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The Lie group of real analytic diffeomorphisms is not real analytic

by

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Abstract. We construct an infinite-dimensional real analytic manifold structure on the space of real analytic mappings from a compact manifold to a locally convex manifold. Here a map is defined to be real analytic if it extends to a holomorphic map on some neighbourhood of the complexification of its domain. As is well known, the construction turns the group of real analytic diffeomorphisms into a smooth locally convex Lie group. We prove that this group is regular in the sense of Milnor.

In the inequivalent "convenient setting of calculus" the real analytic diffeomorphisms even form a real analytic Lie group. However, we prove that the Lie group structure on the group of real analytic diffeomorphisms is in general not real analytic in our sense.

1. Introduction. A classical result by Leslie states that the group of real analytic diffeomorphisms of a compact analytic manifold is an infinite-dimensional Lie group (see [21]). Unfortunately, the proof in [21] contains a gap, as is pointed out in [18]. However, in the "convenient setting of analysis" it is possible to show (cf. [18, 19]):

THEOREM (Kriegl/Michor, 1990). *For a compact real analytic manifold M the group of real analytic diffeomorphisms $\mathrm{Diff}^\omega(M)$ is a real analytic Lie group (in the sense of convenient calculus) modelled on the Silva space* ([1]) *$\mathfrak{X}^\omega(M)$ of real analytic vector fields.*

The theorem above subsumes the earlier (but flawed) result in [21]. This is a nontrivial observation since in general the calculus used in [21] and the

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([1]) A *Silva space* is an inductive limit of a sequence of Banach spaces such that the bonding maps are compact operators. See [9] for more information on Silva spaces (called (LS)-spaces there).

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convenient calculus used in [18, 19] are inequivalent. Before we continue, we have to briefly discuss the different notions of differentiability in locally convex spaces which will be used throughout the present article:

(i) smooth Keller C^r_c-maps ($C^\infty_\mathbb{R}$),
(ii) real analytic mappings ($C^\omega_\mathbb{R}$),
(iii) convenient smooth mappings ($c^\infty_\mathbb{R}$),
(iv) convenient real analytic mappings ($c^\omega_\mathbb{R}$).

In this article, we base our investigation on $C^\infty_\mathbb{R}$-maps, i.e. on Keller's C^r_c-theory (also called Bastiani calculus); see the Appendix or [12] for a streamlined exposition. Building on this and an idea by Milnor, we call a map $f\colon E \supseteq U \to F$ between real locally convex spaces *real analytic* or $C^\omega_\mathbb{R}$ if it extends to a holomorphic map (i.e. a Keller $C^\infty_\mathbb{C}$-map) $\tilde{U} \to F_\mathbb{C}$ on an open neighbourhood $\tilde{U}$ of U in the complexification $E_\mathbb{C}$ of E (cf. Definition A.3 or [12]).

Further, we recall the notions of convenient smooth ($c^\infty_\mathbb{R}$) and convenient real analytic maps ($c^\omega_\mathbb{R}$) (see [19]). A map between complete spaces is *convenient smooth* [*convenient real analytic*] if it maps smooth curves to smooth curves [analytic curves to analytic curves].

In general, the various concepts of smoothness and analyticity are inequivalent. However, it is well known (combine [19, Theorems 4.11 and 10.1] and [9, §9, Satz 6]) that the above notions of differentiability are related as follows:

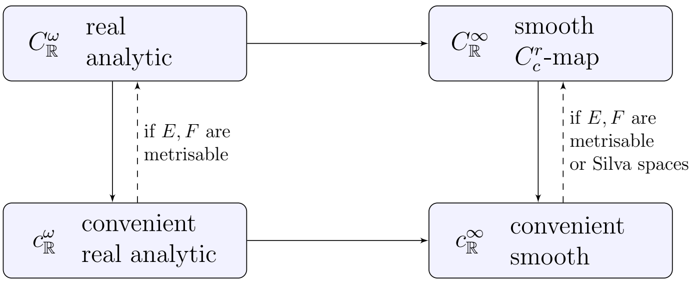

Fig. 1. Relations of different types of differentiability for $f\colon E \to F$ with E, F complete. Dashed arrows indicate conditional implications.

Leslie constructs $\mathrm{Diff}^\omega(M)$ as a Lie group modelled on a Silva space in the setting of Keller's C^r_c-theory. As $C^\infty_\mathbb{R}$-differentiability agrees on Silva spaces with convenient smoothness, [18] subsumes the earlier construction in [21].

In the present paper we study again the Lie group of real analytic diffeomorphisms on a compact real analytic manifold, but this time with a real analytic ($C^\omega_\mathbb{R}$-)structure. Since for Silva spaces real analyticity in our sense is stronger than convenient real analyticity ($c^\omega_\mathbb{R}$), we cannot use the above theorem to conclude that the group is also a $C^\omega_\mathbb{R}$-Lie group.

The paper commences with the construction of a real analytic manifold structure on the set $C^\omega_\mathbb{R}(M,N)$ of real analytic mappings between two analytic manifolds M and N. We assume that M is compact (and hence finite-dimensional), while N may be infinite-dimensional as long as it admits a local addition, which is a very weak assumption (see Definition 2.2). We follow the well known construction of $c^\omega_\mathbb{R}$-structures on spaces of $c^\omega_\mathbb{R}$-mappings (cf. [18]) and obtain the following result.

Theorem A. *Let M, N be real analytic manifolds with M compact and N admitting a local addition. Then $C^\omega_\mathbb{R}(M,N)$ is a real analytic manifold. The manifold structure does not depend on the choice of local addition.*

Here the real analytic structure means real analyticity in the stronger sense explained above ($C^\omega_\mathbb{R}$). The group of diffeomorphisms $\mathrm{Diff}^\omega(M)$ turns out to be open in the $C^\omega_\mathbb{R}$-manifold $C^\omega_\mathbb{R}(M,M)$. Hence the real analytic manifold structure on $C^\omega_\mathbb{R}(M,M)$ induces a real analytic structure on $\mathrm{Diff}^\omega(M)$. To spell it out explicitly, this result is not surprising in any respect. In the inequivalent convenient setting a $c^\omega_\mathbb{R}$-version of Theorem A using the notion of convenient real analytic ($c^\omega_\mathbb{R}$-)maps was already known. However, since $C^\omega_\mathbb{R}$ is stronger than $c^\omega_\mathbb{R}$, it is not possible to adapt the arguments establishing real analyticity to our setting. Moreover, the stronger notion of real analyticity we adopt in this paper requires special care if one wants to deal with spaces $C^\omega_\mathbb{C}(M,N)$ where N is an infinite-dimensional locally convex manifold. To obtain Theorem A for infinite-dimensional manifolds N, we consider complexifications for certain types of infinite-dimensional vector bundles. For finite-dimensional vector bundles this seems to be part of the folklore (see for example [18, 7.1]); an infinite-dimensional analogue has just recently been recorded in [7].

Furthermore, in the convenient setting [19, Theorem 43.3] it has been shown that the c^ω-Lie group $\mathrm{Diff}^\omega(M)$ is regular. To put this result into perspective, recall the notion of regularity for Lie groups:

Let G be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the right translation $\rho_g\colon G \to G$, $x \mapsto xg$, by $g \in G$ to define $v.g := T_\mathbf{1}\rho_g(v) \in T_gG$ for v in $T_\mathbf{1}(G) =: \mathbf{L}(G)$. Following [6] and [15], G is called C^r*-semiregular* if for each C^r-curve $\gamma\colon [0,1] \to \mathbf{L}(G)$ the initial value problem

$$\begin{cases} \eta'(t) = \gamma(t).\eta(t), \\ \eta(0) = \mathbf{1}, \end{cases}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta\colon [0,1] \to G$. If further

$$\mathrm{evol}\colon C^r([0,1], \mathbf{L}(G)) \to G, \qquad \gamma \mapsto \mathrm{Evol}(\gamma)(1),$$

is smooth, we call G *C^r-regular*. If G is C^r-regular and $r \leq s$, then G is also C^s-regular. A C^∞-regular Lie group G is called *regular* (*in the sense of Milnor*). Every finite-dimensional Lie group is C^0-regular. Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [15]; cf. also [19] and the references therein).

In the present situation the model space of the Lie group $\mathrm{Diff}^\omega(M)$ is the Silva space $\mathfrak{X}^\omega(M)$ of real analytic vector fields. Thus C^∞ and c^∞ agree on $\mathfrak{X}^\omega(M)$. However, for regularity as defined above, we have to establish the smoothness of the map evol. Now evol is defined on the function space $C^r([0,1], \mathfrak{X}^\omega(M))$, which is no longer a Silva space although $\mathfrak{X}^\omega(M)$ is one. Thus our notion of smoothness ($C^\infty_\mathbb{R}$) is not equivalent to the notion of convenient smoothness ($c^\infty_\mathbb{R}$) on $C^\infty([0,1], \mathfrak{X}^\omega(M))$, and so the result of Kriegl and Michor does not imply that the Lie group $\mathrm{Diff}^\omega(M)$ is regular in our setting. However, Glöckner's regularity theorem for Silva Lie groups (see [15, Theorem 15.5]) enables us to prove the regularity of $\mathrm{Diff}^\omega(M)$:

Theorem B. *Let M be a compact real analytic manifold. Then the $C^\infty_\mathbb{R}$-Lie group* $\mathrm{Diff}^\omega(M)$ *is C^1-regular.*

Summarizing, we have seen so far that the group $\mathrm{Diff}^\omega(M)$ is a

- regular $c^\omega_\mathbb{R}$-Lie group (Theorem of Kriegl/Michor),
- regular $C^\infty_\mathbb{R}$-Lie group (Theorem B),
- $C^\omega_\mathbb{R}$-manifold (Theorem A).

One would suspect that the real analytic structure turns $\mathrm{Diff}^\omega(M)$ into a $C^\omega_\mathbb{R}$-Lie group, i.e. the group operations are $C^\omega_\mathbb{R}$-mappings (and not just $C^\infty_\mathbb{R}$). In the last part of this paper we investigate this question for the group of analytic diffeomorphisms on the circle $\mathbb{S}^1$. In contrast to the result in the convenient setting, we obtain the following.

Theorem C. *Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ with the canonical real analytic manifold structure. Then the group multiplication of the Lie group* $\mathrm{Diff}^\omega(\mathbb{S}^1)$ *is not real analytic. In particular,* $\mathrm{Diff}^\omega(\mathbb{S}^1)$ *is a $c^\omega_\mathbb{R}$-Lie group but not a $C^\omega_\mathbb{R}$-Lie group.*

Observe that group multiplication is an instructive example of a mapping which is smooth in the sense of Keller's C^r_c-theory, convenient real analytic but not real analytic. The reason for this surprising behaviour is buried in the construction of the model space $\mathfrak{X}^\omega(\mathbb{S}^1)$ of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ and its complexification (cf. Section 4).

This counterexample is tailored to the manifold $\mathbb{S}^1$. Hence it just indicates that $\mathrm{Diff}^\omega(M)$ will in general not be a real analytic Lie group in our sense. Nevertheless, the construction of the counterexample should carry over to the general setting. The authors believe that a similar analysis will show that for an arbitrary compact real analytic manifold M (except the zero-dimensional ones) the group $\mathrm{Diff}^\omega(M)$ is not a real analytic Lie group.

2. The locally convex manifold structure for spaces of analytic maps. In this section we recall the construction of the manifold structure on spaces of analytic functions. The basic idea is not new and follows the exposition in the convenient setting (see [18]). Beyond the Fréchet setting our notion of real analytic maps is not equivalent to the notion in the convenient setting of global analysis. Thus the arguments establishing analyticity in our sense are new and require the complexification of several (infinite-dimensional) vector bundles.

NOTATION 2.1. We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, $\mathbb{K}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. For a normed space $(E, \|\cdot\|)$, $x \in E$ and $R > 0$, we let $B_R^E(x)$ be the open ball of radius R centred at x.

The setting of real analytic mappings used in this paper (cf. [12] and [3]) is briefly recalled in the Appendix. We urge the reader to review the Appendix for more information on the calculus, locally convex manifolds and Lie groups used throughout the paper.

DEFINITION 2.2. Let N be a real analytic manifold modelled on a locally convex space over $\mathbb{R}$. We call a real analytic map $\Sigma\colon TN \supseteq \Omega \to N$ defined on an open neighbourhood Ω of the zero-section in TN a (*real analytic*) *local addition* if

- $(\pi_{TN}, \Sigma)\colon TN \supseteq \Omega \to N \times N$ induces a $C^\omega_{\mathbb{R}}$-diffeomorphism onto an open neighbourhood of the diagonal in $N \times N$,
- $\Sigma(0_x) = x$ for all $x \in N$, where 0_x is the zero-element in the fibre over x.

REMARK 2.3. (1) For finite-dimensional paracompact real analytic manifolds N there always exists a real analytic local addition. It is given by the real analytic Riemannian exponential map $\exp\colon TN \supseteq \Omega \to N$ (see [16] and [18, 7.5]).

(2) Note that also every (possibly infinite-dimensional) (real analytic) Lie group admits a real analytic local addition due to the real analytic group structure and the fact that the tangent bundle is trivial (cf. [19, 42.4]).

Our approach uses complexifications of several vector bundles on M (which are possibly infinite-dimensional). The reader is referred to [7, Section 3] for the notation and details concerning these constructions.

2.4. Let (F, π, M) be a real analytic vector bundle whose typical fibre E is a locally convex vector space, M is finite-dimensional paracompact and

F a locally convex manifold. By [7, Proposition 3.5] (cf. also [18, 7.1] for the finite-dimensional case) the bundle (F, π, M) admits a unique bundle complexification $(F_\mathbb{C}, \pi_\mathbb{C}, M_\mathbb{C})$.

For a compact subset $K \subseteq M$ consider the real vector *space of germs along* K *of real analytic sections*, $\Gamma^\omega_\mathbb{R}(F|K)$. The complexification of $\Gamma^\omega_\mathbb{R}(F|K)$ is given as

$$\Gamma^\omega_\mathbb{R}(F|K)_\mathbb{C} = \Gamma^\omega_\mathbb{C}(F_\mathbb{C}|K)$$

(cf. [18, 7.2]). As the bundle complexification is unique, this construction does not depend on the choice of complexifications.

In the following we identify $\Gamma^\omega_\mathbb{R}(F|K)$ with the corresponding complemented subspace of $\Gamma^\omega_\mathbb{C}(F_\mathbb{C}|K)$ and topologise $\Gamma^\omega_\mathbb{R}(F|K)$ with the subspace topology. If F is a finite-dimensional manifold, Lemma A.16 shows that $\Gamma^\omega_\mathbb{C}(F_\mathbb{C}|K)$ is a Silva space. Since closed subspaces of Silva spaces are Silva spaces by [5, Corollary 8.6.9], $\Gamma^\omega_\mathbb{R}(F|K)$ is a Silva space if $\dim F < \infty$.

2.5 (Canonical charts for $C^\omega_\mathbb{R}(M, N)$)**.** Fix a compact real analytic manifold M and a (possibly infinite-dimensional) real analytic manifold N. We require that N admits a real analytic local addition $\Sigma\colon TN \supseteq \Omega \to N$.

Consider $f \in C^\omega_\mathbb{R}(M, N)$ and define

$$U_f := \{g \in C^\omega_\mathbb{R}(M, N) \mid (f(x), g(x)) \in (\pi_{TN}, \Sigma)(\Omega) \text{ for all } x \in M\}$$

and a map $\Phi_f\colon U_f \to \Gamma^\omega_\mathbb{R}(f^*TN)$ by

$$\Phi_f(\gamma) := \big(\mathrm{id}_M, (\pi_{TN}, \Sigma)^{-1} \circ (f, \gamma)\big).$$

In the following, we identify $f^*\Omega = \{(x, X) \in M \times TN \mid X \in T_{f(x)}N \cap \Omega\}$ with an open submanifold of f^*TN. The topology on $\Gamma^\omega_\mathbb{R}(f^*TN)$ is the subspace topology of $\Gamma^\omega_\mathbb{C}((f^*TN)_\mathbb{C}|M)$. Now $\Gamma^\omega_\mathbb{C}((f^*TN)_\mathbb{C}|M)$ is the locally convex inductive limit of steps whose topology is the compact-open topology (cf. Lemma A.14). We deduce that the topology of $\Gamma^\omega_\mathbb{C}((f^*TN)_\mathbb{C}|M)$ is finer than the compact-open topology, and thus the same holds for $\Gamma^\omega_\mathbb{R}(f^*TN)$. As M is compact, this shows that $\Phi_f(U_f) = \{\sigma \in \Gamma^\omega_\mathbb{R}(f^*TN) \mid \sigma(M) \subseteq f^*\Omega\}$ is an open subset of $\Gamma^\omega_\mathbb{R}(f^*TN)$. Define the real analytic map

$$\tau_f\colon f^*\Omega \to (f \times \mathrm{id}_N)^{-1}(\pi_{TN}, \alpha)(\Omega) \subseteq M \times N, \qquad \tau_f(x, X) := (x, \Sigma(X)).$$

Clearly τ_f is bijective and respects the fibres over M. Its inverse is the real analytic map

$$\tau_f^{-1}(y, z) := \big(y, (\pi_{TN}, \Sigma)^{-1}(f(y), z)\big).$$

By construction this map takes its values in $f^*\Omega \subseteq f^*TN$ and is continuous with respect to the subspace topology on this space induced by $M \times TN$ on the fibre product. We conclude that τ_f is a homeomorphism onto its (open) image, and thus

$$\Omega_{f,g} := \tau_g^{-1}(\tau_f(f^*\Omega)) \subseteq g^*\Omega$$

is open. Let us now compute, for $f, g \in C^\omega_{\mathbb{R}}(M,N)$ and σ in $\Phi_g(U_f \cap U_g)$, a formula for $\Phi_f \circ \Phi_g^{-1}$. Denote by $\pi^*_{TN} g\colon g^*TN \to TN$ the bundle map covering g. Then we obtain

$$(2.1)\qquad \begin{aligned}(\Phi_f \circ \Phi_g^{-1})(\sigma) &= (\mathrm{id}_M, (\pi_{TN}, \Sigma)^{-1} \circ (f, \Sigma \circ (\pi^*_{TN} g) \circ \sigma)) \\ &= \tau_f^{-1} \circ \tau_g \circ \sigma =: (\tau_f^{-1} \circ \tau_g)_*(\sigma).\end{aligned}$$

Here $(\tau_f^{-1} \circ \tau_g)_*$ is defined on $\lfloor M, \Omega_{f,g} \rfloor := \{\sigma \in \Gamma^\omega_{\mathbb{R}}(g^*TN) \mid \sigma(M) \subseteq \Omega_{f,g}\}$, which is an open subset of $\Gamma^\omega_{\mathbb{R}}(g^*TN)$ (2). Thus $\Phi_g(U_f \cap U_g) = \lfloor M, \Omega_{f,g} \rfloor$ is open in the compact-open topology and also in the finer topology on $\Gamma^\omega_{\mathbb{R}}(g^*TN)$.

We will now prove Theorem A. The manifold structure constructed on $C^\omega_{\mathbb{R}}(M,N)$ is a manifold under the extended Definition A.5 of manifolds (i.e. the model space can depend on the chart).

THEOREM 2.6. *Let M, N be real analytic manifolds such that M is compact and N admits a real analytic local addition.*

(1) *The family $(U_f, \Phi_f)_{f \in C^\omega_{\mathbb{R}}(M,N)}$ is a real analytic atlas for $C^\omega_{\mathbb{R}}(M,N)$.*
(2) *The identification topology with respect to the atlas in* (1) *turns $C^\omega_{\mathbb{R}}(M,N)$ into a (Hausdorff) real analytic manifold modelled on the spaces $\Gamma^\omega_{\mathbb{R}}(f^*TN)$ where f runs through $C^\omega_{\mathbb{R}}(M,N)$. If N is finite-dimensional, $C^\omega_{\mathbb{R}}(M,N)$ is a manifold modelled on Silva spaces.*
(3) *The manifold structure does not depend on the choice of local addition.*

Proof. (1) Clearly $(U_f, \Phi_f)_{f \in C^\omega_{\mathbb{R}}(M,N)}$ is an atlas for $C^\omega_{\mathbb{R}}(M,N)$. We have to show that the changes of charts are real analytic. To this end consider $f, g \in C^\omega_{\mathbb{R}}(M,N)$ with $U_f \cap U_g \neq \emptyset$ and fix $\sigma \in \Phi_g(U_f \cap U_g)$. We will construct a holomorphic map on an open σ-neighbourhood in the complexification $\Gamma^\omega_{\mathbb{R}}(g^*TN)_{\mathbb{C}}$ which extends $\Phi_f \circ \Phi_g^{-1}$. To achieve this, we proceed in several steps.

STEP 1: *Complexifications of bundles and local data.* We form the pullback bundles $(f^*TN, f^*\pi_{TN}, M)$ and $(g^*TN, g^*\pi_{TN}, M)$. These are locally convex bundles over a finite-dimensional paracompact base. Hence both admit unique bundle complexifications (cf. 2.4) which we denote by $((f^*TN)_{\mathbb{C}}, (f^*\pi_{TN})_{\mathbb{C}}, M^*)$ and $((g^*TN)_{\mathbb{C}}, (g^*\pi_{TN})_{\mathbb{C}}, M^*)$, respectively. Note that by passing to open subsets in the complexification, we may choose the same complexification M^* of M as base for the complex bundles.

Following 2.4 we identify the bundle complexifications $\Gamma^\omega_{\mathbb{R}}(f^*TN)_{\mathbb{C}} = \Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M)$ and $\Gamma^\omega_{\mathbb{R}}(g^*TN)_{\mathbb{C}} = \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M)$. Hence a real ana-

(2) As is customary, for a smooth or analytic map h we denote by h_* the map defined by $h_*(\gamma) := h \circ \gamma$ on a suitable open subset of a space of mappings.

lytic section $\sigma \in \Phi_g(U_f \cap U_g) \subseteq \Gamma^\omega_{\mathbb{R}}(g^*TN)$ is associated to a germ $\tilde{\sigma} \in \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M)$ of a holomorphic section.

Finally, we fix some local data. As M is compact, we can choose a finite set A with the following properties:

(i) For $\alpha \in A$ there is a bundle trivialisation
$$\kappa^f_\alpha\colon (f^*\pi_{TN})^{-1}(M_\alpha) \to M_\alpha \times E_{\mathbb{C}}$$
of $(f^*TN)_{\mathbb{C}}$ which restricts on the analytic submanifold f^*TN to a bundle trivialisation of f^*TN.

(ii) For $\alpha \in A$ there is a bundle trivialisation $\kappa^g_\alpha\colon (g^*\pi_{TN})^{-1}(M_\alpha) \to M_\alpha \times E_{\mathbb{C}}$ of $(g^*TN)_{\mathbb{C}}$ which restricts on the real analytic submanifold g^*TN to a bundle trivialisation of g^*TN.

(iii) There are compact sets $K_\alpha \subseteq M_\alpha \cap M$ with $M = \bigcup_{\alpha \in A} K_\alpha$.

Step 2: *A suitable $\tilde{\sigma}$-neighbourhood $\mathcal{O}_\sigma$ in $\Gamma^\omega_{\mathbb{R}}(g^*TN)_{\mathbb{C}}$.* As M is compact and thus finite-dimensional, the complexification M^* in Step 1 is paracompact and thus regular as a topological space. From Lemma A.12 we deduce that $(f^*TN)_{\mathbb{C}}$ and $(g^*TN)_{\mathbb{C}}$ are regular as topological spaces. Observe that $\sigma(M)$ is a compact subset of $\Omega_{f,g} \subseteq g^*TN$. Thus by [7, Lemma 2.2(a)] there is an open complex neighbourhood $O_1 \subseteq (g^*TN)_{\mathbb{C}}$ of $\sigma(M)$ on which $\tau_f^{-1} \circ \tau_g|_{\Omega_{f,g}}\colon \Omega_{f,g} \to f^*TN$ extends to a holomorphic map $\phi\colon (g^*TN)_{\mathbb{C}} \supseteq O_1 \to O_2 \subseteq (f^*TN)_{\mathbb{C}}$.

Without loss of generality we can assume $O_1 \cap f^*TN \subseteq \Omega_{f,g}$. Shrinking O_1 further, we may assume that ϕ preserves the fibres of the complex bundle. To see this note that $\tau_f^{-1} \circ \tau_g$ is fibre preserving. Then we compute in pairs of trivialisations which satisfy (i) and (ii) of Step 1. It is easy to construct a holomorphic extension on a complex neighbourhood which preserves the fibres. Finally the identity theorem for real analytic functions shows that we can shrink O_1 so that ϕ is a fibre preserving map of the complex bundle.

As $\sigma(M)$ is contained in O_1, we deduce for each $\alpha \in A$ from (iii) in Step 1 that $\sigma(K_\alpha)$ is contained in the open set
$$U_\alpha := \phi^{-1}((f^*\pi_{TN})^{-1}_{\mathbb{C}}(M_\alpha)) \cap (g^*\pi_{TN})^{-1}_{\mathbb{C}}(M_\alpha) \cap O_1 \subseteq (g^*TN)_{\mathbb{C}}.$$
Now fix $\alpha \in A$ and consider the family $(\kappa^g_\alpha(x))_{x \in K_\alpha}$. Recall that K_α is a compact subset of a finite-dimensional, hence locally compact manifold. Apply now the Wallace theorem [8, 3.2.10] to obtain a finite family $(K_{\alpha,k})_{1 \leq k \leq n_\alpha}$ of compact sets with the following properties:

- For $1 \leq k \leq n_\alpha$ there are open subsets $O_{\alpha,k,1} \subseteq M_\alpha$ and $O_{\alpha,k,2} \subseteq E_{\mathbb{C}}$ such that $\kappa^g_\alpha(\sigma(K_{\alpha,k})) \subseteq O_{\alpha,k,1} \times O_{\alpha,k,2}$. This entails $K_{\alpha,k} \subseteq O_{\alpha,k,1} \subseteq M_\alpha$.
- The open set $O_{\alpha,k} := (\kappa^g_\alpha)^{-1}(O_{\alpha,k,1} \times O_{\alpha,k,2})$ is contained in U_α.
- $K_\alpha \subseteq \bigcup_{1 \leq k \leq n_\alpha} K_{\alpha,k}$.

Repeat this construction for each $\alpha \in A$. Then we can replace $(K_\alpha)_{\alpha\in A}$ with a finite family of compact subsets such that $n_\alpha = 1$ for all $\alpha \in A$ and the above conditions are satisfied. To shorten the notation denote this refinement again by $(K_\alpha)_{\alpha\in A}$ and write $O_{\alpha,i} := O_{\alpha,1,i}$ for $i \in \{1,2\}$ and $O_\alpha := O_{\alpha,1}$.

Observe that by construction $\tilde{\sigma}$ is contained in each of the sets

$$\lfloor K_\alpha, O_\alpha\rfloor := \{s \in \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M) \mid s(K_\alpha) \subseteq O_\alpha\}.$$

Definition A.13 and Lemma A.14 assert that the topology of the locally convex space $\Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}})|M)$ is finer than the compact-open topology. Thus each $\lfloor K_\alpha, O_\alpha\rfloor$ is an open subset of $\Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}})|M)$. Since A is finite, $\mathcal{O}_\sigma = \bigcap_{\alpha\in A}\lfloor K_\alpha, O_\alpha\rfloor$ is an open $\tilde{\sigma}$-neighbourhood. Moreover, each O_α is contained in O_1 and thus property (iii) in Step 1 yields $\mathcal{O}_\sigma \subseteq \lfloor M, O_1\rfloor = \{s \in \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M) \mid s(M) \subseteq O_1\}$. The topology on $\Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M)$ is finer than the compact-open topology, whence $\lfloor M, O_1\rfloor$ is open in $\Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M)$.

Step 3: *Embedding the spaces of complex sections.* With the notation of Lemma A.16 we obtain the topological embeddings

$$\begin{aligned}\Theta_f\colon \Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M) &\to \bigoplus_{\alpha\in A}\operatorname{Hol}(K_\alpha \subseteq M_\alpha^{\mathbb{C}}, E_{\mathbb{C}}),\\ s &\mapsto (I_{\kappa_\alpha^f} \circ \operatorname{res}^M_{K_\alpha}(s))_{\alpha\in A},\\ \Theta_g\colon \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M) &\to \bigoplus_{\alpha\in A}\operatorname{Hol}(K_\alpha \subseteq M_\alpha^{\mathbb{C}}, E_{\mathbb{C}}),\\ s &\mapsto (I_{\kappa_\alpha^g} \circ \operatorname{res}^M_{K_\alpha}(s))_{\alpha\in A}.\end{aligned}$$

The topology on $\operatorname{Hol}(K_\alpha \subseteq M_\alpha, E_{\mathbb{C}})$ is finer than the compact-open topology (Lemma A.15(1) asserts $\operatorname{Hol}(K_\alpha \subseteq M_\alpha, E_{\mathbb{C}}) \cong \Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|K_\alpha)$ and the topology on the latter space has this property). Hence,

$$I_{\kappa_\alpha^g} \circ \operatorname{res}^M_{K_\alpha}(\lfloor K_\alpha, O_\alpha\rfloor) = \lfloor K_\alpha, O_\alpha^2\rfloor \subseteq \operatorname{Hol}(K_\alpha \subseteq M_\alpha, E_{\mathbb{C}})$$

is open. We deduce that $\Theta_g(\mathcal{O}_\sigma)$ is contained in the open neighbourhood $\bigoplus_{\alpha\in A}\lfloor K_\alpha, O_\alpha^2\rfloor \subseteq \bigoplus_{\alpha\in A}\operatorname{Hol}(K_\alpha \subseteq M_\alpha, E_{\mathbb{C}})$.

Step 4: *A holomorphic extension on $\mathcal{O}_\sigma$.* We define the map

$$\phi_*\colon \Gamma^\omega_{\mathbb{C}}((g^*TN)_{\mathbb{C}}|M) \supseteq \lfloor M, O_1\rfloor \to \Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M), \qquad s \mapsto \phi \circ s.$$

Note that ϕ_* makes sense, as ϕ is fibre preserving. Moreover, as $\mathcal{O}_\sigma \subseteq \lfloor M, O_1\rfloor$ is a complex open neighbourhood of $\tilde{\sigma}$, ϕ_* extends $(\tau_f^{-1} \circ \tau_g)_*$ in an open neighbourhood of σ in the complexification.

To see that the map ϕ_* is holomorphic, define maps $f_\alpha\colon \lfloor K_\alpha, O_\alpha^2\rfloor \to \operatorname{Hol}(K_\alpha \subseteq M_\alpha^{\mathbb{C}}, E_{\mathbb{C}})$ via

$$f_\alpha(\gamma) := (\operatorname{pr}_2 \circ \kappa_\alpha^f \circ \phi \circ (\kappa_\alpha^g)^{-1})_*(\operatorname{id}_{K_\alpha}, \gamma).$$

By [13, Proposition 3.3], f_α is a $C^\infty_{\mathbb{C}}$-map and we obtain a map

$$\bigoplus_{\alpha\in A} f_\alpha\colon \bigoplus_{\alpha\in A} \lfloor K_\alpha, O^2_\alpha\rfloor \to \bigoplus_{\alpha\in A}\operatorname{Hol}(K_\alpha M^{\mathbb{C}}_\alpha, E_{\mathbb{C}}).$$

It satisfies $\bigoplus_{\alpha\in A} f_\alpha \circ \Theta_g|_{\mathcal{O}_\sigma} = \Theta_f \circ \phi_*|_{\mathcal{O}_\sigma}$. Since every f_α is $C^\infty_{\mathbb{C}}$, the map $\bigoplus_{\alpha\in A} f_\alpha$ is $C^\infty_{\mathbb{C}}$ by [14, Proposition 4.7]. Recall from Lemma A.16 that Θ_f is a linear topological embedding with closed image. Hence $\phi_*|_{\mathcal{O}_\sigma} = \Theta_f^{-1} \circ \bigoplus_{\alpha\in A} f_\alpha \circ \Theta_g|_{\mathcal{O}_\sigma}$ implies that $\phi_*|_{\mathcal{O}_\sigma}$ is $C^\infty_{\mathbb{C}}$.

We now summarise the results from Steps 1–4. We have seen that the map $(\tau_f^{-1} \circ \tau_g)_*$ extends to a holomorphic map on a neighbourhood of each element in its domain. As real analyticity is a local property, this shows that $(\tau_f^{-1} \circ \tau_g)_*$ is real analytic. Hence (2.1) shows that $\Phi_f \circ \Phi_g^{-1}$ is a real analytic map.

(2) Endow $C^\omega_{\mathbb{R}}(M,N)$ with the identification topology of the atlas constructed in (1). The topological space $C^\omega_{\mathbb{R}}(M,N)$ will be a real analytic locally convex manifold if we can show that the identification topology on $C^\omega_{\mathbb{R}}(M,N)$ is Hausdorff. To see this, it suffices to prove that for all $x \in M$ the point evaluations $\mathrm{ev}_x\colon C^\omega_{\mathbb{R}}(M,N) \to N$, $f \mapsto f(x)$, are continuous with respect to the identification topology. By definition we have to prove that for all $f \in C^\omega_{\mathbb{R}}(M,N)$ and $x \in M$ the composition $\mathrm{ev}_x \circ \Phi_f^{-1}\colon \Phi_f(U_f) \to N$ is continuous. One easily computes that $\mathrm{ev}_x \circ \Phi_f^{-1}(\sigma) = \Sigma \circ \pi^*_{TN}(\sigma(x))$ for every section $\sigma \in \Phi_f(U_f)$. Hence, it suffices to observe that the point evaluations on $\Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M)$ are continuous (since $\Gamma^\omega_{\mathbb{R}}(f^*TN)$ is topologised as a subspace). By definition $\Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M)$ is the inductive limit of the spaces $\Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|W)$ on which the point evaluations are continuous (see Definition A.13). As point evaluations are linear, an inductive limit argument shows that ev_x is continuous on the limit. Consequently, $C^\omega_{\mathbb{R}}(M,N)$ is a Hausdorff topological space.

(3) Let $\Sigma^\#$ be another real analytic local addition on N. Construct new charts $\Phi_f^\#$ and maps $\tau_f^\#$ as in 2.5 with respect to Σ. We have only used the fact that (π_{TN}, Σ) restricts to a diffeomorphism on an open neighbourhood Ω of the zero section in TN. By definition of a local addition the same holds for $\Sigma^\#$. Thus we can define $\tau_f^\# \circ \tau_g^{-1}$ on an open subset $\Omega^\#_{f,g} \subseteq g^*(\Omega \cap \Omega^\#)$ (depending on both Σ and $\Sigma^\#$). Furthermore, we obtain an identity analogous to (2.1) (involving now $\Phi_f^\#$ and $\tau_f^\#$). Note that the arguments and constructions in Steps 1, 3 and 4 of (1) do not depend on Σ, and thus can be copied verbatim. Finally, it is easy to see that also the construction in Step 2 of (1) can be easily adapted (as only the choices of compact and open sets have to be changed). Hence analogous arguments to those in (1) show that the charts constructed with respect to different local additions are compatible, i.e. the resulting changes of charts are real analytic. We

conclude that the construction does not depend on the choice of local addition. ■

Recall that in [22, §10] a smooth manifold structure for $C^\infty_{\mathbb{R}}(M,N)$ has been constructed. Moreover, the construction in [22, Theorem 10.4] carries over to infinite-dimensional manifolds N which admit a local addition ([3]). The manifold $C^\infty_{\mathbb{R}}(M,N)$ is modelled on spaces of smooth sections $\Gamma^\infty(f^*TN)$ for $f \in C^\infty_{\mathbb{R}}(M,N)$ with canonical charts defined analogously to the charts constructed in 2.5. We remark that the notion of differentiability adopted in [22] coincides with the one adopted in this paper. Hence, the set $C^\infty_{\mathbb{R}}(M,N)$ is a smooth manifold and we can copy the arguments in [18, p. 48f.] verbatim to obtain the following results:

2.7 ([18, Theorem 8.3]). *Let M and N be real analytic finite-dimensional manifolds, with M compact. Then the smooth manifold $C^\infty_{\mathbb{R}}(M,N)$ with the structure from* [22, 10.4] *is a real analytic manifold. In fact a real analytic atlas is given by*

$$\Phi_f^\infty\colon C^\infty_{\mathbb{R}}(M,N) \supseteq U_f \to \Gamma^\infty(f^*TN), \qquad g \mapsto (\mathrm{id}_M, (\pi_{TN}, \Sigma)^{-1}(f,g)),$$

where f runs through $C^\omega_{\mathbb{R}}(M,N)$ and U_f is defined as in 2.5 *with respect to the* (*real analytic*) *local addition Σ. As M is compact, the model spaces $\Gamma^\infty(f^*TN)$ are endowed with the compact-open $C^\infty_{\mathbb{R}}$-topology.*

Proposition 2.8. *Let M and N be real analytic manifolds and assume that M is compact. Consider the canonical inclusion $\iota\colon C^\omega_{\mathbb{R}}(M,N) \to C^\infty_{\mathbb{R}}(M,N)$.*

1. *If N is finite-dimensional, then ι is a real analytic map with respect to the real analytic manifold structures of Theorems* 2.6 *and* 2.7.
2. *If N is infinite-dimensional and admits a local addition, then ι is of class $C^\infty_{\mathbb{R}}$ with respect to the smooth structures of Theorem* 2.6 *and* [22, §10] *on $C^\infty(M,N)$.*

Proof. Computing in canonical charts, we see that for $f \in C^\omega_{\mathbb{R}}(M,N)$ the canonical inclusion maps $\mathrm{dom}\,\Phi_f$ into $\mathrm{dom}\,\Phi_f^\infty$. Thus it suffices to consider the local representative $\Phi_f^\infty \circ \iota \circ \Phi_f^{-1}$ which coincides with the restriction of the canonical inclusion $\Lambda\colon \Gamma^\omega_{\mathbb{R}}(f^*TN) \to \Gamma^\infty(f^*TN)$. As Λ is linear, it is sufficient to prove that Λ is continuous. Denote the typical fibre of the vector bundle f^*TN by E. By definition of the compact-open $C^\infty_{\mathbb{R}}$-topology, the topology on $\Gamma^\infty(f^*TN)$ is initial with respect to the linear mappings

$$\theta_\psi^\infty\colon \Gamma^\infty(f^*TN) \to C^\infty_{\mathbb{R}}(M_\psi, E), \qquad X \mapsto \mathrm{pr}_2 \circ \psi \circ X|_{M_\psi},$$

where ψ runs through all (real analytic) bundle trivializations.

([3]) In fact, one has only to replace the Ω-Lemma in the proof of [22, Theorem 10.4] by Glöckner's Ω-Lemma [10, Theorem F.23].

Fixing a trivialisation ψ we will show that $\theta^\infty_\psi \circ \Lambda$ is continuous. In the following we use standard multi-index notation for partial derivatives.

Recall from Definition A.6 that a typical zero-neighbourhood in the space $C^\infty_{\mathbb{R}}(M_\psi, E)$ is of the form

$$\Omega_{\kappa,K,n,p} := \left\{ g \in C^\infty_{\mathbb{R}}(M_\psi, E) \;\middle|\; \begin{array}{l} \sup_{|\alpha|\leq n} P_{\alpha,\kappa,K,p}(g) \\ = \sup_{|\alpha|\leq n} \sup_{x\in K} p\left(\partial^\alpha (g\circ \kappa^{-1})(x)\right) < 1 \end{array} \right\},$$

where $\kappa\colon M_\psi \supseteq U_\kappa \to V_\kappa \subseteq \mathbb{R}^k$ is a real analytic manifold chart, $K \subseteq V_\kappa$ is compact, $n \in \mathbb{N}_0$, and p is a continuous seminorm on E. Fix κ, $K \subseteq V_\kappa$ compact, $n \in \mathbb{N}_0$ and a continuous seminorm p. We construct a zero-neighbourhood in $\Gamma^\omega_{\mathbb{R}}(f^*TN)$ which is mapped by $\theta^\infty_\psi \circ \Lambda$ to $\Omega_{\kappa,K,n,p}$.

The topology on $\Gamma^\omega_{\mathbb{R}}(f^*TN)$ is the subspace topology induced by the complex bundle $\Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M)$. Since ψ is a real analytic bundle trivialisation, the holomorphic extension $\psi_{\mathbb{C}}$ of ψ yields a bundle trivialisation for $(f^*TN)_{\mathbb{C}}$ (cf. [7, Proposition 3.5]). Analogously, we can extend κ to a holomorphic chart $\kappa_{\mathbb{C}}$ of the complexification $M^{\mathbb{C}}_\psi$. Now by a combination of Lemma A.15(2) and (1) we obtain a continuous linear map

$$H\colon \Gamma^\omega_{\mathbb{C}}((f^*TN)_{\mathbb{C}}|M) \to \operatorname{Hol}(K \subseteq M_{\psi_{\mathbb{C}}}, E_{\mathbb{C}}), \qquad X \mapsto \operatorname{pr}_2 \circ \psi_{\mathbb{C}} \circ \operatorname{res}^M_K(X).$$

The space $\operatorname{Hol}(K \subseteq M_{\psi_{\mathbb{C}}}, E_{\mathbb{C}})$ is the inductive limit of the spaces $\operatorname{Hol}(U_k, E_{\mathbb{C}})$ (where $(U_k)_{k\in\mathbb{N}}$ is a fundamental sequence of neighbourhoods of K in $M_{\psi_{\mathbb{C}}}$). Each of the steps carries the compact-open topology, which coincides with the compact-open $C^\infty_{\mathbb{C}}$-topology for holomorphic maps by Lemma A.7. Hence by definition of the locally convex inductive limit, the set

$$O_{\kappa_{\mathbb{C}},K,n,p} := \left\{ g \in \operatorname{Hol}(K \subseteq M_{\psi_{\mathbb{C}}}, E_{\mathbb{C}}) \;\middle|\; \sup_{|\alpha|\leq n} P_{\alpha,\kappa_{\mathbb{C}},K,p_{\mathbb{C}}}(g) < 1 \right\}$$

is an open zero-neighbourhood where we choose a seminorm $p_{\mathbb{C}}$ on $E_{\mathbb{C}}$ which induces the given seminorm p on the real subspace E (this is possible by [4, Section 2]). Note that the partial derivatives in the definition of $O_{K,n,p}$ are taken with respect to complex variables. Since $\psi_{\mathbb{C}}|_{\operatorname{dom}\psi} = \psi$ and $\kappa_{\mathbb{C}}|_{M_\psi \cap U_{\kappa_{\mathbb{C}}}} = \kappa$, the zero-neighbourhood $H^{-1}(O_{\kappa_{\mathbb{C}},K,n,p}) \cap \Gamma^\omega_{\mathbb{R}}(f^*TN)$ is mapped by $\theta^\infty_\psi \circ \Lambda$ into $\Omega_{\kappa,K,n,1}$. ∎

Keller $C^\infty_{\mathbb{K}}$-maps coincide on Silva spaces with smooth mappings in the convenient sense (cf. [18, 1.3]). Thus we can almost ([4]) copy the proof for the following result.

([4]) The authors have not been able to follow the argument given in [19, Theorem 43.3] establishing that the identification topology on $C^\omega_{\mathbb{R}}(M, M)$ is finer than the compact-open $C^\infty_{\mathbb{R}}$-topology. Hence we chose to give another argument for the fact that $\operatorname{Diff}^\omega(M)$ is an open subset in $C^\omega_{\mathbb{R}}(M, N)$.

Proposition 2.9 ([19, Theorem 43.3]). *For a compact real analytic manifold M the group* $\mathrm{Diff}^\omega(M)$ *of all real analytic diffeomorphisms of M is an open submanifold of* $C^\omega_{\mathbb{R}}(M,M)$.

Composition and inversion in this group are smooth, whence $\mathrm{Diff}^\omega(M)$ *is a smooth Lie group modelled on the Silva space $\mathfrak{X}^\omega(M)$. Its Lie algebra is the space $\mathfrak{X}^\omega(M)$ of all real analytic vector fields on M, equipped with the negative of the Lie bracket of vector fields. The associated exponential mapping* $\exp\colon \mathfrak{X}^\omega(M) \to \mathrm{Diff}^\omega(M)$ *is the time* 1 *flow mapping, and it is smooth.*

Proof. Since M is compact and thus finite-dimensional, Proposition 2.8 shows that the canonical inclusion $\iota\colon C^\omega_{\mathbb{R}}(M,M) \to C^\infty_{\mathbb{R}}(M,M)$ is real analytic. Hence ι is continuous and we have $\mathrm{Diff}^\omega(M) = \iota^{-1}(\mathrm{Diff}^\infty(M))$. Now by [22, Theorem 11.11], $\mathrm{Diff}^\infty(M)$ is an open submanifold of $C^\infty_{\mathbb{R}}(M,M)$, whence $\mathrm{Diff}^\omega(M)$ is open in $C^\omega_{\mathbb{R}}(M,M)$. The rest of the proof can be copied verbatim from [19, Theorem 43.4]. ■

Remark 2.10. (1) As shown in [19, Theorem 43.3] the group $\mathrm{Diff}^\omega(M)$ is even a real analytic Lie group in the convenient sense. Beyond the Fréchet setting (e.g. for Silva spaces) our notion of analyticity is not equivalent to the notion of convenient real analyticity. In Section 4 it will turn out that the Lie group $\mathrm{Diff}^\omega(M)$ is not real analytic in our sense.

(2) The topology constructed on $\mathfrak{X}^\omega(M) = \Gamma^\omega_{\mathbb{R}}(\mathrm{id}_M^* TM)$ coincides with the "Van Howe" topology constructed in [21, §4] (this follows from [21, Lemma 4.1]). Thus the Lie group $\mathrm{Diff}^\omega(M)$ is modelled on the same topological vector space as the Lie group constructed in [21]. Although the construction in [21] of the Lie group structure is flawed, the Lie group structure obtained in Proposition 2.9 is precisely the one described in [21].

3. Regularity of the group of analytic diffeomorphisms. In this section we prove Theorem B, that $\mathrm{Diff}^\omega(M)$ is a regular Lie group. This result is new in our setting (see [19, Theorem 43.4] for the corresponding result in the convenient setting) and the proof is based on H. Glöckner's regularity theorem for Silva Lie groups.

Throughout this section, let M be a fixed compact $C^\omega_{\mathbb{R}}$-manifold. Let us first study the differential equation for C^k-regularity of $\mathrm{Diff}^\omega(M)$.

3.1 (The differential equation of C^k-regularity for $\mathrm{Diff}^\omega(M)$)**.** Fix a curve $\gamma \in C^k([0,1], \mathfrak{X}^\omega(M))$. We call a C^{k+1}-curve $\eta\colon [0,1] \to \mathrm{Diff}^\omega(M)$ which solves the differential equation

$$\begin{cases} \eta'(t) = \gamma(t).\eta(t) = T_1\rho_{\eta(t)}(\gamma(t)), \\ \eta(0) = \mathrm{id}_M, \end{cases} \tag{3.1}$$

the *evolution* of γ. Recall that the right translation $\rho_{\eta(t)}$ by the element $\eta(t)$

in the group $\mathrm{Diff}^\omega(M)$ is precomposition with $\eta(t)$. Identifying the tangent spaces $T_f \mathrm{Diff}^\omega(M) = \Gamma^\omega(f^*TM)$ we derive that $T_1\rho_f(X) = X \circ f$. This can be proved exactly as [22, Corollary 10.14]. In loc. cit. only spaces of smooth mappings are considered. Since the manifolds are modelled on Silva spaces, one can alternatively invoke [19, Corollary 42.18] from the convenient setting. Hence the differential equation (3.1) becomes

$$\begin{cases} \eta'(t) = \gamma(t, \eta(t)), \\ \eta(0) = \mathrm{id}_M. \end{cases} \tag{3.2}$$

We can view (3.2) as a differential equation whose right hand side is given by the time dependent vector field γ. Since M is compact, it follows from the theory of ordinary differential equations that the flow $Fl_0^\gamma\colon [0,1] \times M \to M$ of (3.2) is defined on $[0,1] \times M$. Furthermore, for each fixed $t \in [0,1]$ the map $Fl_0^\gamma(t,\cdot)$ is in $\mathrm{Diff}^\omega(M)$ and $Fl_0^\gamma(0,\cdot) = \mathrm{id}_M$ ([5]). We conclude that the evolution of γ is $(Fl_0^\gamma)^\vee\colon [0,1] \to \mathrm{Diff}^\omega(M)$ with $(Fl_0^\gamma)^\vee(t) := Fl_0^\gamma(t,\cdot)$.

Notice that at this point it is not clear whether $(Fl_0^\gamma)^\vee\colon [0,1] \to \mathrm{Diff}^\omega(M)$ is C^{k+1}. In particular, $\mathrm{Diff}^\omega(M)$ will be a C^k-regular Lie group if we can show that $(Fl_0^\gamma)^\vee$ is a C^{k+1}-curve and the evolution

$$\mathrm{evol}\colon C^k([0,1], \mathfrak{X}^\omega(M)) \to \mathrm{Diff}^\omega(M), \qquad X \mapsto (Fl_0^X)^\vee(1) = Fl_0^X(1,\cdot),$$

is smooth. Let us prepare the proof of these results with some auxiliary considerations.

3.2. As M is a finite-dimensional manifold, it admits a complexification $M_\mathbb{C}$ which can be chosen to be a Stein manifold (see [16, §3]). The Stein manifold $M_\mathbb{C}$ can be embedded as a closed complex submanifold of a finite-dimensional complex space $\mathbb{C}^N$ by [17, VII C, Theorem 13]. By [23, Corollary 1], this complex submanifold admits an open neighbourhood $W \subseteq \mathbb{C}^N$ and an open holomorphic retraction $q\colon W \to M_\mathbb{C}$. In the following, we will identify the tangent bundles TM and $TM_\mathbb{C}$ as submanifolds of $T\mathbb{C}^N = \mathbb{C}^N \times \mathbb{C}^N$.

3.3. By Remark 2.3(1), the real manifold M admits a real-analytic local addition $\Sigma\colon \Omega_\Sigma \to M$, whence $(\pi_{TM}, \Sigma)\colon \Omega_\Sigma \to M \times M$ is a $C^\omega_\mathbb{R}$-diffeomorphism onto its open image $(\pi_{TM}, \Sigma)(\Omega_\Sigma) \subseteq M \times M$.

Extend $(\pi_{TM}, \Sigma)^{-1}$ to a holomorphic map $h\colon V_\Sigma \to \mathbb{C}^N \times \mathbb{C}^N$ on an open neighbourhood $V_\Sigma \subseteq W \times W$ of the diagonal Δ_M. Summing up, we obtain a commutative diagram

([5]) See [20, IV, §2]. These results will also later be obtained as a consequence of the proof of regularity.

$$\begin{array}{ccccccc}
\varDelta_M & \xrightarrow{\subseteq} & (\pi_{TM},\varSigma)(\varOmega_\varSigma) & \xrightarrow[\cong]{(\pi_{TM},\varSigma)^{-1}} & \varOmega_\varSigma & \xrightarrow{\subseteq} & TM \\
 & & {\scriptstyle\subseteq}\big\downarrow & & {\scriptstyle\subseteq}\big\downarrow & & \\
 & & V_\varSigma & \xrightarrow{\qquad h \qquad} & \mathbb{C}^N \times \mathbb{C}^N & &
\end{array}$$

The set $V_\varSigma \subseteq W \times W$ is an open neighbourhood of the compact set $\varDelta_M$. Hence there is an $r > 0$ such that $\varDelta_M + (B_r^{\mathbb{C}^N}(0) \times B_r^{\mathbb{C}^N}(0)) \subseteq V_\varSigma$.

Here, $B_r^{\mathbb{C}^N}(0)$ denotes the open r-ball in $\mathbb{C}^N$ with respect to some norm. By replacing this norm by a multiple, we may assume in the following that $r = 1$, i.e.

$$\varDelta_M + (B_1^{\mathbb{C}^N}(0) \times B_1^{\mathbb{C}^N}(0)) \subseteq V_\varSigma.$$

In particular, this implies that $M + B_1^{\mathbb{C}^N}(0) \subseteq W$.

3.4. Consider the following system of fundamental neighbourhoods of the compact set M in the space $W \subseteq \mathbb{C}^N$:

$$U_n := M + B_{1/n}^{\mathbb{C}^N}(0) \subseteq W \subseteq \mathbb{C}^N,$$

together with the corresponding complex Banach spaces

$$F_n := \big(\mathrm{Hol_b}(U_n, \mathbb{C}^N), \|\cdot\|_\infty\big).$$

Since $q\colon W \to M_\mathbb{C}$ is open, the sets $q(U_n) \subseteq M_\mathbb{C}$ form a system of fundamental neighbourhoods of M in $M_\mathbb{C}$. For $n \in \mathbb{N}$, we consider the real Banach space

$$E_n := (\{f \in \mathrm{Hol_b}(q(U_n), \mathbb{C}^N) \mid f(a) \in T_aM \text{ for all } a \in M\}, \|\cdot\|_\infty)$$

which embeds via $\theta_n\colon E_n \to F_n,\ g \mapsto g \circ q$, isometrically into the complex Banach space F_n. We remark that the maps θ_n induce isometric embeddings

$$C([0,1],\theta_n)\colon C([0,1],E_n) \to C([0,1],F_n), \qquad \gamma \mapsto \theta_n \circ \gamma.$$

Furthermore, there is a natural injective linear map $E_n \to E_{n+1} : f \mapsto f|_{q(U_{n+1})}$, which is a compact operator by [18, Theorem 3.4]. Hence, the direct limit of the sequence $(E_n)_{n\in\mathbb{N}}$ is a Silva space which we identify with the Silva space $\mathfrak{X}^\omega(M)$. Using this identification, the limit maps of the inductive limit are

$$j_n\colon E_n \to \varinjlim E_k \cong \mathfrak{X}^\omega(M), \qquad f \mapsto (\mathrm{id}_M, f|_M).$$

To prove the regularity of $\mathrm{Diff}^\omega(M)$ we exploit Glöckner's theorem on regularity of Silva Lie groups [15, Theorem 15.5] whose key point we repeat in the following lemma.

LEMMA 3.5. *The Lie group* $\mathrm{Diff}^\omega(M)$ *is* C^1*-regular if*

$$(\star)\begin{cases} \text{for each } n \in \mathbb{N} \text{ there is a zero-neighbourhood } P_n \subseteq C([0,1],E_n) \text{ such} \\ \text{that the map } (Fl_0^\gamma)^\vee\colon [0,1] \to \mathrm{Diff}^\omega(M),\ t \mapsto Fl_0^\gamma(t,\cdot), \text{ is } C^1 \text{ for} \\ \gamma \in P_n \text{ and } \mathrm{evol}\colon P_n \to \mathrm{Diff}^\omega(M),\ \gamma \mapsto Fl_0^\gamma(1,\cdot), \text{ is continuous.} \end{cases}$$

Proof. Proposition 2.8 shows that the inclusion $\mathrm{Diff}^{\omega}(M) \to \mathrm{Diff}(M)$ is a morphism of Lie groups which separates the points. Now $\mathrm{Diff}(M)$ is a C^0-regular Lie group by [15, Corollary 13.7(a)] and $\mathrm{Diff}^{\omega}(M)$ is modelled on the Silva space $\mathfrak{X}^{\omega}(M) = \varinjlim E_n$. Hence [15, Theorem 15.5] implies that $\mathrm{Diff}^{\omega}(M)$ will be C^1-regular if we can show that for each $n \in \mathbb{N}$ there is an open zero-neighbourhood $P_n \subseteq C([0,1], E_n)$ with the following property: Each element in P_n admits a C^1-evolution and evol is continuous. However, in 3.1 we have seen that for a (time dependent) real analytic vector field the flow solves the differential equation associated to regularity. Thus if the flow induces a C^1-map $(Fl_0^{\gamma})^{\vee}$, this mapping is the evolution of $\gamma \in P_n$. We conclude that $\mathrm{Diff}^{\omega}(M)$ is C^1-regular if condition ($\star$) is satisfied. ∎

Before we can establish ($\star$) from Lemma 3.5, we recall the definition of $C^{r,s}$-mappings from [1].

DEFINITION 3.6. Let E_1, E_2 and F be locally convex spaces, U and V open subsets of E_1 and E_2, respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$.

(1) A mapping $f \colon U \times V \to F$ is called a *$C^{r,s}$-map* if for all $i, j \in \mathbb{N}_0$ such that $i \le r$ and $j \le s$, the iterated directional derivative

$$d^{(i,j)} f(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) := \big(D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f\big)(x, y)$$

exists for all $x \in U$, $y \in V$, $w_1, \ldots, w_i \in E_1$, $v_1, \ldots, v_j \in E_2$ and yields continuous maps

$$d^{(i,j)} f \colon U \times V \times E_1^i \times E_2^j \to F.$$

(2) In (1) all spaces E_1, E_2 and F were assumed to be modelled over the same $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By [1, Remark 4.10] we can instead assume that E_1 is a locally convex space over $\mathbb{R}$ and E_2, F are locally convex spaces over $\mathbb{C}$. Then a map $f \colon U \to F$ is a *$C^{r,s}_{\mathbb{R},\mathbb{C}}$-map* if the iterated differentials $d^{(i,j)} f$ (as in (1)) exist for all $0 \le i \le r$ and $0 \le j \le s$ and are continuous. Here the derivatives in the first component are taken with respect to $\mathbb{R}$, and in the second component with respect to $\mathbb{C}$.

One can extend the definition of $C^{r,s}$- and $C^{r,s}_{\mathbb{R},\mathbb{C}}$-maps to obtain $C^{r,s}$- or $C^{r,s}_{\mathbb{R},\mathbb{C}}$-mappings on closed intervals (see [1, Definition 3.2] for the general case of locally convex domains with dense interior). For further results and details on the calculus of $C^{r,s}$-maps we refer to [1].

With the help of the calculus of $C^{r,s}_{\mathbb{R},\mathbb{C}}$-mappings we can now establish condition ($\star$) from Lemma 3.5. To this end, fix $n \in \mathbb{N}$ and set

$$P_n := B_{1/(4n)}^{C([0,1],E_n)}(0), \qquad Q_n := B_{1/(4n)}^{C([0,1],F_n)}(0).$$

3.7. Consider the map

$$f\colon [0,1]\times U_n\times Q_n\to\mathbb{C}^N,\qquad (t,x,\gamma)\mapsto\gamma(t)(x),$$

where U_n is the fundamental neighbourhood from 3.4. We can rewrite f as $f(t,x,\gamma)=\mathrm{ev}(\mathrm{ev}_1(\gamma,t),x)$ where $\mathrm{ev}_1\colon C([0,1],F_n)\times[0,1]\to F_n$ and $\mathrm{ev}\colon \mathrm{Hol}_{\mathrm{b}}(U_n,\mathbb{C}^N)\times U_n\to\mathbb{C}^N$ are the canonical evaluation maps. Note that the inclusion $\mathrm{Hol}_{\mathrm{b}}(U_n,\mathbb{C}^N)\to\mathrm{Hol}(U_n,\mathbb{C}^N)$ is continuous linear, hence holomorphic. We conclude from [1, Proposition 3.20] that ev_1 is a $C^{0,\infty}_{\mathbb{R},\mathbb{C}}$-map and ev is holomorphic. Thus the chain rule [1, Lemma 3.17] implies that f is of class $C^{0,\infty}_{\mathbb{R},\mathbb{C}}$.

Let us now consider the initial value problem for fixed $x_0\in U_n$,

$$\begin{cases} x'(t)=f(t,x(t),\gamma),\\ x(0)=x_0,\end{cases}\tag{3.3}$$

whose right hand side is given by f from 3.7.

Lemma 3.8. *Let $\gamma\in Q_n$. For every $x_0\in U_{4n}$ the initial value problem* (3.3) *admits a unique maximal solution defined on the whole interval* $[0,1]$. *This solution takes its values in the set U_{2n}.*

Proof. The map f from 3.7 is of class $C^{0,\infty}_{\mathbb{R},\mathbb{C}}$ with respect to $[0,1]\times(U_n\times Q_n)$. Thus (3.3) admits a unique maximal solution $\varphi_{0,x_0,\gamma}$ by [1, Theorem 5.6]. We will now show that this solution takes its values in the compact set $\overline{U_{2n}}$ and hence is globally defined, i.e. defined on $[0,1]$ (cf. [20, IV, §2, Theorem 2.3]). To this end, let t be in the domain of $\varphi_{0,x_0,\gamma}$. Then

$$\begin{aligned}\|\varphi_{0,x_0,\gamma}(t)-x_0\| &= \|\varphi_{0,x_0,\gamma}(t)-\varphi_{0,x_0,\gamma}(0)\| = \Big\|\int_0^t \varphi_{0,x_0,\gamma}'(s)\,ds\Big\|\\ &= \Big\|\int_0^t \gamma(s)(\varphi_{0,x_0,\gamma}(s))\,ds\Big\| \le \int_0^t \underbrace{\|\gamma(s)\|_\infty}_{<1/(4n)}\,ds < \frac{1}{4n}.\end{aligned}$$

Since $x_0\in U_{4n}$, the triangle inequality implies $\varphi_{0,x_0,\gamma}(t)\in U_{2n}\subseteq\overline{U_{2n}}$. ∎

Proposition 3.9. *For a time-dependent vector field $\gamma\in P_n\subseteq C([0,1],E_n)$ the flow induces a C^1-map $(Fl_0^\gamma)^\vee\colon[0,1]\to\mathrm{Diff}^\omega(M)$, $t\mapsto Fl_0^\gamma(t,\cdot)$. Hence,* $\mathrm{Evol}(\gamma)=(Fl_0^\gamma)^\vee$ *for $\gamma\in P_n$*

Proof. Fix $\gamma\in P_n$. We use the holomorphic retraction $q\colon W\to M_{\mathbb{C}}$ (cf. 3.2) to lift γ to an element $\tilde\gamma:=C([0,1],\theta_n)(\gamma)\in Q_n$. Note that by definition this map is given by $\tilde\gamma(t)(x)=\gamma(t)(q(x))$ for all $(t,x)\in[0,1]\times U_n$.

Then the flow $Fl_0^f(\cdot,\tilde\gamma)\colon[0,1]\times U_{4n}\to U_{2n}$ of (3.3) exists for $\tilde\gamma$ by Lemma 3.8 and yields a $C^{1,\infty}_{\mathbb{R},\mathbb{C}}$-map by [1, Proposition 5.9]. The definition of the mapping f shows that on $M \subseteq U_{4n}$ we have

$$Fl_0^f(\cdot,\tilde{\gamma})|_{[0,1]\times M} = Fl_0^\gamma\colon [0,1]\times M \to M. \tag{3.4}$$

Recall from 3.3 and 3.4 that the diffeomorphism $(\pi_{TM},\Sigma)^{-1}$ admits a holomorphic extension $h\colon V_\Sigma \to \mathbb{C}^N \times \mathbb{C}^N$ such that $U_{2n}\times U_{2n} \subseteq V_\Sigma$. Now the chain rules [1, Lemmas 3.17 and 3.18] imply that

$$F_\gamma\colon [0,1]\times U_{4n} \to \mathbb{C}^N, \quad (t,x)\mapsto \mathrm{pr}_2\big(x, \mathrm{pr}_2\circ h \circ (x, Fl_0^f(t,x,\tilde{\gamma}))\big),$$

is a mapping of class $C^{1,\infty}_{\mathbb{R},\mathbb{C}}$. Moreover, since $h|_{M\times M} = (\pi_{TM},\Sigma)^{-1}$ this map satisfies

$$F_\gamma([0,1]\times\{a\}) \subseteq T_aM \subseteq \mathbb{C}^N \quad \text{for each } a\in M. \tag{3.5}$$

Apply now the exponential law [1, Theorem 3.28] for $C^{1,\infty}_{\mathbb{R},\mathbb{C}}$-mappings on finite-dimensional manifolds to obtain a $C^1_{\mathbb{R}}$-map

$$F_\gamma^\vee\colon [0,1]\to \mathrm{Hol}(U_{4n},\mathbb{C}^N), \quad t\mapsto F_\gamma(t,\cdot).$$

As $M \subseteq \overline{U_{6n}} \subseteq U_{4n}$ is compact, restriction induces a continuous linear inclusion $I_n\colon \mathrm{Hol}(U_{4n},\mathbb{C}^N)\to \mathrm{Hol}_{\mathrm{b}}(U_{6n},\mathbb{C}^N)$.

Thus $I_n\circ F_\gamma^\vee\colon [0,1]\to \mathrm{Hol}_{\mathrm{b}}(U_{6n},\mathbb{C}^N) = F_{6n}$ is a $C^1_{\mathbb{R}}$-mapping whose image is contained in the closed real subspace E_{6n} by (3.5). Composing this map with the limit map $j_{6n}\colon E_{6n}\to \mathfrak{X}^\omega(M)$ (see 3.4) we finally obtain a $C^1_{\mathbb{R}}$-mapping

$$H_\gamma\colon [0,1]\to \mathfrak{X}^\omega(M), \quad t\mapsto F_\gamma(t,\cdot)|_M^{TM} \overset{(3.4)}{=} (\pi_{TM},\Sigma)^{-1}\circ(\mathrm{id}_M, Fl_0^\gamma(t,\cdot)).$$

Recall that $\Phi_{\mathrm{id}_M}\colon \mathrm{Diff}^\omega(M)\supseteq U_{\mathrm{id}_M}\to \mathfrak{X}^\omega(M)$, $g\mapsto (\pi_{TM},\Sigma)^{-1}\circ(\mathrm{id}_M,g)$, is a chart for $\mathrm{Diff}^\omega(M)$. By construction H_γ is contained in the image of Φ_{id_M} and thus $\Phi_{\mathrm{id}_M}^{-1}\circ H_\gamma = (Fl_0^\gamma)^\vee$ is a $C^1_{\mathbb{R}}$-map. ∎

By Proposition 3.9 we know that for a fixed curve $\gamma\in Q_n$ the evolution $\mathrm{Evol}(\gamma)$ exists and is given by $(Fl_0^\gamma)^\vee$. To apply Lemma 3.5, it remains to show that the endpoint of $\mathrm{Evol}(\gamma)(1)$ depends continuously on γ.

Proposition 3.10. *The map* $\mathrm{evol}\colon P_n\to \mathrm{Diff}^\omega(M)$, $\gamma\to (Fl_0^\gamma)^\vee(1)$, *is continuous.*

Proof. By Proposition 3.9 the following map makes sense:

$$H\colon U_{4n}\times Q_n\to\mathbb{C}^N, \quad (a,\gamma)\mapsto \mathrm{pr}_2(h(a, Fl_0^f(1,a,\gamma))).$$

We deduce from the proof of Proposition 3.9 that $\mathrm{evol}\colon P_n\to\mathrm{Diff}^\omega(M)$ can be written as $\mathrm{evol} = (\Phi_{\mathrm{id}_M})^{-1}\circ j_{6n}\circ \Psi_P$ where j_{6n} is the limit map (see 3.4) and

$$\Psi_P\colon P_n\to E_{6n}, \quad \gamma\mapsto \big(a\mapsto H(a, C([0,1],\theta_n)(\gamma))\big).$$

Hence, continuity of evol follows as soon as we are able to show that $\Psi_P\colon P_n \to E_{6n}$ is continuous. To show this, consider the commutative diagram

$$\begin{array}{ccccc} C([0,1],E_n) & \xleftarrow{\;\;\supseteq\;\;} & P_n & \xrightarrow{\Psi_P} & E_{6n} \\ {\scriptstyle C([0,1],\theta_n)}\downarrow & & \downarrow & & \downarrow{\scriptstyle \theta_{6n}} \\ C([0,1],F_n) & \xleftarrow{\;\;\supseteq\;\;} & Q_n & \xrightarrow{\Psi_Q} & F_{6n} \end{array}$$

with $\Psi_Q\colon Q_n \to F_{6n},\ \gamma \mapsto (a \mapsto H(a,\gamma))$. As E_{6n} is isometrically embedded in F_{6n}, continuity of Ψ_P will follow from continuity of Ψ_Q. Therefore, we will show that for fixed $\gamma_0 \in Q_n$ the map Ψ_Q is continuous at γ_0.

By [1, Proposition 5.9] the flow $Fl_0^f(1,\cdot)\colon U_{4n} \times Q_n \to \mathbb{C}^N$ is $C^\infty_{\mathbb{C}}$, and hence the map $H\colon U_{4n} \times Q_n \to \mathbb{C}^N$ defined above is $C^\infty_{\mathbb{C}}$ as well. It is well known (see e.g. [2, Proposition 6.3]) that this implies that H is locally Lipschitz. We apply [2, Proposition 6.4] to the compact sets $\overline{U_{6n}} \subseteq U_{4n}$ and $\{\gamma_0\} \subseteq Q_n$ to obtain an open γ_0-neighbourhood W_{γ_0} such that $H|_{\overline{U_{6n}}\times W_{\gamma_0}}\colon \overline{U_{6n}} \times W_{\gamma_0} \to \mathbb{C}^N$ is uniformly Lipschitz continuous with respect to $\gamma \in W_{\gamma_0}$, i.e. there is $L_{\gamma_0} > 0$ such that $W_{\gamma_0} \to \mathbb{C}^N$, $\gamma \mapsto H(a,\gamma)$, is L_{γ_0}-Lipschitz continuous for each $a \in \overline{U_{6n}}$. We conclude that the F_{6n}-valued map

$$\Psi_Q|_{W_{\gamma_0}}\colon W_{\gamma_0} \to F_{6n}, \qquad \gamma \mapsto (a \mapsto H(a,\gamma)),$$

is L_{γ_0}-Lipschitz continuous. This finishes the proof. ∎

We can now summarise the results of this section as follows.

THEOREM 3.11. *The Lie group* $\mathrm{Diff}^\omega(M)$ *is* C^1*-regular.*

Proof. Propositions 3.9 and 3.10 imply that condition $(\star)$ of Lemma 3.5 is satisfied. Thus the assertion follows from that lemma. ∎

4. The group of real analytic diffeomorphisms on the circle is not real analytic. The aim of this section is to prove Theorem C. In particular, this implies that $\mathrm{Diff}^\omega(M)$ is not in general a real analytic Lie group. To this end consider the unit circle $\mathbb{S}^1$ in $\mathbb{C} \cong \mathbb{R}^2$ with its canonical real analytic manifold structure. We begin with preparatory considerations concerning $C^\omega_{\mathbb{R}}(\mathbb{S}^1,\mathbb{S}^1)$.

4.1 (Real analytic local addition on $\mathbb{S}^1$)**.** The manifold $\mathbb{S}^1$ carries the structure of a real analytic (one-dimensional) Lie group, and hence its tangent bundle is trivial via the canonical isomorphism of real analytic vector bundles $\mathbb{S}^1 \times L(\mathbb{S}^1) \to T\mathbb{S}^1$, $(z,v) \mapsto (z, z\cdot v)$. Since the Lie algebra $L(\mathbb{S}^1) = T_1\mathbb{S}^1 = i\mathbb{R}$ is isomorphic to $\mathbb{R}$, we obtain an isomorphism

$$\psi\colon \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1, \qquad (z,r) \mapsto (z, z\cdot ir).$$

The space $\mathfrak{X}^\omega(\mathbb{S}^1)$ consists of all analytic sections of the tangent bundle $T\mathbb{S}^1$, and $C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})$ can be viewed as the analytic sections of the trivial bundle $\mathbb{S}^1\times\mathbb{R}$. Hence, the spaces of sections are isomorphic as locally convex vector spaces. From now on, we identify $T\mathbb{S}^1$ with $\mathbb{S}^1\times\mathbb{R}$ via ψ.

The set $\Omega := \mathbb{S}^1 \times \,]-\pi,\pi[$ is an open neighbourhood of the zero-section in $T\mathbb{S}^1 \cong \mathbb{S}^1\times\mathbb{R}$. Following Remark 2.3(2), the trivialisation of the tangent bundle yields a canonical real analytic local addition via

$$\Sigma\colon \Omega \to \mathbb{S}^1, \qquad (z,r)\mapsto z\cdot e^{ir}.$$

In fact, the map

$$(\pi_{\mathbb{S}^1},\Sigma)\colon \Omega \to \{(z,w)\in\mathbb{S}^1\times\mathbb{S}^1 \mid z\neq -w\}, \qquad (z,r)\mapsto (z, z\cdot e^{ir}),$$

is an analytic diffeomorphism with inverse

$$(\pi_{\mathbb{S}^1},\Sigma)^{-1}\colon \{(z,w)\in\mathbb{S}^1\times\mathbb{S}^1 \mid z\neq -w\}\to\Omega, \qquad (z,w)\mapsto (z,\arg(w/z)),$$

where arg denotes the principal argument in the interval $]-\pi,\pi[$.

4.2 (The composition map of $C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{S}^1)$)**.** Observe that $\mathrm{id}^*_{\mathbb{S}^1}T\mathbb{S}^1 = T\mathbb{S}^1 \cong \mathbb{S}^1\times\mathbb{R}$. Thus, the canonical chart in 2.5 around $\mathrm{id}_{\mathbb{S}^1}$ is given by

$$\begin{aligned}
&\Phi_{\mathrm{id}_{\mathbb{S}^1}} : U_{\mathrm{id}_{\mathbb{S}^1}} \to V_{\mathrm{id}_{\mathbb{S}^1}} \subseteq C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R}), \qquad \gamma\mapsto (z\mapsto \arg(\gamma(z)/z)),\\
&\Phi^{-1}_{\mathrm{id}_{\mathbb{S}^1}} : V_{\mathrm{id}_{\mathbb{S}^1}} \to U_{\mathrm{id}_{\mathbb{S}^1}}, \qquad \eta\mapsto (z\mapsto z\cdot e^{i\eta(z)}).
\end{aligned}$$

In this chart, the composition map looks like

$$\mu\colon C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})\times C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})\to C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R}), \qquad (\eta_1,\eta_2)\mapsto \eta_1\circ E(\eta_2),$$

where $E(\eta)\colon \mathbb{S}^1\to\mathbb{S}^1,\ z\mapsto z\cdot e^{i\eta(z)}$. We will now show that μ is not real analytic in any open neighbourhood of $(0,0)$.

4.3 (The Silva spaces $\mathrm{Hol}(\mathbb{S}^1\subseteq\mathbb{C}^\times,\mathbb{C})$ and $C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})$)**.** Viewing $\mathbb{S}^1$ as a subset of $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$, we may consider $\mathbb{C}^\times$ to be a complexification of $\mathbb{S}^1$. This allows us to fix a fundamental sequence of neighbourhoods

$$U_n := \{z\in\mathbb{C} \mid e^{-1/n} < |z| < e^{1/n}\}$$

of $\mathbb{S}^1$ in its complexification. By [13, 4.2], the complexification of the locally convex space $C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})$ is the Silva space $\mathrm{Hol}(\mathbb{S}^1\subseteq\mathbb{C}^\times,\mathbb{C})$. Now $\mathrm{Hol}(\mathbb{S}^1\subseteq\mathbb{C}^\times,\mathbb{C})$ is the locally convex direct limit of the following sequence of complex Banach spaces:

$$\mathrm{Hol}(\mathbb{S}^1\subseteq\mathbb{C}^\times,\mathbb{C}) = \bigcup_{n\in\mathbb{N}} \mathrm{Hol}_\mathrm{b}(U_n,\mathbb{C}).$$

To shorten the notation we set $E^b_n := \mathrm{Hol}_\mathrm{b}(U_n,\mathbb{C})$.

Proposition 4.4. *The map $\mu\colon C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})\times C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})\to C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})$ introduced in 4.2 is not real analytic on any neighbourhood of $(0,0)\in C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})\times C^\omega_\mathbb{R}(\mathbb{S}^1,\mathbb{R})$.*

Proof. Assume that μ is real analytic in a neighbourhood of $(0,0)$. Then by definition, there exists an open zero-neighbourhood $\Omega \subseteq \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$ and a holomorphic mapping $\mu_{\mathbb{C}} \colon \Omega \times \Omega \to \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$ such that $\mu_{\mathbb{C}}(\eta_1,\eta_2) = \mu(\eta_1,\eta_2)$ whenever $\eta_1,\eta_2 \in \Omega \cap C^\omega_{\mathbb{R}}(\mathbb{S}^1,\mathbb{R})$.

Since the linear map $E^b_1 \to \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$, $f \mapsto f|_{\mathbb{S}^1}$, is continuous, there is $R > 0$ such that the closed ball $\overline{B}^{E^b_1}_R(0)$ is mapped into the open neighbourhood $\Omega \subseteq \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$.

Using this R, we define the following meromorphic function:

$$f \colon \mathbb{C}^\times \setminus \{e^R, e^{-R}\} \to \mathbb{C}, \qquad z \mapsto \frac{1}{z - e^R} + \frac{1}{1/z - e^R}.$$

By construction, this function has the properties

$$f(\overline{z}) = \overline{f(z)} \quad \text{and} \quad f(1/z) = f(z) \quad \text{for all } z.$$

Combining these two properties, we may conclude that whenever $z \in \mathbb{S}^1$, we have $f(z) \in \mathbb{R}$, since

$$\overline{f(z)} = f(\overline{z}) = f(1/z) = f(z).$$

The function f has poles of order 1 at e^R and e^{-R} (and a removable singularity at 0 which is not important for our discussion) and is holomorphic elsewhere.

As a next step, we fix $n \in \mathbb{N}$ such that $1/n < R$ and find $\delta > 0$ such that the closed ball $\overline{B}^{E^b_n}_\delta(0)$ is mapped into the open neighbourhood $\Omega \subseteq \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$. Since the (relatively compact) open set U_n has a positive distance from all the singularities of f, the function f is bounded on U_n. Hence, $f|_{U_n} \in E^b_n$. We fix a scalar $r > 0$ such that $r \cdot f|_{U_n} \in \overline{B}^{E^b_n}_\delta(0)$.

For each complex z, we denote by $z \cdot \mathbb{1}_{U_1}$ the constant function defined on U_1 taking the value z at each point. For $|z| \leq R$, the function $z \cdot \mathbb{1}_{U_1}$ belongs to $\overline{B}^{E^b_1}_R(0)$ which is mapped into Ω when restricting the domain to $\mathbb{S}^1$.

For each $z \in \overline{B}^{\mathbb{C}}_R(0)$, the pair

$$(r \cdot f|_{U_n}, z \cdot \mathbb{1}_{U_1}) \in \overline{B}^{E^b_n}_\delta(0) \times \overline{B}^{E^b_1}_R(0) \subseteq \Omega \times \Omega$$

lies in the domain of the holomorphic map $\mu_{\mathbb{C}} \colon \Omega \times \Omega \to \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$. This allows us to define

$$h \colon B^{\mathbb{C}}_R(0) \to \mathbb{C}, \qquad z \mapsto \mu_{\mathbb{C}}(r \cdot f|_{U_n}, z \cdot \mathbb{1}_{U_1})(1).$$

As a composition of holomorphic mappings, this function is itself holomorphic on the open disc $B^{\mathbb{C}}_R(0) \subseteq \mathbb{C}$.

Now, let $z \in B^{\mathbb{C}}_R(0) \cap \mathbb{R} = \left]-R, R\right[$. Then, we can evaluate $h(z)$ explicitly:

$$\begin{aligned} h(z) &= \mu_{\mathbb{C}}(r \cdot f|_{U_n}, z \cdot \mathbb{1}_{U_1})(1) = \mu(r \cdot f|_{U_n}, z \cdot \mathbb{1}_{U_1})(1) \\ &= r \cdot f \circ E(z \cdot \mathbb{1}_{U_1})(1) = r \cdot f(1 \cdot e^{iz \cdot \mathbb{1}_{U_1}(1)}) \\ &= r \cdot f(e^{iz}). \end{aligned} \tag{4.1}$$

Note that for $|z| < R$, we have
$$|e^{iz}| = e^{\mathrm{Re}(iz)} = e^{-i\,\mathrm{Im}(z)}$$
and since $\mathrm{Im}(z) \in \,]-R, R[$, we conclude that e^{iz} is not one of the singularities of the function f. This shows that the holomorphic function
$$g\colon B_R^{\mathbb{C}}(0) \to \mathbb{C}, \qquad z \mapsto r \cdot f(e^{iz}),$$
makes sense and coincides with h for all real arguments z by (4.1). By the identity theorem for holomorphic functions, we obtain $h \equiv g$ and thus (4.1) holds for all $z \in B_R^{\mathbb{C}}(0)$. In particular, this allows us to conclude that $h(it) = r \cdot f\left(e^{i(it)}\right)$ for all $t \in \,]0, R[$ and hence
$$\mu_{\mathbb{C}}(r \cdot f|_{U_n}, it \cdot \mathbb{1}_{U_1})(1) = r \cdot f(e^{-t}).$$
Now, we take the limit $t \to R$ on both sides and obtain a contradiction, since the left hand side converges to the well-defined number
$$\mu_{\mathbb{C}}(r \cdot f|_{U_n}, iR \cdot \mathbb{1}_{U_1})(1)$$
while the right hand diverges since e^R is a pole of f. ∎

We can now deduce the content of Theorem C.

THEOREM 4.5. *The group multiplication in* $\mathrm{Diff}^\omega(\mathbb{S}^1)$ *is not real analytic. Thus the Lie group* $(\mathrm{Diff}^\omega(\mathbb{S}^1), \circ)$ *is not a real analytic Lie group in our sense.*

Proof. By 2.9, the group $\mathrm{Diff}^\omega(\mathbb{S}^1)$ is an open neighbourhood of $\mathrm{id}_{\mathbb{S}^1}$ in the manifold $C^\omega_{\mathbb{R}}(\mathbb{S}^1, \mathbb{S}^1)$. Pulling back the group multiplication by the canonical chart 4.3, we obtain the mapping μ from 4.2 on some open neighbourhood of $(0,0)$ in $\mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C}) \times \mathrm{Hol}(\mathbb{S}^1 \subseteq \mathbb{C}^\times, \mathbb{C})$. Hence the assertion follows from Proposition 4.4. ∎

Appendix. Locally convex calculus and the topology of the space of germs of analytic maps. In this appendix we recall several well known facts concerning calculus in locally convex spaces. Moreover, we discuss topologies on spaces (of germs) of analytic mappings. These results are well known, but it is sometimes difficult to extract the results and their proofs from the literature. Hence we repeat the results needed together with their proofs for the reader's convenience.

DEFINITION A.1. Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and E, F locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ open. We say a map $f\colon U \to F$ is a $C^r_{\mathbb{K}}$-*map* if it is continuous and the iterated directional derivatives
$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$$
exist for all $k \in \mathbb{N}_0$ with $k \le r$ and $y_1, \ldots, y_k \in E$ and $x \in U$, and the mappings $d^k f\colon U \times E^k \to F$ so obtained are continuous. If f is $C^\infty_{\mathbb{R}}$, we

say that f is *smooth*. If f is $C^\infty_\mathbb{C}$, we say that f is *holomorphic* or *complex analytic* ([6]) and that f is of class $C^\omega_\mathbb{C}$.

Definition A.2 (Complexification of a locally convex space). Let E be a real locally convex topological vector space. We endow the locally convex product $E_\mathbb{C} := E \times E$ with the scalar multliplication

$$(x+iy).(u,v) := (xu - yv, xv + yu) \quad \text{for } x,y \in \mathbb{R},\ u,v \in E.$$

The complex vector space $E_\mathbb{C}$ is called the *complexification* of E. We identify E with the closed real subspace $E \times \{0\}$ of $E_\mathbb{C}$.

Definition A.3. Let E, F be real locally convex spaces and $f\colon U \to F$ defined on an open subset $U \subseteq E$. We call f *real analytic* (or $C^\omega_\mathbb{R}$) if f extends to a $C^\infty_\mathbb{C}$-map $\tilde{f}\colon \tilde{U} \to F_\mathbb{C}$ on an open neighbourhood $\tilde{U}$ of U in the complexification $E_\mathbb{C}$.

For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, being of class $C^r_\mathbb{K}$ is a local condition, i.e. if $f|_{U_\alpha}$ is $C^r_\mathbb{K}$ for every member of an open cover $(U_\alpha)_\alpha$ of its domain, then f is $C^r_\mathbb{K}$ (see [12, pp. 51–52] for the case of $C^\omega_\mathbb{R}$, the other cases are clear by definition). In addition, the composition of $C^r_\mathbb{K}$-maps (if possible) is again a $C^r_\mathbb{K}$-map (cf. [12, Propositions 2.7 and 2.9]).

A.4 ($C^r_\mathbb{K}$-Manifolds and $C^r_\mathbb{K}$-mappings between them)**.** For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, manifolds modelled on a fixed locally convex space can be defined as usual. The model space of a locally convex manifold, and the manifold as a topological space, will always be assumed to be Hausdorff spaces. However, we will not assume that manifolds are second countable or paracompact. Direct products of locally convex manifolds, tangent spaces and tangent bundles as well as $C^r_\mathbb{K}$-maps between manifolds may be defined as in the finite-dimensional setting.

For $C^r_\mathbb{K}$-manifolds M, N we use the notation $C^r_\mathbb{K}(M,N)$ for the set of all $C^r_\mathbb{K}$-maps from M to N. Moreover, we let $\mathrm{Diff}^r_\mathbb{K}(M)$ denote the subset of all $C^r_\mathbb{K}$-diffeomorphisms in $C^r_\mathbb{K}(M,M)$. For $C^\infty_\mathbb{C}$-manifolds, we will also write $\mathrm{Hol}(M,N) := C^\omega_\mathbb{C}(M,N) := C^\infty_\mathbb{C}(M,N)$ for the set of all holomorphic maps from M to N.

Furthermore, for $s \in \{\infty, \omega\}$, we define *locally convex $C^s_\mathbb{K}$-Lie groups* as groups with a $C^s_\mathbb{K}$-manifold structure turning the group operations into $C^s_\mathbb{K}$-maps.

To deal with manifolds of analytic mappings we need to slightly extend the notion of locally convex manifold. This is needed only in Section 2, where we relax the definition of a manifold as follows.

([6]) Recall from [6, Proposition 1.1.16] that $C^\infty_\mathbb{C}$ functions are locally given by series of continuous homogeneous polynomials (cf. [4, 3]). This justifies our abuse of notation.

Definition A.5 (Generalized manifolds). Let M be a Hausdorff topological space.

(1) A pair (U_κ, κ) with $U_\kappa \subseteq M$ open and $\kappa\colon U_\kappa \to V_\kappa \subseteq E_\kappa$ a homeomorphism onto an open subset of a locally convex space E_κ over $\mathbb{K}$ is called a *generalised manifold chart*. Note that the model space may change depending on the chart.

(2) For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ define $C^r_{\mathbb{K}}$-compatibility and $C^r_{\mathbb{K}}$-atlases for generalised manifold charts exactly as in the finite-dimensional case. A generalised $C^r_{\mathbb{K}}$-manifold is a Hausdorff topological space with a $C^r_{\mathbb{K}}$-manifold structure, i.e. an equivalence class of $C^r_{\mathbb{K}}$-atlases induced by an atlas of generalised charts.

Now we discuss the standard topologies on function spaces.

Definition A.6 (The compact open $C^\infty_{\mathbb{K}}$-topology). Let M be a finite-dimensional $C^\infty_{\mathbb{K}}$-manifold of dimension $d \in \mathbb{N}_0$ and let E be any locally convex $\mathbb{K}$-vector space. The *compact-open $C^\infty_{\mathbb{K}}$-topology* on the vector space $C^\infty_{\mathbb{K}}(M, E)$ is the locally convex topology, given by the seminorms

$$P_{\alpha,\phi,K,p}\colon C^\infty_{\mathbb{K}}(M,E) \to [0,\infty[, \qquad \gamma \mapsto \sup_{x\in K} p\big(\partial^\alpha(\gamma \circ \phi^{-1})(x)\big),$$

where p is a continuous seminorm on E, $\alpha \in \mathbb{N}_0^d$ is a multi-index, $\phi\colon U_\phi \to V_\phi$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism of an open subset $U_\phi \subseteq M$ onto an open subset $V_\phi \subseteq \mathbb{K}^d$, and $K \subseteq V_\phi$ is a compact set.

In the case $\mathbb{K} = \mathbb{C}$, the space $C^\infty_{\mathbb{C}}(M, E)$ is endowed just with the compact-open topology. However, it is a well known fact that the compact-open topology coincides in this case with the topology from Definition A.6. For the reader's convenience we give a sketch of the proof.

Lemma A.7. *Let M be a finite-dimensional complex manifold and let E be a complex locally convex vector space. Then the compact-open $C^\infty_{\mathbb{C}}$-topology on the space*

$$\mathrm{Hol}(M,E) = C^\infty_{\mathbb{C}}(M,E),$$

defined in Definition A.6*, agrees with the usual compact-open topology, which is the topology of uniform convergence on compact subsets.*

Proof. Since M is locally compact, we may work in local charts, and hence assume that $M = \Omega \subseteq \mathbb{C}^d$ is an open subset of a finite-dimensional vector space $\mathbb{C}^d$. Let $a \in \Omega$. Then there is $R > 0$ such that $K := \overline{B}_R^{\mathbb{C}^d}(a) \subseteq \Omega$. We fix $r := R/2$. Let $x \in B_r^{\mathbb{C}^d}(a)$ and $v \in \mathbb{C}^d$ be any vector of norm 1. Then by Cauchy's integral formula, we can write the derivative of $\gamma \in \mathrm{Hol}(\Omega, E)$

at x in direction v as

$$d^1\gamma(x,v) = \frac{1}{2\pi i}\int_{|z|=r} \frac{\gamma(x+zv)}{z^2}\,dz.$$

Applying a continuous seminorm p on both sides, we obtain

$$p(d^1\gamma(x,v)) \le \frac{1}{2\pi}\cdot 2\pi r \sup_{|z|=r}\frac{p(\gamma(x+zv))}{r^2} \le \frac{1}{r}\sup_{y\in K} p(\gamma(y)).$$

In particular, if we choose $v = e_j$ (a standard basis vector of $\mathbb{C}^d$), we obtain

$$p\left(\frac{\partial\gamma}{\partial x_j}(x)\right) \le \frac{1}{r}\sup_{y\in K} p(\gamma(y)),$$

and since $x \in B_r^{\mathbb{C}^d}(a)$ was arbitrary, this implies that uniform convergence on K implies uniform convergence on the open ball $B_r^{\mathbb{C}^d}(a)$. As every compact subset $K \subseteq \Omega$ can be covered by finitely many open balls, we have shown that taking the partial derivative $\partial/\partial x_j$ is continuous with respect to the compact-open topology. The case of a derivative with respect to a multi-index follows by induction. ∎

A.8 (The space of bounded holomorphic functions)**.** For a finite-dimensional complex manifold M and complex Banach space E, the *space of bounded holomorphic functions*

$$\mathrm{Hol}_{\mathrm{b}}(M,E) := \{\gamma \in \mathrm{Hol}(M,E) \mid \gamma \text{ is bounded on } M\}$$

is a Banach space with respect to the supremum norm (cf. [3, Proposition 6.5]).

A.9 (Fundamental sequence)**.** Let K be a compact subset of a finite-dimensional $C^\omega_{\mathbb{K}}$ manifold M. Then there always exists a sequence $U_1 \supseteq U_2 \supseteq \cdots$ of metrisable open neighbourhoods of K in M such that:

(1) For each $n \in \mathbb{N}$, the set $\overline{U_{n+1}}$ is compact in U_n.
(2) Each open neighbourhood U of K contains one of the sets U_n, $n \in \mathbb{N}$.
(3) Each connected component of each set U_n intersects the compact set K nontrivially.

Such a sequence will be called a *fundamental sequence* of open neighbourhoods of K in M.

Proof. Note that in general, M need not be metrisable. However, M is locally compact. Thus, the compact set K has a relatively compact neighbourhood U. The closure $\overline{U}$ is compact and locally metrisable, hence metrisable ([7]). Therefore, the compact set K is contained in an open relatively compact and metrisable set U.

([7]) For locally metrisable spaces, metrisability and paracompactness are equivalent.

Using a metric on U, we construct a descending sequence $U_1 \supseteq U_2 \supseteq \cdots$ of open neighbourhoods of K in U such that every neighbourhood of K in M contains some U_n. We can and will always choose a fundamental sequence such that every connected component of each U_n meets K and $\overline{U_{n+1}} \subseteq U_n$ for all n. Furthermore, the closure $\overline{U_{n+1}}$ is contained in $\overline{U}$ for all $n \in \mathbb{N}$, whence it is compact (in M and also in U_n). ■

A.10 (The space of germs of analytic mappings)**.** Let K be a compact subset of a finite-dimensional $C^\omega_{\mathbb{K}}$ manifold M and let E be a locally convex vector space over $\mathbb{K}$.

(1) Let $C^\omega_{\mathbb{K}}(K \subseteq M, E)$ be the space of *germs of* $\mathbb{K}$*-analytic maps along* K of $C^\omega_{\mathbb{K}}$-functions from open K-neighbourhoods in M to E.

Again we set $\mathrm{Hol}(K \subseteq M, E) := C^\omega_{\mathbb{C}}(K \subseteq M, E)$. By abuse of notation the germ of f around K will be denoted as f.

(2) Let $\mathbb{K} = \mathbb{C}$. Consider the directed set $(\mathcal{N}, \subseteq)$ of open neighbourhoods of K in M (partially) ordered by inclusion. Then for $U, W \in \mathcal{N}$ with $W \subseteq U$ we obtain continuous linear maps

$$\mathrm{res}^U_W \colon \mathrm{Hol}(U, E) \to \mathrm{Hol}(W, E), \quad f \mapsto f|_W,$$

which yields an inductive system in the category of complex locally convex spaces. Passing to the limit of the system, we obtain limit maps

$$\mathrm{res}^W_K \colon \mathrm{Hol}(W, E) \to \mathrm{Hol}(K \subseteq M, E), \quad W \in \mathcal{N},$$

assigning to each $f \in \mathrm{Hol}(W, E)$ the associated germ around K. We give $\mathrm{Hol}(K \subseteq M, E)$ the (a priori not necessarily Hausdorff) inductive limit topology of the above inductive system. In Lemma A.11 we will see that this topology is indeed Hausdorff.

(3) Let again $\mathbb{K} = \mathbb{C}$ and fix a fundamental sequence $U_1 \supseteq U_2 \supseteq \cdots$ of K in M. By A.9(3) and the identity theorem for analytic mappings the bonding maps $\mathrm{res}^{U_n}_{U_m}$ are injective for $m \geq n$, and so are the limit maps. Now A.9(2) implies that the direct limit topology on $\mathrm{Hol}(K \subseteq M, E)$ discussed in part (2) equals the direct limit topology of the sequence $(\mathrm{Hol}(U_n, E), \mathrm{res}^{U_n}_{U_{n+1}})_{n \in \mathbb{N}}$.

For E finite-dimensional, the topology on $\mathrm{Hol}(K \subseteq M, E)$ is nicer. By A.9(1) the bonding maps of the inductive limit factor in the obvious way through

$$\mathrm{Hol}(U_n, E) \to \mathrm{Hol}_{\mathrm{b}}(U_{n+1}, E) \to \mathrm{Hol}(U_{n+1}, E).$$

We conclude that $\mathrm{Hol}(K \subseteq M, E) = \varinjlim \mathrm{Hol}_{\mathrm{b}}(U_n, E)$.

The topology on $\mathrm{Hol}(K \subseteq M, \mathbb{C})$ coincides with the inductive topology induced by the system

$$\mathrm{Hol}_{\mathrm{b}}(U_n, \mathbb{C}) \to \mathrm{Hol}(K \subseteq M, \mathbb{C}), \quad g \mapsto \mathrm{res}^{U_n}_K(g), \quad \text{for } n \in \mathbb{N}.$$

In [18, Theorem 3.4] it was proved that the bonding maps of this system are compact, whence $\mathrm{Hol}(K \subseteq M, \mathbb{C})$ becomes a Silva space ([8]). For $k \in \mathbb{N}$ recall from [11, Lemma 3.4] that $\mathrm{Hol}(K \subseteq M, \mathbb{C}^k) \cong \mathrm{Hol}(K \subseteq M, \mathbb{C})^k$ as topological vector spaces. Thus $\mathrm{Hol}(K \subseteq M, \mathbb{C}^k)$ is a Silva space for all $k \in \mathbb{N}$ as a finite locally convex sum of Silva spaces.

LEMMA A.11 ($\mathrm{Hol}(K \subseteq M, E)$ is Hausdorff). *Let K be a compact subset of a finite-dimensional $C^\omega_\mathbb{C}$ manifold M and let E be a complex locally convex vector space. Then* $\mathrm{Hol}(K \subseteq M, E)$ *with the topology of* A.10(2) *is Hausdorff.*

Proof. For each $a \in M$, let $\mathcal{V}_a$ denote the set of all charts around a, i.e. of all $C^\infty_\mathbb{C}$-diffeomorphisms $\phi\colon U_\phi \to V_\phi$ with U_ϕ an open a-neighbourhood in M and V_ϕ open in $\mathbb{C}^d$ with $d = \dim M$. Denote by $\mathcal{N}$ the family of all open K-neighbourhoods in M. For each $W \in \mathcal{N}$, Lemma A.7 yields a continuous linear map

$$\Psi_W\colon \mathrm{Hol}(W, E) \to \prod_{a\in K}\prod_{\phi\in\mathcal{V}_a}\prod_{\alpha\in\mathbb{N}_0^d} E, \qquad \gamma \mapsto \big(\partial^\alpha(\gamma\circ\phi^{-1})(\phi(a))\big).$$

By construction, for $W, U \in \mathcal{N}$ with $W \subseteq U$ we have $\Psi_W \circ \mathrm{res}^U_W = \Psi_U$. Hence on the locally convex limit $\mathrm{Hol}(K \subseteq M, E)$ these maps induce a continuous linear map

$$\Psi\colon \mathrm{Hol}(K \subseteq M, E) \to \prod_{a\in K}\prod_{\phi\in\mathcal{V}_a}\prod_{\alpha\in\mathbb{N}_0^d} E.$$

This map is injective by the identity theorem for holomorphic maps. Hence, we obtain an injective continuous map from $\mathrm{Hol}(K \subseteq M, E)$ into a Hausdorff space, implying the Hausdorff property of the space of germs. ∎

We will now study sections of locally convex vector bundles. Our goal is to topologise spaces of germs of analytic sections around compact subsets.

LEMMA A.12 (Regularity of total spaces of bundles). *Let (F, π, M) be a topological vector bundle, i.e. a vector bundle whose typical fibre E is a topological vector space. The total space F is regular as a topological space if and only if the topological space M is regular.*

Proof. The space M can be embedded in the total space F via the zero-section. Hence, M is regular if F is regular by [8, Theorem 2.1.6]. To show the converse implication, assume that M is regular.

Recall from general topology that F is regular if every open neighbourhood of every point contains a closed neighbourhood of that point. To check this criterion fix $a \in F$ together with an open a-neighbourhood $\Omega \subseteq F$.

([8]) Recall that a Silva space is defined as the inductive limit of a sequence of Banach spaces such that the bonding maps are compact.

Choose a trivialisation $\kappa\colon \pi^{-1}(M_\kappa) \to M_\kappa \times E$ of the bundle with a in $\pi^{-1}(M_\kappa)$. The set $\kappa(\Omega \cap \pi^{-1}(M_\kappa))$ is an open neighbourhood of $\kappa(a)$ in $M \times E$. Hence, we can find open subsets $W \subseteq M_\kappa$ and $V \subseteq E$, respectively, such that

$$\kappa(a) \in W \times V \subseteq \kappa(\Omega \cap \pi^{-1}(M_\kappa)).$$

Now M_κ is regular as a subspace of the regular space M, and E is regular as a topological vector space. Therefore, we may choose W and V so small that $\overline{W} \times \overline{V} \subseteq \kappa(\Omega \cap \pi^{-1}(M_\kappa))$. We obtain an a-neighbourhood $A := \kappa^{-1}(\overline{W} \times \overline{V}) \subseteq F$ contained in Ω. It remains to show that A is closed in F.

Consider an element $\overline{a}$ of the closure $\overline{A}$ of A. Then $\pi(\overline{a}) \in \pi(\overline{A}) \subseteq \overline{\pi(A)} = \overline{\overline{W}} = \overline{W} \subseteq M_\kappa$. As $\overline{a} \in \pi^{-1}(M_\kappa)$ we obtain $\kappa(\overline{a}) \in \kappa(\overline{A}) \subseteq \overline{\kappa(A)} = \overline{\overline{W}} \times \overline{\overline{V}} = \overline{W} \times \overline{V}$. This implies that $\overline{a} \in \kappa^{-1}(\overline{W} \times \overline{V}) = A$, which shows that A is closed and concludes the proof. ∎

Definition A.13. Let (F, π, M) be a $C^\omega_\mathbb{C}$-bundle whose typical fibre E is a complex locally convex vector space and M is finite-dimensional.

(1) Let $\Gamma^\omega_\mathbb{C}(F)$ be the *space of holomorphic sections* of (F, π, M). We topologise $\Gamma^\omega_\mathbb{C}(F)$ with the initial topology with respect to the maps

$$\theta_\psi\colon \Gamma^\omega_\mathbb{C}(F) \to \mathrm{Hol}(M_\psi, E), \qquad X \mapsto \mathrm{pr}_2 \circ \psi \circ X|_{M_\psi}.$$

Here ψ ranges through all bundle trivialisations of F. Note that $\Gamma^\omega_\mathbb{C}(F)$ is Hausdorff as the point evaluations are continuous.

Consider a compact subset $K \subseteq M$.

(2) Let $\mathcal{N}_K$ be the *set of all open K-neighbourhoods.* For $U \in \mathcal{N}_K$ we define the *restricted bundle* $(F|U := \pi^{-1}(U), \pi|^U_{F|U}, U)$.

(3) We denote by $\Gamma^\omega_\mathbb{C}(F|K)$ the *space of germs of sections along* K, i.e. germs of sections in $\Gamma^\omega_\mathbb{C}(F|U)$ where U ranges $\mathcal{N}_K$. Topologise $\Gamma^\omega_\mathbb{C}(F|K)$ as the locally convex inductive limit of the cone $(\Gamma^\omega_\mathbb{C}(F|W) \to \Gamma^\omega_\mathbb{C}(F|K))_{W \in \mathcal{N}_K}$ (where the limit maps send a section to its germ).

At this point it is not clear whether $\Gamma^\omega_\mathbb{C}(F|K)$ is Hausdorff; we will establish it in Lemma A.16.

Lemma A.14. *Let (F, π, M) be a $C^\omega_\mathbb{C}$-bundle whose typical fibre E is a complex locally convex space and M is finite-dimensional. The topology of $\Gamma^\omega_\mathbb{C}(F)$ (cf. Definition* A.13*) coincides with the compact-open topology. Hence a typical subbasis for the topology is*

$$\lfloor L, O \rfloor := \{X \in \Gamma^\omega_\mathbb{C}(F) \mid X(L) \subseteq O\}$$

where L is a compact subset of M, and O is an open subset of F.

Proof. We show first that for every compact set $L \subseteq M$ and open subset $O \subseteq F$ the set $\lfloor L, O \rfloor$ is open in $\Gamma^\omega_\mathbb{C}(F)$. To this end, we will prove that $\lfloor L, O \rfloor$

is a neighbourhood of each fixed $\sigma \in \lfloor L, O \rfloor$. Since $L \subseteq M$ is compact and M is finite-dimensional, there is a finite family of compact sets $K_\alpha \subseteq M$, $1 \leq \alpha \leq m$ such that:

- $L = \bigcup_\alpha K_\alpha$,
- for each α there is a bundle trivialisation ψ_α with $K_\alpha \subseteq M_{\psi_\alpha}$, and
- $(K_\alpha \times \mathrm{pr}_2 \circ \psi_\alpha \circ \sigma(K_\alpha)) \subseteq M_{\psi_\alpha} \times \mathrm{pr}_2 \circ \psi_\alpha(O \cap \operatorname{dom} \psi_\alpha)$.

Recall that the topology on $\Gamma^\omega_{\mathbb{C}}(F)$ is initial with respect to the maps θ_ψ where ψ runs through all bundle trivialisations. Moreover, the range space of θ_ψ carries the compact-open topology. As we are dealing with sections, the following identity holds:

$$\begin{aligned}\Omega_\alpha &:= \lfloor K_\alpha, (\mathrm{pr}_2 \circ \psi_\alpha)^{-1}(\mathrm{pr}_2 \circ \psi_\alpha(O \cap \operatorname{dom} \psi_\alpha))\rfloor \\ &= \theta^{-1}_{\psi_\alpha}(\lfloor K_\alpha, \mathrm{pr}_2 \circ \psi_\alpha(O \cap \operatorname{dom} \psi_\alpha)\rfloor).\end{aligned}$$

In particular, σ is contained in each open set Ω_α for $1 \leq \alpha \leq m$. Note that by construction the set $\bigcap_{1\leq\alpha\leq m} \Omega_\alpha$ is contained in $\lfloor L, O \rfloor$ which proves that $\lfloor L, O \rfloor$ is a neighbourhood of σ.

Conversely, fix a section τ together with an arbitrary τ-neighbourhood Ω in $\Gamma^\omega_{\mathbb{C}}(F)$. By definition of the initial topology, Ω contains an open τ-neighbourhood of the form

$$\bigcap_{1\leq k\leq n} \theta^{-1}_{\psi_k}(\lfloor L_k, U_k \rfloor), \qquad \text{where } L_k \subseteq M_{\psi_k} \text{ is compact and } U_k \subseteq E \text{ is open.}$$

As above, $\theta^{-1}_{\psi_k}(\lfloor L_k, U_k\rfloor) = \lfloor L_k, (\mathrm{pr}_2 \circ \psi_k)^{-1}(U_k)\rfloor$. Thus the assertion follows. ∎

LEMMA A.15. *Let (F, π, M) be a $C^\omega_{\mathbb{C}}$-bundle whose typical fibre E is a complex locally convex space and M is finite-dimensional. Consider a compact subset K of M.*

(1) *If there is a trivialisation $\psi\colon F \supseteq \Omega \to M_\psi \times E$ such that $K \subseteq M_\psi$, then the map $I_\psi\colon \Gamma^\omega_{\mathbb{C}}(F|K) \to C^\omega_{\mathbb{C}}(K \subseteq M_\psi, E)$, $\gamma \mapsto \mathrm{pr}_2 \circ \psi \circ \gamma$, is an isomorphism of locally convex spaces.*
(2) *If $L \subseteq K$ is another compact subset, then the canonical restriction map $\mathrm{res}^K_L\colon \Gamma^\omega_{\mathbb{C}}(F|K) \to \Gamma^\omega_{\mathbb{C}}(F|L)$ is continuous linear.*
(3) *Let $K_1, K_2 \subseteq M$ be compact subsets with $K = K_1 \cup K_2$. Then the map*

$$R := (\mathrm{res}^K_{K_1}, \mathrm{res}^K_{K_2})\colon \Gamma^\omega_{\mathbb{C}}(F|K) \to \Gamma^\omega_{\mathbb{C}}(F|K_1) \oplus \Gamma^\omega_{\mathbb{C}}(F|K_2)$$

is a topological embedding. In particular, if $K_1 \cap K_2 = \emptyset$ then R is an isomorphism of locally convex spaces.

Proof. (1) Let $W \subseteq M_\psi$ be an open neighbourhood of K. From Lemma A.14 we deduce that the linear bijective map $(\mathrm{pr}_2 \circ \psi)_*\colon \Gamma^\omega_{\mathbb{C}}(F|W) \to \mathrm{Hol}(W, E)$, $\sigma \mapsto \mathrm{pr}_2 \circ \psi \circ \sigma$, is an isomorphism of topological vector spaces. Now the assertion follows from an easy inductive limit argument.

(2) Since $\mathrm{res}_U^W : \mathrm{Hol}(W,E) \to \mathrm{Hol}(U,E)$ is continuous linear for all open $U \subseteq W$, the assertion follows from an inductive limit argument.

(3) By (2), R is continuous linear. Clearly R is also injective. If K_1 and K_2 are disjoint, the assertion of the lemma is trivial.

Thus we will now assume that $K_1 \cap K_2 \neq \emptyset$. Fix fundamental sequences $(U_n^i)_{n\in\mathbb{N}}$ of neighbourhoods of K_i for $i \in \{1,2\}$. By construction the sets $U_n := U_n^1 \cup U_n^2$ form a fundamental sequence for K.

It remains to show that the map R is a topological embedding. To this end let Ω be an open zero-neighbourhood in $\Gamma_{\mathbb{C}}^{\omega}(F|K)$. It remains to show that $R(\Omega)$ is an open zero-neighbourhood in $R(\Gamma_{\mathbb{C}}^{\omega}(F|K))$. By [24, p. 109] we may assume that Ω is a zero-neighbourhood of the form

$$\Omega = \bigcup_{n\in\mathbb{N}} \sum_{1\le j\le n} \Omega_j \quad \text{with } \Omega_j \subseteq \Gamma_{\mathbb{C}}^{\omega}(F|U_j) \text{ an open zero-neighbourhood.}$$

Adjust choices to achieve (Lemma A.14) that $\Omega_j = \lfloor L_j, O_j \rfloor$ with $L_j \subseteq U_j$ compact and $O_j \subseteq F$ open. Hence every Ω_j gives rise to two open sets $\lfloor L_j \cap \overline{U_{j+1}^i}, O_j \rfloor \subseteq \Gamma_{\mathbb{C}}^{\omega}(F|U_j^i)$ for $i \in \{1,2\}$. As we deal with sections, one easily obtains the equality

$$R(\Omega) = \bigcup_{n\in\mathbb{N}} \sum_{1\le j\le n} \big(\lfloor L_j \cap \overline{U_{j+1}^1}, O_j\rfloor \times \lfloor L_j \cap \overline{U_{j+1}^2}, O_j\rfloor\big) \cap R(\Gamma_{\mathbb{C}}^{\omega}(F|K)).$$

Since the locally convex space $\Gamma_{\mathbb{C}}^{\omega}(F|K_1) \oplus \Gamma_{\mathbb{C}}^{\omega}(F|K_1)$ is the inductive limit of the system $\Gamma_{\mathbb{C}}^{\omega}(F|U_n^1) \oplus \Gamma_{\mathbb{C}}^{\omega}(F|U_n^2)$, we see that $R(\Omega)$ is open in the subspace topology. Summing up, R is a topological embedding. ∎

LEMMA A.16. *Let (F,π,M) be a $C_{\mathbb{C}}^{\omega}$-bundle whose typical fibre E is a complex locally convex space and M is finite-dimensional. Fix a compact subset $K \subseteq M$ and a finite family $(K_\alpha)_{\alpha\in A}$ of compact subsets of M such that:*

(1) $K = \bigcup_\alpha K_\alpha$,
(2) *for each α there is a bundle trivialisation ψ_α with $K_\alpha \subseteq M_{\psi_\alpha}$.*

With the notation of Lemma A.15 *the mapping*

$$\Theta := (I_{\psi_\alpha} \circ \mathrm{res}_{K_\alpha}^K)_{\alpha\in A} \colon \Gamma_{\mathbb{C}}^{\omega}(F|K) \to \bigoplus_{\alpha\in A} \mathrm{Hol}(K_\alpha \subseteq M_{\psi_\alpha}, E)$$

is a linear topological embedding, whose image is a closed vector subspace. Thus $\Gamma_{\mathbb{C}}^{\omega}(F|K)$ is Hausdorff. If E is finite-dimensional, then $\Gamma_{\mathbb{C}}^{\omega}(F|K)$ is a Silva space.

Proof. Iteratively applying Lemma A.15(3) we obtain a linear topological embedding

$$R_A = (\mathrm{res}_{K_\alpha}^K)_{\alpha\in A} \colon \Gamma_{\mathbb{C}}^{\omega}(F|K) \to \bigoplus_{\alpha\in A} \Gamma_{\mathbb{C}}^{\omega}(F|K_\alpha).$$

Apply Lemma A.15(1) to each summand to see that Θ is a topological embedding. Now $\Gamma^\omega_{\mathbb{C}}(F|K)$ embeds into a product of Hausdorff spaces (cf. Lemma A.11) and thus $\Gamma^\omega_{\mathbb{C}}(F|K)$ is Hausdorff. Moreover, the image $\operatorname{im}\Theta$ is homeomorphic to $\operatorname{im} R_A \subseteq \bigoplus_{\alpha\in A}\Gamma^\omega_{\mathbb{C}}(F|K_\alpha)$.

It is easy to see that the image of R_A is the space

$$\left\{(\gamma_\alpha)_{\alpha\in A}\in\bigoplus_{\alpha\in A}\Gamma^\omega_{\mathbb{C}}(F|K_\alpha)\;\middle|\;\begin{array}{l}\operatorname{res}^{K_\alpha}_{K_\alpha\cap K_\beta}(\gamma_\alpha)=\operatorname{res}^{K_\beta}_{K_\alpha\cap K_\beta}(\gamma_\beta)\\ \text{if } K_\alpha\cap K_\beta\neq\emptyset\end{array}\right\}.$$

As each $\Gamma^\omega_{\mathbb{C}}(F|K_\alpha\cap K_\beta)$ is Hausdorff, the image of R_A is obviously closed.

Let E now be finite-dimensional. Then each $\operatorname{Hol}(K_\alpha\subseteq M_{\psi_\alpha},E)$ is a Silva space by A.10(3). Since any finite locally convex sum of Silva spaces and closed subspaces of Silva spaces is a Silva space by [5, Corollary 8.6.9], the assertion follows. ■

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