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On rigid relation principles in set theory without the axiom of choice

by

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Abstract. We study the deductive strength of the following statements:

RR: *every set has a rigid binary relation*,
HRR: *every set has a hereditarily rigid binary relation*,
SRR: *every set has a strongly rigid binary relation*,

in set theory without the Axiom of Choice. RR was recently formulated by J. D. Hamkins and J. Palumbo, and SRR is a classical (non-trivial) ZFC-result by P. Vopěnka, A. Pultr and Z. Hedrlín.

1. Terminology, background and goals. In this paper, we consider three types of binary relations which we call respectively *rigid*, *hereditarily rigid* and *strongly rigid*. We begin with their definitions.

Definition 1.1. Assume that R is a binary relation on the set X.

- An *automorphism* of the system (X, R) is a one-to-one function π from X onto X such that for all y and z in X, $y\,R\,z$ if and only if $\pi(y)\,R\,\pi(z)$.
- An *endomorphism* (or *homomorphism*) of the system (X, R) is a function $f : X \to X$ such that for all y and z in X, if $y\,R\,z$ then $f(y)\,R\,f(z)$.
- (X, R) is *rigid* if the only automorphism of (X, R) is the identity function on X.
- (X, R) is *hereditarily rigid* if for all $Y \subseteq X$, $(Y, R{\restriction}Y)$ is rigid, where $R{\restriction}Y = R \cap (Y \times Y)$.

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- (X,R) is *strongly rigid* if the only endomorphism of (X,R) is the identity function on X ([1]).

Our main concern will be the existence of these three types of relations in set theory without the Axiom of Choice.

The following definition gives the abbreviations we will use for the Axiom of Choice and several of its consequences.

DEFINITION 1.2.

- AC is the *Axiom of Choice*, i.e. the statement "every family of non-empty sets has a choice function".
- MC is the *Axiom of Multiple Choice*, i.e. the statement "every family $\mathcal{A} = \{A_i : i \in I\}$ of non-empty sets admits a function F with domain I such that for all $i \in I$, $F(i)$ is a non-empty finite subset of A_i".
- $\mathsf{AC}^{\mathsf{fin}}$ is AC restricted to families of non-empty finite sets.
- $\mathsf{AC}^{\mathsf{WO}}$ is AC restricted to families of non-empty well-orderable sets.
- For an infinite well-ordered cardinal number κ, AC_κ denotes AC restricted to κ-sized families of non-empty sets. In particular, for $\kappa = \omega$, AC_ω is the Axiom of Countable Choice.
- $\mathsf{AC}_{\mathsf{WO}}$ is AC restricted to well-ordered families of non-empty sets, i.e. $\mathsf{AC}_{\mathsf{WO}}$ is $(\forall\kappa)\,\mathsf{AC}_\kappa$, where κ denotes an infinite well-ordered cardinal number.
- $\mathsf{AC}^2_{\mathsf{WO}}$ is AC restricted to well-ordered families of 2-element sets.
- DC is the *principle of dependent choices*, i.e. the statement "Let R be a binary relation on a non-empty set A such that $(\forall x \in A)(\exists y \in A)(x\,R\,y)$. Then there is a sequence $(x_n)_{n\in\omega}$ of elements of A such that $x_n\,R\,x_{n+1}$ for all $n \in \omega$".

We also use the abbreviations "ZF" for Zermelo–Fraenkel set theory without the Axiom of Choice, and "ZFA" for ZF with the Axiom of Extensionality weakened to permit the existence of atoms.

The following definitions will be needed in our study of models of set theory.

DEFINITION 1.3.

(1) Assume A is a set. (Frequently, A will be the set of atoms in a model of ZFA.) Then $[A]^{<\omega}$ denotes the set of finite subsets of A.

(2) Assume A is the set of atoms in some model M of $\mathsf{ZFA}+\mathsf{AC}$, $x \in M$ and G is a group of permutations of A. Under these circumstances every element $\phi \in G$ has a unique extension to an $\in$-automorphism of M. Following the usual convention we shall denote this extension

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([1]) We are using the terminology of [2]. This differs from the terminology of [9] where *rigid* has the same meaning as our *strongly rigid*.

also by ϕ. Then $\mathrm{fix}_G(x)$ denotes the group of permutations $\{\phi \in G : \forall t \in x,\, \phi(t) = t\}$. If G is clear from the context, then $\mathrm{fix}_G(x)$ will be denoted by $\mathrm{fix}(x)$.

(3) Under the assumptions in part (2) above, $\mathrm{Sym}_G(x)$ denotes the group of permutations $\{\phi \in G : \phi(x) = x\}$ and, as above, $\mathrm{Sym}_G(x)$ will be denoted by $\mathrm{Sym}(x)$ if G is clear from the context.

We also use the following terms.

Definition 1.4.

(1) An infinite set X is called *amorphous* if X is not the disjoint union of two infinite sets.
(2) A set X is called *Dedekind finite* if X has no countably infinite subsets. Otherwise, X is called *Dedekind infinite*.

Our work is motivated by the paper [2] in which the authors introduce the "rigid relation principle" which is abbreviated RR:

RR: *Every set admits a rigid binary relation.*

In the current paper, we shall also consider the following—stronger than RR—statements which are implicit in [2]:

HRR: *Every set admits a hereditarily rigid binary relation.*
SRR: *Every set admits a strongly rigid binary relation.*

It is easy to see that AC implies RR, since AC is equivalent to the statement "every set can be well-ordered" and it is well-known that well-orders are rigid. (Note that no infinite well-order is strongly rigid.) Since AC is equivalent to the Axiom of Multiple Choice MC in ZF, it follows that, in ZF, MC implies RR. We will show (Theorem 2.2) that MC does not imply RR in ZFA set theory.

While the validity of the implication "$\mathsf{AC} \to \mathsf{RR}$" is easy, this is not the case with the validity of "$\mathsf{AC} \to \mathsf{SRR}$". In fact, this is a non-trivial classical result by P. Vopěnka, A. Pultr and Z. Hedrlín [9] proved in 1965. A simplified proof was given in 2002 by J. Nešetřil [7].

In [2], it is shown that RR is neither equivalent to AC in ZF nor is it provable from the ZF axioms alone.

Here is a summary of the results from [2] and a list of our related results.

Theorem 1.5 ([2]).

(1) RR *is not provable in* ZF. *In particular,* RR *fails in the basic Fraenkel permutation model and via the Jech–Sochor First Embedding Theorem the result can be transferred to* ZF.
(2) $\mathsf{RR} + \neg\mathsf{AC}$ *is relatively consistent with* ZF. *In particular,* RR *holds in the basic Cohen model of* $\mathsf{ZF} + \neg\mathsf{AC}$.

(3) RR *does not follow from* AC_κ *for any (infinite) well-ordered cardinal* κ. *Further,* RR *does not imply* AC_ω, *hence does not imply* DC, *in* ZF.

We prove:

- MC does not imply RR in ZFA. Thus, the statement "there are no amorphous sets" does not imply RR in ZFA (Theorem 2.2 using the second Fraenkel model).
- $\mathsf{AC}_{\mathsf{WO}}$ does not imply RR in ZFA (Theorem 2.2 using the model $\mathcal{N}33$ from [4]). Thus, the principle of dependent choices DC does not imply RR, since $\mathsf{AC}_{\mathsf{WO}}$ implies DC [5, Theorem 8.2]. It follows that RR is a *strong* axiom.
- RR implies $\mathsf{AC}^2_{\mathsf{WO}}$ (Theorem 2.7).
- HRR implies $\mathsf{AC}^{\mathsf{fin}}$ (Theorem 3.1). Thus, HRR implies that there are no amorphous sets (Corollary 3.2).
- HRR implies $\mathsf{AC}^{\mathsf{WO}}$ in every Fraenkel–Mostowski (FM) permutation model (Theorem 3.3).
- HRR does not imply AC in ZF (Theorem 3.4 using the basic Cohen model). *This enhances Theorem* 1.5(2).
- SRR does not imply AC in ZFA (Theorem 3.5 using the Mostowski linear order model). *This answers the question asked in the last sentence of* [2].

The authors of [2] also prove the following about RR for powers of 2 and in particular for $2^\omega \simeq \mathbb{R}$ and its subsets:

THEOREM 1.6 ([2]).

(1) (ZF) *Every set of reals admits a rigid binary relation.*
(2) (ZF) *If a set B has a hereditarily rigid irreflexive binary relation, then every subset $A \subseteq \mathbb{R}\times B$ has a rigid binary relation. In particular, every subset $A \subseteq \mathbb{R} \times \gamma$ for an ordinal γ has a rigid binary relation.*
(3) *It is not provable in* ZF *that every subset of* $2^{\mathbb{R}}$ *has a rigid binary relation.*

We prove:

- In ZF, every set of reals admits a *strongly* rigid binary relation (Theorem 4.1). *This enhances Theorem* 1.6(1) *above.*
- In ZF, for every set X, if X has a strongly rigid binary relation, then 2^X has a strongly rigid binary relation (Theorem 4.4).
- In ZF, for every $n \in \omega$, $\mathcal{P}^n(\omega)$, hence (by Theorem 4.10) $2^{\mathcal{P}^n(\omega)}$, has a strongly rigid binary relation. In particular, $2^{\mathbb{R}}$ has a strongly rigid binary relation (Corollary 4.9).

- In ZF, for every set X, if X has a rigid binary relation, then 2^X has a rigid binary relation (Theorem 4.10).
- In ZF, for every well-ordered cardinal number κ, 2^κ has a rigid binary relation (Corollary 4.11).
- If every subset of $2^{\mathbb{R}}$ has a rigid binary relation, then every countably infinite family of pairs of sets of reals has a choice function (Theorem 4.13(a)). *Thus, Theorem* 1.6(3) *given above is a corollary.*
- If $2^{\mathbb{R}}$ has a hereditarily rigid binary relation, then every family of non-empty finite sets of sets of reals has a choice function, which in turn implies that there exists a non-measurable subset of 2^ω with the product measure (Theorem 4.13(b)).

We now present the main results in our paper divided into three sections: Section 2 on the deductive strength of RR, Section 3 on the hereditary rigidity and the strong rigidity principle of Vopěnka, Pultr and Hedrlín, and Section 4 on rigid hereditarily rigid, and strongly rigid binary relations on Cantor cubes.

2. On the deductive strength of RR. We start this section with the proofs that neither MC nor $\mathsf{AC_{WO}}$ implies RR in ZFA. First, we provide a general property which, if possessed by an FM model M, implies that the only sets in M which admit a rigid binary relation are exactly the well-orderable ones. We refer the reader to [5, Chapter 4] for an extensive treatment of FM models and of the relevant techniques.

To state the property, assume that M is the FM model determined by the group G of permutations of the set A of atoms and the normal filter Γ of subgroups of G. The relevant property is:

(2.1) For every $x \in M$ and every $\phi \in G$ there is a permutation ϕ' of A which is in M and for which $\phi'(x) = \phi(x)$.

Recall here that if $\phi \in G$, then there is a unique extension of ϕ to an $\in$-automorphism of the model M, which we also denote by ϕ. Furthermore, we note that if ϕ' is in M, then so is $\phi'(x)$ for any $x \in M$, since M satisfies the axiom scheme of replacement.

Theorem 2.1. *If M has property* (2.1) *then, in M, every Y that admits a rigid relation is well-orderable.*

Proof. Assume that (2.1) is true of M, that $Y \in M$ and that R is a rigid relation on Y which is in M. Then for some $H \in \Gamma$,

$$\forall \psi \in H, \quad \psi((Y,R)) = (Y,R).$$

We will show, by contradiction, that

$$\forall \psi \in H, \forall t \in Y, \quad \psi(t) = t. \tag{2.2}$$

Assume that for some $\phi \in H$ and some $t \in Y$, $\phi(t) \neq t$. By (2.1), there is a permutation ϕ' of A which is in M and for which $\phi'((Y, R, t)) = \phi((Y, R, t)) = (Y, R, \phi(t))$. Since $\phi'((Y, R)) = (Y, R)$, ϕ' restricted to Y is an automorphism of the system (Y, R) which is in M. But $\phi'(t) = \phi(t) \neq t$, contradicting our assumption that R is rigid on Y.

Equation (2.2) implies that Y is well-orderable in M, finishing the proof of the theorem. ∎

Theorem 2.2.

(1) *The Axiom of Multiple Choice* MC *does not imply* RR *in* ZFA *set theory. Thus, the statement "there are no amorphous sets" does not imply* RR *in* ZFA.

(2) $\mathsf{AC}_{\mathsf{WO}}$ *does not imply* RR *in* ZFA. *Hence, the principle of dependent choices* DC *does not imply* RR *in* ZFA.

Proof. (1) We consider the second Fraenkel permutation model of ZFA, listed as model $\mathcal{N}2$ in [4]. Let us recall its description: One starts with a model $\mathcal{M}$ of ZFA + AC with a set A of atoms which is a countable disjoint union $\bigcup\{A_n : n \in \omega\}$, where $A_n = \{a_n, b_n\}$ for $n \in \omega$. The group G of permutations of A consists of all permutations π such that $\pi(A_n) = A_n$ for each $n \in \omega$. The normal filter Γ of subgroups of G is the filter generated by the filter base $\{\text{fix}(E) : E \in [A]^{<\omega}\}$. (See Definition 1.3.) Then $\mathcal{N}2$ is the FM model determined by $\mathcal{M}$, G and Γ, consisting of all the hereditarily symmetric elements of $\mathcal{M}$, that is, $\mathcal{N}2$ consists of all $x \in \mathcal{M}$ such that x and every element in the transitive closure of x is symmetric, where for $x \in \mathcal{M}$, x is *symmetric* if there is some finite set $E \subseteq A$ such that $\text{fix}(E) \subseteq \text{Sym}(x)$. In this case, E is called a *support* of x.

It is known (see [4]) that MC holds in $\mathcal{N}2$ and that the family $\mathcal{A} = \{A_n : n \in \omega\}$ (which is countable in $\mathcal{N}2$) admits no partial choice function in the model, i.e. $\mathcal{A}$ has no infinite subfamily with a choice function in the model. It follows that A is not a well-orderable set in $\mathcal{N}2$.

We show now that $\mathcal{N}2$ satisfies property (2.1). To this end, let $x \in \mathcal{N}2$ and let $\phi \in G$. Let E be a support of x and let ϕ' be any permutation of A which agrees with ϕ on E and moves only finitely many atoms. Clearly, $\phi' \in G$, $\phi'(x) = \phi(x)$, and also ϕ' is in $\mathcal{N}2$, since the finite set $E' = \{a \in A : \phi'(a) \neq a\}$ is a support of ϕ'. By Theorem 2.1, the set A of atoms does not admit a rigid binary relation in $\mathcal{N}2$, hence RR fails in $\mathcal{N}2$.

The second assertion of (1) follows from the fact that MC implies that there are no amorphous sets (recall here Lévy's characterization of MC in [6]: MC is equivalent to "every infinite set has a well-ordered partition into non-empty finite sets").

(2) We shall use the P. Howard/H. Rubin/J. Rubin permutation model $\mathcal{N}33$ in [4]: The set A of atoms is countably infinite. $\preceq$ is a dense linear order

on A without endpoints. Thus $(A, \preceq)$ is order isomorphic to $(\mathbb{Q}, \leq)$, the rationals with its usual ordering. G is the group of all order automorphisms on $(A, \preceq)$ and Γ is the normal filter of subgroups of G generated by the filter base $\{\mathrm{fix}(E) : E \text{ is a bounded subset of } A\}$.

Using standard techniques of FM models, it can be verified that A is a non-well-orderable set. On the other hand, $\mathsf{AC_{WO}}$ is true in $\mathcal{N}33$ (see [4]). Thus, we only need to show that RR fails in this model. By Theorem 2.1, it suffices to show that $\mathcal{N}33$ has property (2.1). To this end, let $x \in \mathcal{N}33$ with support E and let $\phi \in G$. By the definition of Γ, there exist atoms e_1 and e_2 such that $e_1 \prec e_2$ and $E \subseteq (e_1, e_2) = \{a \in A : e_1 \prec a \prec e_2\}$. Since ϕ is an order automorphism of A, it follows that $\phi(e_1) \prec \phi(e_2)$ and $\phi(E) \subseteq (\phi(e_1), \phi(e_2))$. Let $a, b \in A$ be such that $a \prec \min\{e_1, e_2, \phi(e_1), \phi(e_2)\}$ and $b \succ \max\{e_1, e_2, \phi(e_1), \phi(e_2)\}$. Let $\phi' \in G$ agree with ϕ on the closed interval $[e_1, e_2]$ of A and let $\phi' \restriction (-\infty, a] \cup [b, \infty)$ be the identity mapping. Then under ϕ', $[a, e_1]$ is order isomorphic to $[a, \phi(e_1)]$, and $[e_2, b]$ is order isomorphic to $[\phi(e_2), b]$. Then $\phi'(x) = \phi(x)$ (since ϕ' and ϕ agree on the support $E \subseteq [e_1, e_2]$ of x), and $\phi' \in \mathcal{N}33$, since $[a, b]$ is a support of ϕ'. Thus, $\mathcal{N}33$ satisfies property (2.1), and since A is not well-orderable, Theorem 2.1 shows that A does not admit a rigid relation in $\mathcal{N}33$. Therefore RR fails in this model, as required. ∎

Remark 2.3. In [2, Theorem 2.1] it is shown that RR fails for the amorphous set A of atoms in the basic Fraenkel model $\mathcal{N}1$ in [4] (A is countably infinite (in the ground model where AC holds), G is the group of all permutations of A, and Γ is the finite support normal filter). We note here that $\mathcal{N}1$ satisfies property (2.1) since, as in the case of the second Fraenkel model $\mathcal{N}2$, given an $x \in \mathcal{N}1$ and a permutation $\phi \in G$, there is a permutation ϕ' of A which belongs to G, agrees with ϕ on a support of x, and moves only finitely many atoms. Therefore $\phi' \in \mathcal{N}1$. Since A is not well-orderable in $\mathcal{N}1$, Theorem 2.1 implies that A does not admit a rigid relation in that model.

The next lemma is mainly for use in the forthcoming Theorem 2.7, but it can also be used to prove Theorems 3.1 and 3.3 of the next section.

Lemma 2.4. (ZF) *Assume*

(1) *$\mathcal{Z}$ is a set such that every element of $\mathcal{Z}$ is an ordered pair (X, S) where X is a well-orderable set and S is a rigid relation on X, and*
(2) *there is a function $\preceq$ whose domain is $\{X : \exists S,\ (X, S) \in \mathcal{Z}\}$ such that for every X in the domain of $\preceq$, $\preceq(X)$ is a well-ordering of $\mathcal{P}(|X| \times |X|)$ (here $|X|$ denotes the initial ordinal with the same cardinality as X).*

Then there is a function F with domain $\mathcal{Z}$ such that for all (X, S) in $\mathcal{Z}$, $F(X, S) \in X$.

Proof. For each $(X,S)\in\mathcal{Z}$ and $t\in X$, we let $C(t,(X,S))$ be the $\preceq(X)$-least element U of $\mathcal{P}(|X|\times|X|)$ such that there is an isomorphism ψ of the relational systems (X,S) and $(|X|,U)$ such that $\psi(t)=0$. We claim that for all t_1 and t_2 in X, if $t_1\neq t_2$ then $C(t_1,(X,S))\neq C(t_2,(X,S))$. For if $C(t_1,(X,S))=C(t_2,(X,S))=U$ then there are isomorphisms ψ_1 and ψ_2 from (X,S) onto $(|X|,U)$ for which $\psi_1(t_1)=0$ and $\psi_2(t_2)=0$. But then $\psi=\psi_2^{-1}\circ\psi_1$ is an isomorphism from (X,S) onto (X,S) for which $\psi(t_1)=t_2$. Since (X,S) is rigid this implies that $t_1=t_2$.

Hence we can define $F(X,S)=$ the element t of X for which $C(t,(X,S))$ is $\preceq(X)$-least. ∎

The next two lemmas will only be used in the proof of Theorem 2.7. Their proofs are similar to that of Lemma 2.4. In these lemmas and in Theorem 2.7, if $z=\{a,b\}$ is a 2-element set we use T_z for the transposition (a,b). The intended domain of the permutation T_z will always be some finite set containing z. It will vary but will always be clear from the context.

Lemma 2.5. *Let $\mathcal{Y}$ be a set such that every element of $\mathcal{Y}$ is an ordered triple (x,w,S) where x and w are (unordered) pairs, S is a relation on $x\cup w$ and*

(1) $x\cap w=\emptyset$,
(2) *neither T_x nor $T_x\circ T_w$ is an automorphism of the relational system $(x\cup w,S)$.*

Then there is a function H with domain $\mathcal{Y}$ such that $H(x,w,S)\in x$ for every $(x,w,S)\in\mathcal{Y}$.

Proof. The proof is similar to that of Lemma 2.4. We let $\mathcal{S}$ be the set of all relations on $\{0,1,2,3\}$. Let $\preceq$ be a fixed well-ordering of the finite set $\mathcal{S}$. Assume that $\mathcal{Y}$ satisfies the hypotheses of the lemma. For each $(x,w,S)\in\mathcal{Y}$ and each element t of x we let $C(t,(x,w,S))$ be the $\preceq$-least element U of $\mathcal{S}$ for which there is an isomorphism ψ of the systems $(x\cup w,S)$ and $(\{0,1,2,3\},U)$ such that $\psi(x)=\{0,1\}$, $\psi(w)=\{2,3\}$ and $\psi(t)=0$. We claim that for all (x,w,S) in $\mathcal{S}$, if t_1 and t_2 are in x and $C(t_1,(x,w,S))=C(t_2,(x,w,S))$ then $t_1=t_2$. Assume the hypotheses of the claim. Then there are isomorphisms ψ_1 and ψ_2 of $(x\cup w,S)$ and $(\{0,1,2,3\},U)$ such that $\psi_1(x)=\psi_2(x)=\{0,1\}$, $\psi_1(w)=\psi_2(w)=\{2,3\}$, $\psi_1(t_1)=0$ and $\psi_2(t_2)=0$. It follows that $\psi=\psi_2^{-1}\circ\psi_1$ is an automorphism of $(x\cup w,S)$ such that $\psi(x)=x$, $\psi(w)=w$ and $\psi(t_1)=t_2$. From the first two of the equalities we conclude that ψ is one of T_x, T_w, $T_x\circ T_w$ or the identity on $x\cup w$. By assumption (2) of the lemma, ψ is neither T_x nor $T_x\circ T_w$. In the remaining two cases $\psi{\restriction}x$ is the identity so that $t_2=\psi(t_1)=t_1$.

We can therefore define $H(x,w,S)$ to be the $t\in x$ for which $C(t,(x,w,S))$ is $\preceq$-least. ∎

LEMMA 2.6. *Let $\mathcal{W}$ be a set such that every element of $\mathcal{W}$ is an ordered triple (x, w, S) where x and w are two-element sets, S is a relation on $x \cup w$ and*

(1) $x \cap w = \emptyset$,
(2) $T_x \circ T_w$ *is an automorphism of* $(x \cup w, S)$ *but* T_x *is not.*

Then there is a function J defined on $\mathcal{W}$ such that for all (x, w, S) in $\mathcal{W}$, $J(x, w, S)$ is a one-to-one function from w onto x.

Proof. As in the proof of Lemma 2.5 we let $\mathcal{S}$ be the set of all relations on $\{0, 1, 2, 3\}$ and we let $\preceq$ be a fixed well-ordering of the set $\mathcal{S}$. Assume that $\mathcal{W}$ satisfies the hypotheses of the lemma. For each triple $(x, w, S) \in \mathcal{W}$ and each one-to-one function g from w onto x we let $C(g, (x, w, S))$ be the $\preceq$-least element U of $\mathcal{S}$ for which there is an isomorphism ψ of the systems $(x \cup w, S)$ and $(\{0, 1, 2, 3\}, U)$ such that $\psi(x) = \{0, 1\}$, $\psi(w) = \{2, 3\}$ and $\psi(g) = \{(2, 0), (3, 1)\}$.

We claim that for all (x, w, S) in $\mathcal{S}$, if g_1 and g_2 are one-to-one functions from w onto x and $C(g_1, (x, w, S)) = C(g_2, (x, w, S))$ then $g_1 = g_2$. Assume the hypotheses of the claim. Let $C(g_1, (x, w, S)) = C(g_2, (x, w, S)) = U$. Then there are isomorphisms ψ_1 and ψ_2 of $(x \cup w, S)$ and $(\{0, 1, 2, 3, \}, U)$ such that $\psi_1(x) = \psi_2(x) = \{0, 1\}$, $\psi_1(w) = \psi_2(w) = \{2, 3\}$ and $\psi_1(g_1) = \psi_2(g_2) = \{((2, 0), (3, 1)\}$. It follows that $\psi = \psi_2^{-1} \circ \psi_1$ is an automorphism of $(x \cup w, S)$ such that $\psi(x) = x$, $\psi(w) = w$ and $\psi(g_1) = g_2$. From the first two equalities we conclude that ψ is one of T_x, T_w, $T_x \circ T_w$ or the identity on $x \cup w$. By assumption (2) of the lemma, ψ is not T_x. It also follows from (2) that ψ is not T_w since if T_w and $T_x \circ T_w$ were automorphisms then T_x would be. Therefore ψ is either the identity on $x \cup w$ or $T_x \circ T_w$. Both fix any one-to-one function from w onto x, so $g_2 = \psi(g_1) = g_1$.

We define $J(x, w, S)$ to be the one-to-one function g from w onto x for which $C(g, (x, w, S))$ is least. ∎

THEOREM 2.7. RR *implies that every well-ordered collection of two-element sets has a choice function.*

Proof. Let $\mathcal{A}$ be a well-ordered family of pairwise disjoint two-element sets and assume that R is a rigid binary relation on $\bigcup \mathcal{A}$. As above, for each $z \in \mathcal{A}$, we let T_z denote the transposition (a, b) where $z = \{a, b\}$.

We note that for z and z' in $\mathcal{A}$, if $z \neq z'$ then $T_z \circ T_{z'} = T_{z'} \circ T_z$ and

$$T_z \circ T_{z'} \restriction z = T_z \quad \text{and} \quad T_{z'} \circ T_z \restriction z' = T_{z'}. \tag{2.3}$$

DEFINITION 2.8. A non-empty finite sequence $\langle z_i \rangle_{i=1}^n$ of distinct elements of $\mathcal{A}$ is called a *good sequence* if either

(1) $n = 1$ and T_{z_1} is an automorphism of the relational system $(z_1, R \restriction z_1)$, or

(2) $n > 1$ and for all i such that $1 \le i < n$, $T_{z_i} \circ T_{z_{i+1}}$ is an automorphism of $(z_i \cup z_{i+1}, R{\restriction}(z_i \cup z_{i+1}))$ and T_{z_i} is not an automorphism of $(z_i \cup z_{i+1}, R{\restriction}(z_i \cup z_{i+1}))$.

Note that in item (1), T_{z_1} is a permutation of z_1, while in (2) both T_{z_i} and $T_{z_{i+1}}$ represent permutations of $z_i \cup z_{i+1}$. Also note that under the conditions in (2) the permutation $T_{z_{i+1}}$ is not an automorphism of $(z_i \cup z_{i+1}, R{\restriction}(z_i \cup z_{i+1}))$. Further by (2.3),

(2.4) T_{z_i} and $T_{z_{i+1}}$ are automorphisms of $(z_i, R{\restriction}z_i)$ and $(z_{i+1}, R{\restriction}z_{i+1})$ respectively.

Lemma 2.9. *For all $y \in \mathcal{A}$, either*

(1) *T_y is not an automorphism of $(y, R{\restriction}y)$, or*
(2) *there is a $z \in \mathcal{A}$ and a good sequence $\langle z_i\rangle_{i=1}^n$, $n > 1$, such that $z_1 = y$ and neither T_{z_n} nor $T_{z_n} \circ T_z$ is an automorphism of $(z_n \cup z, R{\restriction}(z_n \cup z))$.*

Proof. Assume there is a y in $\mathcal{A}$ for which both (1) and (2) are false. We will arrive at a contradiction by constructing a non-identity automorphism Φ of $(\bigcup\mathcal{A}, R)$.

We first let $\mathcal{A}_0 = \{z \in \mathcal{A} :$ there is a good sequence $\langle z_i\rangle_{i=1}^n$ such that $z_1 = y$ and $z_n = z\}$ and then define $\Phi : \bigcup\mathcal{A} \to \bigcup\mathcal{A}$ by

$$\Phi(a) = \begin{cases} b & \text{if } \exists z \in \mathcal{A}_0,\ z = \{a, b\}, \\ a & \text{otherwise.} \end{cases} \tag{2.5}$$

The fact that Φ is a one-to-one function from $\bigcup\mathcal{A}$ onto $\bigcup\mathcal{A}$ follows from our assumption that the elements of $\mathcal{A}$ are pairwise disjoint. We leave the details to the reader.

We now argue that Φ is an automorphism of the system $(\bigcup\mathcal{A}, R)$. Assume that Φ is not an automorphism; then there are elements a and b of $\bigcup\mathcal{A}$ such that either

$$a\,R\,b \quad\text{and}\quad \Phi(a)\not R\,\Phi(b), \qquad\text{or} \tag{2.6}$$

$$a \not R\, b \quad\text{and}\quad \Phi(a)\,R\,\Phi(b). \tag{2.7}$$

If (2.7) holds then letting $a' = \Phi(a)$ and $b' = \Phi(b)$ we get $\Phi(a') = a$ and $\Phi(b') = b$. Therefore (2.7) is equivalent to "$a'\,R\,b'$ and $\Phi(a')\not R\,\Phi(b')$". So it suffices to complete the proof under the assumption of (2.6). From (2.6) we see that either $\Phi(a) \ne a$ or $\Phi(b) \ne b$. Therefore we may do a proof by cases using the following outline:

Case 1: a and b are in the same element z of $\mathcal{A}$.

Case 2: a and b are in different elements of $\mathcal{A}$.
Subcase 1: $\Phi(a) \ne a$ and $\Phi(b) = b$.
Subcase 2: $\Phi(b) \ne b$ and $\Phi(a) = a$.
Subcase 3: $\Phi(a) \ne a$ and $\Phi(b) \ne b$.

Case 1: *a and b are in the same element z of $\mathcal{A}$.* In this case, since Φ moves either a or b, the set $z = \{a, b\}$ is in $\mathcal{A}_0$ and $\Phi(a) = b$ and $\Phi(b) = a$. Therefore $\Phi\restriction z = T_z$ so we can conclude from (2.6) that

$$a\,R\,b \quad \text{and} \quad T_z(a)\,\not{R}\,T_z(b). \tag{2.8}$$

We also deduce from the definition of Φ (and the fact that Φ moves a) that there is a good sequence $\langle z_i\rangle_{i=1}^n$ with $z_1 = y$ and $z_n = z$. By (2.4), T_z is an automorphism of $(z, R\restriction z)$, which contradicts (2.8).

Case 2: *a and b are in different elements of $\mathcal{A}$.* In this case there are elements a' and b' such that $\{a, a'\}$ and $\{b, b'\}$ are in $\mathcal{A}$ and $\{a, a'\} \cap \{b, b'\} = \emptyset$.

Subcase 1: *$\Phi(a) \neq a$ and $\Phi(b) = b$.* Here we use the definition of Φ to conclude that:

(1) There is a good sequence $\langle z_i\rangle_{i=1}^n$ such that $z_1 = y$ and $z_n = \{a, a'\}$.
(2) The sequence $\langle z_1, \ldots, z_n, z\rangle$, where $z = \{b, b'\}$, is not good. (Otherwise $\Phi\restriction z$ would be T_z.)
(3) $\Phi\restriction\{a, a'\} = \Phi\restriction z_n = T_{z_n}$.
(4) Since $z_n \cap z = \emptyset$, $T_{z_n}\restriction z$ is the identity, so $\Phi\restriction z = T_{z_n}\restriction z$.

By using (3) and (4), equation (2.6) becomes "$a\,R\,b$ and $T_{z_n}(a)\,\not{R}\,T_{z_n}(b)$". Therefore

$$T_{z_n} \text{ is not an automorphism of } (z_n \cup z, R\restriction(z_n \cup z)). \tag{2.9}$$

Since $\langle z_1, \ldots, z_n, z\rangle$ has length greater than 1 and is not good, condition (2) of the definition of "good" must fail for this sequence. Since $\langle z_i\rangle_{i=1}^n$ is good, (2) must fail because either T_{z_n} is an automorphism of $(z_n \cup z, R\restriction(z_n \cup z))$, or $T_{z_n} \circ T_z$ is not an automorphism of $(z_n \cup z, R\restriction(z_n \cup z))$. By (2.9) the latter must hold. Combining this fact with (2.9) we have a sequence $\langle z_i\rangle_{i=1}^n$ and an element z for which condition (2) of the lemma is true. This is a contradiction.

Subcase 2: *$\Phi(a) = a$ and $\Phi(b) \neq b$.* The argument in this case is very similar to that of the previous case and we leave it to the reader.

Subcase 3: *$\Phi(a) \neq a$ and $\Phi(b) \neq b$.* In this case (by the definition of Φ) there are two good sequences $\langle z_i\rangle_{i=1}^n$ and $\langle w_j\rangle_{j=1}^m$ with $z_1 = w_1 = y$, $z_n = \{a, a'\}$ and $w_m = \{b, b'\}$. Further $\Phi\restriction z_n \cup w_m = T_{z_n} \circ T_{w_m}$. Applying this equality to (2.6) we conclude that $a\,R\,b$ and $T_{z_n} \circ T_{w_m}(a)\,\not{R}\,T_{z_n} \circ T_{w_m}(b)$. So $T_{z_n} \circ T_{w_m}$ is not an automorphism of $(z_n \cup w_m, R\restriction(z_n \cup w_m))$. It follows that at least one of T_{z_n} or T_{w_m} is not an automorphism of this relational system. If the first possibility holds then $w_m \in \mathcal{A}$ and $\langle z_i\rangle_{i=1}^n$ provide an example for case (2) of the lemma, a contradiction. Similarly, if T_{w_m} is not an automorphism then $z_n \in \mathcal{A}$ and $\langle w_j\rangle_{j=1}^m$ provide such an example. ∎

To complete the proof of the theorem we let $\mathcal{Z} = \{(y, R\restriction y) : y \in \mathcal{A}$ and $R\restriction y$ is rigid$\}$. The set $\mathcal{Z}$ satisfies hypothesis (1) of Lemma 2.4 and we can obtain a function $\preceq$ satisfying hypothesis (2) of the lemma by choosing some well-ordering $\trianglelefteq$ of $\mathcal{P}(2 \times 2) = \mathcal{P}(\{0,1\} \times \{0,1\})$ and letting $\preceq(y) = \trianglelefteq$ for any y which is a first component of an element of $\mathcal{Z}$. Let F be a function satisfying the conclusion of the lemma.

Similarly, if we let

$$\mathcal{Y} = \{(x, w, R\restriction(x \cup w)) : x, w \in \mathcal{A},\ x \neq w,\ \text{and neither } T_x \text{ nor } T_x \circ T_w \\ \text{is an automorphism of } (x \cup w, R\restriction(x \cup w)\}$$

then $\mathcal{Y}$ satisfies the hypotheses of Lemma 2.5. We let H be a function satisfying the conclusion.

Finally we let

$$\mathcal{W} = \{(x, w, R\restriction(x \cup w)) : x, w \in \mathcal{A},\ x \neq w,\ \text{and } T_x \circ T_w \text{ is} \\ \text{an automorphism of } (x \cup w, R\restriction(x \cup w)) \text{ but } T_x \text{ is not}\}.$$

$\mathcal{W}$ satisfies the hypotheses of Lemma 2.6 and we let J be a function satisfying the conclusion.

Our goal in this section is to use H and J to define a function K whose domain is the set of pairs $(\langle z_i\rangle_{i=1}^n, z)$ where $\langle z_i\rangle_{i=1}^n$ is a good sequence and z is an element of $\mathcal{A}$ such that neither T_{z_n} nor $T_{z_n} \circ T_z$ is an automorphism of $(z_n \cup z, R\restriction(z_n \cup z))$. (See condition (2) of Lemma 2.9.) We also want $K(\langle z_i\rangle_{i=1}^n, z) \in z_1$. It follows from the definition of "good" that for $1 \le i < n$, $(z_i, z_{i+1}, R\restriction(z_i \cup z_{i+1})) \in \mathcal{W}$, so

(2.10) $J(z_i, z_{i+1}, R\restriction(z_i \cup z_{i+1}))$ is a one-to-one function from z_{i+1} onto z_i.

To simplify the notation we let $J_i = J(z_i, z_{i+1}, R\restriction(z_i \cup z_{i+1}))$ for $1 \le i < n$. Secondly we note that $(z_n, z, R\restriction(z_n \cup z))$ is in $\mathcal{Y}$, so

$$H((z_n, z, R\restriction(z_n \cup z))) \in z_n. \tag{2.11}$$

We can now define $K(\langle z_i\rangle_{i=1}^n, z) = J_1 \circ \cdots \circ J_{n-1}(H((z_n, z, R\restriction(z_n \cup z))))$. By (2.10) and (2.11), we have $K(\langle z_i\rangle_{i=1}^n, z) \in z_1$.

As a final step in the proof we define a choice function CH for $\mathcal{A}$. Since $\mathcal{A}$ is well-orderable, the set of finite sequences of elements of $\mathcal{A}$ is well-orderable. Choose one such ordering $\le$. Assume that $y \in \mathcal{A}$. If condition (1) of Lemma 2.9 holds then $(y, R\restriction y) \in \mathcal{Z}$ and we let $\mathsf{CH}(y) = F(y, R\restriction y) \in y$. If condition (2) of Lemma 2.9 holds then we let $\langle z_1, \ldots, z_n, z\rangle$ be the $\le$-least sequence for which $z_1 = y$, $\langle z_i\rangle_{i=1}^n$ is good and neither T_{z_n} nor $T_{z_n} \circ T_z$ is an automorphism of $(z_n \cup z, R\restriction(z_n \cup z))$. Then $(\langle z_i\rangle_{i=1}^n, z)$ is in the domain of K and we let $\mathsf{CH}(y) = K(\langle z_i\rangle_{i=1}^n, z)$. As noted above, this is an element of z_1 and therefore an element of y since $y = z_1$. ∎

3. On the hereditary rigidity and the Strong Rigidity Principle of Vopěnka, Pultr and Hedrlín. We start this section with the proofs that the principle HRR implies the Axiom of Choice for families of non-empty finite sets and that it is not equivalent to AC in ZF.

THEOREM 3.1. HRR *implies* $\mathsf{AC}^{\mathsf{fin}}$.

Proof. Assume HRR and assume that $\mathcal{A}$ is a family of finite sets. By HRR there is a hereditarily rigid relation R on $\bigcup\mathcal{A}$. Let $\mathcal{Z} = \{(A, R\restriction A) : A \in \mathcal{A}\}$. Then $\mathcal{Z}$ satisfies hypothesis (1) of Lemma 2.4. Further, hypothesis (2) is also satisfied since $\bigcup_{n\in\omega} \mathcal{P}(n \times n)$ is countable and therefore well-orderable, say by $\trianglelefteq$. So for each $A \in \mathcal{A}$ we can define $\preceq(A) = \trianglelefteq\restriction|A|$.

Let F be the function given by the conclusion of Lemma 2.4. We define a choice function G on $\mathcal{A}$ by $G(A) = F(A, R\restriction A)$. ∎

J. Truss [8, Theorem 3] showed that if for some natural number $n > 1$, the set $[X]^n$ of all n-element subsets of a set X has a choice function, then X is finite or not amorphous. From this and Theorem 3.1, we immediately obtain the following corollary.

COROLLARY 3.2. HRR *implies that there are no amorphous sets.*

THEOREM 3.3. *In every Fraenkel–Mostowski model of* ZFA*, the statement* HRR *implies* $\mathsf{AC}^{\mathsf{WO}}$ (AC *for families of non-empty well-orderable sets*).

Proof. The proof proceeds as in Theorem 3.1 except that hypothesis (2) of Lemma 2.4 is now true because in every Fraenkel–Mostowski model there is a function (actually a proper class) $\preceq$ such that for every well-orderable set X, $\preceq(X)$ is a well-ordering of $\mathcal{P}(|X| \times |X|)$ (see [3]). ∎

Our next result strengthens Theorem 1.5(2) (see the introduction) and states that every set in the basic Cohen forcing model admits a *hereditarily* rigid binary relation. Thus HRR is not equivalent to the full Axiom of Choice in ZF set theory.

THEOREM 3.4. HRR *holds in the basic Cohen model of* ZF + ¬AC (*model* $\mathcal{M}1$ *in* [4]) *and in the Mostowski linearly ordered permutation model of* ZFA + ¬AC (*model* $\mathcal{N}3$ *in* [4]). *Therefore,* HRR *is not equivalent to* AC *in* ZF.

Proof. Let $\mathcal{M}$ be the basic Cohen model and let A be the set of the countably many added generic reals. It is known (see [5, Lemma 5.25]) that for every set $X \in \mathcal{M}$, there is, in $\mathcal{M}$, an ordinal γ and a one-to-one function $f : X \to [A]^{<\omega} \times \gamma$. The same holds in the Mostowski model (see [5, Lemma 4.6]), except that A in that model denotes the set of its atoms.

We show that HRR holds in $\mathcal{M}$. To this end, let $X \in \mathcal{M}$. By [5, Lemma 5.25], there is a well-ordered partition $\{X_\alpha : \alpha \in \gamma\}$, γ some ordinal number, of X into Dedekind finite sets. Without loss of generality we may view each X_α as a subset of $[A]^{<\omega}$, hence X has a linear order in $\mathcal{M}$, say $\prec$. We define a partial order R on X as follows: If $x, y \in X_\alpha$ for some $\alpha < \gamma$, then $x\,R\,y$ if and only if $x \prec y$. If $x \in X_\alpha$ and $y \in X_\beta$, $\alpha \neq \beta$, then $x\,R\,y$ if and only if $\alpha \in \beta$. In this case, we shall write $x < y$ instead of $x\,R\,y$. It is clear that R is a linear order on X.

We show that R is a hereditarily rigid relation on X. In fact, due to the definition of R, it is easy to see that it suffices to show that R is a rigid relation on X. To this end, let $f : X \to X$ be an automorphism of the relational system $\langle X, R\rangle$, and toward a contradiction assume that f is not the identity mapping on X. For every $x \in X$, let $\mathrm{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$ be the orbit of x under f. We first observe the following facts:

(1) Since for each $\alpha < \gamma$, X_α is Dedekind finite, it follows that for every $x \in X$ and for every $\alpha < \gamma$, $\mathrm{Orb}(x) \cap X_\alpha$ is a finite set.
(2) There is an ordinal $\alpha < \gamma$ and an element $x \in X_\alpha$ with $f(x) \neq x$ such that $\mathrm{Orb}(x)$ is infinite. Otherwise, that is, if all the orbits of f were finite, then since R is a linear order on X, f would necessarily fix X pointwise, contradicting our assumption on f.

By observation (2), let α_0 be the least $\alpha \in \gamma$ such that there exists an $x \in X_\alpha$ with $f(x) \neq x$ and $\mathrm{Orb}(x)$ is infinite. By (1), it then follows that for some $n \in \mathbb{Z}$, we have $x < f^n(x)$ or $f^n(x) < x$ (that is, x and $f^n(x)$ belong to distinct elements of the partition $\{X_\alpha : \alpha \in \gamma\}$ of X). Without loss of generality, assume that $x < f(x)$ (that is, $f(x) \in X_\beta$ for some $\beta > \alpha_0$). By the property of the ordinal α_0, observation (1) and the fact that f is an automorphism, it follows that $\{f^{-n}(x) : n \in \mathbb{N}\}$ is necessarily a countably infinite subset of X_{α_0}. This contradicts the fact that X_{α_0} is Dedekind finite. Thus, f is the identity mapping and R is a rigid binary relation on X. ∎

Theorem 3.5. SRR *is not provable in* ZF *set theory.*

Proof. SRR implies RR and the latter is not provable in ZF by [2, Theorem 2.1]. ∎

Next, we prove that SRR is not equivalent to AC in ZFA. First, we need the following lemma.

Lemma 3.6. *If R is a strongly rigid binary relation on a set X and $|X| > 1$, then R is irreflexive, i.e. there is no $y \in X$ for which $y\,R\,y$.*

Proof. Assuming the hypotheses and the existence of a $y_0 \in X$ for which $y_0\,R\,y_0$ the function f defined by $f(x) = y_0$ is a non-identity endomorphism of the system (X, R). ∎

Theorem 3.7. SRR *holds in the Mostowski linear order model of* ZFA + ¬AC*. Thus,* SRR *is not equivalent to* AC *in* ZFA*.*

Proof. The Mostowski linear order model M (model $\mathcal{N}3$ in [4]) is constructed from a model M' of ZFA + AC with a countable set A of atoms using an ordering $\leq$ of A chosen so that $(A, \leq)$ is order isomorphic to the set of rational numbers with the usual ordering. We let G be the group of all order isomorphisms of $(A, \leq)$. Then M is the Fraenkel–Mostowski model determined by G using finite supports. We now give a slightly more detailed description of M and introduce some useful notation.

If $\phi \in G$ then, as in Definition 1.3, ϕ also denotes its unique extension to an $\in$-automorphism of the model M'. Note that if E is a finite subset of A then $\mathrm{fix}(E) = \mathrm{Sym}(E)$. An element x of M' is *symmetric* if there is some finite $E \subseteq A$ such that $\mathrm{fix}(E) \subseteq \mathrm{Sym}(x)$. We then call E a *support* of x. The element x is called *hereditarily symmetric* if x and every element in the transitive closure of x is symmetric. The model M consists of all the hereditarily symmetric elements of M'.

The following facts will be useful:

(1) Every element of M has a minimum (under $\subseteq$) support which we denote by $\mathrm{supp}(x)$. For $x \in M$, $\mathrm{supp}(x)$ has the property that

$$\forall \phi \in G,\ \phi(x) = x \text{ if and only if } \phi(\mathrm{supp}(x)) = \mathrm{supp}(x) \text{ if and only if } \phi \in \mathrm{fix}(\mathrm{supp}(x)). \tag{3.1}$$

(2) Similarly, if $x \in M$ and $E \subseteq A$ is finite then there is a minimum finite subset F of A with the properties

$$F \cap E = \emptyset \text{ and } \forall \phi \in \mathrm{fix}(E),\ \phi(x) = x \text{ if and only if } \phi(F) = F \text{ if and only if } \phi \in \mathrm{fix}(F). \tag{3.2}$$

We call this set F the *support* of x *relative to* E and denote it by $\mathrm{supp}_E(x)$. (It is fairly easy to show that $\mathrm{supp}_E(x) = \mathrm{supp}(x) \setminus E$.)

(3) If $x \in M$ and there is some finite $H \subseteq A$ such that $\mathrm{fix}(H) \subseteq \mathrm{fix}(x)$ (that is, $\mathrm{supp}(t) \subseteq H$ for all $t \in x$) then x is well-orderable in M and for every well-ordering $\prec$ of x which is in M', H is a support of $\prec$. In fact, H is a support of every relation on x which is in M'.

For the proof of the theorem let X be a set in M and let $E = \mathrm{supp}(X)$. Let $S = \{F \subseteq A : \exists y \in X,\ F = \mathrm{supp}_E(y)\}$. It follows from (3.2) that $S \in M$. For each $F \in S$ let $X_F = \{y \in X : \mathrm{supp}_E(y) = F\}$. Then $C = \{X_F : F \in S\}$ is a partition of X. Note that $F \mapsto X_F$ is a one-to-one function from S onto C and is in M. Both of these follow from the fact that

$$\mathrm{supp}_E(X_F) = F. \tag{3.3}$$

We need to find a strongly rigid relation on X in the model M. We first consider the easy case in which S is finite. Then $\bigcup S$ is a finite subset of A. Further, since every element of X is supported by $E \cup F$ for some element F of S, $E \cup \bigcup S$ is a support of every element of X. By fact (3) above, $E \cup \bigcup S$ is a support of every well-ordering of X which is in M'. At this point we need the result of Vopěnka, Pultr and Hedrlín [9] that under AC every set admits a strongly rigid binary relation. In particular, since AC holds in M', there is a strongly rigid relation defined on X which is in M'. By the last sentence in (3), this relation is in M.

Now we consider the case where S is infinite. Since every set in M is linearly orderable in M (see [5, Section 4.5] for example) we let $\prec$ be a strict linear ordering of C which is in M. We shall use $\prec$ to construct a strongly rigid relation $\mathcal{R}$ on X.

The construction will also require a function R on $C = \{X_F : F \in S\}$ with the properties

(3.4) R is in M and $\operatorname{supp}(R) \subseteq E$,

(3.5) for each $X_F \in C, R(X_F)$ is a strongly rigid relation on X_F,

(3.6) for each $X_F \in C$, if $|X_F| = 1$ then $R(X_F) = \emptyset$.

Let ORB be the set of fix(E)-orbits of elements of C. That is, ORB $= \{\{\phi(X_F) : \phi \in \operatorname{fix}(E)\} : X_F \in C\}$.

We claim that in order to construct R it will suffice to construct for each $U \in$ ORB a function R_U with domain U for which

(1) R_U is in M and $\operatorname{supp}(R_U) \subseteq E$,
(2) for each $X_F \in U$, $R_U(X_F)$ is a strongly rigid relation on X_F,
(3) for each $X_F \in U$, if $|X_F| = 1$ then $R_U(X_F) = \emptyset$.

To prove the claim we note that ORB is a partition of C with support E. Therefore, assuming that the R_Us have been constructed as above, $R = \bigcup_{U \in \mathrm{ORB}} R_U$ will be a function satisfying (3.4)–(3.6).

Assume $U \in$ ORB. We construct R_U as follows: Choose an element X_{F_0} of U. By the definition of X_{F_0}, each element y of X_{F_0} has $\operatorname{supp}_E(y) = F_0$. As in the proof for S finite, there is a strongly rigid relation r on X_{F_0} which is in M and for which $\operatorname{supp}_E(r) \subseteq F_0$ and we may choose $r = \emptyset$ if $|X_{F_0}| = 1$. We are now in a position to define R_U:

(3.7) $$R_U = \{\phi(X_{F_0}, r) : \phi \in \operatorname{fix}(E)\} = \{(\phi(X_{F_0}), \phi(r)) : \phi \in \operatorname{fix}(E)\}.$$

Since fix(E) is closed under composition it is clear that $\psi(R_U) = R_U$ for all $\psi \in \operatorname{fix}(E)$, and therefore $\operatorname{supp}(R_U) \subseteq E$.

To argue that R_U is a function assume that both $(\phi_1(X_{F_0}), \phi_1(r))$ and $(\phi_2(X_{F_0}), \phi_2(r))$ are in R_U (where ϕ_1 and ϕ_2 are in fix(E)) and that $\phi_1(X_{F_0}) = \phi_2(X_{F_0})$. Then $\phi_2^{-1}\phi_1(X_{F_0}) = X_{F_0}$. By (3.3), $\operatorname{supp}_E(X_{F_0}) = F_0$ and therefore by (3.2), $\phi_2^{-1}\phi_1(F_0) = F_0$. Since $\operatorname{supp}_E(r) \subseteq F_0$ we conclude that $\phi_2^{-1}\phi_1(r) = r$ and so $\phi_1(r) = \phi_2(r)$.

Finally, we argue that for all $X_F \in U$, $R_U(X_F)$ is a strongly rigid relation on X_F. If $X_F \in U$ there is a $\phi \in \operatorname{fix}(E)$ such that $X_F = \phi(X_{F_0})$ and $R_U(X_F) = \phi(r)$. Since r is a relation on X_{F_0}, $\phi(r) = R_U(X_F)$ is a relation on $\phi(X_{F_0}) = X_F$. Since $\phi(r) = R_U(X_F)$ it also follows that ϕ restricted to X_{F_0} is an order isomorphism of the relational systems (X_{F_0}, r) and $(X_F, R_U(X_F))$ which is in the model M since $E \cup F_0 \cup \phi(F_0)$ is a support (of ϕ restricted to X_{F_0}). As r is strongly rigid on X_{F_0}, $R_U(X_F)$ is strongly rigid on X_F.

This completes the construction of a function R on $\{X_F : F \in S\}$ satisfying (3.4)–(3.6).

Our final step is to (use R and $\prec$ to) define a relation $\mathcal{R}$ on X and to argue that $\mathcal{R}$ is strongly rigid. We define $\mathcal{R}$ as follows: If y_1 and y_2 are in X then

$$\begin{aligned} y_1 \mathrel{\mathcal{R}} y_2 \text{ if and only if } &(\exists X_F \in C,\ y_1, y_2 \in X_F \text{ and } y_1 \mathrel{R(X_F)} y_2), \text{ or} \\ &\big(\exists X_{F_1}\ \exists X_{F_2} \in C,\ X_{F_1} \neq X_{F_2}, \\ &\qquad y_1 \in X_{F_1},\ y_2 \in X_{F_2} \text{ and } X_{F_1} \prec X_{F_2}\big). \end{aligned}$$

The relation $\mathcal{R}$ is in M since it is supported by the union of a support for R and a support for $\prec$. Some immediate properties of $\mathcal{R}$ are given by the following lemma.

LEMMA 3.8.

(1) *If y_1 and y_2 are both in X_F for some $X_F \in C$ then $y_1 \mathrel{\mathcal{R}} y_2$ if and only if $y_1 \mathrel{R(X_F)} y_2$. (This and the following item use the definition of $\mathcal{R}$ and the fact that C is a partition of X.)*
(2) *If y_1 and y_2 are in different elements of C, say X_{F_1} and X_{F_2}, then $y_1 \mathrel{\mathcal{R}} y_2$ if and only if $X_{F_1} \prec X_{F_2}$.*
(3) *If y_1 and y_2 are in different elements of C then either $y_1 \mathrel{\mathcal{R}} y_2$ or $y_2 \mathrel{\mathcal{R}} y_1$. (By the second part of the definition of $\mathcal{R}$ and the fact that $\prec$ is a linear ordering.)*
(4) *For every endomorphism $f : X \to X$ of $(X, \mathcal{R})$,*

$$\forall X_F \in C,\ \text{if } \exists y \in X_F,\ f(y) \neq y \text{ then } \exists z \in X_F,\ f(z) \notin X_F$$

(since $\mathcal{R}$ restricted to X_F is $R(X_F)$ which is strongly rigid).

We now argue that $\mathcal{R}$ is strongly rigid. Assume that it is not. Then there is an endomorphism f of $(X, \mathcal{R})$ and an element $y_0 \in X$ such that $f(y_0) \neq y_0$. Let K be a support of f. We define a sequence $\langle z_n, X_{H_n} \rangle_{n \in \omega}$ with the following properties:

(1) $X_{H_n} \in C$,
(2) $z_n \in X_{H_n}$,
(3) $f(z_n) \notin X_{H_n}$,

by recursion (but we do not claim that the sequence is in M).

For $n = 0$ let $\mathrm{supp}_E(y_0) = H_0$; then $y_0 \in X_{H_0}$. By Lemma 3.8(4) there is a $z \in X_{H_0}$ such that $f(z) \notin X_{H_0}$ and we let $z_0 = z$.

Assume that z_n and X_{H_n} have been defined satisfying (1)–(3). We let $H_{n+1} = \mathrm{supp}_E(f(z_n))$ so that $f(z_n) \in X_{H_{n+1}}$. We claim that there is an element z_{n+1} of $X_{H_{n+1}}$ such that $f(z_{n+1}) \notin X_{H_{n+1}}$. To prove the claim we consider two cases. First if $f(f(z_n)) \notin X_{H_{n+1}}$ we can let $z_{n+1} = f(z_n)$. On the other hand, if $f(f(z_n)) \in X_{H_{n+1}}$, then since $f(z_n) \in X_{H_{n+1}} \setminus X_{H_n}$, we have $X_{H_{n+1}} \neq X_{H_n}$. It follows from this and Lemma 3.8(3) that either $z_n \mathcal{R} f(z_n)$ or $f(z_n) \mathcal{R} z_n$. Since f is an endomorphism we see that

$$f(z_n) \text{ is related to } f(f(z_n)) \text{ by } \mathcal{R}. \tag{3.8}$$

Since $f(z_n)$ and $f(f(z_n))$ are both in $X_{H_{n+1}}$ we can use Lemma 3.8(1) to deduce that

$$f(z_n) \text{ is related to } f(f(z_n)) \text{ by } R(X_{H_{n+1}}). \tag{3.9}$$

Therefore $R(X_{H_{n+1}}) \neq \emptyset$. By Lemma 3.6 no element of $X_{H_{n+1}}$ is related to itself by $R(X_{H_{n+1}})$ and in particular (using (3.9)) $f(z_n) \neq f(f(z_n))$. Now it follows from Lemma 3.8(4) that there is at least one element z of $X_{H_{n+1}}$ such that $f(z) \notin X_{H_{n+1}}$, and we let z_{n+1} be one of these. This completes the definition of the sequence $\langle z_n, X_{H_n}\rangle$.

It follows from the definition that

$$\forall n \in \omega, \quad f(z_n) \in X_{H_{n+1}}. \tag{3.10}$$

Further from conditions (2) and (3) and the fact that $f(z_n) \in X_{H_{n+1}}$ we may conclude that

$$X_{H_n} \neq X_{H_{n+1}}. \tag{3.11}$$

Therefore, since $\prec$ is a linear ordering C, it follows that X_{H_n} is related by $\prec$ to $X_{H_{n+1}}$. In particular, X_{H_0} is related to X_{H_1} by $\prec$. For the remainder of the proof we assume

$$X_{H_0} \prec X_{H_1}. \tag{3.12}$$

The proof for $X_{H_1} \prec X_{H_0}$ is similar and is left to the reader.

LEMMA 3.9. *For all $n \in \omega$, $X_{H_n} \prec X_{H_{n+1}}$.*

Proof. The proof is by induction. Assume that $n \in \omega$ and for all $k \in \omega$ if $k < n$ then $X_{H_k} \prec X_{H_{k+1}}$. We will show that $X_{H_n} \prec X_{H_{n+1}}$. If $n = 0$ then "$X_{H_n} \prec X_{H_{n+1}}$" is our assumption (3.12). If $n \neq 0$ then $n - 1 \in \omega$ and therefore by our induction assumption $X_{H_{n-1}} \prec X_{H_n}$. Since $z_{n-1} \in X_{H_{n-1}}$ and $z_n \in X_{H_n}$ we have $z_{n-1} \mathcal{R} z_n$. As f is an endomorphism we get $f(z_{n-1}) \mathcal{R} f(z_n)$. Since $f(z_{n-1}) \in X_{H_n}$ and $f(z_n) \in X_{H_{n+1}}$ we can use Lemma 3.8(2) to conclude that $X_{H_n} \prec X_{H_{n+1}}$. ∎

It follows from the lemma and the fact that $\prec$ is a linear ordering that $\langle X_{H_n}\rangle_{n\in\omega}$ is a sequence of pairwise distinct elements of C. Since $H \mapsto X_H$ is a one-to-one function from S onto C, $\langle H_n\rangle_{n\in\omega}$ is also a sequence of pairwise distinct finite subsets of A.

We now argue that there must be an $n_0 \in \omega$ with $H_{n_0+1} \nsubseteq E \cup K \cup H_{n_0}$. ($K$ is the finite support of f and E is the finite support of X chosen earlier.) If there is no such n_0 then $H_{n+1} \subseteq E \cup K \cup H_n$ for every $n \in \omega$. Then by induction $H_n \subseteq E \cup K \cup H_0$ for all $n \in \omega$. This is impossible since $E \cup K \cup H_0$ is finite and so has only finitely many subsets.

We obtain a contradiction by considering z_{n_0} which is in $X_{H_{n_0}}$ and $f(z_{n_0})$ which is in $X_{H_{n_0+1}}$. Since $H_{n_0+1} \nsubseteq E \cup K \cup H_{n_0}$ we may choose an element a of A in $H_{n_0+1} \setminus (E \cup K \cup H_{n_0})$ and a permutation $\phi \in \mathrm{fix}(E \cup K \cup H_{n_0})$ for which $\phi(a) \neq a$. Since $\mathrm{supp}_E(f(z_{n_0})) = H_{n_0+1}$, we have $\phi(f(z_{n_0})) \neq f(z_{n_0})$. Since $\phi \in \mathrm{fix}(E \cup K \cup H_{n_0})$, ϕ fixes both z_{n_0} and f. This is a contradiction since a permutation which fixes both f and z_{n_0} must fix $f(z_{n_0})$. ∎

4. Rigid, hereditarily rigid, and strongly rigid binary relations on Cantor cubes. We begin with the observation that both HRR and RR are related to the corresponding formulations restricted to powers of 2. In particular:

(1) HRR if and only if for every set X, 2^X has a hereditarily rigid binary relation. (This is straightforward, since X can be considered as a subset of 2^X.)
(2) RR if and only if for every set X, every subset of 2^X has a rigid binary relation.

It is also clear that RR implies

(∗) For every set X, 2^X has a rigid binary relation.

We do not know whether "(∗) $\to$ RR" holds in ZF. Note that (∗) is not provable in set theory without choice, since it fails in the basic Fraenkel permutation model M for the set A of atoms. In particular (see [4], [5]), 2^A is Dedekind finite in M, and since M satisfies "for every $x \in M$ and every $\phi \in G$ there is a permutation ϕ' of A which is in M and for which $\phi'(x) = \phi(x)$" (see Remark 2.3), it follows from Theorem 2.1 that 2^A does not admit a rigid binary relation.

As mentioned in the introduction, the statement "every subset of 2^ω has a rigid binary relation" is provable in ZF set theory (see [2, Theorem 2.1]). Here, we shall give *a sharper result*: "every subset of 2^ω has a strongly rigid binary relation" is *also a theorem of* ZF.

THEOREM 4.1. (ZF) *Every set of reals has a strongly rigid binary relation.*

Proof. We shall work with the Cantor cube 2^ω instead of $\mathbb{R}$. So let $A \subseteq 2^\omega$. We consider the following two cases:

Case 1: *A is Dedekind infinite.* Let $\{b_n : n \in \omega\}$ be an enumeration of a countably infinite subset $B \subseteq A$. Without loss of generality assume that $A \setminus B \neq \emptyset$. (If $A = B$, i.e. A is countably infinite, then the relational system $\langle B, S \rangle$, where S is the binary relation on B defined below, is strongly rigid. See the argument below.)

First we fix an integer $n^* > 1$ and we also let $\{p_n : n \in \omega\}$ be an enumeration of the set $\mathrm{Fn}(\omega, 2)$ of all partial finite functions from ω into 2. Now we define a binary relation S on A as follows:

(1) $(b_0, b_{n^*}) \in S$ and $(b_n, b_{n+1}) \in S$ for all $n \in \omega$.
(2) For $a \in A \setminus B$ and for $n \in \omega$ we require $(b_n, a) \in S$ if and only if $p_n \subseteq a$.

We assert that S is a strongly rigid relation on A. First note that $S \restriction B$ is strongly rigid since any homomorphism of $\langle B, S \restriction B \rangle$ would have to fix b_0, hence b_{n^*}, and consequently every element of B. Indeed, let $g : B \to B$ be a homomorphism of $\langle B, S \restriction B \rangle$. Suppose, toward a contradiction, that $g(b_0) \neq b_0$. There are the following cases:

(a) $g(b_0) = b_{n^*}$. Then $(b_0, b_{n^*}) \in S \Rightarrow (b_{n^*}, g(b_{n^*})) \in S$, hence $g(b_{n^*}) = b_{n^*+1}$. But then $(b_{n^*-1}, b_{n^*}) \in S$, hence $g(b_{n^*-1}) = b_{n^*}$. From this, it easily follows that $g(b_m) = b_{m+1}$ for all $m < n^*$. Now, $(b_0, b_1) \in S$, so $(g(b_0), g(b_1)) \in S$ and consequently $(b_{n^*}, b_2) \in S$. Since $2 \leq n^*$, we have reached a contradiction.
(b) $g(b_0) = b_j$ *for some* $j < n^*$. Then $j \leq n^* - 1$, hence necessarily $g(b_{n^*}) = b_k$ for some $k \leq n^*$, since $(b_0, b_{n^*}) \in S$ implies $(g(b_0), g(b_{n^*})) \in S$. So by the definition of S and the fact that $j < n^*$, it follows that $g(b_{n^*}) = b_k$ for some $k \leq n^*$. Similarly to case (a), one then shows that $(b_m, b_n) \in S$ for $n \leq m$, which is a contradiction.
(c) $g(b_0) = b_j$ *for some* $j > n^*$. It follows that $g(b_{n^*}) = b_{j+1}$ (for $(b_0, b_{n^*}) \in S \Rightarrow (g(b_0), g(b_{n^*})) \in S$, hence $(b_j, g(b_{n^*})) \in S$, and consequently $g(b_{n^*}) = b_{j+1}$), $j + 1 > n^*$, and it is not hard to verify that then $(b_m, b_k) \in S$ for some m, k such that $n^* \leq k < m$, which is a contradiction.

It follows that $g(b_0) = b_0$, hence necessarily $g(b_{n^*}) = b_{n^*}$. (Note that $(b_0, b_{n^*}) \in S \Rightarrow (b_0, g(b_{n^*})) \in S$, hence $g(b_{n^*}) = b_1$ or $g(b_{n^*}) = b_{n^*}$. If $g(b_{n^*}) = b_1$, then $g(b_{n^*-1}) = b_0$, so we would have $(b_{n^*-2}, b_{n^*-1}) \in S \Rightarrow (g(b_{n^*-2}), b_0) \in S$ (recall that $n^* > 1$) which is impossible since there is no $m \in \omega$ such that $(b_m, b_0) \in S$.) We then easily obtain $g(b_m) = b_m$ for all $m \in \omega \setminus \{0, n^*\}$.

We proceed now with the proof that S is strongly rigid on A. To this end, let $f : A \to A$ be a homomorphism of the system $\langle A, S \rangle$.

We prove first that $f(b_n) \in B$ for all $n \in \omega$. If not, let $n \in \omega$ be such that $f(b_n) \in A \setminus B$. Then $(b_n, b_{n+1}) \in S$, hence $(f(b_n), f(b_{n+1})) \in S$. But for no $(a, b) \in S$ is a an element of $A \setminus B$, therefore $f(b_n) \in B$. Since $S{\restriction}B$ is strongly rigid, it follows that $f(b_n) = b_n$ for all $n \in \omega$.

Next we show that $f(y) = y$ for all $y \in A \setminus B$. We first argue that $f(y) \in A \setminus B$ for all $y \in A \setminus B$. Assume on the contrary that $y \in A \setminus B$ is such that $f(y) = b_n$ for some $n \in \omega$. It is clear that there is an infinite subset $M \subseteq \omega$ such that $p_m \subseteq y$ for all $m \in M$ (note that for all $k \in \omega$, $y{\restriction}k \subseteq y$, and for $k, k' \in \omega$ with $k \neq k'$, we have $y{\restriction}k \neq y{\restriction}k'$) hence $(b_m, y) \in S$ for all $m \in M$. But then, for all $m \in M$, $(f(b_m), f(y)) \in S$, hence $(b_m, b_n) \in S$ since f fixes every element of B and $f(y) = b_n$. This is clearly a contradiction, due to the fact that M is infinite and the definition of $S{\restriction}B$. Thus $f[A \setminus B] \subseteq A \setminus B$, as required.

To end the proof, assume toward a contradiction that $f(y) \neq y$ for some $y \in A \setminus B$. Let n_0 be the minimum $n \in \omega$ such that $p_n \subseteq y$ but $p_n \nsubseteq f(y)$. Then $(b_{n_0}, y) \in S$, hence $(b_{n_0}, f(y)) \in S$. Thus $p_{n_0} \subseteq f(y)$, since $f(y) \in A \setminus B$. This is a contradiction, hence $f(y) = y$. It follows that S is a strongly rigid binary relation on A, as required.

CASE 2: *A is a Dedekind finite set.* Since A is also strictly linearly orderable (that is, linearly orderable by an irreflexive relation), say by the lexicographical order $<_{\text{lex}}$ on 2^ω, it can be easily verified that the only homomorphism of $\langle A, <_{\text{lex}}\rangle$ is the identity, hence $<_{\text{lex}}$ is a strongly rigid relation on A. ∎

REMARK 4.2. As mentioned in Section 1, a new simplified proof of SRR (in ZFC) is provided in [7]. As with the original proof in [9], it is shown that every ordinal admits a strongly rigid binary relation and the key point of both proofs (in [7] and [9]) is choosing for each ordinal β with countable cofinality—in a given ordinal α—a sequence $(\beta_n)_{n\in\omega}$ converging to β. This procedure is not choice free. Now, if we restrict attention to $\aleph_1$, that is, the given ordinal α is equal to $\aleph_1$, then in view of Theorem 4.1 and its proof, we may avoid choosing (converging) sequences for ordinals in $\aleph_1$ with countable cofinality. In particular, in order to show that $\aleph_1$ has a strongly rigid binary relation, we only need to assume that there is an injection $f : \aleph_1 \to \mathbb{R}$. Then Theorem 4.1 applies. Note that it is not provable in ZF that $\aleph_1$ can be embedded into $\mathbb{R}$ (see [4, Form 170]).

Since, by Theorem 4.1, no choice form is required in order to prove that 2^ω has a strongly rigid binary relation, it is natural to ask if, in ZF, $2^{\mathcal{P}(\omega)}$ (hence $2^{\mathbb{R}}$), or $2^{\mathcal{P}^n(\omega)}$ for any positive integer n, has a strongly rigid binary relation. We will answer these questions *affirmatively*: see Corollary 4.9. First we establish, *in* ZF, the following general result: *If a set X has a strongly rigid binary relation, then so does its power set $\mathcal{P}(X)$*. To start with the proof, we need the following lemma.

Lemma 4.3. *If* (X_1, R_1) *and* (X_2, R_2) *are strongly rigid relational systems where* $X_1 \cap X_2 = \emptyset$, $|X_1| > 1$ *and* $|X_2| > 1$ *then* $(X_1 \cup X_2, R_1 \cup R_2 \cup \{(b,a) : a \in X_1 \text{ and } b \in X_2\})$ *is strongly rigid.*

Proof. Assume that the hypotheses hold and let $R = R_1 \cup R_2 \cup \{(b,a) : a \in X_1 \text{ and } b \in X_2\}$. We note that if $x\,R\,y$ then it cannot be the case that $x \in X_1$ and $y \in X_2$.

Assume that $f : X_1 \cup X_2 \to X_1 \cup X_2$ is an endomorphism. To prove that f is the identity it will suffice to show that $f(a) \in X_1$ for all $a \in X_1$ and $f(b) \in X_2$ for all $b \in X_2$ (since (X_1, R_1) and (X_2, R_2) are rigid). We will prove the first of these statements; the proof of the second is similar.

Toward a contradiction assume that $a_0 \in X_1$ and $f(a_0) \in X_2$. Since $b\,R\,a_0$ for every $b \in X_2$, we conclude that $f(b)\,R\,f(a_0)$ for every such b. Using the remark in the second sentence of the proof and the fact that $f(a_0) \in X_2$ we conclude that $f(b) \in X_2$ for all $b \in X_2$. Therefore $f{\restriction}X_2$ is an endomorphism of (X_2, R_2). In order to complete the proof we only have to show that $f{\restriction}X_2$ is not the identity function on X_2. We will prove this by showing that $f(f(a_0)) \neq f(a_0)$.

We first note that $f(a_0)\,R\,a_0$ since $f(a_0) \in X_2$ and $a_0 \in X_1$. Therefore $f(f(a_0))\,R\,f(a_0)$. By Lemma 3.6 (that is, R is irreflexive since it is strongly rigid) and the assumption that $|X_2| > 1$ we get $f(f(a_0)) \neq f(a_0)$. ∎

Theorem 4.4. *If* X *admits a strongly rigid binary relation, then so does* $\mathcal{P}(X)$.

Proof. Assume that R_0 is a strongly rigid relation on X. Since every finite set admits a strongly rigid relation it suffices to prove the theorem when X is infinite.

As a first step we choose two disjoint subsets X_1 and X_2 of $\mathcal{P}(X)$ such that $X_1 \cap X_2 = \emptyset$, $|X_1| = |X|$, $|X_2| = |X|$, $\emptyset \in X_1$ and $X \in X_2$, and we let $\phi_1 : X \to X_1$ and $\phi_2 : X \to X_2$ be one-to-one surjective functions. For example, we could choose two distinct elements a and b of X, let $X_1 = \{\{a,x\} : x \in X \setminus \{b\}\} \cup \{\emptyset\}$, $X_2 = \{\{b,x\} : x \in X \setminus \{a\}\} \cup \{X\}$, and define

$$\phi_1(x) = \begin{cases} \{a,x\} & \text{if } x \in X \setminus \{b\}, \\ \emptyset & \text{if } x = b, \end{cases} \qquad \phi_2(x) = \begin{cases} \{b,x\} & \text{if } x \in X \setminus \{a\}, \\ X & \text{if } x = a. \end{cases}$$

Next we define relations R_1 and R_2 on X_1 and X_2 respectively so that (X_1, R_1) and (X_2, R_2) are copies of (X, R_0). More specifically, we let $R_1 = \phi_1(R_0) = \{(\phi(x), \phi(y)) : (x,y) \in R_0\}$ and $R_2 = \phi_2(R_0)$. By Lemma 4.3 the relation $R = R_1 \cup R_2 \cup \{(b,a) : a \in X_1 \text{ and } b \in X_2\}$ is a strongly rigid relation on $X_1 \cup X_2$. We also note that $\phi_2 \circ \phi_1^{-1}$ is an isomorphism of the systems (X_1, R_1) and (X_2, R_2). Using R we define a relation S on $\mathcal{P}(X)$ to be the union of the following three sets:

- R,
- $\{(\phi_1(a), y) : y \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$ and $a \in y\}$ (which is a subset of $X_1 \times (\mathcal{P}(X) \setminus (X_1 \cup X_2)))$,
- $\{(y, \phi_2(a)) : y \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$ and $a \notin y\}$ (which is a subset of $(\mathcal{P}(X) \setminus (X_1 \cup X_2)) \times X_2)$.

We note that by the definition of S, if $y \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$ then for all z, $z\ S\ y$ implies $z \in X_1$ and $y\ S\ z$ implies $z \in X_2$. We claim that S is a strongly rigid relation on $\mathcal{P}(X)$.

Indeed, assume that f is an endomorphism of $(\mathcal{P}(X), S)$. We will show that f is the identity function on $\mathcal{P}(X)$. As a first step we show that $f{\restriction}X_1 \cup X_2$ is the identity. This is the content of the following lemma and its corollary.

Lemma 4.5. *For all $x \in X_1 \cup X_2$, $f(x) \in X_1 \cup X_2$.*

Proof. For contradiction, assume that $x_0 \in X_1 \cup X_2$ is an element for which $f(x_0) \notin X_1 \cup X_2$.

Case 1: $x_0 \in X_1$. In this case $x\ S\ x_0$ for every $x \in X_2$. It follows that

$$\forall x \in X_2, \quad f(x)\ S\ f(x_0). \tag{4.1}$$

Since $f(x_0) \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$, the remark following the definition of S implies

$$\forall x \in X_2, \quad f(x) \in X_1. \tag{4.2}$$

Composing f with the isomorphism $\phi_2 \circ \phi_1^{-1}$ we have $(f \circ \phi_2 \circ \phi_1^{-1})(X_1) = f(\phi_2\phi_1^{-1}(X_1)) = f(X_2) \subseteq X_1$. Hence $f \circ \phi_2 \circ \phi_1^{-1}$ is a homomorphism of (X_1, R_1). (This uses the fact that $S \cap R_1 = R_1$.) Since (X_1, R_1) is strongly rigid we conclude that $f(\phi_2(\phi_1^{-1}(x))) = x$ for all $x \in X_1$. Therefore for any $a \in X$, letting $a = \phi_1^{-1}(x)$, we conclude that

$$\forall a \in X, \quad f(\phi_2(a)) = \phi_1(a). \tag{4.3}$$

Since $f(x_0) \notin X_1 \cup X_2$ and $X \in X_1 \cup X_2$ there is some $a_0 \in X$ such that $a_0 \notin f(x_0)$. By the definition of S we have

$$\phi_1(a_0)\ \not S\ f(x_0). \tag{4.4}$$

Since $\phi_1(a_0) \in X_1$ and $x_0 \in X_2$ we find (using the definition of S again) that $\phi_2(a_0)\ S\ x_0$. Since f is a homomorphism, we have $f(\phi_2(a_0))\ S\ f(x_0)$. By (4.3) this is the same as $\phi_1(a_0)\ S\ f(x_0)$, contrary to (4.4).

Case 2: $x_0 \in X_2$. The proof in this case is similar and uses the fact that $\emptyset \in X_1 \cup X_2$. ■

Since $(X_1 \cup X_2, R)$ is strongly rigid, we have the following corollary of Lemma 4.5.

Corollary 4.6. *For all $x \in X_1 \cup X_2$, $f(x) = x$.*

LEMMA 4.7. *For all* $y \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$, $f(y) \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$.

Proof. Assume that the lemma is false and that $y_0 \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$ has $f(y_0) \in X_1 \cup X_2$.

CASE 1: $f(y_0) \in X_1$. Since $X \in X_1 \cup X_2$ we may choose $a \in X$ such that $a \notin y_0$. By the definition of S, $y_0 \; S \; \phi_2(a)$. It follows that $f(y_0) \; S \; f(\phi_2(a))$, which gives $f(y_0) \; S \; \phi_2(a)$ (using the corollary above). But this contradicts the definition of S since $f(y_0) \in X_1$ and $\phi_2(a) \in X_2$.

CASE 2: $f(y_0) \in X_2$. The proof is similar to the proof in Case 1 and is left to the reader. ∎

The following lemma will complete the proof of the theorem.

LEMMA 4.8. *For all* $y \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$, $f(y) = y$.

Proof. Toward a contradiction, assume there is a $y_1 \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$ for which $f(y_1) \neq y_1$. Let $y_2 = f(y_1)$. Then by Lemma 4.7, we have $y_2 \in \mathcal{P}(X) \setminus (X_1 \cup X_2)$. Since $y_1 \neq y_2$ there are two cases to consider:

CASE 1: *There is an element* $a \in y_1 \setminus y_2$. In this case $\phi_1(a) \; S \; y_1$ and $\phi_1(a) \not S \; y_2$ by the definition of S. From the first of these we deduce that $f(\phi_1(a)) \; S \; f(y_1)$. Now using $f(\phi_1(a)) = \phi_1(a)$ we arrive at the contradiction $\phi_1(a) \; S \; f(y_1)$.

CASE 2: *There is an element* $a \in y_2 \setminus y_1$. This is similar to the previous case and is left to the reader. ∎

The proof of the theorem is complete. ∎

COROLLARY 4.9. (ZF) *For every* $n \in \omega$, $\mathcal{P}^n(\omega)$, *hence* (*by Theorem* 4.4) $2^{\mathcal{P}^n(\omega)}$, *has a strongly rigid binary relation. In particular,* $2^{\mathbb{R}}$ *has a strongly rigid binary relation.*

Proof. Use Theorem 4.4 and an easy induction. ∎

Our next result, Theorem 4.10, is the analogue of Theorem 4.4 for rigid binary relations. That is, we prove that if a set X has a rigid binary relation, then in ZF, so does $\mathcal{P}(X)$. In addition, our proof will implicitly suggest a simplification of the proof of [2, Theorem 2.1] (see Remark 4.12(2)). We shall also take the opportunity to point out a slightly different argument for [2, Theorem 2.1], which is applied directly to $\mathbb{R}$ rather than to 2^ω (see Remark 4.12(3)). We do this in order to extract possible new ideas.

THEOREM 4.10. (ZF) *For every set* X, *if* X *has a rigid binary relation, then so does* 2^X.

Proof. Assume that a set X admits a rigid binary relation S. We shall identify each element x of X with the element of 2^X which is the characteristic function of $\{x\}$. This will simplify the notation. We define a binary relation R on 2^X as follows:

- For $x, y \in X$, we define $(x, y) \in R$ if and only if $(x, y) \in S$.
- For $x \in X$ and $y \in 2^X \setminus X$, define $(x, y) \in R$ if and only if $y(x) = 1$.

We assert that R is a rigid relation on 2^X. To see this, let f be an automorphism of the relational system $\langle 2^X, R\rangle$. We will show that f is the identity mapping. First, note that $f(x) \in X$ for all $x \in X$. If not, let $x \in X$ be such that $f(x) = y \in 2^X \setminus X$. Pick any function $z \in 2^X \setminus X$ such that $z(x) = 1$. Then $(x, z) \in R$ and thus $(f(x), f(z)) \in R$ or $(y, f(z)) \in R$. But there is no pair $(a, b) \in R$ such that $a \in 2^X \setminus X$, and we have reached a contradiction. Similarly, one shows that $f{\restriction}X$ is onto X (if $x \in X$, then there is a $y \in 2^X$ such that $f(y) = x$; if $y \notin X$, then taking an element z as before, we find that $(x, z) \in R$, hence $(y, f^{-1}(z)) \in R$, a contradiction, hence $y \in X$). It follows that $f{\restriction}X$ is an automorphism of $\langle X, R{\restriction}X\rangle$. Furthermore, as $R{\restriction}X \equiv S$ and S is rigid on X, we conclude that $f(x) = x$ for all $x \in X$.

Next we show that $f(y) = y$ for all $y \in 2^X \setminus X$. Assume on the contrary that $y \in 2^X \setminus X$ is such that $f(y) \neq y$. Since f is the identity mapping on X and f is a permutation of 2^X, we have $f(y) \in 2^X \setminus X$. Let $x \in X$ be such that $f(y)(x) \neq y(x)$. Without loss of generality assume that $y(x) = 1$, hence $f(y)(x) = 0$. It follows that $(x, y) \in R$ and $(x, f(y)) \notin R$. However, since f is an automorphism of $\langle 2^X, R\rangle$, we have

$$(x, y) \in R \;\rightarrow\; (f(x), f(y)) \in R \;\rightarrow\; (x, f(y)) \in R,$$

a contradiction. Thus $f(y) = y$, as required. ∎

Corollary 4.11. (ZF) *For every well-ordered cardinal number κ, 2^κ has a rigid binary relation.*

Proof. This follows from Theorem 4.10 and the fact that well-orders are rigid. ∎

Remark 4.12. 1. Using ideas from the proof of Theorem 4.10 one may simplify (part of) the definition and the proof of the rigid relation R in the proof of [2, Theorem 1.1], for a Dedekind infinite set $A \subseteq 2^\omega$, as follows: For $y \notin Z = \{z^*, z_0, z_1, \ldots\} \subseteq A$, $R(x, y)$ holds if and only if $x = z_n$ for some n and $y(n) = 1$.

2. We present a modification of the proof of [2, Theorem 1.1], adjusted to $\mathbb{R}$ and not to 2^ω. Let A be an infinite uncountable subset of $\mathbb{R}$. Suppose first that A has a countably infinite subset $B = \{b_n : n \in \omega\}$ and let $\mathbb{Q} = \{q_n : n \in \omega\}$ be an enumeration of the rationals. We define a binary relation R on A as follows: For all $n \in \mathbb{N}$, $(b_n, b_{n+1}) \in R$ and $(b_n, b_0) \in R$. In addition, we require $(b_0, b_0) \in R$. For $a \in A \setminus B$ we require $(b_n, a) \in R$ if and only if $q_n < a$, where $<$ is the usual ordering on $\mathbb{R}$. Now we show that R is

rigid. Let f be an automorphism of $\langle A, R\rangle$. As in the proof of [2, Theorem 1.1], one verifies that f fixes B pointwise. Now let $a \in A \setminus B$ and suppose, for contradiction, that $f(a) = c \neq a$. Without loss of generality assume that $a < c$ and let $n \in \omega$ be such that $a < q_n < c$. Then $(b_n, c) \in R$, hence $(b_n, f(a)) \in R$ and thus $(b_n, a) \in R$. This means that $q_n < a$, a contradiction. Therefore, $f(a) = a$ and R is rigid as asserted.

The case where A is a Dedekind finite subset of $\mathbb{R}$ can be treated as in the proof of [2, Theorem 1.1] or Theorem 4.1 of the current paper.

The next natural question is whether the statement "every subset of $2^{\mathcal{P}(\omega)}$ has a rigid binary relation" (in other words, the statement "every set of sets of reals has a rigid binary relation") is still provable in ZF. In [2, top of p. 397], the answer is found to be negative. Here, we go one step further, giving some information regarding the placement of "every subset of $2^{\mathcal{P}(\omega)}$ has a rigid binary relation" in the hierarchy of weak choice principles. In particular, based on the proof of Theorem 2.7, we will see that "every subset of $2^{\mathcal{P}(\omega)}$ has a rigid binary relation" implies the Axiom of Countable Choice for pairs of sets of reals (see Theorem 4.13). Since the latter weak choice principle fails in the second Cohen symmetric model for $\mathsf{ZF} + \neg\mathsf{AC}$ (see [4]), we will also see as a by-product that "every subset of $2^{\mathcal{P}(\omega)}$ has a rigid binary relation" is *not provable* in ZF.

Recall that the statement "$2^{\mathbb{R}}$ has a strongly rigid binary relation" *is* a theorem of ZF (see Corollary 4.9).

THEOREM 4.13.

(a) *If every subset of $2^{\mathbb{R}}$ has a rigid binary relation, then every countably infinite family of pairs of sets of reals has a choice function. Thus, the statement "every subset of $2^{\mathbb{R}}$ has a rigid binary relation" is not provable in* ZF.

(b) *If $2^{\mathbb{R}}$ has a hereditarily rigid binary relation, then every family of non-empty finite sets of sets of reals has a choice function, which in turn implies that there exists a non-measurable subset of 2^{ω} with the product measure.*

Proof. (a) Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a countably infinite family of pairs of sets of reals. We may assume that for all $i \in \omega$, if $A_i = \{x, y\}$, then $x \setminus y \neq \emptyset$ and $y \setminus x \neq \emptyset$ (otherwise we could choose from A_i the set $\bigcap A_i$). Thus, we can further assume that for each $i \in \omega$, the elements of A_i are disjoint. In addition, since $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ we may also assume that $\mathcal{A}$ is pairwise disjoint (otherwise, replace each A_i with $\{x \times \{i\} : x \in A_i\}$).

Since $|\mathcal{P}(\mathbb{R})| = |2^{\mathbb{R}}|$, we may view $\bigcup \mathcal{A}$ as a subset of $2^{\mathbb{R}}$. By our assumption, $\bigcup \mathcal{A}$ has a rigid binary relation R. Following now the proof of Theorem 2.7, we conclude that $\mathcal{A}$ has a choice function.

For the second assertion, in the second Cohen forcing model (model $\mathcal{M}7$ in [4]) there is a countably infinite family of pairs of sets of reals which does not have a partial choice function in the model, i.e. a choice function whose domain is some infinite subset of $\mathcal{A}$ (see [4], [5]). Thus, the statement "every subset of $2^{\mathbb{R}}$ has a rigid binary relation" fails in Cohen's second model.

(b) The first implication of (b) follows from Theorem 3.1, while the second one follows from [5, Problem 10, p. 7]. ∎

5. Summary. The following diagram summarizes results of the paper on the rigidity principles RR, HRR and SRR. In the diagram, a solid arrow "$\rightarrow$", either horizontal or vertical, means that the implication holds in ZF. A negated solid arrow "$\nrightarrow$", either horizontal or diagonal, means "non-implication in ZFA". The negated arrow "$\not\Uparrow$" means "non-implication in ZF".

$$
\begin{array}{ccccc}
\boxed{\mathsf{AC}} & \rightarrow & \boxed{\mathsf{SRR}} & \not\searrow & \\
\downarrow\not\Uparrow & & \downarrow & & \\
\boxed{\mathsf{HRR}} & \rightarrow & \boxed{\mathsf{RR}} & \nleftarrow & \boxed{\mathsf{MC}, \mathsf{AC}_{\mathsf{WO}}, \mathsf{DC}} \\
\downarrow & & \downarrow & & \\
\boxed{\mathsf{AC}^{\mathsf{fin}}} & \rightarrow & \boxed{\mathsf{AC}^2_{\mathsf{WO}}} & & \\
\downarrow & & \uparrow & & \\
\boxed{\nexists \text{ amorphous sets}} & \nrightarrow & \boxed{\mathsf{RR}} & &
\end{array}
$$

Implications and non-implications between rigidity principles and certain choice forms

6. Problems

1. What is the relationship of RR or SRR with BPI (the Boolean Prime Ideal Theorem, i.e. the statement "every non-trivial Boolean algebra has a prime ideal") or with OP (the Ordering Principle, i.e. the statement "every set can be linearly ordered". It is known (see [4]) that $\mathsf{BPI} \rightarrow \mathsf{OP}$).
2. What is the relationship between HRR and SRR?
3. Does RR or SRR imply "there are no amorphous sets"?
4. Is the statement "2^ω (equivalently, $\mathbb{R}$) has a hereditarily rigid binary relation" provable in ZF?
5. Is the statement "for every well-ordered cardinal κ, every subset of 2^κ has a rigid binary relation" provable in ZF? The latter question is also addressed in [1]. (Note that the statement is *true in every FM model*, since in each such model the power set of a well-orderable set is well-orderable: see [4].)

6. Is the statement "for every set X, if X has a hereditarily rigid binary relation, then 2^X has a hereditarily rigid binary relation" provable in ZF? (Note that the latter statement implies the statement of problem 5, so a possible negative answer to problem 5 yields a negative answer to the current problem.)
7. Given a set X, is it true (in ZF) that if 2^X has a rigid binary relation, then X also has such a relation?
8. Is it provable in ZF that every well-ordered set admits a strongly rigid binary relation?

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