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A HILBERT-TYPE INTEGRAL INEQUALITY WITH A HYBRID KERNEL AND ITS APPLICATIONS

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Abstract. We prove a multi-parameter Hilbert-type integral inequality with a hybrid kernel. We describe the best constant in the inequality in terms of hypergeometric functions. Some equivalent forms of the inequalities are also studied. By specifying parameter values we obtain results proved by other authors as well as many new inequalities.

1. Introduction. Let $\theta(x)$ (> 0) be a measurable function on $(0,\infty)$ and $\rho \geq 1$. Denote

$$L^\rho(0,\infty) := \Big\{h;\ \|h\|_\rho := \Big(\int_0^\infty |h(x)|^\rho\,dx\Big)^{1/\rho} < \infty\Big\},$$

$$L^\rho_\theta(0,\infty) := \Big\{h;\ \|h\|_{\rho,\theta} := \Big(\int_0^\infty \theta(x)|h(x)|^\rho\,dx\Big)^{1/\rho} < \infty\Big\}.$$

If $f, g \geq 0$, $f, g \in L^2(0,\infty)$, $\|f\|_2, \|g\|_2 > 0$, we have (see [1])

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\|f\|_2\|g\|_2, \tag{1.1}$$

where the constant factor π is best possible. Inequality (1.1) is known as the *Hilbert integral inequality*. In 1925, Hardy–Riesz gave an extension of (1.1) by introducing a pair of conjugate indices (p,q) (see [1]). Namely, if $p > 1$, $1/p + 1/q = 1$, $f \in L^p(0,\infty)$, $g \in L^q(0,\infty)$, $\|f\|_p, \|g\|_q > 0$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is optimal. Inequality (1.2) is referred to as *Hardy–Hilbert's integral inequality*, which is important in analysis and applications (see [2]). Recently, Yang et al. (see [8], [11], [9], [6]) studied

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the integral kernels $\frac{1}{(x+y)^\lambda}$, $\frac{1}{|x-y|^\lambda}$, $\frac{1}{(\max\{x,y\})^\lambda}$, $(\min\{x,y\})^\lambda$, by introducing an independent parameter λ; the corresponding Hilbert-type integral inequalities are as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < B(\lambda/2,\lambda/2)\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{1.3}$$

where $\lambda > 0$, $\varphi(x) = x^{1-\lambda}$, and $B(u,v)$ is Euler's beta function;

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}\,dx\,dy < 2B(\lambda/2,1-\lambda)\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{1.4}$$

where $0 < \lambda < 1$ and $\varphi(x) = x^{1-\lambda}$;

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\max\{x,y\})^\lambda}\,dx\,dy < \frac{4}{\lambda}\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{1.5}$$

where $\lambda > 0$ and $\varphi(x) = x^{1-\lambda}$; and

$$\int_0^\infty\int_0^\infty (\min\{x,y\})^\lambda f(x)g(y)\,dx\,dy < \frac{4}{\lambda}\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{1.6}$$

where $\lambda > 0$, $\varphi(x) = x^{1+\lambda}$. All constants in these inequalities are optimal.

At the same time, some Hilbert-type integral inequalities have been obtained by suitably mixing these kernels (see [10], [7]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda}(\min\{x,y\})^\lambda}\,dx\,dy < B(1/2-\lambda,\lambda)\|f\|_2\|g\|_2, \tag{1.7}$$

where $0 < \lambda < 1/2$, and

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda-1}f(x)g(y)}{(\max\{x,y\})^\lambda}\,dx\,dy < B(1/2,\lambda)\|f\|_2\|g\|_2, \tag{1.8}$$

where $\lambda > 0$. Again the constants are best possible.

In this paper, by introducing some parameters, and using weight functions and techniques of real analysis, a Hilbert-type integral inequality and its equivalent form with a hybrid kernel $k(x,y) := \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}$ are given. Moreover, we prove that the constant factors linked to the hypergeometric function are optimal. As applications, some interesting results are obtained by selecting special parameter values.

2. Some lemmas. We need the following special functions (see [3]): the Γ-function

$$\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}\,du \qquad (z > 0), \tag{2.1}$$

and the β-function: if $u, v > 0$,

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt = \int_0^1 \frac{t^{u-1}+t^{v-1}}{(1+t)^{u+v}}\,dt = \int_0^1 (1-t)^{u-1}t^{v-1}\,dt = B(v,u). \tag{2.2}$$

If $\mathrm{Re}(\gamma_3) > \mathrm{Re}(\gamma_2) > 0$ and $|\arg(1-z)| < \pi$, the following function is called the *hypergeometric function*:

$$F(\gamma_1,\gamma_2,\gamma_3,z) = \frac{\Gamma(\gamma_3)}{\Gamma(\gamma_2)\Gamma(\gamma_3-\gamma_2)}\int_0^1 t^{\gamma_2-1}(1-t)^{\gamma_3-\gamma_2-1}(1-zt)^{-\gamma_1}\,dt. \tag{2.3}$$

LEMMA 2.1. *For $p > 1$, $\lambda_1 > -1$, $\lambda_3, \lambda_4 \ge 0$, and $\lambda_2+\lambda_3+\lambda_4 > \lambda_1$, define the weight function*

$$\omega(p,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) := \int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\,\frac{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{x^{p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2q}}\,dy \qquad (y>0),$$

where $1/p+1/q = 1$. Then

$$\omega(p,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) = C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1},$$

where

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \frac{2\Gamma\big(\frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}\big)\Gamma(\lambda_1+1)}{\Gamma\big(\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1\big)} \times F\bigg(\lambda_3, \frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}, \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1, -1\bigg). \tag{2.4}$$

Proof. Set $y/x = t$. Then

$$\begin{aligned}
&\omega(p,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x)\\
&= \int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\,\frac{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{x^{p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2q}}\,dy\\
&= x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\int_0^\infty \frac{|1-t|^{\lambda_1}(\min\{1,t\})^{\lambda_2}t^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{(1+t)^{\lambda_3}(\max\{1,t\})^{\lambda_4}}\,dt\\
&= x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\\
&\quad\times\bigg[\int_0^1 \frac{(1-t)^{\lambda_1}t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1-2)/2}}{(1+t)^{\lambda_3}}\,dt + \int_1^\infty \frac{(t-1)^{\lambda_1}t^{(\lambda_3-\lambda_1-\lambda_2-\lambda_4-2)/2}}{(1+t)^{\lambda_3}}\,dt\bigg]\\
&= 2x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\int_0^1 \frac{(1-t)^{\lambda_1}t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1-2)/2}}{(1+t)^{\lambda_3}}\,dt
\end{aligned}$$

$$= 2x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\int_0^1 t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1)/2-1}$$
$$\times (1-t)^{((\lambda_1+\lambda_2+\lambda_3+\lambda_4)/2+1)-(\lambda_2+\lambda_3+\lambda_4-\lambda_1)/2-1}(1+t)^{-\lambda_3}\,dt$$
$$= C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}.$$

LEMMA 2.2. *For* $p>1$, $1/p+1/q=1$, $\lambda_1>-1$, $\lambda_3,\lambda_4\ge 0$, $\lambda_2+\lambda_3+\lambda_4>\lambda_1$, $0<\varepsilon<(\lambda_2+\lambda_3+\lambda_4-\lambda_1)/(2q)$, *and* ε *small enough, define the real functions as follows:*

$$\tilde f_\varepsilon(x) := \begin{cases} 0, & x\in(0,1),\\ x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-\varepsilon/p}, & x\in[1,\infty),\end{cases}$$

$$\tilde g_\varepsilon(y) := \begin{cases} 0, & y\in(0,1),\\ y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-\varepsilon/q}, & y\in[1,\infty).\end{cases}$$

Then

$$\Big[\int_0^\infty x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\tilde f_\varepsilon^p(x)\,dx\Big]^{1/p} \times \Big[\int_0^\infty y^{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\tilde g_\varepsilon^q(y)\,dy\Big]^{1/q}\varepsilon = 1, \tag{2.5}$$

$$\tilde I\varepsilon := \varepsilon\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\tilde f_\varepsilon(x)\tilde g_\varepsilon(y)\,dx\,dy > C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)(1-o(1)) \quad (\varepsilon\to 0^+). \tag{2.6}$$

Proof. We easily obtain

$$\Big[\int_0^\infty x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\tilde f_\varepsilon^p(x)\,dx\Big]^{1/p}\Big[\int_0^\infty y^{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}\tilde g_\varepsilon^q(y)\,dy\Big]^{1/q}\varepsilon$$
$$= \Big[\int_1^\infty x^{-1-\varepsilon}\,dx\Big]^{1/p}\Big[\int_0^1 y^{-1-\varepsilon}\,dy\Big]^{1/q}\varepsilon = 1.$$

Since $F(t) = \frac{(1-t)^{1+\lambda_1}t^{(\lambda-2+\lambda_3+\lambda_4-\lambda_1)/2-(\varepsilon+\sqrt[3]{\varepsilon})/q}}{(1+t)^{\lambda_3}}$ is continuous in $(0,1]$, and $\lim_{t\to 0^+}F(t)=0$, there exists $M>0$ such $F(t)\le M$. Hence, by Fubini's theorem (see [4]),

$$\tilde I\varepsilon = \varepsilon\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\tilde f(x)\tilde g(y)\,dx\,dy$$
$$= \varepsilon\int_1^\infty x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-\varepsilon/p}\,dx\Big[\int_1^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \times y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-\varepsilon/q}\,dy\Big]$$

$$
\begin{aligned}
&= \varepsilon \int_1^\infty x^{-1-\varepsilon}\,dx \Big[\int_{1/x}^\infty \frac{|1-t|^{\lambda_1}(\min\{1,t\})^{\lambda_2} t^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-\varepsilon/q}}{(1+t)^{\lambda_3}(\max\{1,t\})^{\lambda_4}}\,dt\Big] \\
&= \varepsilon \int_1^\infty x^{-1-\varepsilon}\,dx \Big[\int_0^1 \frac{(1-t)^{\lambda_1} t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1-2)/2-\varepsilon/q}}{(1+t)^{\lambda_3}}\,dt \\
&\qquad\qquad + \int_1^\infty \frac{(t-1)^{\lambda_1} t^{(\lambda_3-\lambda_1-\lambda_2-\lambda_4-2)/2-\varepsilon/q}}{(1+t)^{\lambda_3}}\,dt\Big] \\
&\quad - \varepsilon \int_1^\infty x^{-1-\varepsilon}\,dx \int_0^{1/x} \frac{(1-t)^{\lambda_1} t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1-2)/2-\varepsilon/q}}{(1+t)^{\lambda_3}}\,dt \\
&= \frac{\Gamma\big(\frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}-\frac{\varepsilon}{q}\big)\Gamma(\lambda_1+1)}{\Gamma\big(\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1-\frac{\varepsilon}{q}\big)} \\
&\qquad \times F\Big(\lambda_3, \frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}-\frac{\varepsilon}{q}, \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1-\frac{\varepsilon}{q}, -1\Big) \\
&\quad + \frac{\Gamma\big(\frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}+\frac{\varepsilon}{q}\big)\Gamma(\lambda_1+1)}{\Gamma\big(\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1+\frac{\varepsilon}{q}\big)} \\
&\qquad \times F\Big(\lambda_3, \frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}+\frac{\varepsilon}{q}, \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1+\frac{\varepsilon}{q}, -1\Big) \\
&\quad - \varepsilon \int_1^\infty x^{-1-\varepsilon}\,dx \int_0^{1/x} \frac{(1-t)^{\lambda_1} t^{(\lambda_2+\lambda_3+\lambda_4-\lambda_1-2)/2-\varepsilon/q}}{(1+t)^{\lambda_3}}\,dt \\
&> \frac{2\Gamma\big(\frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}\big)\Gamma(\lambda_1+1)}{\Gamma\big(\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1\big)} \\
&\qquad \times F\Big(\lambda_3, \frac{\lambda_2+\lambda_3+\lambda_4-\lambda_1}{2}, \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2}+1, -1\Big) + o_1(1) \\
&\quad - \varepsilon \int_1^\infty x^{-1}\Big[\int_0^{x^{-1}} \frac{(1-t)^{1+\lambda_1} t^{(\lambda-2+\lambda_3+\lambda_4-\lambda_1)/2-(\varepsilon+\sqrt[3]{\varepsilon})/q}}{(1+t)^{\lambda_3}}\,\frac{t^{-1+\sqrt[3]{\varepsilon}/q}}{1-t}\,dt\Big]\,dx \\
&\ge C(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + o_1(1) - M\varepsilon \int_1^\infty x^{-1}\Big[\int_0^{x^{-1}} \frac{t^{-1+\sqrt[3]{\varepsilon}/q}}{1-t}\,dt\Big]\,dx \\
&= C(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + o_1(1) - M\varepsilon \sum_{k=0}^\infty \int_1^\infty x^{-1}\Big[\int_0^{x^{-1}} t^{-1+k+\sqrt[3]{\varepsilon}/q}\,dt\Big]\,dx \\
&= C(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + o_1(1) - M\varepsilon \sum_{k=0}^\infty \frac{1}{(k+\sqrt[3]{\varepsilon}/q)^2} \\
&= C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)(1-o(1)) \qquad (\varepsilon\to 0^+).
\end{aligned}
$$

3. Main results and applications

THEOREM 3.1. *If* $p > 1$, $1/p+1/q = 1$, $\lambda_1 > -1$, $\lambda_3, \lambda_4 \geq 0$, $\lambda_2+\lambda_3+\lambda_4 > \lambda_1$, $\varphi(x) = x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}$, $\psi(y) = y^{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}$, *and* $f \in L^p_\varphi(0,\infty)$, $g \in L^q_\psi(0,\infty)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, *then*

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f(x)g(y)\,dx\,dy < C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.1}$$

where the constant factor $C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ *defined in* (2.4) *is best possible.*

Proof. From Hölder's inequality, Fubini's theorem, and Lemma 2.1, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f(x)g(y)\,dx\,dy \qquad (3.2)\\
&= \int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \left[\frac{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2p)}}{x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2q)}}\right]\\
&\quad\times\left[\frac{x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2q)}}{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2p)}}\right] f(x)g(y)\,dx\,dy\\
&\leq \left[\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2} f^p(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \frac{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{x^{p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2q)}}\,dx\,dy\right]^{1/p}\\
&\quad\times\left[\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2} g^q(y)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} \frac{x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{y^{(q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2))/(2p)}}\,dx\,dy\right]^{1/q}\\
&= \left\{\int_0^\infty \omega(p,\lambda_1,\lambda_2,\lambda_3,\lambda_4,x) f^p(x)\,dx\right\}^{1/p}\left\{\int_0^\infty \omega(q,\lambda_1,\lambda_2,\lambda_3,\lambda_4,y) g^q(y)\,dy\right\}^{1/q}\\
&= C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\|f\|_{p,\varphi}\|g\|_{q,\psi}.
\end{aligned}$$

Now assume equality holds in (3.2). Then according to [5], there exist two nonzero constants A and B such that

$$A\frac{y^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{x^{p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2q)}} f^p(x) = B\frac{x^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2}}{y^{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/(2p)}} g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

Thus

$$Ax^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2} f^p(x) = By^{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2} g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

Assuming that $A \neq 0$, there exists $y > 0$ such that

$$x^{-p(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1} f^p(x) = [y^{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2} g^q(y)] \frac{B}{Ax} \quad \text{a.e. in } (0,\infty),$$

which contradicts the fact that $0 < \|f\|_{p,\varphi} < \infty$. Thus inequality (3.2) is strict.

If the constant factor $C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ in (3.1) is not optimal, then there exists a positive $K < C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ such that (3.1) is still valid if we replace $C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ by K. Hence by (2.5) and (2.6), we have

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)(1-o(1)) < K.$$

Letting $\varepsilon \to 0^+$, we get $K \geq C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, which contradicts the fact that $K < C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Therefore $C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ in (3.1) is the best constant.

THEOREM 3.2. *Under the conditions of Theorem* 3.1, *we have*

$$\int_0^\infty y^{\frac{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2+1}{q-1}} dy \left\{ \int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} f(x)\,dx \right\}^p < C^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\|f\|^p_{p,\varphi}, \tag{3.3}$$

where the constant factor $C^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is best possible.

Proof. Define

$$[f(x)]_n := \min\{n, f(x)\} = \begin{cases} f(x), & f(x) < n, \\ n, & f(x) \geq n. \end{cases}$$

Since $0 < \|f\|_{p,\varphi} < \infty$, there exists $n_0 \in \mathbb{N}$ such that $0 < \int_{1/n}^n \varphi(x)[f(x)]_n^p\,dx < \infty$ $(n \geq n_0)$. Setting

$$g_n(y) := y^{\frac{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2+1}{q-1}} \left[\int_{1/n}^n \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} [f(x)]_n\,dx \right]^{p/q} \quad (1/n < y < n,\ n \geq n_0),$$

when $n \geq n_0$, by (3.1), we obtain

$$\begin{aligned} 0 &< \int_{1/n}^n \psi(y) g_n^q(y)\,dy \\ &= \int_{1/n}^n y^{\frac{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2+1}{q-1}} \left\{ \int_{1/n}^n \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}} [f(x)]_n\,dx \right\}^p dy \end{aligned} \tag{3.4}$$

$$= \int_{1/n}^{n}\int_{1/n}^{n} \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}[f(x)]_n g_n(y)\,dx\,dy$$
$$< C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Big\{\int_{1/n}^{n}\varphi(x)[f(x)]_n^p\,dx\Big\}^{1/p}\Big\{\int_{1/n}^{n}\psi(y)g_n^q(y)\,dy\Big\}^{1/q}.$$

Moreover, by (3.4) we have

$$0 < \int_{1/n}^{n}\psi(y)g_n^q(y)\,dy < C^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\int_0^\infty \varphi(x)f^p(x)\,dx < \infty, \tag{3.5}$$

and it follows that $0 < \|f\|_{p,\varphi} < \infty$. For $n \to \infty$, by (3.1), both (3.4) and (3.5) still keep the form of strict inequalities. Hence we have inequality (3.3).

It remains to show that the constant $C^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is optimal. Assume that (3.3) holds with some $K > 0$. Then by Hölder's inequality and Fubini's theorem, we have

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}f(x)g(y)\,dx\,dy$$
$$= \int_0^\infty \Big\{y^{\frac{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2+1}{p(q-1)}}\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\,dx\Big\}$$
$$\times\Big\{y^{\frac{-q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2-1}{p(q-1)}}g(y)\Big\}\,dy$$
$$\le \Big\{\int_0^\infty y^{\frac{q(\lambda_3+\lambda_4-\lambda_1-\lambda_2-2)/2+1}{q-1}}\,dy\Big[\int_0^\infty \frac{|x-y|^{\lambda_1}(\min\{x,y\})^{\lambda_2}f(x)}{(x+y)^{\lambda_3}(\max\{x,y\})^{\lambda_4}}\,dx\Big]^p\Big\}^{1/p}\|g\|_{q,\psi}$$
$$< K^{1/p}\|f\|_{p,\varphi}\|g\|_{q,\psi}.$$

Now from Theorem 3.1 it follows that $K^{1/p} \le C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Therefore $C^p(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is optimal in (3.3). ∎

Selecting some special parameter values in (3.1) and (3.3), and using mathematics software to calculate, the related results of the references are obtained, and new Hilbert-type inequalities and their equivalent forms are given.

EXAMPLE 3.1. Setting $p = q = 2$, and respectively letting $\lambda_1 = \lambda_2 = \lambda_4 = 0$, $\lambda_3 = \lambda > 0$; $\lambda_1 = -\lambda$ $(0 < \lambda < 1)$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$; $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = \lambda > 0$; $\lambda_1 = \lambda_3 = \lambda_4 = 0$, $\lambda_2 = \lambda > 0$, by (2.4) we get $C(0,0,\lambda,0) = B(\lambda/2,\lambda/2)$; $C(-\lambda,0,0,0) = 2B(\lambda/2,1-\lambda)$; $C(0,0,0,\lambda) = 4/\lambda$; $C(0,\lambda,0,0) = 4/\lambda$. Hence, by Theorems 3.1 and 3.2, we obtain (1.3)–(1.6) and their equivalent forms.

EXAMPLE 3.2. Letting $\lambda_1 = \alpha$, $\lambda_2 = \beta$, $\lambda_3 = \lambda_4 = 0$ and $\alpha > -1$, $\beta > \alpha$, by (2.4) we have $C(\alpha, \beta, 0, 0) = 2B(\alpha+1, (\beta-\alpha)/2)$. For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty |x-y|^\alpha(\min\{x,y\})^\beta f(x)g(y)\,dx\,dy < 2B\left(\alpha+1,\frac{\beta-\alpha}{2}\right)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.6}$$

$$\int_0^\infty y^{\frac{q(\beta-\alpha-2)/q+1}{q-1}}\,dy\left[\int_0^\infty |x-y|^\alpha(\min\{x,y\})^\beta f(x)\,dx\right]^p < \left[2B\left(\alpha+1,\frac{\beta-\alpha}{2}\right)\right]^p\|f\|^p_{p,\varphi}, \tag{3.7}$$

where $\varphi(x) = x^{p(\alpha+\beta+2)/2-1}$, $\psi(y) = y^{q(\alpha+\beta+2)/2-1}$. Moreover, the constants $2B(\alpha+1,(\beta-\alpha)/2)$ and $\left[2B(\alpha+1,(\beta-\alpha)/2)\right]^p$ are optimal in (3.6) and (3.7), respectively. In particular taking $p = q = 2$, $\alpha = \lambda - 1$, $\beta = -\lambda$, we get $0 < \lambda < 1/2$, $C(\lambda-1,-\lambda,0,0) = B(1/2-\lambda,\lambda)$, $\varphi(x) = \psi(y) = 1$. By (3.6) and (3.7), we get (1.7) and its equivalent form.

EXAMPLE 3.3. Letting $\lambda_1 = \alpha$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = \beta$ and $\alpha > -1$, $\beta > \alpha$, by (2.3) we have $C(\alpha, 0, 0, \beta) = 2B(\alpha+1, (\beta-\alpha)/2)$. For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^\alpha}{(\max\{x,y\})^\beta} f(x)g(y)\,dx\,dy < 2B\left(\alpha+1,\frac{\beta-\alpha}{2}\right)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.8}$$

$$\int_0^\infty y^{\frac{q(\beta-\alpha-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{|x-y|^\alpha f(x)}{(\max\{x,y\})^\beta}\,dx\right]^p < \left[2B\left(\alpha+1,\frac{\beta-\alpha}{2}\right)\right]^p\|f\|^p_{p,\varphi}, \tag{3.9}$$

where $\varphi(x) = x^{-p(\beta-\alpha-2)/2-1}$, $\psi(y) = y^{-q(\beta-\alpha-2)/2-1}$. Moreover, the constants $2B(\alpha+1,(\beta-\alpha)/2)$ and $\left[2B(\alpha+1,(\beta-\alpha)/2)\right]^p$ are optimal in (3.8) and (3.9), respectively. In particular taking $p = q = 2$, $\alpha = \lambda-1$, $\beta = \lambda > 0$, we get $C(\lambda-1,0,0,\lambda) = B(1/2,\lambda)$, $\varphi(x) = \psi(y) = 1$. By (3.8) and (3.9), we obtain (1.8) and its equivalent form.

EXAMPLE 3.4. Letting $\lambda_1 = \alpha$, $\lambda_2 = \lambda_4 = 0$, $\lambda_3 = \beta$ and $\alpha > -1$, $\beta > \alpha$, by (2.4) we get

$$C(\alpha,0,\beta,0) = \frac{2\Gamma(\alpha+1)\Gamma\left(\frac{\beta-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}+1\right)}F\left(\beta,\frac{\beta-\alpha}{2},\frac{\alpha+\beta}{2}+1,-1\right).$$

For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we obtain the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^\alpha}{(x+y)^\beta} f(x)g(y)\,dx\,dy < C(\alpha,0,\beta,0)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.10}$$

$$\int_0^\infty y^{\frac{q(\beta-\alpha-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{|x-y|^\alpha f(x)}{(x+y)^\beta}\,dx\right]^p < C^p(\alpha,0,\beta,0)\|f\|^p_{p,\varphi}, \tag{3.11}$$

where $\varphi(x) = x^{-p(\beta-\alpha-2)/2-1}$ and $\psi(y) = y^{-q(\beta-\alpha-2)/2-1}$. Moreover, the constants $C(\alpha,0,\beta,0)$ and $C^p(\alpha,0,\beta,0)$ are optimal in (3.10) and (3.11), respectively. In particular taking $p = q = 2$, $\alpha = -1/2$, $\beta = 1$, we get $C(-1/2,0,1,0) = \pi$, $\varphi(x) = 1/\sqrt{x}$, and obtain the equivalent inequalities

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)\sqrt{|x-y|}}\,dx\,dy < \pi\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{3.12}$$

$$\int_0^\infty \sqrt{y}\,dy\left[\int_0^\infty \frac{f(x)}{(x+y)\sqrt{|x-y|}}\,dx\right]^2 < \pi^2\|f\|^2_{2,\varphi}, \tag{3.13}$$

where the constants π and π^2 are optimal in (3.12) and (3.13), respectively.

Example 3.5. Letting $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \alpha$, $\lambda_3 = \beta$ and $\beta > 0$, $\alpha+\beta > 0$, by (2.4) we get

$$C(0,\alpha,\beta,0) = \frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right).$$

For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\begin{aligned}\int_0^\infty\int_0^\infty &\frac{(\min\{x,y\})^\alpha f(x)g(y)}{(x+y)^\beta}\,dx\,dy \\ &< \frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)\|f\|_{p,\varphi}\|g\|_{q,\psi},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\int_0^\infty y^{\frac{q(\alpha+\beta-2)/q+1}{q-1}}\,dy&\left[\int_0^\infty \frac{(\min\{x,y\})^\alpha f(x)}{(x+y)^\beta}\,dx\right]^p \\ &< \left[\frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)\right]^p\|f\|^p_{p,\varphi},\end{aligned} \tag{3.15}$$

where $\varphi(x) = x^{-p(\beta-\alpha-2)/2-1}$ and $\psi(y) = y^{-q(\beta-\alpha-2)/2-1}$. Moreover, the constants $\frac{4}{\alpha+\beta}F\big(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\big)$ and $\big[\frac{4}{\alpha+\beta}F\big(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\big)\big]^p$ are optimal in (3.14) and (3.15), respectively. In particular taking $p = q = 2$ and $\alpha = \beta = 1/2$ we get $C(0,1/2,1/2,0) = 2\ln(3+2\sqrt{2})$ and $\varphi(x) = x$, and obtain the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \sqrt{\frac{\min\{x,y\}}{x+y}} f(x)g(y)\,dx\,dy < 2\ln(3+2\sqrt{2})\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{3.16}$$

$$\int_0^\infty dy\left[\int_0^\infty \sqrt{\frac{\min\{x,y\}}{x+y}} f(x)\,dx\right]^2 < 4\ln^2(3+2\sqrt{2})\|f\|^2_{2,\varphi}, \tag{3.17}$$

where the constants $2\ln(3+2\sqrt{2})$ and $4\ln^2(3+2\sqrt{2})$ are optimal in (3.16) and (3.17), respectively.

EXAMPLE 3.6. Letting $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = \alpha$, $\lambda_4 = \beta$ and $\beta > 0$, $\alpha+\beta > 0$, by (2.4) we get $C(0,\alpha,0,\beta) = 4/(\alpha+\beta)$. For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{(\min\{x,y\})^\alpha f(x)g(y)}{(\max\{x,y\})^\beta}\,dx\,dy < \frac{4}{\alpha+\beta}\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.18}$$

$$\int_0^\infty y^{\frac{q(\alpha+\beta-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{(\min\{x,y\})^\alpha f(x)}{(\max\{x,y\})^\beta}\,dx\right]^p < \left[\frac{4}{\alpha+\beta}\right]^p \|f\|^p_{p,\varphi}, \tag{3.19}$$

where $\varphi(x) = x^{-p(\beta-\alpha-2)/2-1}$ and $\psi(y) = y^{-q(\beta-\alpha-2)/2-1}$. Moreover, the constants $\frac{4}{\alpha+\beta}$ and $\left[\frac{4}{\alpha+\beta}\right]^p$ are optimal in (3.18) and (3.19), respectively.

EXAMPLE 3.7. Letting $\lambda_1 = \lambda_2 = 0$, $\lambda_4 = \alpha$, $\lambda_3 = \beta$ and $\beta > 0$, $\alpha+\beta > 0$, by (2.4) we get $C(0,0,\beta,\alpha) = \frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)$. For $f \in L^p_\varphi(0,\infty)$ and $g \in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\begin{aligned} &\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\max\{x,y\})^\alpha (x+y)^\beta}\,dx\,dy \\ &\qquad < \frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\int_0^\infty y^{\frac{q(\alpha+\beta-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{f(x)}{(\max\{x,y\})^\alpha (x+y)^\beta}\,dx\right]^p \\ &\qquad < \left[\frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)\right]^p \|f\|^p_{p,\varphi}, \end{aligned} \tag{3.21}$$

where $\varphi(x) = x^{-p(\alpha+\beta-2)/2-1}$ and $\psi(y) = y^{-q(\alpha+\beta-2)/2-1}$. Moreover, the constants $\frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)$ and $\left[\frac{4}{\alpha+\beta}F\left(\beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+1, -1\right)\right]^p$ are optimal in (3.20) and (3.21), respectively. Some special inequalities can be obtained by selecting specific α, β; e.g. letting $p = q = 2$, $\alpha = \beta = 1/2$, we get $C(0,0,1/2,1/2) = 2\ln(3+2\sqrt{2})$ and $\varphi(x) = \psi(y) = 1$, and obtain the equivalent inequalities

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\sqrt{(x+y)\max\{x,y\}}}\,dx\,dy < 2\ln(3+2\sqrt{2})\|f\|_2\|g\|_2, \tag{3.22}$$

$$\int_0^\infty dy\left[\int_0^\infty \frac{f(x)}{\sqrt{(x+y)\max\{x,y\}}}\,dx\right]^2 < 4\ln^2(3+2\sqrt{2})\|f\|_2^2, \tag{3.23}$$

where the constants $2\ln(3+2\sqrt{2})$ and $4\ln^2(3+2\sqrt{2})$ are optimal in (3.22) and (3.23), respectively.

EXAMPLE 3.8. Letting $\lambda_1=\alpha$, $\lambda_2=0$, $\lambda_3=\beta$, $\lambda_4=\gamma$ and $\alpha>-1$, $\beta>0$, $\beta+\gamma>\alpha$ by (2.4), we have

$$C(\alpha,0,\beta,\gamma)=\frac{\Gamma\big(\frac{\beta+\gamma-\alpha}{2}\big)\Gamma(\alpha+1)}{\Gamma\big(\frac{\alpha+\beta+\gamma}{2}+1\big)}F\Big(\beta,\frac{\beta+\gamma-\alpha}{2},\frac{\alpha+\beta+\gamma}{2}+1,-1\Big).$$

If $f\in L^p_\varphi(0,\infty)$ and $g\in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi},\|g\|_{q,\psi}>0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^\alpha f(x)g(y)}{(x+y)^\beta(\max\{x,y\})^\gamma}\,dx\,dy < C(\alpha,0,\beta,\gamma)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.24}$$

$$\int_0^\infty y^{\frac{q(\beta+\gamma-\alpha-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{|x-y|^\alpha f(x)}{(x+y)^\beta(\max\{x,y\})^\gamma}\,dx\right]^p < C^p(\alpha,0,\beta,\gamma)\|f\|_{p,\varphi}^p, \tag{3.25}$$

where $\varphi(x)=x^{-p(\beta+\gamma-\alpha-2)/2-1}$ and $\psi(y)=y^{-q(\beta+\gamma-\alpha-2)/2-1}$. Moreover, the constants $C(\alpha,0,\beta,\gamma)$ and $C^p(\alpha,0,\beta,\gamma)$ are optimal in (3.24) and (3.25), respectively. Some special inequalities can be obtained by selecting specific α,β,γ; e.g. letting $p=q=2$, $\alpha=\beta=\gamma=1$, we get $C(1,0,1,1)=2\pi-4$ and $\varphi(x)=\psi(y)=1$, and obtain the equivalent inequalities

$$\int_0^\infty\int_0^\infty \frac{|x-y|}{(x+y)\max\{x,y\}}f(x)g(y)\,dx\,dy < (2\pi-4)\|f\|_2\|g\|_2, \tag{3.26}$$

$$\int_0^\infty dy\left[\int_0^\infty \frac{|x-y|}{(x+y)\max\{x,y\}}f(x)\,dx\right]^2 < (2\pi-4)^2\|f\|_2^2, \tag{3.27}$$

where the constants $2\pi-4$ and $(2\pi-4)^2$ are optimal in (3.26) and (3.27), respectively.

EXAMPLE 3.9. Letting $\lambda_1=\alpha$, $\lambda_2=\beta$, $\lambda_3=0$, $\lambda_4=\gamma$ and $\alpha>-1$, $\gamma>0$, $\beta+\gamma>\alpha$, by (2.4) we get $C(\alpha,\beta,0,\gamma)=2B\big(\frac{\beta+\gamma-\alpha}{2},\alpha+1\big)$. If $f\in L^p_\varphi(0,\infty)$ and $g\in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi},\|g\|_{q,\psi}>0$, by Theorems 3.1

and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^\alpha(\min\{x,y\})^\beta}{(\max\{x,y\})^\gamma} f(x)g(y)\,dx\,dy < 2B\Big(\frac{\beta+\gamma-\alpha}{2},\alpha+1\Big)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.28}$$

$$\int_0^\infty y^{\frac{q(\beta+\gamma-\alpha-2)/q+1}{q-1}}\,dy\bigg[\int_0^\infty \frac{|x-y|^\alpha(\min\{x,y\})^\beta}{(\max\{x,y\})^\gamma} f(x)\,dx\bigg]^p < \Big[2B\Big(\frac{\beta+\gamma-\alpha}{2},\alpha+1\Big)\Big]^p\|f\|_{p,\varphi}^p, \tag{3.29}$$

where $\varphi(x)=x^{-p(\gamma-\alpha-\beta-2)/2-1}$ and $\psi(y)=y^{-q(\gamma-\alpha-\beta-2)/2-1}$. Moreover, the constants $2B\big(\frac{\beta+\gamma-\alpha}{2},\alpha+1\big)$ and $\big[2B\big(\frac{\beta+\gamma-\alpha}{2},\alpha+1\big)\big]^p$ are optimal in (3.28) and (3.29), respectively.

EXAMPLE 3.10. Letting $\lambda_1=\alpha$, $\lambda_2=\beta$, $\lambda_3=\gamma$, $\lambda_4=0$ and $\alpha>-1$, $\beta>0$, $\beta+\gamma>\alpha$, by (2.4) we get

$$C(\alpha,\beta,\gamma,0)=\frac{\Gamma\big(\frac{\beta+\gamma-\alpha}{2}\big)\Gamma(\alpha+1)}{\Gamma\big(\frac{\alpha+\beta+\gamma}{2}+1\big)}F\Big(\beta,\frac{\beta+\gamma-\alpha}{2},\frac{\alpha+\beta+\gamma}{2}+1,-1\Big).$$

If $f\in L^p_\varphi(0,\infty)$ and $g\in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi},\|g\|_{q,\psi}>0$, by Theorems 3.1 and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^\alpha(\min\{x,y\})^\beta}{(x+y)^\gamma} f(x)g(y)\,dx\,dy < C(\alpha,\beta,\gamma,0)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.30}$$

$$\int_0^\infty y^{\frac{q(\beta+\gamma-\alpha-2)/q+1}{q-1}}\,dy\bigg[\int_0^\infty \frac{|x-y|^\alpha(\min\{x,y\})^\beta}{(x+y)^\gamma} f(x)\,dx\bigg]^p < C^p(\alpha,\beta,\gamma,0)\|f\|_{p,\varphi}^p, \tag{3.31}$$

where $\varphi(x)=x^{-p(\gamma-\alpha-\beta-2)/2-1}$ and $\psi(y)=y^{-q(\gamma-\alpha-\beta-2)/2-1}$. Moreover, the constants $C(\alpha,\beta,\gamma,0)$ and $C^p(\alpha,\beta,\gamma,0)$ are optimal in (3.30) and (3.31), respectively.

EXAMPLE 3.11. Letting $\lambda_1=0$, $\lambda_2=\alpha$, $\lambda_3=\beta$, $\lambda_4=\gamma$ and $\beta,\gamma>0$, $\alpha+\beta+\gamma>0$, by (2.4) we get

$$C(0,\alpha,\beta,\gamma)=\frac{4}{\alpha+\beta+\gamma}F\Big(\beta,\frac{\alpha+\beta+\gamma}{2},\frac{\alpha+\beta+\gamma}{2}+1,-1\Big).$$

If $f\in L^p_\varphi(0,\infty)$ and $g\in L^q_\varphi(0,\infty)$ with $\|f\|_{p,\varphi},\|g\|_{q,\psi}>0$, by Theorems 3.1

and 3.2 we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{(\min\{x,y\})^\alpha f(x)g(y)}{(x+y)^\beta(\max\{x,y\})^\gamma}\,dx\,dy < C(0,\alpha,\beta,\gamma)\|f\|_{p,\varphi}\|g\|_{q,\psi}, \tag{3.32}$$

$$\int_0^\infty y^{\frac{q(\alpha+\beta+\gamma-2)/q+1}{q-1}}\,dy\left[\int_0^\infty \frac{(\min\{x,y\})^\alpha f(x)}{(x+y)^\beta(\max\{x,y\})^\gamma}\,dx\right]^p < C^p(0,\alpha,\beta,\gamma)\|f\|_{p,\varphi}^p, \tag{3.33}$$

where $\varphi(x) = x^{-p(\beta+\gamma-\alpha-2)/2-1}$ and $\psi(y) = y^{-q(\beta+\gamma-\alpha-2)/2-1}$. Moreover, the constants $C(0,\alpha,\beta,\gamma)$ and $C^p(0,\alpha,\beta,\gamma)$ are optimal in (3.32) and (3.33), respectively. Some special inequalities can be obtained by selecting specific α,β,γ; e.g. letting $p = q = 2$, $\alpha = \beta = \gamma = 1$, we get $C(0,1,1,1) = 4-\pi$ and $\varphi(x) = \psi(y) = 1$, and obtain the equivalent inequalities

$$\int_0^\infty\int_0^\infty \frac{\min\{x,y\}}{(x+y)\max\{x,y\}} f(x)g(y)\,dx\,dy < (4-\pi)\|f\|_2\|g\|_2, \tag{3.34}$$

$$\int_0^\infty dy\left[\int_0^\infty \frac{\min\{x,y\}}{(x+y)\max\{x,y\}} f(x)\,dx\right]^2 < (4-\pi)^2\|f\|_2^2, \tag{3.35}$$

where the constants $4-\pi$ and $(4-\pi)^2$ are optimal in (3.34) and (3.35), respectively.

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