

Abelian pro-countable groups and orbit equivalence relations

by

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Abstract. We study a class of abelian groups that can be defined as Polish pro-countable groups, as non-archimedean groups with a compatible two-sided invariant metric or as quasi-countable groups, i.e., closed subdirect products of countable discrete groups, endowed with the product topology.

We show that for every non-locally compact, abelian quasi-countable group G there exists a closed $L \leq G$ and a closed, non-locally compact $K \leq G/L$ which is a direct product of discrete countable groups. As an application we prove that for every abelian Polish group G of the form H/L , where $H, L \leq \text{Iso}(X)$ and X is a locally compact separable metric space (in particular, for every abelian, quasi-countable group G), the following holds: G is locally compact iff every continuous action of G on a Polish space Y induces an orbit equivalence relation that is reducible to an equivalence relation with countable classes.

1. Introduction. We study groups that can be characterized in the following three ways. First, they can be defined as Polish, pro-countable groups, i.e., inverse limits of countable families of discrete countable groups. Also, they can be viewed as non-archimedean groups (i.e., Polish groups with a neighborhood basis at the identity consisting of open subgroups), that admit an invariant metric. Finally, they can be defined as *quasi-countable* groups, i.e., closed countable subdirect products of discrete countable groups, endowed with the product topology. In the context of this paper, the latter characterization is the most natural, especially in case we want to put some additional requirements on the discrete building blocks.

As a matter of fact, quasi-countable groups have been studied before (see, e.g., [5], [6], [7]). However, it seems that no one has considered them so far as quasi-countable groups, that is, taking advantage of the fact that

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they are built out of countable groups in a nice way. Our approach allows for applying tools coming from the theory of discrete groups; for example, we use a classical theorem of Kulikov saying that for every abelian torsion group G there exists a direct sum $L \leq G$ of cyclic groups such that K/L is divisible. Most of the results presented in this paper concern non-locally compact, abelian quasi-countable groups. In particular, we prove a structure theorem to the effect that for every non-locally compact, abelian quasi-countable group G there exists a closed $L \leq G$ and a closed $K \leq G/L$ such that K is a non-locally compact *direct* product of discrete countable groups.

As an application of our study, we consider connections between the structure of abelian quasi-countable groups and properties of orbit equivalence relations induced by their continuous actions. Recall that for Polish spaces X, Y and equivalence relations E, F defined on X, Y , respectively, we say that E is *Borel reducible* to F if there exists a Borel mapping $f : X \rightarrow Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y) \quad \text{for } x, y \in X$$

(see [3] for more information on this notion). An important class of equivalence relations are the *orbit equivalence relations*, that is, relations E_α of the form

$$xE_\alpha y \Leftrightarrow \exists g \in G (\alpha(g, x) = y) \quad \text{for } x, y \in X,$$

where α is a continuous action of G on a Polish space X . It turns out that there are deep connections between the structure of G and properties of orbit equivalence relations induced by G . For example, S. Solecki [12] proved that a Polish group G is non-compact if and only if there exists a continuous action α of G such that E_0 is reducible to E_α (where E_0 is the relation of eventual identity on the space $2^{\mathbb{N}}$ of all 0-1 sequences). Even more interestingly, A. Kechris [8] proved that if G is locally compact, then every orbit equivalence relation induced by G is reducible to an equivalence relation with countable classes. A natural question arises whether the converse holds. This problem has been remaining ‘stubbornly open’, as G. Hjorth put it, since the early 1990s.

In this paper, we show that the converse does hold for abelian quasi-countable groups. Then, applying certain results of S. Gao, A. Kechris, A. Kwiatkowska, and S. Solecki, as well as Pontryagin duality, we generalize it to abelian Polish groups of the form G/L , where G, L are closed subgroups of $\text{Iso}(X)$ for some locally compact separable metric space X .

Let us mention that this partially complements a theorem proved by A. Thompson [15], which says that the converse of Kechris’s theorem holds for *non-cli* groups, that is, Polish groups that do not admit a compatible, left-invariant complete metric. As Polish abelian groups are cli, Thompson’s theorem does not apply in the present context.

2. Notation and basic facts. A topological space is *Polish* if it is separable and completely metrizable. A topological group is Polish if its group topology is Polish. In the following, we consider only Polish groups, and their Polish (equivalently, closed) subgroups.

It is well known (see [3, Theorem 2.2.10]) that if G is a Polish group, and H is a closed normal subgroup of G , then G/H is a Polish group in the quotient topology. The following fact seems to be standard but we could not find any reference for it.

LEMMA 1. *Let G be a non-locally compact Polish group, and let H be a closed normal subgroup of G . If H is locally compact, then G/H is not locally compact.*

Proof. Suppose that G/H is locally compact. Let π be the projection of G onto G/H . Fix a compatible left-invariant metric d on G , an open neighborhood of the identity $U \subseteq G/H$ whose closure is compact in G/H , and $\epsilon > 0$ such that the closure of $B(e, 3\epsilon) \cap H$ is compact, where $B(e, 3\epsilon)$ is the open ball in G with center at the identity e and radius 3ϵ .

Let $\{x_n\}$ be a sequence in $\pi^{-1}[U] \cap B(e, \epsilon)$. Then $\{\pi(x_n)\}$ contains a subsequence that converges in G/H to some gH with $d(g, e) \leq \epsilon$. In particular, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\text{dist}(y_n, gH) \rightarrow 0$. Fix $h_n \in H$ such that $d(y_n, gh_n) \rightarrow 0$. Then, for sufficiently large n ,

$$d(e, y_n) \leq \epsilon, \quad d(y_n, gh_n) = d(g^{-1}y_n, h_n) < \epsilon,$$

so we get

$$g^{-1}y_n \in B(e, 2\epsilon), \quad h_n \in B(e, 3\epsilon).$$

But the closure of $H \cap B(e, 3\epsilon)$ is compact, so there is a convergent subsequence in $\{h_n\}$, and thus a convergent subsequence in $\{y_n\}$. This implies that G is locally compact, which is a contradiction. ■

A group G is called *non-archimedean* if it is Polish and has a neighborhood basis at the identity consisting of open subgroups. Equivalently, non-archimedean groups are closed subgroups of the group S_∞ of all permutations of the natural numbers, with the pointwise convergence topology.

Let \mathcal{D} be a class of discrete countable groups. A Polish group G is called *quasi- \mathcal{D}* if it is a subdirect product of groups from \mathcal{D} , that is,

$$G \leq \prod_n G_n,$$

where $G_n \in \mathcal{D}$ for $n \in \mathbb{N}$, $\prod_n G_n$ is endowed with the product topology, and all the projections of G on G_n are surjective. Thus we can talk about quasi-countable groups, quasi-divisible groups, quasi-dsc groups (where dsc stands for ‘direct sum of finite cyclic groups’), etc. Similarly, a Polish group G

is called pro- \mathcal{D} if it is the inverse limit of an inverse system of groups from \mathcal{D} . Clearly, every Polish pro- \mathcal{D} group is quasi- \mathcal{D} .

Let $G \leq \prod_n G_n$. For $g \in G$, we denote by $g(n)$ the value of g on its n th coordinate; $\pi_n[G]$ denotes the projection of G on G_n , $G_{\langle n \rangle}$ denotes the group

$$G_{\langle n \rangle} = \{g \in G : g(k) = e \text{ for } k \leq n\},$$

and $G_{\langle -1 \rangle} = G$. Clearly, in the product topology, the family $\{G_{\langle n \rangle}\}$ forms a neighborhood basis at the identity consisting of normal subgroups of G .

If, additionally, $\pi_n[G] = G_n$ for each n , we say that the family $\{G_n\}$ is adequate for G .

LEMMA 2. *Let G be a Polish group. The following conditions are equivalent:*

- (1) G is quasi-countable,
- (2) G is pro-countable,
- (3) G has a countable neighborhood basis at the identity consisting of open normal subgroups,
- (4) G is non-archimedean, and there exists a compatible two-sided invariant metric on G .

Proof. In order to show (1) \Rightarrow (2), fix a countable family $\{N_n\}$ of open normal subgroups of G that form a neighborhood basis at the identity satisfying and $N_{n+1} \leq N_n$. Let f_n be the natural projection of G/N_{n+1} onto G/N_n . Clearly, $\{G/N_n, f_n\}$ is as required.

The implication (2) \Rightarrow (3) is immediate, and the implications (3) \Rightarrow (4) and (4) \Rightarrow (1) follow from the fact that G has a neighborhood basis at the identity consisting of open normal subgroups iff G is non-archimedean and admits a compatible invariant metric (see, e.g., [3, Exercise 2.1.4]). ■

Let us mention that analogous characterizations can be proved for classes of quasi-countable groups that are sufficiently regular, e.g., quasi-torsion groups or quasi-dsc groups.

An element g of a quasi-countable group G is called *pro- p* if $\overline{\langle g \rangle}$ is a pro- p group, or equivalently, there is a prime p_0 such that the order of $g(n)$ is a power of p_0 for every n . If G is abelian, and p_0 is a fixed prime, the *p_0 -Sylow* subgroup of G is defined as the group of all pro- p_0 elements in G . This agrees with the standard terminology used in the theory of pro-finite groups (see [11]). Observe that every Sylow subgroup of G is closed.

A cyclic group of order n is denoted by $\mathbb{Z}(n)$. For a fixed prime p , the *Prüfer group* $\mathbb{Z}(p^\infty)$ is the unique p -group in which the number of p th roots of every element is exactly p .

For a Polish group G , and a continuous action α of G on a Polish space X , the symbol E_α denotes the orbit equivalence relation induced on X by G

via α , that is,

$$xE_\alpha y \Leftrightarrow \exists g \in G (\alpha(g, x) = y)$$

for $x, y \in X$. For two equivalence relations E, F defined on Polish spaces X, Y , respectively, we say that E is *reducible* to F , and write $E \leq F$, if there exists a Borel mapping $f : X \rightarrow Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

If there exists an injective f as above, we write $E \sqsubseteq F$.

Let $E_0, E_0^\mathbb{N}$ denote the equivalence relations on $2^\mathbb{N}, (2^\mathbb{N})^\mathbb{N}$, respectively, defined by

$$xE_0y \Leftrightarrow \exists m \forall n \geq n (x(n) = y(n)),$$

$$xE_0^\mathbb{N}y \Leftrightarrow \forall n (x(n)E_0y(n)).$$

It is well known that if $E_0^\mathbb{N} \leq E$, then E is not reducible to an equivalence relation with countable classes (see, e.g., [3, Theorem 8.5.2] and [3, Exercise 8.4.3]).

We will need two theorems that are important tools in the theory of Borel reducibility.

THEOREM 3 (Mackey, Hjorth, [3, Theorem 3.5.2]). *Let G be a Polish group, and let H be a closed subgroup of G . Let β be a continuous action of H on a Polish space X . There exists a Polish space Y , and a continuous action α of G on Y such that*

- (1) X is a closed subset of Y ,
- (2) $\alpha(g, x) = \beta(g, x)$ for all $g \in H$ and $x \in X$,
- (3) every α -orbit in Y contains exactly one β -orbit in X .

In particular, $E_\beta \sqsubseteq E_\alpha$.

THEOREM 4 (Solecki [12]). *Let G be a non-compact Polish group. There exists a Polish space X and a continuous action α of G on X such that*

$$E_0 \sqsubseteq E_\alpha.$$

3. Main results

LEMMA 5. *Let G be a non-archimedean group, and let $g \in G$. If the group $\overline{\langle g \rangle}$ is not discrete, then it is pro-cyclic. In particular, there exist distinct primes p_k , $k \in \mathbb{N}$, $\mathbb{N} \in \mathbb{N} \cup \{\mathbb{N}\}$, and pairwise commuting pro- p_k elements $g_k \in \overline{\langle g \rangle}$ such that*

$$g = \lim_k g_0 \dots g_k.$$

Proof. Set $L = \overline{\langle g \rangle}$. Since L is abelian, there exists an invariant metric on L . By Lemma 2, L is quasi-countable. Without loss of generality we can assume that $L \leq \prod_n L_n$, where each L_n is discrete and countable.

Since $\langle g \rangle$ is not discrete, it is easy to see that $\langle g(n) \rangle \leq L_n$ must be finite for every n . Therefore we can assume that each L_n is finite, that is, $\prod_n L_n$ and L are pro-finite. Since g topologically generates L , the group L is pro-cyclic. Now it is well known (see, e.g., [11, Theorem 2.7.2]) that pro-cyclic groups are direct products of their Sylow subgroups. ■

Let G_n , $n \in \mathbb{N}$, be groups with metrics d_n . Define

$$c_0(G_n, d_n) = \left\{ g \in \prod_n G_n : d_n(g(n), e) \rightarrow 0 \right\}$$

with the supremum metric

$$d_s(g, h) = \sup\{d_n(g(n), h(n)) : n \in \mathbb{N}\}.$$

LEMMA 6. *Let G_n , $n \in \mathbb{N}$, be Polish groups with two-sided invariant compatible metrics d_n . Then the group $c_0(G_n, d_n)$ with metric d_s is Polish.*

Proof. Fix countable dense sets $D_n \subseteq G_n$ for each $n \in \mathbb{N}$. It is easy to see that the set

$$D = \{g \in c_0(G_n, d_n) : \forall n \ g(n) \in D_n \text{ and } \forall^\infty n \ g(n) = e\}$$

is countable and dense in $c_0(G_n, d_n)$.

Fix now a Cauchy sequence $\{g_m\}$ in $c_0(G_n, d_n)$. Clearly, $\{g_m(n)\}$ is Cauchy in d_n for every fixed n . Because the metrics d_n are complete (see [3, Exercise 2.2.4]), for every fixed n there exists $g(n) \in G_n$ such that $g_m(n)$ converge to $g(n)$. Because the sequence $\{g_m\}$ is Cauchy in d_s , the element $g = (g(0), g(1), \dots)$ is in $c_0(G_n, d_n)$, and $\{g_m\}$ converges to g in d_s . Therefore d_s induces a Polish topology on $c_0(G_n, d_n)$.

In order to see that group operations are continuous in d_s , note that d_s is also two-sided invariant. Therefore

$$d_s(gh, g'h') \leq d_s(g, h) + d(g', h')$$

for all $g, g', h, h' \in c_0(G_n, d_n)$ (see, e.g., [3, Exercise 2.1.6]). This easily implies that multiplication is continuous with respect to d_s . Continuity of the inverse directly follows from the invariance of d_s . ■

LEMMA 7. *Let G be an abelian quasi-torsion group with a two-sided invariant metric d . Let $\{H_m\}$ be the family of all Sylow subgroups of G . Then $G = c_0(H_m, d_m)$, where d_m is the restriction of d to H_m .*

Proof. We show that the mapping $h \mapsto \sum_m h_m$ is a well defined isomorphism $\phi : c_0(H_m, d_m) \rightarrow G$. Fix $h_m \in H_m$ such that $h_m \rightarrow 0$. Fix $\epsilon > 0$ and an open subgroup $V \leq G$ such that $\text{dist}(V, 0) < \epsilon$. Let m_0 be such that $h_m \in V$ for $m \geq m_0$. Then

$$\sum_{m=m_0}^{m_1} h_m \in V$$

for every $m_1 > m_0$, which implies

$$d(h_0 + \cdots + h_{m_1}, h_0 + \cdots + h_{m_0}) = d(h_{m_0+1} + \cdots + h_{m_1}, 0) < \epsilon.$$

Since ϵ was arbitrary, $\sum_m h_m$ is convergent, and ϕ is well defined. An analogous argument shows that ϕ is a continuous homomorphism.

Let $\{G_n\}$ be an adequate family for G consisting of torsion groups. Observe that if $h_m \in H_m$, $m = 0, \dots, l$, then

$$h_0(n) + \cdots + h_l(n) = 0 \Leftrightarrow h_0(n) = \cdots = h_l(n) = 0$$

for every n (here, $h_m(n)$ is the value of the n th coordinate of h_m in $\prod_n G_n$). This easily implies that for $h_m \in H_m$, $m \in \mathbb{N}$, such that $h_m \rightarrow 0$ we have

$$\sum_m h_m = 0 \Leftrightarrow h_m = 0 \text{ for every } m,$$

that is, ϕ is injective. By Lemma 5, for every $g \in G$ there exist $h_m \in H_m$, $m \in \mathbb{N}$, such that $g = \sum_m h_m$, that is, ϕ is also surjective. But this implies that ϕ is open (see [3, Theorem 2.3.3]), so ϕ is an isomorphism. ■

The same argument can be used to prove the following.

LEMMA 8. *Let G be an abelian quasi-torsion group with a two-sided invariant metric d , and let H_0, H_1, \dots be subgroups of G such that:*

- (1) $\text{diam}(H_n) \rightarrow 0$,
- (2) each H_n is a pro- π_n -group for some set π_n of primes,
- (3) $\pi_n \cap \pi_{n'} = \emptyset$ if $n \neq n'$.

Then $\overline{\bigoplus_n H_n} = \prod_n H_n \leq G$.

Proof. Observe that $c_0(H_n, d_n) = \prod_n H_n$, where d_n is the restriction of d to H_n , and prove that $\phi : \prod_n H_n \rightarrow \overline{\bigoplus_n H_n}$ defined by $h \mapsto \sum_n h(n)$ is an isomorphism. ■

Now we prove two special cases of the main structure theorem which are of some independent interest.

LEMMA 9. *Let G be an abelian quasi-torsion group. Suppose that G is non-locally compact, and all Sylow subgroups of G are locally compact. Then there exist non-compact groups K_n , $n \in \mathbb{N}$, such that*

$$\prod_n K_n \leq G.$$

Proof. Let $\{G_n\}$ be an adequate family for G consisting of torsion groups. Because G is non-locally compact, we can assume that the G_n are such that $\pi_n[G_{\langle m \rangle}]$ is infinite for every m and $n > m$. Let $\{H_m\}$ be the family of all Sylow subgroups of G . We consider the following two cases.

CASE 1: *For every n there exist $n' \geq n$ and m such that $H_m \cap G_{\langle n' \rangle}$ is non-compact.* Observe that since each H_m is locally compact, for every m there

exist only finitely many n such that $H_m \cap G_{\langle n \rangle}$ is non-compact. Therefore we can find an infinite sequence (k_n, l_n) , $n \in \mathbb{N}$, such that $k_n \neq k_{n'}$ if $n \neq n'$, (l_n) is strictly increasing, and

$$K_n = H_{k_n} \cap G_{\langle l_n \rangle}$$

is non-compact. Lemma 8 implies that $\prod_n K_n \leq G$.

CASE 2: *There exists n such that $G_{\langle n \rangle} \cap H_m$ is compact for every m .* After replacing G with $G_{\langle n \rangle}$, which is also non-locally compact, we can assume that every H_m is compact.

Suppose that there exists an infinite set $A \subseteq \mathbb{N}$ such that for every infinite $B \subseteq A$ there exist infinitely many n such that

$$\bigoplus_{m \in B} H_m \cap G_{\langle n \rangle}$$

is non-compact. Then we can partition A into infinite sets B_n , and find a strictly increasing sequence (l_n) such that each

$$K_n = \bigoplus_{m \in B_n} H_m \cap G_{\langle l_n \rangle}$$

is non-compact, and, as in Case 1, $\prod_n K_n \leq G$.

Otherwise, we set

$$A_0 = \{m \in \mathbb{N} : \pi_0[H_m] \text{ is not trivial}\}.$$

By our assumption on the form of the adequate family $\{G_n\}$, and by compactness of all H_m , the set A_0 is infinite. Obviously, for every infinite $B \subseteq A_0$ the set

$$K_0 = \bigoplus_{m \in B} H_m$$

is then non-compact. On the other hand, since there is no set $A \subseteq \mathbb{N}$ as above, there must exist an infinite $B_0 \subseteq A_0$ and l_0 such that

$$\bigoplus_{m \in B_0} H_m \cap G_{\langle l_0 \rangle}$$

is compact. Then, again, the set A_1 defined by

$$A_1 = \{m \in \mathbb{N} \setminus B_0 : \pi_{l_0+1}[H_m \cap G_{\langle l_0 \rangle}] \text{ is not trivial}\}$$

is infinite, and the group

$$\bigoplus_{m \in B} H_m \cap G_{\langle l_0 \rangle}$$

is non-compact for every infinite $B \subseteq A_1$. We can find $B_1 \subseteq A_1$ and $l_1 \in \mathbb{N}$ in the same way we have found B_0 and l_0 .

In this manner, we construct pairwise disjoint infinite sets $B_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$, and a strictly increasing sequence (l_n) so that

$$K_n = \overline{\bigoplus_{m \in B_n} H_m \cap G_{\langle l_n \rangle}}$$

is non-compact for all n . An application of Lemma 8 completes the proof. ■

Since each K_n in Lemma 9 is a non-compact quasi-countable group, we can find closed $L_n \leq K_n$ such that K_n/L_n is infinite and discrete. Therefore we get the following corollary.

COROLLARY 10. *Let G be an abelian quasi-torsion group. Suppose that G is non-locally compact, and all Sylow subgroups of G are locally compact. Then there exists a closed $L \leq G$ and infinite discrete groups K_n , $n \in \mathbb{N}$, such that*

$$\prod_n K_n \leq G/L.$$

LEMMA 11. *Let G be an abelian and torsion quasi-countable group. If G is non-locally compact, then there exists an infinite discrete subgroup $L \leq G$.*

Proof. Let $\{G_n\}$ be an adequate family for G , and for $n, m \in \mathbb{N} \setminus \{0\}$, let

$$A_{n,m} = \{g \in G : o(g) \leq m \text{ and } \forall k \leq m (kg \in G_{\langle n \rangle} \rightarrow kg = 0)\}.$$

Observe that each $A_{n,m}$ is closed. Since G is torsion, $\bigcup_{n,m} A_{n,m} = G$, and there exist n_0, m_0 such that A_{n_0, m_0} is not meager. Let $H \leq G$ be the subgroup generated by all elements of order $\leq m_0$ in G .

Note that non-local compactness of G implies that if $A_{n_0, m_0} \subseteq H$ were compact, then it would be meager. Therefore there must exist $n_1 \geq n_0$ such that $A_{n_0, m_0}/G_{\langle n_1 \rangle}$ is an infinite set.

We now construct L . Set $L_0 = \langle h \rangle$ for some fixed element $h \in A_{n_0, m_0}$, and suppose that a finite group L_k , where $k \in \mathbb{N}$, has already been constructed so that

$$L_k \cap G_{\langle n_1 \rangle} = \{0\}.$$

The group $L_k/G_{\langle n_1 \rangle}$ is finite, while $A_{n_0, m_0}/G_{\langle n_1 \rangle}$ is infinite. Moreover, $H/G_{\langle n_1 \rangle}$ has bounded exponent, so it is a direct sum of finite cyclic groups (see, e.g., [2, Theorem 17.2]). Therefore there is $h \in A_{n_0, m_0} \setminus L_k$ such that $\langle L_k, h \rangle$ is isomorphic to $L \oplus \langle h \rangle$, and, by the definition of the sets $A_{n,m}$,

$$\langle h \rangle \cap (L_k + G_{\langle n_1 \rangle}) = \{0\}.$$

But then

$$\langle L_k, h \rangle \cap G_{\langle n_1 \rangle} = \{0\},$$

so we can set $L_{k+1} = \langle L_k, h \rangle$ and $L = \bigcup_k L_k$. Clearly, L is infinite and discrete. ■

COROLLARY 12. *Let G be an abelian quasi-countable group which is torsion or such that in every neighborhood of 0 there exists an element generating an infinite discrete group. If G is non-locally compact, then there exist infinite discrete $L_n \leq G$, $n \in \mathbb{N}$, such that*

$$\prod_n L_n \leq G.$$

Proof. Let $\{G_n\}$ be an adequate family for G . By Lemma 11, there exists an infinite discrete $K_n \leq G_{\langle n \rangle}$ for every $n \in \mathbb{N}$. For every n fix $f_0(n) \in \mathbb{N}$ such that

$$K_n \cap G_{\langle f_0(n) \rangle} = \{0\},$$

and set $f(n) = (f_0)^n(0)$. Clearly,

$$K_{f(n)} \cap K_{f(n')} = \{0\}$$

if $n \neq n'$, and the sum $\sum_n h(n)$ is convergent for every $h \in \prod_n K_{f(n)}$. A proof similar to that of Lemma 7 shows that every convergent sequence in $\bigoplus_n K_{f(n)}$ is of the form $\sum_n h(n)$ for a unique $h \in \prod_n K_{f(n)}$. Thus, we can set $L_n = K_{f(n)}$. ■

Let us state Kulikov’s theorem.

THEOREM 13 (Kulikov). *Let G be an abelian torsion group. Then there exists a dsc group $L \leq G$ such that G/L is divisible.*

LEMMA 14. *Suppose that G is an abelian quasi-torsion group. If G is non-locally compact, then either there exists n such that $G_{\langle n \rangle}$ is quasi-dsc, or there exists a closed subgroup $L \leq G$ such that G/L is quasi-divisible and non-locally compact.*

Proof. Let $\{G_n\}$ be an adequate family for G consisting of torsion groups. Applying Kulikov’s theorem, for each n fix a dsc group L_n such that G_n/L_n is divisible. By [2, §18, Theorem 1], if A is a torsion abelian group, $B \leq A$ is a dsc group, and A/B is finite, then A is a dsc group. Therefore there are two cases to consider:

CASE 1: *There exists m such that $\pi_n[G_{\langle m \rangle}]/L_n$ is finite for every $n \geq m$.* By the above remark, $\pi_n[G_{\langle m \rangle}]$ is a dsc group for every $n \geq m$, that is, $G_{\langle m \rangle}$ is quasi-dsc.

CASE 2: *For every m there exists $n > m$ such that $\pi_n[G_{\langle m \rangle}]/L_n$ is infinite.* Then $G/\prod_n L_n$ is non-locally compact. Set $L = \prod_n L_n$. ■

In the following we assume that every dsc group G comes equipped with a fixed basis $\{x_m\}$, that is, a family of elements $x_m \in G$ such that

$$G = \bigoplus_m \langle x_m \rangle.$$

For $x \in G$ and $A \subseteq G$, we define

$$\text{supp}_G(x) = \{m \in \mathbb{N} : \text{the projection of } x \text{ onto } \langle x_m \rangle \text{ is non-trivial}\},$$

$$\text{supp}_G(A) = \bigcup \{\text{supp}(x) : x \in A\},$$

and, for $E \subseteq \mathbb{N}$,

$$x \upharpoonright E = \pi(x), \quad A \upharpoonright E = \pi[A],$$

where π is the projection of G onto $\bigoplus_{m \in E} \langle x_m \rangle$.

LEMMA 15. *Let G be an abelian quasi-dsc group. If G is non-locally compact, then there exist an adequate family $\{G_n\}$ for G , $L_n \leq G_n$, and infinite $K_n \leq G_n$, $n \in \mathbb{N}$, such that*

- (1) *each L_n is a direct summand in G_n ,*
- (2) *$\prod_n K_n \leq G / \prod_n L_n$.*

Proof. Because G is non-locally compact, by combining factors we can fix an adequate family $\{G_n\}$ such that $\pi_{n+1}[G_{\langle n \rangle}]$ is infinite for every n .

Fix a sequence $\{n_k\}$ of natural numbers such that each natural number appears in it infinitely many times. We will find a sequence $g_k \in G$, $k \in \mathbb{N}$, such that, for each k ,

- (a) $g_k \in G_{\langle n_k-1 \rangle}$,
- (b) $g_k(n_k) \neq 0$,
- (c) $\text{supp}_{G_n}(g_k(n)) \cap \text{supp}_{G_n}(g_{k'}(n)) = \emptyset$ for any $n \leq k$ and $k' < k$.

For $k = 0$ fix any $g_0 \in G_{\langle n_0-1 \rangle}$ with $g_0(n_0) \neq 0$, and suppose that $g_0, \dots, g_{l-1} \in G$ satisfying conditions (a)–(c) for $k < l$ have already been constructed.

For every $n \leq l$ define

$$E_n = \bigcup_{k < l} \text{supp}_{G_n}(g_k(n)).$$

For every $m \in \mathbb{N}$ fix $h_m \in G_{\langle n_l-1 \rangle}$ such that $h_m(n_l) \neq h_{m'}(n_l)$ if $m \neq m'$. Since each E_n is finite, by the pigeon-hole principle there exist distinct $m, m' \in \mathbb{N}$ such that

$$h_m(n) \upharpoonright E_n = h_{m'}(n) \upharpoonright E_n$$

for $n \leq l$. Let $g_l = h_m - h_{m'}$. It is easy to see that g_l satisfies conditions (a)–(c).

For $n \in \mathbb{N}$ define

$$K_n = \langle \{g_k(n) : k \in \mathbb{N} \text{ such that } n_k = n\} \rangle,$$

$$F_n = \text{supp}_{G_n}(K_n), \quad L_n = G_n \upharpoonright (\mathbb{N} \setminus F_n).$$

By conditions (b) and (c), the group K_n is an infinite subgroup of G_n . Moreover, obviously,

$$\text{supp}_{G_n}(K_n) \cap \text{supp}_{G_n}(L_n) = \emptyset,$$

and L_n is a direct summand in G_n .

We show that for every $h \in \prod_n K_n$ there is $g \in G$ such that for all n ,

$$g(n) \upharpoonright F_n = h(n).$$

Fix $h \in \prod_n K_n$. For each n , find $k_0, \dots, k_m, l_0, \dots, l_m \in \mathbb{N}$ such that

$$n_{k_i} = n, \quad h(n) = l_0 g_{k_0}(n) + \dots + l_m g_{k_m}(n),$$

and set

$$\gamma_n = l_0 g_{k_0} + \dots + l_m g_{k_m}.$$

Note that condition (c) implies that

$$\text{supp}(g_k(n)) \cap F_n = \emptyset$$

for any k with $n_k \neq n$. Also, condition (a) gives $\gamma_n \in G_{\langle n-1 \rangle}$, so the limit $\gamma = \sum_n \gamma_n$ exists. Therefore

$$\gamma(n) \upharpoonright F_n = h(n)$$

for every n . Since $h \in \prod_n K_n$ was arbitrary, we get

$$\prod_n K_n \leq G / \prod_n L_n. \blacksquare$$

THEOREM 16. *Suppose that G is an abelian quasi-torsion group. If G is non-locally compact, then there exist a closed $L \leq G$ and infinite discrete groups $K_n, n \in \mathbb{N}$, such that*

$$\prod_n K_n \leq G/L.$$

Proof. If there exists n such that $G_{\langle n \rangle}$ is quasi-dsc, we can use Lemma 15. Otherwise, we apply Lemma 14 to find $L \leq G$ such that $H = G/L$ is quasi-torsion, quasi-divisible, and non-locally compact. Let $\{H_n\}$ be an adequate family for H consisting of torsion divisible groups. Because H is non-locally compact, we can assume that $\pi_m[H_{\langle n \rangle}]$ is infinite for every n and $m > n$.

It is well known that every torsion divisible group is isomorphic to a direct sum of Prüfer groups.

Suppose that for every n there exists $f(n) > n$ and a Prüfer group $K'_{f(n)}$ which is a direct summand in $H_{f(n)}$ and is such that the projection of $\pi_{f(n)}[H_{\langle n \rangle}]$ on $K'_{f(n)}$ is surjective. Let $L'_{f(n)} \leq H_{f(n)}$ be such that

$$H_{f(n)} = K'_{f(n)} \oplus L'_{f(n)}.$$

Set $K_{f^m(0)} = K'_{f^m(0)}$ for $m \in \mathbb{N}$, and $L_{f^m(0)} = L'_{f^m(0)}$ for $m \in \mathbb{N}$, and set $L_m = H_m$ if m is not of the form $f^n(0)$ for some n . It is easy to see that

$$H / \prod_n L_n = \prod_n K_n,$$

and $\prod_m K_m$ is non-locally compact.

Otherwise, there exists m such that for every $n > m$, the projection of $\pi_n[H_{\langle m \rangle}]$ on every Prüfer group which is a direct summand in H_n , is a finite, cyclic group. Then $\pi_n[H_{\langle m \rangle}]$ is a subgroup of a direct sum of cyclic groups. But a subgroup of a dsc group is also a dsc group (see, e.g., [2, Theorem 18.1]), so $H_{\langle m \rangle}$ is quasi-dsc. Again, we can apply Lemma 15. ■

THEOREM 17. *Let G be an abelian quasi-countable group. If G is non-locally compact, then there exist a closed $L \leq G$ and infinite discrete groups K_n , $n \in \mathbb{N}$, such that*

$$\prod_n K_n \leq G/L.$$

Moreover, if G is torsion or in every neighborhood of 0 there exists an element generating an infinite discrete group, we can take $L = \{0\}$.

Proof. If G is torsion or in every neighborhood of 0 there exists an element generating an infinite discrete group, we can apply Corollary 12. Otherwise, by Lemma 5, there exists an open subgroup H in G all of whose elements generate pro-cyclic subgroups, which means that H is quasi-torsion. Since H is also non-locally compact, we can assume that G itself is quasi-torsion, and apply Theorem 16. ■

Next, we would like to analyse what happens if we require that L in Theorem 17 be a pro-cyclic group. It turns out that this gives a little more insight into the structure of abelian quasi- p groups. Let us start with some auxiliary results.

LEMMA 18. *Let G be a pro- p , pro-cyclic group, let $g \in G$ be a topological generator of G , and let $H \leq G$ be closed and non-trivial. Then $ng \in H$ for some $n > 0$.*

Proof. By our assumption,

$$G = \varprojlim \mathbb{Z}(p^n),$$

where $\mathbb{Z}(p^n)$ is the cyclic group of order p^n . For $n \in \mathbb{N}$, let π_n be the projection of G onto its n th coordinate. Observe that for any closed $H_1, H_2 \leq G$ either $H_1 \leq H_2$ or $H_2 \leq H_1$. Indeed,

$$\pi_n(H_1) \leq \pi_n(H_2) \quad \text{or} \quad \pi_n(H_2) \leq \pi_n(H_1)$$

for every $n \in \mathbb{N}$, and

$$\pi_{n+1}(H_1) \leq \pi_{n+1}(H_2) \rightarrow \pi_n(H_1) \leq \pi_n(H_2).$$

By closedness of H_1 and H_2 , the claim follows.

Note also that

$$\bigcap_n \overline{\langle ng \rangle} = \{0\},$$

which implies that if $H \neq \{0\}$, then there exists $n > 0$ such that $\overline{\langle ng \rangle} \leq H$. Thus we get $ng \in H$. ■

COROLLARY 19. *Suppose G is an abelian quasi- p group, and $g, g' \in G$. Then*

$$\overline{\langle g \rangle} \cap \langle g' \rangle = \{0\} \quad \text{implies} \quad \overline{\langle g \rangle} \cap \overline{\langle g' \rangle} = \{0\}.$$

THEOREM 20. *Let G be an abelian quasi- p group. Then one of the following holds:*

- (1) *There exists a pro-cyclic $H \leq G$ such that G/H contains a clopen subgroup with bounded exponent.*
- (2) *The set $D \subseteq G^2$ defined by*

$$(g, g') \in D \Leftrightarrow \overline{\langle g \rangle} \cap \overline{\langle g' \rangle} = \{0\}$$

is comeager in G^2 .

Proof. If the group of torsion elements is non-meager in G , there exists n such that the group generated by elements of order $\leq n$ is non-meager, and hence open in G . In this case, (1) holds for $H = \{0\}$.

If there exists $g \in G$ such that $H = \overline{\langle g \rangle}$ is compact, and for non-meager many $g' \in G$ there is $n > 0$ such that $ng' \in H$, then G/H contains a clopen subgroup with bounded exponent.

Otherwise, we have

$$\forall g \in G \forall^* g' \in G (\overline{\langle g \rangle} \cap \langle g' \rangle = \{0\}),$$

so, by Corollary 19,

$$\forall g \in G \forall^* g' \in G (\overline{\langle g \rangle} \cap \overline{\langle g' \rangle} = \{0\}).$$

Then, by the Kuratowski–Ulam theorem, the set D , defined as in the statement of the lemma, is comeager. ■

4. Applications

THEOREM 21. *Suppose that G is an abelian quasi-countable group. Then G is locally compact if and only if every continuous action of G on a Polish space X is reducible to an equivalence relation with countable classes.*

Proof. The implication from left to right follows from [8]. To show the other direction, fix K_n , $n \in \mathbb{N}$, and L as in Theorem 17. By Theorem 4, there exist continuous actions α_n of K_n on some Polish spaces Y_n such that $E_0 \subseteq E_{\alpha_n}$. Then

$$E_0^{\mathbb{N}} \subseteq E_{\alpha''},$$

where α'' is the action of $\prod_n K_n$ on $\prod_n Y_n$ defined by

$$\alpha''(g, y) = (\alpha_n(g(n), y(n)))$$

for $g \in \prod_n K_n$ and $y \in \prod_n Y_n$. By Theorem 3, the action α'' can be extended to a continuous action α' of G/L on some Polish space X so that

$$E_{\alpha''} \subseteq E_{\alpha'}.$$

Define α as

$$\alpha(g, x) = \alpha'(g/L, x).$$

Then $E_0^{\mathbb{N}} \subseteq E_{\alpha}$, so E_{α} is not reducible to an equivalence relation with countable classes. ■

It is well known that every Polish group is isomorphic to some $G \leq \text{Iso}(X)$, where X is a Polish metric space, and $\text{Iso}(X)$ is the group of all isometries of X with the topology of pointwise convergence. Using Pontryagin duality, a characterization of isometry groups of locally compact spaces due to S. Gao and A. Kechris, and a result of A. Kwiatkowska and S. Solecki, we are able to show the following.

THEOREM 22. *Let X be a locally compact separable metric space, and let $G, L \leq \text{Iso}(X)$ be Polish groups such that G/L is abelian. Then G/L is locally compact if and only if every continuous action of G/L on a Polish space X is reducible to an equivalence relation with countable classes.*

Proof. By [14, Corollary 1.3], there exists a locally compact separable metric space X such that $G/L \leq \text{Iso}(X)$, so we can assume that L is trivial and G is abelian. As before, we will prove the implication from right to left by showing that if G is non-locally compact, then there exists a continuous action α of G such that

$$E_0^{\mathbb{N}} \subseteq E_{\alpha}.$$

By [4, Theorem 6.3], there exist closed subgroups $K_n \leq S_{\infty}$ and locally compact groups $L_n, n \in \mathbb{N}$, such that

$$G \leq \prod_n (K_n \times L_n^{\mathbb{N}}),$$

where each K_n acts on $L_n^{\mathbb{N}}$ by permuting coordinates. Since G is abelian, the definition of the semidirect product implies that without loss of generality we can assume that all the groups K_n are abelian.

Fix a neighborhood basis $\{U_n\}$ at the identity in G , and let $H_n, n \in \mathbb{N}$, be the clopen groups generated by U_n . If

$$H = \bigcap_n H_n$$

is locally compact, then, by Lemma 1, G/H is non-locally compact, and the family $\{(G/H)/(H_n/H)\}$ witnesses that G/H is an abelian quasi-countable group. Thus, we can apply Theorem 21 to G/H . As every action of G/H gives rise to an action of G , this finishes the proof when H is locally compact.

Otherwise, $H \leq G$ is non-locally compact, and clearly the actions of K_n on $L_n^{\mathbb{N}}$ are trivial when restricted to elements of H . Therefore

$$H \leq \left(\prod_n K_n \right) \oplus \left(\prod_n L_n \right),$$

and since G is abelian, we can assume that L_n are abelian as well.

If the projection of H on $\prod_n K_n$ is non-locally compact, we apply Theorem 21. Otherwise, the projection of H on $\prod_n L_n$ must be non-locally compact. It is a well known result in the theory of locally compact abelian groups, following from Pontryagin duality, that every such group has an open subgroup topologically isomorphic to $\mathbb{R}^k \oplus C$ for some $k \geq 0$ and compact group C (see, e.g., [10, Theorem 25]). For every L_n , fix such an $R_n = \mathbb{R}^k$, and $C_n = C$.

Suppose that $H \cap \prod_n R_n$ is locally compact. Then $H / \prod_n R_n$ is non-locally compact, and since all C_n are compact, $H / \prod_n (R_n \oplus C_n)$ is non-locally compact as well. But each $R_n \oplus C_n$ is open in L_n , so $L_n / (R_n \oplus C_n)$ is countable. In other words, $H / \prod_n (R_n \oplus C_n)$ is a quasi-countable group, and we can apply Theorem 21 once more.

The last possibility to consider is that $H_0 = H \cap \prod_n R_n$ is non-locally compact. By [1, Theorem 2],

$$H_0 \cong \mathbb{Z}^k \oplus \mathbb{R}^l,$$

where $k, l \in \mathbb{N} \cup \{\mathbb{N}\}$, and, by non-local compactness of H_0 , at least one of them is \mathbb{N} . Applying Theorem 4 to each element of the infinite product, we can find an action β of H_0 such that $E_0^{\mathbb{N}} \subseteq E_\beta$. Using Theorem 3, we extend β to an action α of G so that $E_0^{\mathbb{N}} \subseteq E_\alpha$. ■

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