsto (a \otimes b)E,$$

and $\pi_C \circ \theta_C$ is the identity map on ${}^C A \otimes A_C$.

Proof. (i) For all elements $a, b \in A$ and $x \in B$ we have

$$E(xa \otimes b) = E(a \otimes S_B(x)b)$$

and therefore the map $a \otimes b \mapsto E(a \otimes b)$ is a well-defined map θ_B from the balanced tensor product ${}_B A \otimes A^B$ to $A \otimes A$. Furthermore, for all elements $a, b \in A$ and $y \in C$, we have

$$\begin{aligned} \pi_B(\theta_B(a \otimes yb)) &= \pi_B(E(a \otimes yb)) \\ &= \pi_B(a \otimes S_B(E_{(1)})E_{(2)}yb) = \pi_B(a \otimes yb) \end{aligned}$$

where we are using the Sweedler type notation $E = E_{(1)} \otimes E_{(2)}$. Remark that

$$E(1 \otimes C) \subseteq B \otimes C \quad \text{and} \quad S_B(E_{(1)})E_{(2)}y = y$$

when $y \in C$ as shown in Proposition 1.8 of [VD2] (see also a remark at the end of the preliminary section). Finally, because $CA = A$, we get $\pi_B(\theta_B(a \otimes b)) = a \otimes b$ in ${}_B A \otimes A^B$ for all $a, b \in A$.

(ii) The proof here is completely similar. Now we use that

$$(1 \otimes y)E = (S_C(y) \otimes 1)E$$

for all $y \in C$ and that $xE_{(1)}S_C(E_{(2)}) = x$ for all $x \in B$ (again see Proposition 1.8 of [VD2] and a remark at the end of the preliminary section). ■

It follows from this result that the map $\theta_B : {}_B A \otimes A^B \rightarrow A \otimes A$ is injective and its range is $E(A \otimes A)$. Similarly the map $\theta_C : {}^C A \otimes A_C \rightarrow A \otimes A$ is injective with range $(A \otimes A)E$.

Now we prove similar results for the balanced tensor products

$$A_B \otimes {}_B A, \quad A_C \otimes {}_C A, \quad {}_B A \otimes A_B, \quad {}^C A \otimes A_C.$$

First we need the following analogue of (part of) the result reviewed in Proposition 2.6 in the case of a weak multiplier Hopf algebra.

PROPOSITION 3.10. *There exist idempotents F_1, F_2, F_3 and F_4 in $M(A \otimes A^{\text{op}})$ given by*

$$\begin{aligned} F_1 &= (\iota \otimes S_C)E & \text{and} & & F_3 &= (\iota \otimes S_B^{-1})E \\ F_2 &= (S_B \otimes \iota)E & \text{and} & & F_4 &= (S_C^{-1} \otimes \iota)E. \end{aligned}$$

Proof. Recall that $E(x \otimes 1) \in B \otimes C$ for all $x \in B$. Then $E(xa \otimes 1) \in A \otimes C$ and as $BA = A$ we find $E(A \otimes 1) \subseteq A \otimes C$. It follows that $(\iota \otimes S_C)E$ is a well-defined multiplier F_1 in $M(A \otimes A^{\text{op}})$. Similarly for the other elements F_2, F_3 and F_4 . ■

Remark that the identity map from $A \otimes A$ to itself is an anti-isomorphism from $A \otimes A^{\text{op}}$ to $A^{\text{op}} \otimes A$ that extends to a natural anti-isomorphism from $M(A \otimes A^{\text{op}})$ to $M(A^{\text{op}} \otimes A)$.

Now we find the following results about the four other balanced tensor products.

PROPOSITION 3.11. *We can define the four maps*

$$\begin{aligned}
 a \otimes b &\mapsto (a \otimes 1)F_1(1 \otimes b) && \text{on} && A_B \otimes_B A \\
 a \otimes b &\mapsto (a \otimes 1)F_2(1 \otimes b) && \text{on} && A_C \otimes_C A \\
 a \otimes b &\mapsto (1 \otimes b)F_3(a \otimes 1) && \text{on} && {}_B A \otimes A_B \\
 a \otimes b &\mapsto (1 \otimes b)F_4(a \otimes 1) && \text{on} && {}_C A \otimes A_C,
 \end{aligned}$$

all of them with range in $A \otimes A$. For each of them, when composed with the corresponding canonical projection map, we find the identity map on the corresponding balanced tensor product.

Proof. In the first case, we use e.g. that

$$(x \otimes 1)F_1 = (1 \otimes S_C)((x \otimes 1)E) \in B \otimes B$$

and that $xE_{(1)}S_C(E_{(2)}) = x$ for all $x \in B$. In the second case we use that

$$F_2(1 \otimes y) = (S_B \otimes \iota)(E(1 \otimes y)) \in C \otimes C$$

and that $S_B(E_{(1)})E_{(2)}y = y$ for all $y \in C$. For the third case, we need

$$F_3(x \otimes 1) = (\iota \otimes S_B^{-1})(E(x \otimes 1)) \in B \otimes B$$

and $S_B^{-1}(E_{(2)})E_{(1)}x = x$ for all $x \in B$. Finally in the last case we use

$$(1 \otimes y)F_4 = (S_C^{-1} \otimes \iota)((1 \otimes y)E) \in C \otimes C$$

and $yE_{(2)}S_C^{-1}(E_{(1)}) = y$ for all $y \in C$. ■

Again, as a consequence these four maps are bijections of these balanced tensor product with appropriate subspaces of $A \otimes A$.

4. From weak multiplier Hopf algebras to multiplier Hopf algebroids. In this section we start with a regular weak multiplier Hopf algebra (A, Δ) . We set

$$B = \varepsilon_s(A) \subseteq M(A) \quad \text{and} \quad C = \varepsilon_t(A) \subseteq M(A)$$

where ε_s and ε_t are the source and target maps. We know that B and C are non-degenerate idempotent algebras, sitting nicely in the multiplier algebra $M(A)$ of A . They are commuting subalgebras of $M(A)$. The canonical idempotent E of the weak multiplier Hopf algebra is a separability idempotent in $M(B \otimes C)$. The associated antipodal maps $S_B : B \rightarrow C$ and $S_C : C \rightarrow B$ are the corresponding restrictions of the extension of the antipode S to the multiplier algebra $M(A)$. Because of this, it cannot be confusing if we sometimes drop the indices here and simply use S for these two associated maps. We refer to the preliminary section for more details.

We thus obtain in a canonical way a quantum graph with local units (A, B, C, S_B, S_C) as in Example 3.6.

The *aim of this section* is to show that the coproduct Δ induces a left and a right coproduct on A making it into a regular multiplier Hopf algebroid as defined in [T-VD]. We will also express the data of the multiplier Hopf algebroid in terms of the various data of the given weak multiplier Hopf algebra.

In the *next section*, we find a necessary and sufficient condition for a multiplier Hopf algebroid to arise in this way.

4.1. The left and the right coproducts Δ_B and Δ_C . We first consider the balanced tensor products ${}_B A \otimes A^B$ and ${}^C A \otimes A_C$ as in Notation 3.7 of the previous section.

The first space ${}_B A \otimes A^B$ is endowed with two actions of A , obtained by multiplication from the right in the first and in the second factor. Note that this space is non-degenerate as a right A -module with respect to either of these two actions because A has local units. We use ${}_B A \overline{\otimes} A^B$ for the *extended module*. It can be characterized by the property that for any $z \in {}_B A \overline{\otimes} A^B$, we have well-defined elements

$$z(1 \otimes a) \quad \text{and} \quad z(a \otimes 1)$$

in ${}_B A \otimes A^B$ for all $a \in A$. Similarly we use ${}^C A \overline{\otimes} A_C$ for the extended module of ${}^C A \otimes A_C$ where elements $z \in {}^C A \overline{\otimes} A_C$ are characterized by the property that

$$(a \otimes 1)z \quad \text{and} \quad (1 \otimes a)z$$

are in ${}^C A \otimes A_C$ for all $a \in A$.

For a more detailed treatment of the notion of an extended module, sometimes also called the completed module, we refer to Section 2 of [VD1].

NOTATION 4.1. We denote by

$${}_B A \overline{\times} A^B \subseteq {}_B A \overline{\otimes} A^B$$

the subspace formed by all elements $z \in {}_B A \overline{\otimes} A^B$ satisfying

$$z(x \otimes 1) = z(1 \otimes S(x))$$

for all $x \in B$. Remark that we extend the module action to the multiplier algebras.

Similarly, we denote by

$${}^C A \overline{\times} A_C \subseteq {}^C A \overline{\otimes} A_C$$

the subspace formed by all elements $z \in {}^C A \overline{\otimes} A_C$ satisfying

$$(1 \otimes y)z = (S(y) \otimes 1)z$$

for all $y \in C$.

There is a natural embedding of this subspace into $\text{End}({}_B A \overline{\otimes} A^B)$ by letting an element z of ${}_B A \overline{\times} A^B$ act on ${}_B A \otimes A^B$ by

$$z\pi_B(a \otimes b) := z(a \otimes b)$$

for $a, b \in A$ where as before, π_B denotes the canonical quotient map from $A \otimes A$ onto ${}_B A \otimes A^B$. In the right hand side, $z(a \otimes b)$ is the result of the right action of $a \otimes b$ on z . The action of ${}_B A \overline{\times} A^B$ on ${}_B A \otimes A^B$ is well-defined, precisely because of the condition imposed on elements of ${}_B A \overline{\times} A^B$. It is also not hard to see that composition of maps induces a product on ${}_B A \overline{\times} A^B$, making it into an associative algebra.

Similarly, the space ${}^C A \overline{\times} A_C$ embeds naturally into $\text{End}({}^C A \otimes A_C)$ and will be regarded as an algebra with respect to the *reversed* composition of maps.

The spaces ${}_B A \overline{\times} A^B$ and ${}^C A \overline{\times} A_C$ are the target spaces for the left and the right coproducts respectively in what follows. Denote by ι_B and ι_C the identity map on B or C , respectively.

PROPOSITION 4.2. *Define $\Delta_B : A \rightarrow {}_B A \overline{\otimes} A^B$ by*

$$\Delta_B(a)(1 \otimes b) = \pi_B(\Delta(a)(1 \otimes b)) \quad \text{and} \quad \Delta_B(a)(c \otimes 1) = \pi_B(\Delta(a)(c \otimes 1))$$

for $a, b, c \in A$. Here again π_B is the canonical projection of $A \otimes A$ onto the quotient space ${}_B A \overline{\otimes} A^B$. Then Δ_B has range in ${}_B A \overline{\times} A^B$ and $(A, B, \iota_B, S_B, \Delta_B)$ is a left multiplier bialgebroid in the sense of Definition 2.6 of [T-VD].

Similarly, define $\Delta_C : A \rightarrow {}^C A \overline{\otimes} A_C$ by

$$(c \otimes 1)\Delta_C(a) = \pi_C((c \otimes 1)\Delta(a)) \quad \text{and} \quad (1 \otimes b)\Delta_C(a) = \pi_C((1 \otimes b)\Delta(a))$$

where now we use π_C for the canonical projection of $A \otimes A$ onto the quotient space ${}^C A \overline{\otimes} A_C$. Then Δ_C has range in ${}^C A \overline{\times} A_C$ and $(A, C, \iota_C, S_C, \Delta_C)$ is a right multiplier bialgebroid in the sense of Definition 4.1 of [T-VD].

The tuple $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ is a multiplier bialgebroid in the sense of Definition 5.1 of [T-VD].

Proof. The proof is rather straightforward. The different steps are as follows.

(i) First note that since A has local units and $AB = BA = A = CA = AC$, conditions (1), (2) in Definition 2.6 and in Definition 4.1 in [T-VD] hold.

(ii) Next observe that $\Delta_B(a)$ is a well-defined element in the extended module ${}_B A \overline{\otimes} A^B$. To show this, first the elements $\Delta_B(a)(1 \otimes b)$ and $\Delta_B(a)(c \otimes 1)$, given by the formulas in the formulation, are considered and then the obvious compatibility relations are verified. All of this is essentially trivial.

(iii) It has to be shown that $\Delta_B(a)$, defined in this way, really belongs to the subspace ${}_B A \overline{\times} A^B$. This follows from the fact that

$$\Delta(a)(x \otimes 1) = \Delta(a)(1 \otimes S(x))$$

for all $a \in A$ and $x \in B$.

(iv) The map Δ_B will be a homomorphism from A to the algebra ${}_B A \overline{\times} A^B$ because Δ is a homomorphism.

(v) The behavior of Δ_B on B and C is determined by the behavior of Δ itself on these subalgebras.

(vi) Finally, coassociativity of Δ_B will follow from coassociativity of Δ , written in the form

$$(\Delta \otimes \iota)(\Delta(a)(1 \otimes b))(c \otimes 1 \otimes 1) = (\iota \otimes \Delta)(\Delta(a)(c \otimes 1))(1 \otimes 1 \otimes b)$$

for all $a, b, c \in A$. We have to project onto the appropriate balanced subspace of $A \otimes A \otimes A$ and use that

$$\Delta(xa) = (1 \otimes x)\Delta(a) \quad \text{and} \quad \Delta(ya) = (y \otimes 1)\Delta(a)$$

when $a \in A$, $x \in B$ and $y \in C$.

The same arguments are used to show that Δ_C is a right coproduct.

The compatibility of the two coproducts is again obtained from the coassociativity of Δ , now formulated as

$$\begin{aligned} (c \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(a)(1 \otimes b)) &= (\iota \otimes \Delta)(c \otimes 1)\Delta(a)(1 \otimes 1 \otimes b) \\ (1 \otimes 1 \otimes b)(\iota \otimes \Delta)(\Delta(a)(c \otimes 1)) &= (\Delta \otimes \iota)((1 \otimes b)\Delta(a))(c \otimes 1 \otimes 1) \end{aligned}$$

for all $a, b, c \in A$. Also here, we have to project onto the appropriate quotient space of $A \otimes A \otimes A$. ■

Next we consider the *canonical linear maps* associated with these coproducts. Let us first recall the definitions.

NOTATION 4.3. For the left coproduct Δ_B we have the maps \widetilde{T}_λ and \widetilde{T}_ρ going from $A \otimes A$ to ${}_B A \otimes A^B$ and defined by

$$\widetilde{T}_\lambda(a \otimes b) = \Delta_B(b)(a \otimes 1) \quad \widetilde{T}_\rho(a \otimes b) = \Delta_B(a)(1 \otimes b)$$

for $a, b, c \in A$; see Section 2 of [T-VD]. For the right coproduct Δ_C we have the maps $\widetilde{\lambda}T$ and $\widetilde{\rho}T$, going from $A \otimes A$ to ${}^C A \otimes A_C$ and defined by

$$\widetilde{\lambda}T(a \otimes b) = (a \otimes 1)\Delta_C(b) \quad \widetilde{\rho}T(a \otimes b) = (1 \otimes b)\Delta_C(a)$$

for $a, b, c \in A$; see Section 4 of [T-VD].

It is useful to compare these four maps with the canonical maps T_1, T_2, T_3 and T_4 as used in [VD-W1]. These are maps from $A \otimes A$ to itself, given by the formulas

$$\begin{aligned} T_1(a \otimes b) &= \Delta(a)(1 \otimes b) & \text{and} & & T_2(a \otimes b) &= (a \otimes 1)\Delta(b) \\ T_3(a \otimes b) &= (1 \otimes b)\Delta(a) & \text{and} & & T_4(a \otimes b) &= \Delta(b)(a \otimes 1). \end{aligned}$$

See Section 1 in [VD-W1]. We get from one set to the other by applying the appropriate quotient maps, sometimes combined with the flip.

As a consequence of the results in the previous section on the various balanced tensor products (Proposition 3.9 and Proposition 3.11), we find the following.

PROPOSITION 4.4. *The maps $\widetilde{T}_\lambda, \widetilde{T}_\rho, \widetilde{\lambda}T, \widetilde{\rho}T$ defined above factorize to bijective maps*

$$\begin{aligned} T_\lambda : A^B \otimes {}^B A &\rightarrow {}_B A \otimes A^B & \text{and} & & T_\rho : A_B \otimes B_A &\rightarrow {}_B A \otimes A^B \\ \widetilde{\lambda}T : A_C \otimes {}^C A &\rightarrow {}^C A \otimes A_C & \text{and} & & \widetilde{\rho}T : A^C \otimes C_A &\rightarrow {}^C A \otimes A_C. \end{aligned}$$

Proof. Consider the canonical map $T_1 : A \otimes A \rightarrow A \otimes A$, given by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b).$$

We know that the range of this map is $E(A \otimes A)$ and that the kernel is the space $(A \otimes 1)(1 - F_1)(1 \otimes A)$. On the other hand, from Proposition 3.9 we know that the projection map $\pi_B : A \otimes A \rightarrow {}_B A \otimes A^B$, restricted to $E(A \otimes A)$ is an isomorphism. Similarly from Proposition 3.11 we know that the projection map $\pi : A \otimes A \rightarrow A_B \otimes B_A$, restricted to $(A \otimes 1)(1 - F_1)(1 \otimes A)$ is also an isomorphism. It follows that we have

a bijective map $T_\rho : A_B \otimes_B A \rightarrow {}_B A \otimes A^B$ making the following diagram commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{T_1} & A \otimes A \\ \pi \downarrow & & \downarrow \pi_B \\ A_B \otimes_B A & \xrightarrow{T_\rho} & {}_B A \otimes A^B \end{array}$$

A similar proof can be given for the other cases, using the appropriate results obtained in Proposition 3.9 and Proposition 3.11. ■

COROLLARY 4.5. *Let (A, Δ) be a regular weak multiplier Hopf algebra. Then the canonical maps of the multiplier bialgebroid $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ constructed above are bijective.*

To verify that this multiplier bialgebroid is a regular multiplier Hopf algebra, the technical condition (1) in Definition 5.2 of [T-VD] has to be checked. For that we will use the counits of the multiplier bialgebroid in the next subsection.

4.2. The left and the right counit of the associated multiplier algebraoid. The source and the target maps ε_s and ε_t of the original weak multiplier Hopf algebra (A, Δ) can be used to construct a left counit ε_B and a right counit ε_C for the associated multiplier bialgebroid as follows.

PROPOSITION 4.6. *Let (A, Δ) be a regular weak multiplier Hopf algebra. Then the associated left multiplier bialgebroid $(A, B, \iota_B, S_B, \Delta_B)$ has a unique and surjective left counit ε_B , which is the map from A to B defined by*

$$\varepsilon_B(a) = \sum_{(a)} a_{(2)} S^{-1}(a_{(1)}).$$

Here we use the Sweedler notation for the original coproduct Δ and S is the original antipode of (A, Δ) .

Proof. Define ε_B as above. We see that $\varepsilon_B(a) = S^{-1}(\varepsilon_t(a))$ and this belongs to $S^{-1}(C)$. It follows that ε_B is a map from A to B . By definition of B and C , it is surjective, and by (2.3), it satisfies

$$\varepsilon_B(xa) = x\varepsilon_B(a) \quad \text{and} \quad \varepsilon_B(S_B(x)a) = \varepsilon_B(a)x$$

for all $x \in B$ and $a \in A$. In particular, the spaces I^t and I^s considered in Lemma 3.8 in [T-VD] are equal to B . Proposition 3.9 in [T-VD] implies that the left multiplier bialgebroid has a unique left counit. We show that this left counit is the map ε_B .

We have

$$(\varepsilon_B \otimes \iota)(\Delta(a)(1 \otimes b)) = \sum_{(a)} a_{(2)} S^{-1}(a_{(1)}) \otimes a_{(3)} b$$

in $A \otimes A$ for all $a, b \in A$. Because $\varepsilon_B(xa) = x\varepsilon_B(a)$ for $x \in B$ and $a \in A$, it is easily seen that, in the multiplier Hopf algebraoid framework, this formula reads as

$$(\varepsilon_B \otimes \iota)T_\rho(a \otimes b) = \sum_{(a)} a_{(2)} S^{-1}(a_{(1)}) \otimes a_{(3)} b \tag{4.1}$$

where $T_\rho(a \otimes b) = \Delta_B(a)(1 \otimes b)$ in ${}_B A \otimes A^B$ (cf. Notation 4.3 above) and where the expression (4.1) is considered in the balanced tensor product ${}_B B \otimes A^B$, defined via the relations $xx' \otimes a = x' \otimes S_B(x)a$ for $a \in A$ and $x, x' \in B$. This space is naturally identified with $CA \subseteq A$ via the map

$$x \otimes a \mapsto S_B(x)A.$$

If we apply this identification map to (4.1) above, we end up with

$$\sum_{(a)} a_{(1)} S(a_{(2)}) a_{(3)} b = ab.$$

This shows that ε_B satisfies the first equation of (3.2) in Definition 3.1 of [T-VD]. Since T_ρ is surjective, this equation determines ε_B . ■

We have a similar *right-handed* result.

PROPOSITION 4.7. *Let (A, Δ) be a regular weak multiplier Hopf algebra. Then the associated right multiplier bialgebroid $(A, C, \iota_C, S_C, \Delta_C)$ has a unique and surjective right counit ε_C , which is the map from A to C given by*

$$\varepsilon_C(a) = \sum_{(a)} S^{-1}(a_{(2)}) a_{(1)}.$$

Remark that we have here

$$\varepsilon_C(ay) = \varepsilon_C(a)y \quad \text{and} \quad \varepsilon_C(aS_C(y)) = y\varepsilon_C(a)$$

for all $y \in C$ and $a \in A$. Moreover we have

$$\begin{aligned} (\varepsilon_C \otimes \iota)_\rho T(a \otimes b) &= (\varepsilon_C \otimes \iota)((1 \otimes b)\Delta_C(a)) \\ &= \sum_{(a)} S^{-1}(a_{(2)}) a_{(1)} \otimes ba_{(3)} \end{aligned} \tag{4.2}$$

in the balanced tensor product ${}_C C \otimes A_C$, defined by the relations $yy' \otimes a = y' \otimes ay$ where $a \in A$ and $y, y' \in C$. If this space is identified with A via $y \otimes a \mapsto ay$, the expression (4.2) yields

$$\sum_{(a)} ba_{(3)} S^{-1}(a_{(2)}) a_{(1)} = ba.$$

This is precisely proving the first equation of (4.8) in Definition 4.3 of [T-VD].

Since the left and the right counit are surjective, the multiplier bialgebroid constructed above satisfies the technical condition (1) of Definition 5.2 in [T-VD] and is a regular multiplier Hopf algebroid:

THEOREM 4.8. *Let (A, Δ) be a regular weak multiplier Hopf algebra. Then the associated multiplier bialgebroid $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ is a regular multiplier Hopf algebroid.*

4.3. What about the antipode? It should be no surprise that the antipode of the multiplier Hopf algebroid is the same as the original antipode. And it is instructive to verify the various diagrams in Section 5 of [T-VD]. Consider e.g. the two diagrams in Definition 5.4 of [T-VD].

For the first one, we have

$$\mu(S \otimes \iota)T_\rho(a \otimes b) = \sum_{(a)} S(a_{(1)})a_{(2)}b$$

where S is the antipode of the algebroid and where we use the Sweedler notation for $\Delta_B(a)$. We use μ for the multiplication. On the other hand we have

$$\mu(S_C \varepsilon_C \otimes \iota)(a \otimes b) = \sum_{(a)} S_C(S^{-1}(a_{(2)})a_{(1)})b = \sum_{(a)} S(a_{(1)})a_{(2)}b$$

where now S is the original antipode and with the Sweedler notation for the original coproduct on A . We see that we get the same expressions.

For the second diagram, we have similar formulas. On the one hand, we have

$$\mu(\iota \otimes S)_\lambda T(a \otimes b) = \sum_{(b)} ab_{(1)}S(b_{(2)})$$

with the antipode of the algebroid and the Sweedler notation for $\Delta_C(b)$. On the other hand, we find

$$\mu(\iota \otimes S_B \varepsilon_B)(a \otimes b) = \sum_{(b)} aS_B(b_{(2)}S^{-1}(b_{(1)})) = \sum_{(b)} ab_{(1)}S(b_{(2)})$$

where now we have the original antipode, as well as the Sweedler notation for the original coproduct. Again we find the same expressions.

5. Multiplier Hopf algebroids arising from weak multiplier Hopf algebra. In this section we will give a *necessary and sufficient condition* for a regular multiplier Hopf algebroid to come from a regular weak multiplier Hopf algebra via the procedure described in the previous section.

The starting point in this section is a quantum graph with local units (A, B, C, S_B, S_C) as considered in the second item of Section 3. So A is an algebra with local units, B and C are two commuting subalgebras, sitting nicely in $M(A)$, and

$$S_B : B \rightarrow C \quad \text{and} \quad S_C : C \rightarrow B$$

are anti-isomorphisms which need not be mutually inverse.

The basic extra assumption we will need for the main theorem was formulated already in Section 3.3:

ASSUMPTION. *We require that the quantum graph (A, B, C, S_B, S_C) satisfies the equivalent conditions (i)–(iii) of Lemma 3.8, that is, the following equivalent conditions hold:*

- B and C are separable Frobenius algebras with separating linear functionals φ_B and φ_C whose modular automorphism groups σ_B and σ_C coincide with the given automorphisms $S_B^{-1}S_C^{-1}$ and $S_B S_C$, respectively;
- there exists a (regular) separability idempotent $E \in M(B \otimes C)$ such that the associated antipodal maps coincide with S_B and S_C , respectively, that is,

$$E(x \otimes 1) = E(1 \otimes S_B(x)) \quad \text{and} \quad (1 \otimes y)E = (S_C(y) \otimes 1)E$$

for all $x \in B$ and $y \in C$.

Before we continue, we need to add some remarks.

REMARKS 5.1.

(i) It will not be sufficient only to require that B is separable Frobenius. It can occur that there is a separating linear functional φ on B as in Definition 3.1, but that it cannot be chosen so that its modular automorphism σ is the inverse of the given automorphism $S_C S_B$ of B . Indeed, any automorphism σ of B can occur by making appropriate choices for C and S_B, S_C . And if σ does not leave the center invariant, it cannot be the modular automorphism of a faithful functional. See also Example 6.3 (iii) in the next section.

(ii) The extra assumption only depends on the pair of the algebra B and the given automorphism $S_C S_B$. It does not refer to how B and C are realized as commuting subalgebras of $M(A)$.

5.1. The main result. Now we assume that moreover A carries two coproducts Δ_B and Δ_C so that the tuple

$$(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$$

is a regular multiplier Hopf algebra. We will proceed from these assumptions and try to construct an underlying regular weak multiplier Hopf algebra (A, Δ) .

The first step in the procedure is to associate two coproducts Δ and Δ' on A . The first one will be induced from Δ_B while the second one from Δ_C .

Recall that the left coproduct Δ_B is a map from A to ${}_B A \overline{\otimes} A^B$. So by definition we have elements

$$\Delta_B(a)(1 \otimes b) \quad \text{and} \quad \Delta_B(a)(c \otimes 1)$$

in ${}_B A \otimes A^B$ for all a, b, c in A . See e.g. the first item in Section 4.

Now consider also $A \otimes A$ with the two actions of A , given by right multiplication in the first and in the second factor. This gives rise to the extended module, denoted by $A \overline{\otimes} A$, whose elements ξ are characterized by the fact that

$$\xi(1 \otimes b) \quad \text{and} \quad \xi(c \otimes 1)$$

are elements in $A \otimes A$ for all b and c in A . Here we can view $A \overline{\otimes} A$ as sitting in $\text{End}(A \otimes A)$ in the obvious way. This will clearly make $A \overline{\otimes} A$ into an algebra. In fact, it is a subalgebra of the algebra $L(A \otimes A)$ of left multipliers of $A \otimes A$. The situation is similar as for ${}_B A \overline{\otimes} A^B$, or rather ${}_B A \overline{\times} A^B$, as in Section 4.

In Proposition 3.9 we have seen that the map θ_B from ${}_B A \otimes A^B$ to $A \otimes A$, given by $\theta_B(a \otimes b) = E(a \otimes b)$ is well-defined and that $\pi_B \circ \theta_B$ is the identity map on ${}_B A \otimes A^B$ where π_B is the quotient map from $A \otimes A$ to ${}_B A \otimes A^B$.

We are now ready to obtain the first coproduct $\Delta : A \rightarrow M(A \otimes A)$.

LEMMA 5.2. *There is a homomorphism $\Delta : A \rightarrow A \overline{\otimes} A$ given by*

$$\Delta(a)(1 \otimes b) = \theta_B(\Delta_B(a)(1 \otimes b)) \quad \text{and} \quad \Delta(a)(c \otimes 1) = \theta_B(\Delta_B(a)(c \otimes 1)) \quad (5.1)$$

for $a, b, c \in A$. It satisfies $\Delta(a) = E\Delta(a) = \Delta(a)E$ for all $a \in A$. It is coassociative in the sense that

$$(\Delta \otimes \iota)(\Delta(a)(1 \otimes b))(c \otimes 1 \otimes 1) = ((\iota \otimes \Delta)(\Delta(a)(c \otimes 1))(1 \otimes 1 \otimes b) \quad (5.2)$$

holds for all a, b, c in A .

Proof. (i) From the definition of the map θ_B it is clear that

$$\theta_B(\xi(1 \otimes b)) = \theta_B(\xi)(1 \otimes b) \quad \text{and} \quad \theta_B(\xi(c \otimes 1)) = \theta_B(\xi)(c \otimes 1)$$

for all $\xi \in {}_B A \otimes A^B$ and $b, c \in A$. On the other hand, because $\Delta_B(a) \in {}_B A \overline{\otimes} A^B$, it follows that

$$(\Delta_B(a)(1 \otimes b))(c \otimes 1) = (\Delta_B(a)(c \otimes 1))(b \otimes 1)$$

for all $a, b, c \in A$. If we now apply θ_B to this last equation, it will follow that $\Delta(a)$ is well-defined in $A \overline{\otimes} A$ by the formulas in (5.1).

(ii) By the definition of $\Delta(a)$ we have $\Delta(a)(1 \otimes b) = \theta_B(\Delta_B(a)(1 \otimes b))$ for all $a, b \in A$. Because the range of θ_B is precisely $E(A \otimes A)$, it is clear that $E\Delta(a)(1 \otimes b) = \Delta(a)(1 \otimes b)$. This holds for all b and so $E\Delta(a) = \Delta(a)$ for all a . On the other hand we have

$$\Delta_B(a)E(p \otimes q) = \Delta_B(a)(p \otimes S_B(E_{(1)})E_{(2)}q) = \Delta_B(a)(p \otimes q)$$

for all $a, p, q \in A$ because $\Delta_B(a) \in {}_B A \overline{\otimes} A^B$. As before, we are using the Sweedler type notation for E , the fact that the left leg of E belongs to B and that $S_B(E_{(1)})E_{(2)} = 1$. See the remark at the end of the preliminary section. If we now apply θ_B to the above equation, we arrive at

$$\Delta(a)E(p \otimes q) = \Delta(a)(p \otimes q)$$

for all a, p, q and hence $\Delta(a)E = \Delta(a)$ for all a .

(iii) Now we show that Δ is a homomorphism. We take a, a' and b in A and start with the formula

$$\Delta_B(aa')(1 \otimes b) = \Delta_B(a)\Delta_B(a')(1 \otimes b) \tag{5.3}$$

that holds in ${}_B A \otimes A^B$. If we apply θ_B on the left hand side of this equation, we find $\Delta(aa')(1 \otimes b)$ by the definition of Δ . We now also want to apply θ_B to the right hand side of (5.3). We claim that $\theta_B(\Delta_B(a)\xi) = \Delta(a)\theta_B(\xi)$ for all $a \in A$ and $\xi \in {}_B A \otimes A^B$. Then we find by applying θ_B on (5.3)

$$\begin{aligned} \Delta(aa')(1 \otimes b) &= \theta_B(\Delta_B(a)\Delta_B(a')(1 \otimes b)) \\ &= \Delta(a)\theta_B(\Delta_B(a')(1 \otimes b)) = \Delta(a)\Delta(a')(1 \otimes b) \end{aligned}$$

and we are done.

To prove the claim, observe that

$$\theta_B(\Delta_B(a)(p \otimes q)) = \Delta(a)(p \otimes q) = \Delta(a)E(p \otimes q) = \Delta(a)\theta_B(p \otimes q)$$

for all $a \in A$ and $p, q \in A$. Remark that in the first and the last expression, we view $p \otimes q$ in ${}_B A \otimes A^B$ and in the second and the third, we consider it as sitting in $A \otimes A$.

(iv) Finally we prove coassociativity of Δ as in (5.2). As the first leg of E belongs to the algebra B , we will have

$$(\Delta_B \otimes \iota)(E(p \otimes q)) = (1 \otimes E)(\Delta_B \otimes \iota)(p \otimes q)$$

for all $p, q \in A$. Now take $a, b, c \in A$. Then

$$\begin{aligned} (\Delta \otimes \iota)(\Delta(a)(1 \otimes b))(c \otimes 1 \otimes 1) \\ &= (E \otimes 1)(\Delta_B \otimes \iota)(E(\Delta_B(a)(1 \otimes b))(c \otimes 1 \otimes 1)) \\ &= (E \otimes 1)(1 \otimes E)(\Delta_B \otimes \iota)(\Delta_B(a)(1 \otimes b))(c \otimes 1 \otimes 1). \end{aligned}$$

Similarly we will find

$$\begin{aligned} ((\iota \otimes \Delta)(\Delta(a)(c \otimes 1))(1 \otimes 1 \otimes b)) \\ &= (1 \otimes E)(\iota \otimes \Delta_B)(E(\Delta_B(a)(c \otimes 1))(1 \otimes 1 \otimes b)) \\ &= (1 \otimes E)(E \otimes 1)(\iota \otimes \Delta_B)(\Delta_B(a)(c \otimes 1))(1 \otimes 1 \otimes b). \end{aligned}$$

As $(E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1)$ in $M(A \otimes A \otimes A)$, coassociativity of Δ follows from coassociativity of Δ_B . ■

The proof of coassociativity in item (iv) above needs more and finer arguments. We have e.g. made no distinction between $E(p \otimes q)$ and $\theta_B(p \otimes q)$ where in the first place, we consider $p \otimes q$ as sitting in $A \otimes A$ while in the second case it is considered as an element in ${}_B A \otimes A^B$. On the other hand, the result is so obvious and to give a more precise argument will not really clarify the proof.

In a completely similar fashion, we can associate another coproduct Δ' on A . Now we use $A \overline{\otimes}' A$ for the extended module of elements η satisfying and characterized by the requirement that

$$(1 \otimes b)\eta \quad \text{and} \quad (c \otimes 1)\eta$$

belong to $A \otimes A$ for all $b, c \in A$. Recall that the right coproduct Δ_C is a map from A to ${}^C A \overline{\otimes} A_C$. We will now use the map $\theta_C : {}^C A \overline{\otimes} A_C \rightarrow A \otimes A$ given by $\theta_C(a \otimes b) = (a \otimes b)E$.

LEMMA 5.3. *There is a homomorphism $\Delta' : A \rightarrow A \overline{\otimes}' A$ given by*

$$(1 \otimes b)\Delta'(a) = \theta_C((1 \otimes b)\Delta_C(a)) \quad \text{and} \quad (c \otimes 1)\Delta'(a) = \theta_C((c \otimes 1)\Delta_C(a))$$

for $a, b, c \in A$. We have $\Delta'(a) = E\Delta'(a) = \Delta'(a)E$ for all a . The coproduct Δ' is coassociative in the sense that

$$(c \otimes 1 \otimes 1)(\Delta' \otimes \iota)((1 \otimes b)\Delta'(a)) = (1 \otimes 1 \otimes b)(\iota \otimes \Delta')(c \otimes 1)\Delta'(a))$$

holds for all a, b, c in A .

The proof of Lemma 5.3 is completely similar as the one of Lemma 5.2.

We would like that these two coproducts coincide. And because $\Delta(a)$ is a left multiplier and $\Delta'(a)$ a right multiplier of $A \otimes A$, it then will follow that Δ maps A into $M(A \otimes A)$ for all $a \in A$. So we will eventually have that Δ is a (regular) coproduct on A .

We will of course need the given relation between the left coproduct Δ_B and the right coproduct Δ_C of the multiplier Hopf algebroid. However, before we are able to use this result to obtain the equality of Δ and Δ' on A , we need the counit for each of these two coproducts. They are obtained in the Lemma 5.4 below. We will also need *an extra assumption*. This will be discussed later, see Proposition 5.9 and the subsequent remarks there.

First recall the counital maps $\varepsilon_B : A \rightarrow B$ and $\varepsilon_C : A \rightarrow C$ obtained in Theorem 5.6 of [T-VD] for the given multiplier Hopf algebroid. They satisfy, among other properties, the following equalities:

$$\varepsilon_B(xa) = x\varepsilon_B(a) \quad \text{and} \quad \varepsilon_B(S_B(x)a) = \varepsilon_B(a)x \quad (5.4)$$

$$\varepsilon_C(ay) = \varepsilon_C(a)y \quad \text{and} \quad \varepsilon_C(aS_C(y)) = y\varepsilon_C(a) \quad (5.5)$$

where $a \in A$, $x \in B$ and $y \in C$. See a remark before Definition 5.2 of [T-VD].

We will need this to obtain the counits in the following lemma.

LEMMA 5.4. *Define $\varepsilon : A \rightarrow \mathbb{C}$ by $\varepsilon = \varphi_B \circ \varepsilon_B$ and $\varepsilon' : A \rightarrow \mathbb{C}$ by $\varepsilon' = \varphi_C \circ \varepsilon_C$ where φ_B and φ_C are the linear functionals on B and C respectively characterized by the formulas*

$$(\varphi_B \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi_C)E = 1.$$

Then ε is a counit for Δ and ε' is a counit for Δ' .

Proof. (i) First consider the characterizing property of the left counit ε_B as in the first equality in (3.2) of [T-VD]. The map $x \otimes q \mapsto S_B(x)q$ from $B \otimes A$ to A will map $(\varepsilon_B \otimes \iota)T_\rho(a \otimes b)$ to ab for all $a, b \in A$. Recall that $T_\rho(a \otimes b) = \Delta_B(a)(1 \otimes b)$. It follows that

$$\begin{aligned} (\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) &= (\varphi_B \otimes \iota)(\varepsilon_B \otimes \iota)(E(\Delta_B(a)(1 \otimes b))) \\ &= (\varphi_B \otimes \iota)(E(\varepsilon_B \otimes \iota)(\Delta_B(a)(1 \otimes b))). \end{aligned}$$

We have used that the left leg of E belongs to B and that $\varepsilon_B(xp) = x\varepsilon_B(p)$ when $x \in B$ and $p \in A$. If we write $(\varepsilon_B \otimes \iota)(\Delta_B(a)(1 \otimes b))$ as $\sum_i x_i \otimes q_j$, we see that

$$E(\varepsilon_B \otimes \iota)(\Delta_B(a)(1 \otimes b)) = \sum_i E(x_i \otimes q_i) = \sum_i E(1 \otimes S_B(x_i)q_i)$$

and we know that this is $E(1 \otimes ab)$. Therefore, if we apply $\varphi_B \otimes \iota$ we get ab and hence all together we find

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab$$

for all a, b in A .

(ii) Next consider the second equality in (3.2) of [T-VD]. Now the map $p \otimes x \mapsto xp$ from $A \otimes B$ to A will map $(\iota \otimes \varepsilon_B)T_\lambda(c \otimes a)$ to ac for all $a, c \in A$. Recall that $T_\lambda(c \otimes a) = \Delta_B(a)(c \otimes 1)$. It follows that

$$\begin{aligned} (\iota \otimes \varepsilon)(\Delta(a)(c \otimes 1)) &= (\iota \otimes \varphi_B)(\iota \otimes \varepsilon_B)(E(\Delta_B(a)(c \otimes 1))) \\ &= (\iota \otimes \varphi_C)(\iota \otimes S_B \circ \varepsilon_B)(E(\Delta_B(a)(c \otimes 1))) \end{aligned}$$

Now

$$S_B(\varepsilon_B(y)) = S_B(\varepsilon_B(p))S_B^{-1}(y) = yS_B(\varepsilon_B(p))$$

for all $y \in C$ and $p \in A$. Using now that the right leg of E is in C , we find that the above expression equals

$$(\iota \otimes \varphi_C)(E(\iota \otimes S_B \circ \varepsilon_B)(\Delta_B(a)(c \otimes 1))).$$

We now write $(\iota \otimes \varepsilon_B)(\Delta_B(a)(c \otimes 1))$ as $\sum_i p_i \otimes x_i$. Then we see that

$$E(\iota \otimes S_B \circ \varepsilon_B)(\Delta_B(a)(c \otimes 1)) = \sum_i E(p_i \otimes S_B(x_i)) = \sum_i E(x_i p_i \otimes 1)$$

and this is $E(ac \otimes 1)$. If we apply $\iota \otimes \varphi_C$ we find ac and all together we get

$$(\iota \otimes \varepsilon)(\Delta(a)(c \otimes 1)) = ac$$

for all $a, c \in A$.

We see from (i) and (ii) that ε is a counit for the coproduct Δ on A .

A completely similar argument will give that ε' , defined as $\varphi_C \circ \varepsilon_C$, is a counit for the coproduct Δ' on A . Here we will need the module properties of ε_C as recalled before the formulation of the lemma, as well as equation (4.8) in Definition 4.3 of [T-VD]. ■

Again, we should be more precise as we indicated already after the proof of Lemma 5.2. Moreover we are not even working with genuine coproducts as we just have that $\Delta(a)$ is a left multiplier and $\Delta'(a)$ is a right multiplier of $A \otimes A$ for any $a \in A$. Fortunately, we have taken care of these restrictions. We also do this for the next statement.

We know that the counits are unique provided the coproducts are full. Fullness of the coproducts Δ and Δ' will be a consequence of the following result.

LEMMA 5.5. *We have*

$$\begin{aligned} \Delta(A)(1 \otimes A) &= \Delta(A)(A \otimes 1) = E(A \otimes A) \\ (1 \otimes A)\Delta'(A) &= (A \otimes 1)\Delta'(A) = (A \otimes A)E. \end{aligned}$$

Proof. Because we are working with a regular multiplier Hopf algebroid, all canonical maps are bijective by assumption (see Definition 5.2 of [T-VD]). In particular, the ranges of the maps T_ρ and T_λ are all of ${}_B A \otimes A^B$. On the other hand, we know that the map θ_B is an isomorphism of ${}_B A \otimes A^B$ to the subspace $E(A \otimes A)$ of $A \otimes A$. Combining these two results will give

$$\Delta(A)(1 \otimes A) = E(A \otimes A) \quad \text{and} \quad \Delta(A)(A \otimes 1) = E(A \otimes A).$$

Similarly, using that the maps ${}_\rho T$ and ${}_\lambda T$ have range all of ${}^C A \otimes A_C$ and that θ_C is an isomorphism from ${}^C A \otimes A_C$ to $(A \otimes A)E$, we find

$$(1 \otimes A)\Delta'(A) = (A \otimes A)E \quad \text{and} \quad (A \otimes 1)\Delta'(A) = (A \otimes A)E. \quad \blacksquare$$

This indeed implies fullness of the coproducts. Assume e.g. that V is a subspace of A so that $\Delta(A)(1 \otimes A) \subseteq V \otimes A$. Then we have $E(A \otimes A) \subseteq V \otimes A$. Because the left leg of E is all of B , it follows that $BA \subseteq V$. But as we know that $BA = A$, we get $V = A$. Similarly for the right leg of Δ and for the legs of Δ' . Hence these counits obtained in Lemma 5.4 are unique.

This property already takes care of the condition on the ranges of the canonical maps for a regular weak multiplier Hopf algebra. See Definition 1.14 in [VD-W1]. And observe that we want to obtain a *regular* weak multiplier Hopf algebra so that we have to consider not only the ranges of the canonical maps T_1 and T_2 , but also of the maps T_3 and T_4 . Of course, we still need to argue that Δ and Δ' coincide (see further).

Next we show that the coproducts have the expected behavior on the legs of E . This is condition (ii) in Definition 1.14 of [VD-W1].

LEMMA 5.6. *We have*

$$(\iota \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1)$$

and similarly for the other coproduct Δ' .

Recall first that we have not yet obtained the equality of Δ and Δ' . This implies not only that we have to prove these results for the two, but also that we have to be a bit more careful. At this moment, we only know that $\Delta(a)$ is a left multiplier and that $\Delta'(a)$ is right multiplier of $A \otimes A$. This means that formulas should be given a meaning by multiplying, left or right, and in the appropriate way, with elements of A .

We know the behavior of Δ_B and Δ_C on B and on C . See the formulas (5.2) in Section 5 of [T-VD]. This will imply the same for the coproducts Δ and Δ' . We will e.g. get that $\Delta(y) = E(y \otimes 1)$ for $y \in C$ and as C is the right leg of E , this will imply

$$(\iota \otimes \Delta)E = (1 \otimes E)(E \otimes 1).$$

In fact, these two results are the same. All the other cases are obtained in a completely similar fashion. It is also possible to use that the two legs of E commute.

Next we look again at the canonical maps T_1, T_2, T_3 and T_4 associated with the coproducts Δ and Δ' on A by the formulas

$$\begin{aligned} T_1(a \otimes b) &= \Delta(a)(1 \otimes b) & \text{and} & & T_2(a \otimes b) &= (a \otimes 1)\Delta'(b) \\ T_3(a \otimes b) &= (1 \otimes b)\Delta'(a) & \text{and} & & T_4(a \otimes b) &= \Delta(b)(a \otimes 1) \end{aligned}$$

where $a, b \in A$.

In Lemma 5.5 we have obtained the ranges of these maps. The range of T_1 and T_4 is $E(A \otimes A)$ whereas the range of T_2 and T_3 is $(A \otimes A)E$. In the next lemma, we obtain the kernels of these maps.

LEMMA 5.7. *We have*

$$\begin{aligned} \text{Ker}(T_1) &= (A \otimes 1)(1 - F_1)(1 \otimes A) & \text{and} & & \text{Ker}(T_2) &= (A \otimes 1)(1 - F_2)(1 \otimes A) \\ \text{Ker}(T_3) &= (1 \otimes A)(1 - F_3)(A \otimes 1) & \text{and} & & \text{Ker}(T_4) &= (1 \otimes A)(1 - F_4)(A \otimes 1) \end{aligned}$$

where F_1, F_2, F_3 and F_4 are as in Proposition 2.6.

Proof. Consider the map T_1 . It is related with the map T_ρ in the following commuting diagram (see the proof of Proposition 4.4):

$$\begin{array}{ccc} A \otimes A & \xrightarrow{T_1} & A \otimes A \\ \pi \downarrow & & \downarrow \pi_B \\ A_B \otimes_B A & \xrightarrow{T_\rho} & {}_B A \otimes A^B \end{array}$$

So if $\xi \in A \otimes A$ and if $T_1(\xi) = 0$, we have $T_\rho(\pi_B(\xi)) = 0$ and because T_ρ is assumed to be injective, we find that also $\pi(\xi) = 0$. On the other hand, if $\pi(\xi) = 0$, we have $\pi_B(T_1(\xi)) = 0$ and because π_B is injective on the range of T_1 we must have $T_1(\xi) = 0$.

So we see that the kernel of T_1 is the same as the kernel of π . From Proposition 3.11 we know that the kernel of π is equal to $(A \otimes 1)(1 - F_1)(1 \otimes A)$ where $F_1 = (\iota \otimes S_C)E$. This proves the first formula of the lemma.

A similar argument works in the three other cases. For T_2 we need the map λT , in the case of T_3 we need ρT and for T_4 , the relevant map is T_λ . See Notation 4.3 and the subsequent remark. In all cases, we again need the results of Proposition 3.11. ■

So far we have not considered any relation between the left and the right structures. We now proceed and try to show that the two coproducts Δ and Δ' coincide. First we need the joint coassociativity rules.

LEMMA 5.8. *For all $a, b, c \in A$ we have*

$$(c \otimes 1 \otimes 1)(\Delta' \otimes \iota)(\Delta(a)(1 \otimes b)) = (\iota \otimes \Delta)((c \otimes 1)\Delta'(a))(1 \otimes 1 \otimes b) \quad (5.6)$$

$$(\Delta \otimes \iota)((1 \otimes b)\Delta'(a))(c \otimes 1 \otimes 1) = (1 \otimes 1 \otimes b)(\iota \otimes \Delta')(\Delta(a)(c \otimes 1)). \quad (5.7)$$

Proof. We start from the joint coassociativity for the left and the right coproducts of the original multiplier Hopf algebroid. The first one says that

$$(c \otimes 1 \otimes 1)(\Delta_C \otimes \iota)(\Delta_B(a)(1 \otimes b)) = (\iota \otimes \Delta_B)((c \otimes 1)\Delta_C(a))(1 \otimes 1 \otimes b) \quad (5.8)$$

holds in ${}^C A \otimes_B A_C \otimes A^B$ for all $a, b, c \in A$; see equations (5.3) in [T-VD]. Here the triple balanced tensor product ${}^C A \otimes_B A_C \otimes A^B$ is defined by the relations

$$\begin{aligned} p \otimes xq \otimes r &= p \otimes q \otimes S_B(x)r \\ p \otimes qy \otimes r &= pS_C(y) \otimes q \otimes r \end{aligned}$$

where $p, q, r \in A$ and $x \in B$ and $y \in C$.

We now proceed as in the proof of the coassociativity of the coproducts Δ and Δ' on A (item (iv)) in the proof of Lemma 5.2. For all $a, b, c \in A$ we have

$$\begin{aligned} (c \otimes 1 \otimes 1)(\Delta' \otimes \iota)(\Delta(a)(1 \otimes b)) &= (c \otimes 1 \otimes 1)(\Delta_C \otimes \iota)(E\Delta_B(a)(1 \otimes b))(E \otimes 1) \\ &= (c \otimes 1 \otimes 1)(1 \otimes E)(\Delta_C \otimes \iota)(\Delta_B(a)(1 \otimes b))(E \otimes 1) \\ &= (1 \otimes E)(c \otimes 1 \otimes 1)(\Delta_C \otimes \iota)(\Delta_B(a)(1 \otimes b))(E \otimes 1) \end{aligned}$$

where we have used that the first leg of E belongs to B . On the other hand we have

$$\begin{aligned} (\iota \otimes \Delta)((c \otimes 1)\Delta'(a))(1 \otimes 1 \otimes b) &= (1 \otimes E)(\iota \otimes \Delta_B)((c \otimes 1)\Delta_C(a)E)(1 \otimes 1 \otimes b) \\ &= (1 \otimes E)(\iota \otimes \Delta_B)((c \otimes 1)\Delta_C(a))(E \otimes 1)(1 \otimes 1 \otimes b) \\ &= (1 \otimes E)(\iota \otimes \Delta_B)((c \otimes 1)\Delta_C(a))(1 \otimes 1 \otimes b)(E \otimes 1) \end{aligned}$$

where now we used that the right leg of E belongs to C . So coassociativity as in (5.6) follows now from coassociativity as in (5.7).

Similarly coassociativity as in (5.7) will follow from the other joint coassociativity rule

$$(\Delta_B \otimes \iota)((1 \otimes b)\Delta_C(a))(c \otimes 1 \otimes 1) = (1 \otimes 1 \otimes b)(\iota \otimes \Delta_C)(\Delta_B(a)(c \otimes 1))$$

in equation (5.3) of [T-VD]. ■

The remark made after the proof of Lemma 5.2 is also relevant here.

Finally we are now able to show equality of Δ and Δ' . However we need *an extra condition*. We will comment on it after the proof of the next proposition.

PROPOSITION 5.9. *Assume that the counits ε and ε' are equal. Then also the coproducts Δ and Δ' coincide in the sense that $p(\Delta(a)q) = (p\Delta'(a))q$ for all $a \in A$ and $p, q \in A \otimes A$. In particular, Δ will map A to $M(A \otimes A)$.*

Proof. Consider coassociativity as in (5.6):

$$(c \otimes 1 \otimes 1)(\Delta' \otimes \iota)(\Delta(a)(1 \otimes b)) = (\iota \otimes \Delta)((c \otimes 1)\Delta'(a))(1 \otimes 1 \otimes b).$$

If we apply ε' on the middle leg of the left hand side of this equality, we find

$$(c \otimes 1)\Delta(a)(1 \otimes b).$$

If on the other hand, we apply ε on the middle leg of the right hand side of this equality, we find

$$(c \otimes 1)\Delta'(a)(1 \otimes b).$$

So when $\varepsilon = \varepsilon'$ it follows that $\Delta = \Delta'$. ■

Let us look closer at the extra condition on ε and ε' .

LEMMA 5.10. *The counits $\varepsilon, \varepsilon'$ and the antipode S satisfy*

$$\varepsilon \circ S = \varepsilon' \quad \text{and} \quad \varepsilon = \varepsilon' \circ S.$$

Thus, the counits ε and ε' are equal if and only if one of them is invariant under the antipode.

Proof. We have $\varepsilon_C \circ S = S_B \circ \varepsilon_B$ by Corollary 5.12 in [T-VD] and $\varphi_C = \varphi_B \circ S_B^{-1}$ by Proposition 3.3. Therefore,

$$\varepsilon' \circ S = \varphi_C \circ \varepsilon_C \circ S = \varphi_B \circ \varepsilon_B = \varepsilon.$$

Moreover, $S_C \circ \varepsilon_C = \varepsilon_B \circ S$ by Corollary 5.12 in [T-VD] and

$$\varphi_C = \varphi_C \circ \sigma_C = \varphi_C \circ S_B \circ S_C = \varphi_B \circ S_C$$

by Proposition 3.3 again, whence

$$\varepsilon \circ S = \varphi_B \circ \varepsilon_B \circ S = \varphi_C \circ \varepsilon_C = \varepsilon'. \quad \blacksquare$$

Remark that the condition $\varepsilon = \varepsilon'$ involves the linear functionals φ_B and φ_C on the base algebras B and C , but not only that. Indeed, the counits ε and ε' are given by $\varphi_B \circ \varepsilon_B$ and $\varphi_C \circ \varepsilon_C$. So, the condition $\varepsilon = \varepsilon'$ involves also the counital maps ε_B and ε_C of the given multiplier Hopf algebroid. A similar remark can be made when the condition is formulated as the invariance of ε under the antipode S .

It is possible to give an example of a regular multiplier Hopf algebroid with a base algebra B that is separable Frobenius, such that the faithful linear functional φ_B has the right modular automorphism, but where the associated counit ε is not invariant under the given antipode S . In fact, such an example already exists in the framework of Hopf algebroids. We will give such an example in the next section, see Proposition 6.4 and 6.5.

We now formulate the main result of this section. For completeness and the convenience of the reader, we include all the details. For the notation and concepts in the formulation, we refer to the preceding.

THEOREM 5.11. *Assume that A is an algebra with local units, that B and C are commuting subalgebras of $M(A)$ satisfying $AB = BA = A$ and $AC = CA = A$, and that there is a (regular) separability idempotent $E \in M(B \otimes C)$. Denote by*

$$S_B : B \rightarrow C \quad \text{and} \quad S_C : C \rightarrow B$$

the anti-isomorphisms characterized by

$$E(x \otimes 1) = E(1 \otimes S_B(x)) \quad \text{and} \quad (1 \otimes y)E = (S_C(y) \otimes 1)E$$

for $x \in B$ and $y \in C$, and by φ_B and φ_C the unique linear functionals on B and C satisfying

$$(\varphi_B \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi_C)E = 1.$$

Finally we assume the existence of coproducts

$$\Delta_B : A \rightarrow {}_B A \overline{\times} A^B \quad \text{and} \quad \Delta_C : A \rightarrow {}^C A \overline{\times} A_C$$

that turn the quantum graph with local units

$$(A, B, C, S_B, S_C)$$

into a regular multiplier Hopf algebroid.

Then there exist coproducts Δ and Δ' on A with values in the left and the right multiplier algebra of $A \otimes A$ respectively, defined by

$$\Delta(a)(1 \otimes b) = E(\Delta_B(a)(1 \otimes b)) \quad \text{and} \quad (c \otimes 1)\Delta'(a) = ((c \otimes 1)\Delta_C(a))E$$

where $a, b, c \in A$. These two coproducts are full in the sense that the left and right legs of each of them equal A .

The composition $\varphi_B \circ \varepsilon_B$ is a counit ε on (A, Δ) and the composition $\varphi_C \circ \varepsilon_C$ is a counit ε' for (A, Δ') . These counits ε and ε' are the same if and only if one of them is invariant for the antipode S of the original multiplier Hopf algebroid. In that case, the coproducts Δ and Δ' coincide and the pair (A, Δ) is a regular weak multiplier Hopf algebra. Its canonical idempotent is E and its antipode is the same as the original antipode S of the multiplier Hopf algebroid.

Proof. (i) We have seen that (A, B, C, S_B, S_C) is a quantum graph with local units in Example 3.6.

(ii) In Proposition 3.9 we have shown that there are well-defined maps

$$\theta_B : {}_B A \otimes A^B \rightarrow A \otimes A \quad \text{and} \quad \theta_C : {}^C A \otimes A_C \rightarrow A \otimes A$$

given by

$$\theta_B(p \otimes q) = E(p \otimes q) \quad \text{and} \quad \theta_C(p \otimes q) = (p \otimes q)E$$

for $p, q \in A$. The map θ_B is a bijection from ${}_B A \otimes A^B$ to $E(A \otimes A)$ and θ_C is bijective from ${}^C A \otimes A_C$ to $(A \otimes A)E$. They allow us to define Δ and Δ' on A by

$$\begin{aligned}\Delta(a)(1 \otimes b) &= \theta_B(\Delta_B(a)(1 \otimes b)) \\ (c \otimes 1)\Delta'(a) &= \theta_C((c \otimes 1)\Delta_C(a))\end{aligned}$$

for $a, b, c \in A$. Remark that $\Delta(a)$ is a left multiplier of $A \otimes A$ and $\Delta'(a)$ is a right multiplier of $A \otimes A$ for all a . In Lemma 5.2 and Lemma 5.3 we have seen that Δ and Δ' are coproducts. As a consequence of Lemma 5.5, we could show that these coproducts are full in the sense that their two legs are all of A .

(iii) In Lemma 5.4 we have seen that ε and ε' , defined on A as $\varphi_B \circ \varepsilon_B$ and $\varphi_C \circ \varepsilon_C$, are counits for Δ and Δ' respectively.

(iv) In Proposition 5.9 we showed that the equality $\varepsilon = \varepsilon'$ yields that also $\Delta = \Delta'$. In fact, this last equality means that Δ and Δ' satisfy e.g.

$$(c \otimes 1)(\Delta(a)(1 \otimes b)) = ((c \otimes 1)\Delta'(a))(1 \otimes b)$$

for all $a, b, c \in A$. Therefore, $\Delta(a)$ and $\Delta'(a)$ determine the same multiplier of $A \otimes A$. Then $\Delta : A \rightarrow M(A \otimes A)$ is a regular and full coproduct on A as defined in Definitions 1.1 and 1.4 of [VD-W1]. The functional ε is a counit for this coproduct Δ as defined in Definition 1.3 of [VD-W1]. It is unique because Δ is full.

We have seen in Lemma 5.10 that the condition $\varepsilon = \varepsilon'$ is equivalent with the requirement that ε or ε' is invariant under the antipode S of the multiplier Hopf algebroid.

(v) In Lemma 5.5 we have seen that the ranges of the canonical maps T_1 and T_2 are $E(A \otimes A)$ and $(A \otimes A)E$, respectively. This takes care of condition (i) of Definition 1.14 in [VD-W1] (definition of a weak multiplier Hopf algebra). We want to show that (A, Δ) is regular and so we also need condition (i) of Definition 1.14 in [VD-W1] for the pair (A^{op}, Δ) . For this we need that also the range of the canonical map T_3 is $(A \otimes A)E$ and that of T_4 is $E(A \otimes A)$. This is also part of Lemma 5.5.

(vi) In Lemma 5.6 we found

$$(\iota \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1).$$

This is condition (ii) in Definition 1.14 of [VD-W1].

(vii) Finally, we need to see that the kernels of the four canonical maps T_1, T_2, T_3 and T_4 all satisfy the requirement as in Definition 1.14 of [VD-W1]. This is essentially shown in Lemma 5.7. The result there is given in terms of the idempotents F_i in $M(A \otimes A^{\text{op}})$ whereas in Definition 1.14 of [VD-W1], the condition is in terms of the idempotent maps G_i . However, e.g. in Proposition 4.5 of [VD-W1], we find the relation between these two.

So we see that all together, we get a regular weak multiplier Hopf algebra (A, Δ) .

To show that the antipode of the weak multiplier Hopf algebra is the same as the antipode of the original multiplier Hopf algebroid, one can give various arguments. At the end of the previous section, we have seen that the antipode of the multiplier Hopf algebroid associated to a given weak multiplier Hopf algebra is the same as the original antipode. Then the result follows because the antipode of a multiplier Hopf algebroid is unique. ■

Of course, to make this last argument complete, one has to argue that the two procedures, the one described in Section 4 (from a weak multiplier Hopf algebra to a multiplier Hopf algebroid) and the one developed in this section (from a multiplier Hopf algebroid to a weak multiplier Hopf algebra), are essentially inverses of each other. However, this should be immediately clear from the two constructions.

There is only one subtle detail to observe. If one starts with a weak multiplier Hopf algebra, then the left and the right counit of the associated regular multiplier Hopf algebroid, constructed in Propositions 4.6 and 4.7, will automatically be surjective. But in the definition of a regular multiplier Hopf algebroid, this surjectivity is not required. If one passes from a regular multiplier Hopf algebroid $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ to a weak multiplier Hopf algebra (A, Δ) as in Theorem 5.11, and then to the associated regular multiplier Hopf algebroid as in Theorem 4.8, the algebras B and C will get replaced by the images of the left and the right counit, respectively. Compare also with Lemma 3.7 in [T-VD].

COROLLARY 5.12. *Let $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ be a regular multiplier Hopf algebroid with antipode S and left and right counits ε_B and ε_C . Assume that the following conditions hold:*

- (i) *the algebra A has local units;*
- (ii) *the algebra B is a separable Frobenius algebra;*
- (iii) *there exists a separating linear functional φ_B for B such that the composition $\varepsilon := \varphi_B \circ \varepsilon_B$ is invariant under the antipode in the sense that $\varepsilon \circ S = \varepsilon$;*
- (iv) *the left counit ε_B is surjective, that is, $\varepsilon_B(A) = B$.*

Then there exist

- (a) *a (regular) separability idempotent $E \in M(B \otimes C)$ and unique linear functionals φ_B and φ_C on B and C satisfying*

$$\begin{aligned} E(x \otimes 1) &= E(1 \otimes S_B(x)), & (\varphi_B \otimes \iota)E &= 1, \\ (1 \otimes y)E &= (S_C(y) \otimes 1)E, & (\iota \otimes \varphi_C)E &= 1; \end{aligned}$$

- (b) *a unique coproduct Δ on A such that*

$$\Delta(a)(1 \otimes b) = E(\Delta_B(a)(1 \otimes b)) \quad \text{and} \quad (c \otimes 1)\Delta(a) = ((c \otimes 1)\Delta_C(a))E$$

for all $a, b, c \in A$.

The pair (A, Δ) is a regular weak multiplier Hopf algebra with separability idempotent E , counit $\varepsilon = \varphi_B \circ \varepsilon_B$ and antipode S .

Proof. We show that condition (iii) above implies condition (i) in Lemma 3.8. Then Lemma 3.8 implies (a) so that we can apply Theorem 5.11 and obtain (b). Let $a \in A$ and $x, x' \in B$. Then the relations (5.4) and the assumption $\varepsilon \circ S = \varepsilon$ imply

$$\begin{aligned} \varphi_B(x\varepsilon_B(a)) &= \varepsilon(xa) = \varepsilon(S(a)S(x)) \\ \text{and} \quad \varphi_B(\varepsilon_B(a)S_B^{-1}(S_C^{-1}(x))) &= \varepsilon(S_C^{-1}(x)a) = \varepsilon(S(a)x). \end{aligned}$$

But by surjectivity of ε_B and equation (3.5) in Proposition 3.5 of [T-VD], $\varepsilon(S(a)S(x)) = \varepsilon(S(a)x)$. Thus, the modular automorphism of φ_B is equal to $S_B^{-1}S_C^{-1}$. ■

6. Examples and special cases. In this section we will give examples and discuss some special cases. The aim is to illustrate various aspects of our results in the previous sections.

Let H be a group acting from the left on a space X and denote the action as $h \times x \mapsto h.x$ for $h \in H$ and $x \in X$. The set of triples

$$G = \{(y, h, x) \mid x, y \in X, h \in H \text{ and } y = h.x\}$$

is a groupoid when the product of elements (y, h, u) and (v, k, x) is defined for $u = v$ by

$$(y, h, u)(u, k, x) = (y, hk, x)$$

where $x, y, u, v \in X$ and $h, k \in H$. This example of a groupoid illustrates very well the fact that the structure is a combination of two other ones, say that of a space and that of a group. Similarly, in the case of a weak multiplier Hopf algebra, as well as in that of a multiplier Hopf algebroid, there are two distinguished aspects. There is the base algebra on the one hand, corresponding to the 'space' aspect, and the Hopf-like structure on the other hand, corresponding to the 'group' aspect.

It is also clear from what we see in this paper that the relation of a weak multiplier Hopf algebra and its associated multiplier Hopf algebroid is situated mainly on the level of the base algebra, i.e. the 'space' ingredient. For this reason, we start our examples with weak multiplier Hopf algebras and multiplier Hopf algebroids where the second aspect, the 'group' ingredient, is trivial.

6.1. Examples with a trivial Hopf-like structure. Recall first the following example of a regular weak multiplier Hopf algebra from Section 3 in [VD-W2].

EXAMPLE 6.1. Let B and C be non-degenerate idempotent algebras and assume that E is a separability idempotent in $M(B \otimes C)$. Let A be the algebra $C \otimes B$. We identify B and C with their natural images in $M(A)$ and hence we write yx for $y \otimes x$ in A where (also further) $x \in B$ and $y \in C$. We have that the images of B and C in $M(A)$ commute and so $xy = yx$. There is a full and regular coproduct Δ on A defined by

$$\Delta(yx) = (y \otimes 1)E(1 \otimes x)$$

for $x \in B$ and $y \in C$. The pair (A, Δ) is a regular weak multiplier Hopf algebra. The source and target maps are given by

$$\varepsilon_s(yx) = S_C(y)x \quad \text{and} \quad \varepsilon_t(yx) = yS_B(x).$$

The antipode is given by $S(yx) = S_B(x)S_C(y)$. The counit is given by the formulas $\varepsilon(yx) = \varphi_C(yS_B(x))$ and $\varepsilon(yx) = \varphi_B(S_C(y)x)$. For details we refer to Proposition 3.3 of [VD-W2].

We now obtain the following result.

PROPOSITION 6.2. *Consider the regular weak multiplier Hopf algebra (A, Δ) from Example 6.1 and the associated regular multiplier Hopf algebroid as in Section 4 of this paper. The spaces ${}_B A \otimes A^B$ and ${}^C A \otimes A_C$ are naturally identified with $C \otimes A$ and $A \otimes B$ respectively via the maps*

$$y'x' \otimes yx \mapsto y' \otimes S_B(x')yx \quad \text{and} \quad y'x' \otimes yx \mapsto y'x'S_C(y) \otimes x$$

whenever $x, x' \in B$ and $y, y' \in C$. The left and right coproducts Δ_B and Δ_C then both take the form

$$\Delta_B(yx) = y \otimes x \quad \text{and} \quad \Delta_C(yx) = y \otimes x$$

for $x \in B$ and $y \in C$ where in the first case $y \otimes x$ is considered in $C \otimes M(A)$ and in the second case in $M(A) \otimes B$. The counital maps ε_B and ε_C of the associated regular multiplier Hopf algebroid are given by

$$\varepsilon_B(xy) = xS_B^{-1}(y) \quad \text{and} \quad \varepsilon_C(xy) = S_C^{-1}(x)y$$

for $x \in B$ and $y \in C$.

The proof is straightforward.

We can verify the formulas we get in Proposition 4.6 for the relation between the source and target maps ε_s and ε_t , as we have in Example 6.1, and the counital maps ε_B and ε_C as found in Proposition 4.6 and 4.7 in Section 4 of this paper. Indeed we get

$$\begin{aligned} \varepsilon_B(xy) &= S^{-1}(\varepsilon_t(yx)) = S_B^{-1}(yS_B(x)) = xS_B^{-1}(y) \\ \varepsilon_C(xy) &= S^{-1}(\varepsilon_s(yx)) = S_C^{-1}(S_C(y)x) = S_C^{-1}(x)y \end{aligned}$$

for $x \in B$ and $y \in C$ and we see that the formulas in Example 6.1 and in Proposition 6.2 are in accordance with the result of Proposition 4.6 and 4.7.

The above example is considered already in Section 7 of [T-VD].

We now use the above example to consider some examples of multiplier Hopf algebroids that do not arise from a weak multiplier Hopf algebra. For this we first remark that any pair of non-degenerate algebras B and C with anti-isomorphisms $S_B : B \rightarrow C$ and $S_C : C \rightarrow B$ give rise to a regular multiplier Hopf algebroid as in Proposition 6.2. See Section 7 of [T-VD].

EXAMPLE 6.3.

(i) Take any non-degenerate idempotent algebra B . Let $C = B^{\text{op}}$, take for S_B the identity map and for S_C its inverse. This will give a multiplier Hopf algebroid. If B is not separable Frobenius, it cannot come from a weak multiplier Hopf algebra.

(ii) Again take a non-degenerate and idempotent algebra B and $C = B^{\text{op}}$. Take again for S_B the identity map, but now let $S_C = \sigma^{-1}S_B^{-1}$ where σ is a given automorphism of B . Again this will give a regular multiplier Hopf algebroid. If B is not separable Frobenius, it cannot come from a weak multiplier Hopf algebra.

(iii) Consider the situation as in (ii). If B is separable Frobenius, but if the automorphism σ does not leave the center of B invariant, the regular multiplier Hopf algebroid still has no underlying weak multiplier Hopf algebra. Indeed, for any faithful linear functional φ_B on B with a modular automorphism σ_B , this modular automorphism σ_B will leave the center of B invariant and so it cannot be the given automorphism σ .

(iv) Finally, if B is separable Frobenius, we know that any faithful linear functional will be a separating linear function. Therefore if in (iii) the automorphism is a modular automorphism of some faithful linear functional, then the given multiplier Hopf algebroid has an underlying weak multiplier Hopf algebra as in this paper. Also conversely, for such a weak multiplier Hopf algebra constructed from a pair of algebras B and C with anti-isomorphism S_B and S_C as in Example 6.1, we know that B is separable Frobenius and

that there is a separating functional φ with a modular automorphism that coincides with the inverse of the automorphism $S_C S_B$ of B .

It is still not clear if, in the case of a separable Frobenius algebra B , any automorphism leaving the center invariant will be the modular automorphism of a separating linear functional. If this is the case, the situation as in Example 6.3 would be completely understood.

6.2. More multiplier Hopf algebroids without an underlying weak multiplier Hopf algebra. With the next example, we show that there are cases where the base algebra of a regular multiplier Hopf algebroid satisfies all the required conditions but where the two coproducts Δ and Δ' do not coincide. Hence also in this case, the multiplier Hopf algebroid will not result from a weak multiplier Hopf algebra.

We start with any regular weak multiplier Hopf algebra (A, Δ) . We assume that u, v are invertible elements in the multiplier algebra $M(B)$ of the base algebra B satisfying

$$E(vu \otimes 1)E = E. \tag{6.1}$$

This condition is fulfilled if u and v are each others inverses. However, there are other cases. Indeed, consider $v' = S_B(v)$ and $u' = S_C^{-1}(u)$ so that

$$E(vu \otimes 1)E = E(v \otimes 1)(u \otimes 1)E = E(1 \otimes v')(1 \otimes u')E = E(1 \otimes v'u')E.$$

We see that condition (6.1) is also fulfilled if u' and v' are each others inverses. If S_B and S_C are each others inverses, this happens only if u and v are inverses of each other, but otherwise, it can happen that u' and v' are inverses of each other while u and v are not.

In [VD3] the following is proven. We use the notation as above.

PROPOSITION 6.4. *Define $\Delta'(a) = (u \otimes 1)\Delta(a)(v \otimes 1)$ for all a in A . Then (A, Δ') is again a regular weak multiplier Hopf algebra. The canonical idempotent E' of (A, Δ') is $(u \otimes 1)E(v \otimes 1)$. The counit ε' for the new pair (A, Δ') is given by $a \mapsto \varepsilon(u^{-1}av^{-1})$ or equivalently by $a \mapsto \varepsilon(u'^{-1}av'^{-1})$ where ε is the counit for Δ . The new antipode S' is given by $S'(a) = uS(vav^{-1})u^{-1}$ where S is the antipode of the original pair (A, Δ) . For the new source and target maps ε'_s and ε'_t , we have*

$$\varepsilon'_s(a) = u\varepsilon_s(u^{-1}a) \quad \text{and} \quad \varepsilon'_t(a) = \varepsilon_t(av'^{-1})v' \quad \text{for all } a.$$

Observe that the base algebra B is the same for the two weak multiplier Hopf algebras (A, Δ) and (A, Δ') . The same is true for the base algebra C . However, the associated anti-isomorphisms S'_B and S'_C are not the same as S_B and S_C . Instead they are given by

$$S'_B(x) = S_B(vxv^{-1}) \quad \text{and} \quad S'_C(y) = uS_C(y)u^{-1}$$

for $x \in B$ and $y \in C$.

In [VD3] it is also shown that we have *mixed coassociativity*. This means that

$$(\Delta' \otimes \iota)\Delta(a) = (\iota \otimes \Delta)\Delta'(a) \quad \text{and} \quad (\Delta \otimes \iota)\Delta'(a) = (\iota \otimes \Delta')\Delta(a)$$

for all $a \in A$. These two relations make sense in the multiplier algebra $M(A \otimes A \otimes A)$, either by covering with the necessary elements of A or by extending the maps involved.

Then the two coproducts are used to obtain a regular multiplier Hopf algebroid (see Theorem 2.3 in [VD3]). We use the terminology from [T-VD].

PROPOSITION 6.5. *Consider the quantum graph (A, B, C, S_B, S'_C) . Define a left coproduct Δ_B on A as in Proposition 4.2, using the coproduct Δ and define a right coproduct Δ'_C on A as in Proposition 4.2, using the new coproduct Δ' . Then $(A, B, C, S_B, S_C, \Delta_B, \Delta'_C)$ is a regular multiplier Hopf algebroid.*

The proof is straightforward and details are found in [VD3]. The antipode S_r of this multiplier Hopf algebroid is given by $S_r(a) = uS(a)u^{-1}$ for $a \in A$ where S is the original antipode of (A, Δ) (see Proposition 2.5 in [VD3]).

It is also clear that the regular multiplier Hopf algebroid obtained in this way will not come from a single weak multiplier Hopf algebra, except if the two coproducts Δ and Δ' are the same, that is if the elements u and v are central and each others inverses.

7. Conclusions and further research. In Section 4 of this paper, we have seen how any regular weak multiplier Hopf algebra gives rise, in a natural way, to a regular multiplier Hopf algebroid. On the other hand, in Section 5, we have obtained various criteria for a regular multiplier Hopf algebroid to arise in this way.

There are several conditions and of a different nature. In the first place, the base algebra B has to be separable Frobenius. This however is not sufficient. Indeed, the two anti-isomorphisms S_B and S_C are given, as well as the automorphism σ_B of B , defined as the inverse of the composition $S_C S_B$ of the two anti-isomorphisms. If now this automorphism does not leave the center invariant, it cannot be the modular automorphism of a faithful functional on B . This means that, even if B is separable Frobenius, the given regular multiplier Hopf algebroid will not have an underlying weak multiplier Hopf algebra. We have seen in Section 6 that this situation can occur.

Besides of these conditions on the base algebras B and C and the given anti-isomorphisms S_B and S_C , there is still another condition, of a completely different nature. Even if all the necessary conditions on these objects are fulfilled, the left and right coproducts Δ_B and Δ_C may give rise to different coproducts Δ and Δ' on the algebra A . Also in this case, there is no underlying weak multiplier Hopf algebra. Again, we have seen in Section 6 that this can occur.

This takes us to two interesting questions. First, is it possible to formulate a condition on the given multiplier Hopf algebroid, not related in any way with the conditions on the base algebra, that guarantees the equality of these coproducts if they exist? The second question goes in the other direction. Is it possible to develop a more general theory of weak multiplier Hopf algebras where two different coproducts Δ and Δ' are allowed, related with each other only by the joint coassociativity rules? Would any such theory make sense? We see that there are examples.

In other words, one may wonder if either the axioms of a weak multiplier Hopf algebra are too restrictive or if the axioms of a multiplier Hopf algebroid are too general.

Finally, there are two other aspects that need to be considered. First, there is the theory of integrals, both in the case of regular weak multiplier Hopf algebras and for regular multiplier Hopf algebroids. The relation between the two should not give any surprises. Secondly, there is the non-regular case that has to be examined. Of course, before this can be done, it would be necessary first to develop the more general theory

of non-regular multiplier Hopf algebroids. On the other hand, the obvious link between the non-regular weak multiplier Hopf algebras and the multiplier Hopf algebroids, might be the source of inspiration to do this.

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