BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 63, No. 3, 2015

FUNCTIONS OF A COMPLEX VARIABLE

Generalized Hilbert Operators on Bergman and Dirichlet Spaces of Analytic Functions

by

Sunanda NAIK and Karabi RAJBANGSHI

Presented by Józef SICIAK

Summary. Let f be an analytic function on the unit disk $\mathbb{D}$. We define a generalized Hilbert-type operator $\mathcal{H}_{a,b}$ by

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 \frac{f(t)(1-t)^b}{(1-tz)^{a+1}}\, dt,$$

where a and b are non-negative real numbers. In particular, for $a = b = \beta$, $\mathcal{H}_{a,b}$ becomes the generalized Hilbert operator $\mathcal{H}_\beta$, and $\beta = 0$ gives the classical Hilbert operator $\mathcal{H}$. In this article, we find conditions on a and b such that $\mathcal{H}_{a,b}$ is bounded on Dirichlet-type spaces S^p, $0 < p < 2$, and on Bergman spaces A^p, $2 < p < \infty$. Also we find an upper bound for the norm of the operator $\mathcal{H}_{a,b}$. These generalize some results of E. Diamantopolous (2004) and S. Li (2009).

1. Introduction. Let $H(\mathbb{D})$ denote the class of all analytic functions in the unit disc $\mathbb{D}$ of the complex plane. For $0 < p < \infty$, the *Bergman space* A^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p\, dm(z) < \infty,$$

where $dm(z) = \pi^{-1} r\, dr\, d\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$. We refer to [DS1] and [Z2] for Bergman spaces.

2010 *Mathematics Subject Classification*: 30H20, 47B38, 30A10, 47B35.
Key words and phrases: generalized Hilbert operators, integral operators, Bergman spaces, Dirichlet spaces.
Received 28 July 2015; revised 8 December 2015.
Published online 4 January 2016.

DOI: 10.4064/ba8031-1-2016

Let $p \in \mathbb{R}$ and $f \in H(\mathbb{D})$ with the Taylor expansion $f(z) = \sum_{n=1}^{\infty} a_n z^n$. We say that f belongs to the space S^p if

$$\|f\|_{S^p}^2 = \sum_{n=1}^{\infty} (n+1)^p |a_n|^2 < \infty.$$

S^p is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{n=1}^{\infty} (n+1)^p a_n \overline{b_n},$$

where $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ (see [S]). The spaces S^0 and S^{-1} are the Hardy space H^2 and the Bergman space A^2, respectively, and S^1 is the Dirichlet space $\mathcal{D}$ (see [L]).

For $0 < r < 1$ and $f = \sum_{n=1}^{\infty} a_n z^n \in H(\mathbb{D})$, we define

$$M_2(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right)^{1/2} = \left(\sum_{n=1}^{\infty} |a_n|^2 r^{2n}\right)^{1/2}.$$

If $0 < p < 2$ and $f \in S^p$, then

$$\begin{aligned} (1.1) \qquad c_p\|f\|_{S^p}^2 &\le |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-p} \, dm(z) \\ &= |f(0)|^2 + 2\int_0^1 r(1-r^2)^{1-p} M_2^2(r, f') \, dr \le C_p \|f\|_{S^p}^2. \end{aligned}$$

The optimal constants c_p and C_p are given in the Appendix.

In 2009, S. Li and S. Stević [LS] for $\beta \ge 0$ defined the operator

$$\mathcal{H}_\beta(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)} a_k\right) z^n,$$

which they called a *generalized Hilbert operator*. For $\beta = 0$ this is the classical Hilbert operator $\mathcal{H}$. In [LS] the authors proved the boundedness of generalized Hilbert operators on Hardy spaces on the polydisc. In [L], S. Li proved the boundedness of generalized Hilbert operators on Dirichlet-type spaces S^p for $0 < p < 1$.

In this article, we extend the class of generalized Hilbert operators. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and a, b be non-negative real numbers. We define

$$\mathcal{H}_{a,b}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\Gamma(n+a+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+b+2)} a_k\right) z^n,$$

and call it a *generalized Hilbert-type operator*. Note that $\mathcal{H}_{a,b} = \mathcal{H}_\beta$ for $a = b = \beta$ and $\mathcal{H}_{a,b} = \mathcal{H}$ for $a = b = 0$.

A simple computation shows that $\mathcal{H}_{a,b}$ has a representation as an integral-type operator:

$$\mathcal{H}_{a,b}f(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 \frac{f(t)(1-t)^b}{(1-tz)^{a+1}}\, dt.$$

In particular, $a = b = \beta$ gives

$$\mathcal{H}_\beta(f)(z) = \int_0^1 \frac{f(t)(1-t)^\beta}{(1-tz)^{\beta+1}}\, dt$$

and $\beta = 0$ gives

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\, dt.$$

In [L], S. Li proved the boundedness of $\mathcal{H}_\beta$ on S^p, $0 < p < 1$. The main objective of this article is to prove the boundedness of $\mathcal{H}_{a,b}$ on Dirichlet and Bergman spaces for some p, a, b. In Theorem 2.1 we extend the result of S. Li by proving the boundedness of $\mathcal{H}_{a,b}$ on S^p, $0 < p < 2$, and we give an estimate of the norm $\|\mathcal{H}_{a,b}\|_{S^p}$. In Theorem 2.2 conditions on a, b, p are given which ensure the boundedness of $\mathcal{H}_{a,b}$ on A^p together with an estimate of its norm.

2. Main results. Throughout this article, $B(x, y)$ denotes the usual Beta function defined for $x, y > 0$ by

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\, ds.$$

THEOREM 2.1. *Suppose $a, b \geq 0$ and $0 < p < 2$. Then $\mathcal{H}_{a,b}$ is bounded on S^p and*

$$c_p\|\mathcal{H}_{a,b}f\|^2_{S^p} \leq C_1^2(a,b,p)\bigg[\bigg(\frac{1}{b+(p-1)/2}\bigg)^2 + \frac{2^{2-p}(a+1)^2C_1(b,p)}{(2a+p+1)(2a+p+2)}\bigg]C_p\|f\|^2_{S^p},$$

where

$$C_1(a,b,p) = \frac{\Gamma(a+1)2^{(4-p)/2}}{\Gamma(b+1)} \quad \textit{and} \quad C_1(b,p) = B^2\bigg(b + \frac{p-1}{2}, \frac{1-p}{2}\bigg).$$

THEOREM 2.2. *Let $p > 2$ and $b \geq a \geq 0$ with $|b - a - 1/p| < 1/p$. Then $\mathcal{H}_{a,b}$ is bounded on A^p and*

$$\|\mathcal{H}_{a,b}(f)\|_{A^p} \leq C(a,b)B\frac{\Gamma(a+1)2^{b-a}}{\Gamma(b+1)}\|f\|_{A^p},$$

where

$$B = B\left(\frac{2}{p} + a - b, b - \frac{2}{p} + 1\right),$$

$$C(a,b) = \begin{cases} 2^{a-b} & \text{if } 4 \le p < \infty, \\ \left(\dfrac{2^{7-p}}{9(p-2+p(b-a))} + 2^{4-p}\right)^{1/p} & \text{if } 2 < p < 4. \end{cases}$$

In order to prove Theorem 2.1 we establish the following lemma.

LEMMA 2.3. *Let* $0 < p < 2$ *and* $f \in S^p$. *Then for any* $z \in \mathbb{D}$,

$$|f(z)| \le 2^{2-p/2} C_p^{1/2} \left(\frac{1}{1-|z|}\right)^{(3-p)/2} \|f\|_{S^p}.$$

Proof. By (1.1) we have

$$C_p \|f\|_{S^p}^2 \ge 2 \int_0^1 u(1-u^2)^{1-p} M_2^2(u, f')\, du + |f(0)|^2.$$

Hence and by the increasing property of integral mean we obtain

$$\begin{aligned} C_p \|f\|_{S^p}^2 &\ge 2 \int_0^1 u(1-u^2)^{1-p} M_2^2(u^2, f')\, du + |f(0)|^2 \\ &= \int_0^1 (1-u)^{1-p} M_2^2(u, f')\, du + |f(0)|^2 \\ &\ge \int_{(1+|z|)/2}^{(3+|z|)/4} (1-u)^{1-p} M_2^2(u, f')\, du + |f(0)|^2 \\ &\ge \left(\frac{1-|z|}{2}\right)^{1-p} M_2^2\left(\frac{1+|z|}{2}, f'\right) \int_{(1+|z|)/2}^{(3+|z|)/4} du + |f(0)|^2 \\ &= \frac{1}{2^{3-p}} (1-|z|)^{2-p} M_2^2\left(\frac{1+|z|}{2}, f'\right) + |f(0)|^2. \end{aligned} \tag{2.1}$$

Applying the Cauchy integral formula to f^2 we get

$$(1-|z|)|f(z)|^2 < 2M_2^2\left(\frac{1+|z|}{2}, f\right). \tag{2.2}$$

The second equality in the definition of $M_2(r,f)$ easily implies that $M_2^2(r,f) \le |f(0)|^2 + M_2^2(r,f')$ for all $0 < r < 1$. Hence we obtain

$$M_2^2\left(\frac{1+|z|}{2}, f\right) \le |f(0)|^2 + M_2^2\left(\frac{1+|z|}{2}, f'\right).$$

By the previous inequality and (2.2), we have

$$M_2^2\left(\frac{1+|z|}{2}, f'\right) \geq \frac{1}{2}|f(z)|^2(1-|z|) - |f(0)|^2.$$

Hence inequality (2.1) gives

$$C_p\|f\|_{S^p}^2 \geq \frac{1}{2^{4-p}}|f(z)|^2(1-|z|)^{3-p},$$

which is the required result. ∎

Proof of Theorem 2.1. Differentiating the integral representation of $\mathcal{H}_{a,b}$ we get

$$|(\mathcal{H}_{a,b}f)'(z)| \leq \frac{\Gamma(a+2)}{\Gamma(b+1)} \int_0^1 \left| \frac{f(t)t(1-t)^b}{(1-tz)^{a+2}} \right| dt.$$

Now,

$$c_p\|\mathcal{H}_{a,b}f\|_{S^p}^2 \leq |\mathcal{H}_{a,b}f(0)|^2 + 2\int_0^1 r(1-r^2)^{1-p} M_2^2((\mathcal{H}_{a,b}f)', r)\, dr.$$

Minkowski's inequality together with the triangular inequality gives

$$\begin{aligned} M_2((\mathcal{H}_{a,b}f)', r) &\leq \frac{\Gamma(a+2)}{\Gamma(b+1)} \int_0^1 \left[\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(t)t(1-t)^b}{(1-tre^{i\theta})^{(a+2)}} \right|^2 d\theta \right]^{1/2} dt \\ &\leq \frac{\Gamma(a+2)}{\Gamma(b+1)} \int_0^1 |f(t)|t(1-t)^b(1-tr)^{-(a+2)}\, dt. \end{aligned}$$

Hence,

$$c_p\|\mathcal{H}_{a,b}f\|_{S^p}^2 \leq |\mathcal{H}_{a,b}f(0)|^2 + I, \tag{2.3}$$

where

$$I = 2\int_0^1 r(1-r^2)^{1-p} \left(\frac{\Gamma(a+2)}{\Gamma(b+1)} \int_0^1 |f(t)|t(1-t)^b(1-tr)^{-(a+2)}\, dt \right)^2 dr.$$

Using Lemma 2.3 we obtain

$$|\mathcal{H}_{a,b}f(0)|^2 \leq \left(\frac{\Gamma(a+1)2^{2-p/2}}{\Gamma(b+1)\left(b+\frac{p-1}{2}\right)} \right)^2 C_p\|f\|_{S^p}^2.$$

Moreover,

$$\begin{aligned} I \leq 2\left(\frac{\Gamma(a+2)}{\Gamma(b+1)} \right)^2 \int_0^1 & r(1-r^2)^{1-p}(1-r)^{-2(a+2)} \\ &\times \left(\int_0^1 \frac{2^{2-p/2}C_p^{1/2}\|f\|_{S^p}}{(1-t)^{(3-p)/2}} t(1-t)^b\, dt \right)^2 dr \end{aligned}$$

$$
\begin{aligned}
&\le 2^{5-p}\bigg(\frac{\Gamma(a+2)}{\Gamma(b+1)}\bigg)^2 C_p\|f\|_{S^p}^2\\
&\quad\times\bigg[\int_0^1 r(1-r^2)^{1-p}(1-r)^{-2(a+2)}\,dr\bigg]\bigg(\int_0^1(1-t)^{b+(p-3)/2}t\,dt\bigg)^2\\
&\le \frac{2^{6-2p}}{(2a+p+1)(2a+p+2)}\bigg(\frac{\Gamma(a+2)}{\Gamma(b+1)}\bigg)^2\\
&\quad\times\bigg(\int_0^1(1-t)^{b+(p-3)/2}t^{(-1-p)/2}\,dt\bigg)^2 C_p\|f\|_{S^p}^2\\
&= \frac{2^{6-2p}}{(2a+p+1)(2a+p+2)}\bigg(\frac{\Gamma(a+2)}{\Gamma(b+1)}\bigg)^2\\
&\quad\times B^2\bigg(b+\frac{p-1}{2},\frac{1-p}{2}\bigg)C_p\|f\|_{S^p}^2.
\end{aligned}
$$

Therefore inequality (2.3) gives

$$
\begin{aligned}
&c_p\|\mathcal{H}_{a,b}f\|_{S^p}^2\\
&\quad\le\bigg[\bigg(\frac{\Gamma(a+1)2^{2-p/2}}{\Gamma(b+1)\big(b+\frac{p-1}{2}\big)}\bigg)^2+\frac{2^{6-2p}}{(2a+p+1)(2a+p+2)}\bigg(\frac{\Gamma(a+2)}{\Gamma(b+1)}\bigg)^2 C_1(b,p)\bigg]\\
&\qquad\times C_p\|f\|_{S^p}^2\\
&\quad= C_1^2(a,b,p)\bigg[\bigg(\frac{1}{b+\frac{p-1}{2}}\bigg)^2+\frac{2^{2-p}(a+1)^2C_1(b,p)}{(2a+p+1)(2a+p+2)}\bigg]C_p\|f\|_{S^p}^2,
\end{aligned}
$$

where $C_1(a,b,p)$ and $C_1(b,p)$ are as in the statement. ∎

Remark. If $a=b=\beta$, Theorem 2.1 gives the boundedness of $\mathcal{H}_\beta$ on S^p for $0<p<2$, which extends [L, Theorem 1]. In particular, for $\beta=0$, $\mathcal{H}$ is bounded on S^p for $0<p<2$.

We recall the following result, to be used in the proof of Theorem 2.2.

Lemma 2.4 ([D, p. 1069]). *Let $2<p<\infty$ and $f\in A^p$. Then for any $z\in\mathbb{D}$,*

$$
|f(z)|\le\bigg(\frac{1}{1-|z|^2}\bigg)^{2/p}\|f\|_{A^p}.
$$

Proof of Theorem 2.2. For $z\in\mathbb{D}$, we choose the path

$$
\zeta(t)=\zeta_z(t)=\frac{t}{(t-1)z+1},\qquad 0\le t\le 1,
$$

i.e a circular arc in $\mathbb{D}$ joining 0 to 1. A change of variable in the integral representation of $\mathcal{H}_{a,b}$ gives

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 f\left(\frac{t}{(t-1)z+1}\right) \times \frac{(1-t)^b(1-z)^{b-a}}{(1+(t-1)z)^{b-a+1}}\, dt.$$

We define a weighted composition operator T_t as follows:

$$T_t(f)(z) = f(\phi_t(z))\omega_t^{b-a+1}(z)$$

where

$$\phi_t(z) = \frac{t}{(t-1)z+1} \quad \text{and} \quad \omega_t(z) = \frac{1}{(t-1)z+1}.$$

Then

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 T_t(f)(z)(1-t)^b(1-z)^{b-a}\, dt.$$

We first estimate the norm of T_t. Proceeding much as in the proof [D, Lemma 2], for $4 \le p < \infty$ we get

$$\|T_t(f)\|_{A^p} \le \frac{t^{2/p+a-b-1}}{(1-t)^{2/p}} \|f\|_{A^p}, \tag{2.4}$$

and for $2 < p < 4$ we get

$$\|T_t(f)\|_{A^p} \le \left(\frac{2^{7-p+p(a-b)}}{9(p-2+p(b-a))} + 2^{4-p+p(a-b)}\right)^{1/p} \frac{t^{2/p+a-b-1}}{(1-t)^{2/p}} \|f\|_{A^p}. \tag{2.5}$$

Now we estimate the norm

$$\|\mathcal{H}_{a,b}(f)\|_{A^p} = \frac{\Gamma(a+1)}{\Gamma(b+1)} \left(\int_{\mathbb{D}} \left|\int_0^1 T_t(f)(1-t)^b(1-z)^{b-a}\, dt\right|^p dm(z)\right)^{1/p}.$$

Applying Minkowski's inequality gives

$$\|\mathcal{H}_{a,b}(f)\|_{A^p} \le \frac{\Gamma(a+1)2^{b-a}}{\Gamma(b+1)} \int_0^1 \|T_t(f)\|_{A^p}(1-t)^b\, dt.$$

For $4 \le p < \infty$, using (2.4) we get

$$\|\mathcal{H}_{a,b}(f)\|_{A^p} \le \frac{\Gamma(a+1)2^{b-a}}{\Gamma(b+1)} B\|f\|_{A^p}.$$

For $2 < p < 4$, using (2.5) we get

$$\|\mathcal{H}_{a,b}(f)\|_{A^p} \le \frac{\Gamma(a+1)}{\Gamma(b+1)} \left(\frac{2^{7-p}}{9(p-2+p(b-a))} + 2^{4-p}\right)^{1/p} B\|f\|_{A^p}.$$

Remark. For $a = b = \beta$, Theorem 2.2 gives a new result on the boundedness of $\mathcal{H}_\beta$ on A^p for $2 < p < \infty$. In particular when $\beta = 0$, we obtain [D, Theorem 1].

3. Appendix. Here we give the calculations which give the optimal values for c_p and C_p. Let $f = \sum_{n=0}^{\infty} a_n z^n$. Then

$$|f(0)|^2 + 2\int_0^1 r(1-r^2)^{1-p} M_2^2(r, f')\,dr = |a_0|^2 + 2\int_0^1 r(1-r^2)^{1-p} M_2^2(r, f')\,dr$$
$$= |a_0|^2 + \sum_{n=1}^{\infty} |na_n|^2 \int_0^1 (1-r)^{1-p} r^{n-1}\,dr = |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 \frac{n(n!)\Gamma(2-p)}{\Gamma(n+2-p)}$$
$$= |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 (n+1)^p \frac{n(n!)}{(2-p)(3-p)\cdots(n+1-p)(n+1)^p}.$$

Let

$$L_n = \frac{(n-1)(n-1)!}{(2-p)\cdots(n-p)n^p} \quad \text{for } n \ge 2.$$

It is clear that the optimal choice for c_p is $\min(1, \inf_{n\ge2} L_n)$, and for C_p it is $\max(1, \sup_{n\ge2} L_n)$. We will show that L_n is an increasing sequence, which means that

$$\frac{L_{n+1}}{L_n} = \left(\frac{n}{n+1}\right)^{p+1} \left(\frac{n}{n-1}\right) \left(\frac{1}{1-\frac{p}{n+1}}\right) \ge 1.$$

Indeed, the last inequality is the same as

$$\left(1 - \frac{1}{n+1}\right)^{p+2} \ge 1 - \frac{p+2}{n+1} + \frac{2p}{(n+1)^2}.$$

Therefore it is enough to prove that

$$(1-x)^r \ge 1 - rx + (2r-4)x^2 \quad \text{for } 2 < r < 4 \text{ and } 0 < x \le 1/2.$$

For some $0 \le \theta \le x \le 1/2$, by the Taylor formula we get

$$(1-x)^r = 1 - rx + \frac{r(r-1)}{2}x^2 - \frac{r(r-1)(r-2)}{6}x^3(1-\theta)^{r-3}.$$

Thus the preceding inequality will be proved if we show that for $2 < r < 4$,

$$\frac{r(r-1)}{2} \ge 2r - 4 + \frac{r(r-1)(r-2)}{12},$$

which is the same as

$$(4-r)(r^2 - 5r + 12) \ge 0.$$

This is true for all $r < 4$. Therefore

$$\inf_{n\ge2} L_n = \frac{1}{(2-p)2^p}.$$

Using the Gauss formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)} \quad \text{for } x > 0,$$

we get

$$\sup_{n\geq 2} L_n = \lim_{n\to\infty} L_n = \Gamma(2-p).$$

Thus

$$c_p = \min\left\{1, \frac{1}{(2-p)2^p}\right\} \quad \text{and} \quad C_p = \max\{1, \Gamma(2-p)\}.$$

Acknowledgements. The authors thank Stanisław Kwapień for his valuable comments and suggestions which improved the presentation of the paper.

References

[D] E. Diamantopolous, *Hilbert matrix on Bergman spaces*, Illinois J. Math. 48 (2004), 1067–1078.

[DS] E. Diamantopolous and A. Siskakis, *Composition operators and the Hilbert matrix*, Studia Math. 140 (2000), 191–198.

[DJV] M. Dostanić, M. Jevtić and D. Vukotić, *Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type*, J. Funct. Anal. 254 (2008), 2800–2815.

[DS1] P. L. Duren and A. P. Schuster, *Bergman Spaces*, Math. Surveys Monogr. 100, Amer. Math. Soc., Providence, RI, 2004.

[HLP] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1988.

[L] S. Li, *Generalized Hilbert operator on Dirichlet-type space*, Appl. Math. Comput. 214 (2009), 304–309.

[LS] S. Li and S. Stević, *Generalized Hilbert operator and Fejér–Riesz type inequalities on the polydisc*, Acta Math. Sci. Ser. B 29 (2009), 191–200.

[S] A. Shields, *Weighted shift operators and analytic function theory*, in: Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, RI, 1974, 49–128.

[SS] S. Stević, *Hilbert operator on the polydisc*, Bull. Inst. Math. Acad. Sinica 31 (2003), 135–142.

[Z1] X. Zhu, *A class of integral operators on weighted Bergman spaces with a small parameter*, Indian J. Math. 50 (2008), 381–388.

[Z2] K. Zhu, *Operator Theory in Function Spaces*, 2nd ed., Math. Surveys Monogr. 138, Amer. Math. Soc., Providence, RI, 2007.

Sunanda Naik, Karabi Rajbangshi
Department of Applied Sciences
Gauhati University
Guwahati 781-014, India
E-mail: spn20@yahoo.com
karabi26@gmail.com