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Finite Embeddability of Sets and Ultrafilters

by

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Summary. A set A of natural numbers is finitely embeddable in another such set B if every finite subset of A has a rightward translate that is a subset of B. This notion of finite embeddability arose in combinatorial number theory, but in this paper we study it in its own right. We also study a related notion of finite embeddability of ultrafilters on the natural numbers. Among other results, we obtain connections between finite embeddability and the algebraic and topological structure of the Stone–Čech compactification of the discrete space of natural numbers. We also obtain connections with nonstandard models of arithmetic.

1. Introduction. The notion of finite embeddability of sets of natural numbers arose naturally in combinatorial number theory [4] (see also [9] where the notion is implicitly used). The present paper is a study of the basic properties of this notion and a closely related one in the realm of ultrafilters on the set $\mathbb{N}$ of natural numbers. These notions have also been considered in [7]. Additional information about them has appeared in [8], and more will appear in a planned sequel to this paper.

Definition 1 ([4, §4]). For $A, B \subseteq \mathbb{N}$, we say that A is *finitely embeddable* in B and we write $A \leq_{\mathrm{fe}} B$ if each finite subset F of A has a rightward translate $F + k$ included in B.

We use the standard notations $A + k = \{a + k : a \in A\}$ and $A - k = \{x \in \mathbb{N} : x + k \in A\}$ when $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$. We also use the standard conventions that the set $\mathbb{N}$ of natural numbers contains 0 and that each natural number n is identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors.

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Remark 2. Our definition of finite embeddability differs from that in [4] in that we work in $\mathbb{N}$ and use only rightward shifts. The corresponding Definition 1.3 in [4] worked with subsets of $\mathbb{Z}$ and allowed shifts in both directions.

Definition 3. For ultrafilters $\mathcal{U}$, $\mathcal{V}$ on $\mathbb{N}$, we say that $\mathcal{U}$ is *finitely embeddable* in $\mathcal{V}$ and we write $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$ if, for each set $B \in \mathcal{V}$, there is some $A \in \mathcal{U}$ such that $A \leq_{\mathrm{fe}} B$.

It is clear that both of the relations $\leq_{\mathrm{fe}}$, one on subsets of $\mathbb{N}$ and one on ultrafilters, are reflexive and transitive.

In Sections 2 and 3, we study finite embeddability primarily from a combinatorial point of view, with occasional mentions of topological aspects. The connection between finite embeddability and nonstandard models, though crucial for the original motivation of finite embeddability, has been postponed to Section 4, in order to make most of our results accessible to readers unfamiliar with nonstandard methods. On the other hand, readers who are comfortable with nonstandard models can read Section 4 without first working through the preceding sections.

2. Finite embeddability of sets of natural numbers. The following theorem summarizes some equivalent formulations of finite embeddability of sets of natural numbers. Additional equivalent characterizations in terms of nonstandard models will be given in Section 4.

Theorem 4. *For any $A, B \subseteq \mathbb{N}$, the following are equivalent:*

(1) $A \leq_{\mathrm{fe}} B$.
(2) *The family $\{B - a : a \in A\}$ has the finite intersection property.*
(3) *There exists an ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that A is a subset of the "leftward $\mathcal{V}$-shift" of B, namely*

$$B - \mathcal{V} = \{x \in \mathbb{N} : B - x \in \mathcal{V}\}.$$

(4) *There exists an ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that $A = B' - \mathcal{V}$ for some subset B' of B.*
(5) *The basic open sets $\overline{A}$ and $\overline{B}$ in the Stone–Čech compactification $\beta\mathbb{N}$ satisfy $\overline{A} + \mathcal{V} \subseteq \overline{B}$ for some ultrafilter $\mathcal{V} \in \beta\mathbb{N}$.*
(6) *Some superset of A is in the topological closure, in the power set $\mathcal{P}(\mathbb{N})$, of the set $\{B - k : k \in \mathbb{N}\}$ of leftward shifts of B.*
(7) *A is in the topological closure of the set of leftward shifts of some subset B' of B.*

Before beginning the proof, we clarify the notation and terminology used in this proposition, and we comment on some alternative ways to view parts of it.

In item (5), we identify the Stone–Čech compactification of $\mathbb{N}$ with the set of ultrafilters on $\mathbb{N}$, where natural numbers are identified with the corresponding principal ultrafilters. The basic open sets are defined as $\bar{A} = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$, and this notation is justified by the fact that $\bar{A}$ is also the closure in $\beta\mathbb{N}$ of the set $A \subseteq \mathbb{N} \subseteq \beta\mathbb{N}$.

The operation of addition of natural numbers is extended to $\beta\mathbb{N}$ by defining

$$\mathcal{U} + \mathcal{V} = \big\{X \subseteq \mathbb{N} : \{k : X - k \in \mathcal{V}\} \in \mathcal{U}\big\}.$$

See [5] for extensive information about this operation (and its analogs for other semigroups). The notation $\bar{A} + \mathcal{V}$ in item (5) of the proposition means $\{\mathcal{U} + \mathcal{V} : \mathcal{U} \in \bar{A}\}$.

In items (6) and (7), the power set $\mathcal{P}(\mathbb{N})$ is to be understood as topologized as the product $\{0,1\}^{\mathbb{N}}$ via the identification of subsets of $\mathbb{N}$ with their characteristic functions; here $\{0,1\}$ is given the discrete topology. So a small neighborhood of a subset A of $\mathbb{N}$ consists of those $X \subseteq \mathbb{N}$ that share a long initial segment with A.

The leftward shifts of a set, as used in items (6) and (7), constitute the orbit of that set under the transformation $X \mapsto X - 1$ of the space $\mathcal{P}(\mathbb{N})$. So these two items can be reformulated in terms of orbit closures for this transformation.

To the authors' knowledge, the notion of "leftward $\mathcal{V}$-shift" in item (3) was first considered by Peter Krautzberger in his thesis [6]; independently, Mathias Beiglböck introduced it for his ultrafilter proof of Jin's theorem [1].

This item is a first indication that finite embeddability is related to ultrafilters; that relation plays an important role later in this paper.

Proof of Theorem 4. We shall first prove (1)⇔(2), then (3)⇔(5), and finally the cycle (4)⇒(3)⇒(6)⇒(1)⇒(7)⇒(4).

(1)⇔(2) is immediate from the definitions, because k is in a finite intersection $(B - a_1) \cap \cdots \cap (B - a_n)$ if and only if $\{a_1, \ldots, a_n\} + k \subseteq B$.

(3)⇒(5). If $\mathcal{V}$ is as in (3), and if $\mathcal{U} \in \bar{A}$, then $\{x \in \mathbb{N} : B - x \in \mathcal{V}\}$, being a superset of A, is in $\mathcal{U}$. This means that $B \in \mathcal{U} + \mathcal{V}$, and so $\mathcal{U} + \mathcal{V} \in \bar{B}$. Since $\mathcal{U}$ was arbitrary in $\bar{A}$, we have (5).

(5)⇒(3). Apply (5) to the principal ultrafilters $\mathcal{U}_a$ concentrated at points $a \in A$. These are in $\bar{A}$, so we have $B \in \mathcal{U}_a + \mathcal{V}$. But this means that $B - a \in \mathcal{V}$, as required for (3).

(4)⇒(3). If $\mathcal{V}$ and B' are as in (4), then $A = B' - \mathcal{V} \subseteq B - \mathcal{V}$.

(3)⇒(6). Every leftward $\mathcal{V}$-shift $B - \mathcal{V}$, as defined in (3), is the limit along $\mathcal{V}$ of the leftward shifts $B - k$. So the set of all these leftward $\mathcal{V}$-shifts, for all $\mathcal{V}$, is the topological closure mentioned in (6). Finally, the specific $B - \mathcal{V}$ asserted to exist in (3) serves as the superset of A required in (6).

(6)⇒(1). Let F be any finite subset of A and therefore also of the superset A' mentioned in (6). By definition of the topological closure, there must be a leftward shift $B-k$ whose characteristic function agrees on F with that of A'. Therefore $F \subseteq B-k$.

(1)⇒(7). According to the assumption (1), for each $n \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ such that $(A \cap n)+k \subseteq B$. (Recall that we use the usual convention identifying a natural number n with its set of predecessors.) If one and the same k works for arbitrarily large n (and therefore for all n), then $A+k \subseteq B$. Setting $B' = A+k$, we have (7), because $B' \subseteq B$ and $A = B'-k$ is in the set of leftward shifts of B' (not just in its closure).

So assume from now on that, for each k, there is an upper bound on the n's for which $(A \cap n)+k \subseteq B$. Then, for each $n \in \mathbb{N}$, there must exist arbitrarily large $k \in \mathbb{N}$ with $(A \cap n)+k \subseteq B$. Indeed, let n be given and consider any $m \in \mathbb{N}$; we shall find an appropriate (for n) $k \geq m$. By assumption, for each $k < m$, we have an upper bound b_k on the n's for which this k works; fix some N larger than all of these m bounds and larger than our given n. There is a k that works for N and thus also for our given n, and this k must, by choice of N, be $\geq m$.

Now define a subset B' of B as the union of finite pieces of the form $(A \cap n)+k_n$, where the k_n's are chosen (for $n \in \mathbb{N}$) inductively as follows. Suppose we already have k_j for all $j < n$. Then choose k_n so that $k_n > k_j + j$ for all $j < n$ and $(A \cap n)+k_n \subseteq B$. This ensures that (the characteristic function of) A agrees on $[0,n)$ with that of a leftward shift $B'-k_n$ of B'. Since this happens for each n, A is in the closure of the set of leftward shifts of B'.

(7)⇒(4). The closure of the set of leftward shifts $B'-k$ consists of exactly the sets $B'-\mathcal{V}$ for all ultrafilters $\mathcal{V}$ on $\mathbb{N}$. ∎

Remark 5. Peter Krautzberger independently proved the equivalence of items (1), (2), and (5) in Theorem 4. After learning of the second author's proof that $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{U}+\mathcal{V}$, he also proved Theorem 11 below.

Remark 6. Suppose that, when we topologize $\mathcal{P}(\mathbb{N}) \cong \{0,1\}^{\mathbb{N}}$ as a product, we do not give $\{0,1\}$ the discrete topology but rather the Sierpiński topology in which $\{1\}$ is open but $\{0\}$ is not. Then $A \leq_{\mathrm{fe}} B$ if and only if A is in the closure of the set of leftward shifts of B. The reason is that, with this new topology, closed sets are automatically downward closed with respect to $\subseteq$.

As mentioned in the introduction, the notion of finite embeddability was originally motivated by considerations from combinatorial number theory. To give an idea of this motivation, we list in the following proposition some facts from [4, Propositions 4.1 and 4.2]; for the relevant definitions, see also [5].

Proposition 7. *Let A and B be sets of natural numbers.*

(1) *A is maximal with respect to $\leq_{\mathrm{fe}}$ if and only if it is thick, i.e., it includes arbitrarily long intervals.*
(2) *If $A \leq_{\mathrm{fe}} B$ and A is piecewise syndetic, then B is also piecewise syndetic.*
(3) *If $A \leq_{\mathrm{fe}} B$ and A contains a k-term arithmetic progression, then also B contains a k-term arithmetic progression.*
(4) *If $A \leq_{\mathrm{fe}} B$ then the upper Banach densities satisfy* $\mathrm{BD}(A) \leq \mathrm{BD}(B)$.
(5) *If $A \leq_{\mathrm{fe}} B$ then $A - A \subseteq B - B$.*
(6) *If $A \leq_{\mathrm{fe}} B$ then $\bigcap_{t\in G}(A-t) \leq_{\mathrm{fe}} \bigcap_{t\in G}(B-t)$ for every finite $G \subseteq \mathbb{N}$.*

We remark in connection with item (2) that "piecewise" is essential there. The property of being syndetic is not in general preserved upward under $\leq_{\mathrm{fe}}$. Intuitively, this is because the definition of $A \leq_{\mathrm{fe}} B$ does nothing to prevent the occurrence of long gaps in B. For similar reasons, "upper" is essential in item (4); asymptotic density is not monotone with respect to $\leq_{\mathrm{fe}}$.

3. Finite embeddability of ultrafilters. The results of the preceding section exhibit some connections between finite embeddability and the additive structure of ultrafilters. In the present section, we first reformulate those connections in some corollaries and a theorem, and then we establish some additional connections.

Definition 8. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. A set $B \subseteq \mathbb{N}$ is *$\mathcal{U}$-rich* if there is an $A \in \mathcal{U}$ with $A \leq_{\mathrm{fe}} B$.

Notice that Definition 3 of finite embeddability of ultrafilters says that $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$ if and only if every set in $\mathcal{V}$ is $\mathcal{U}$-rich.

Corollary 9. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. A set B is $\mathcal{U}$-rich if and only if $B \in \mathcal{U} + \mathcal{W}$ for some ultrafilter $\mathcal{W}$ on $\mathbb{N}$.*

Proof. The equivalence of (1) and (3) in Proposition 4 shows that B is $\mathcal{U}$-rich if and only if there is an ultrafilter $\mathcal{W}$ on $\mathbb{N}$ such that $B - \mathcal{W} \in \mathcal{U}$. But this is precisely what is required for $B \in \mathcal{U} + \mathcal{W}$. ∎

Corollary 10. *For any ultrafilter $\mathcal{U}$, the family of $\mathcal{U}$-rich sets is partition-regular. That is, if a $\mathcal{U}$-rich set is partitioned into finitely many pieces, then at least one of the pieces is $\mathcal{U}$-rich.*

Proof. The preceding corollary shows that the family of $\mathcal{U}$-rich sets is the union of a collection of ultrafilters, and it is easy to see that any such union is partition-regular. ∎

Theorem 11. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. Then $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$ if and only if $\mathcal{V}$ is in the closure, in $\beta\mathbb{N}$, of the set of sums $\{\mathcal{U} + \mathcal{W} : \mathcal{W} \in \beta\mathbb{N}\}$.*

Proof. Each of the following statements is clearly equivalent to the next:

- $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$.
- Every $B \in \mathcal{V}$ is $\mathcal{U}$-rich.
- Every $B \in \mathcal{V}$ is in $\mathcal{U} + \mathcal{W}$ for some ultrafilter $\mathcal{W}$ on $\mathbb{N}$.
- Every basic neighborhood $\overline{B}$ of $\mathcal{V}$ in $\beta\mathbb{N}$ contains a sum $\mathcal{U} + \mathcal{W}$.
- $\mathcal{V}$ is in the closure of the set $\{\mathcal{U} + \mathcal{W} : \mathcal{W} \in \beta\mathbb{N}\}$ of sums. ∎

The preceding theorem implies, in particular, that the upward cone $\{\mathcal{V} : \mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}\}$ determined by any ultrafilter $\mathcal{U}$ in the $\leq_{\mathrm{fe}}$ ordering is a closed subset of $\beta\mathbb{N}$.

This fact can also be seen by inspecting the form of the definition of $\leq_{\mathrm{fe}}$. To say that $\mathcal{V}$ is in this cone is to say "For all $B \in \mathcal{V}$, $\Phi(B)$" where the statement $\Phi(B)$ does not mention $\mathcal{V}$. Any set of ultrafilters with a definition of this form is closed. Furthermore, it can be expressed as the set of all ultrafilters extending the filter generated by the complements of all sets B that do not satisfy $\Phi(B)$.

In the case at hand, the upward cone $\{\mathcal{V} : \mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}\}$ consists of all extensions of the filter $\mathcal{F}_{\mathcal{U}}$ generated by those sets whose complements are not $\mathcal{U}$-rich. In view of Corollary 10, this description can be simplified, because the sets whose complements are not $\mathcal{U}$-rich already constitute a filter $\mathcal{F}_{\mathcal{U}}$; there is no need to form the filter they generate.

The theorem provides another description of $\mathcal{F}_{\mathcal{U}}$ as an intersection of ultrafilter sums,

$$\mathcal{F}_{\mathcal{U}} = \bigcap_{\mathcal{W} \in \beta\mathbb{N}} (\mathcal{U} + \mathcal{W}).$$

The preceding results have related finite embeddability of $\mathcal{U}$ with sums in which $\mathcal{U}$ appears as the left summand. Since addition is not commutative in $\beta\mathbb{N}$, the question arises whether there are similar results with $\mathcal{U}$ as the right summand. Most of the preceding results do not have such analogs, but the basic fact (a special case of Theorem 11) that $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{U} + \mathcal{W}$ does, and in fact we get a somewhat stronger conclusion.

THEOREM 12. *For any ultrafilters $\mathcal{V}$ and $\mathcal{W}$ on $\mathbb{N}$, the following are equivalent:*

(1) *Each set in $\mathcal{W}$ includes a rightward translate of a whole set from $\mathcal{V}$.*
(2) *$\mathcal{W} = \mathcal{U} + \mathcal{V}$ for a suitable $\mathcal{U}$.*

Proof. (1)⇒(2). The family $\mathcal{F} = \{B - \mathcal{V} : B \in \mathcal{W}\}$ of the leftward $\mathcal{V}$-shifts of sets in $\mathcal{W}$ has the finite intersection property. Indeed, for each $B \in \mathcal{W}$, by hypothesis $A + k \subseteq B$ for suitable $A \in \mathcal{V}$ and $k \in \mathbb{N}$, and so $k \in B - \mathcal{V} \neq \emptyset$. Moreover, given finitely many $B_1, \ldots, B_n \in \mathcal{W}$, the intersection $\bigcap_{i=1}^{n}(B_i - \mathcal{V}) = (\bigcap_{i=1}^{n} B_i) - \mathcal{V}$ is in $\mathcal{F}$, and therefore is nonempty. So we can extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. For every $B \in \mathcal{W}$ one has $B - \mathcal{V} \in \mathcal{F} \subseteq \mathcal{U}$, so $B \in \mathcal{U} + \mathcal{V}$, and the equality $\mathcal{W} = \mathcal{U} + \mathcal{V}$ follows.

(2)⇒(1). The definition of $\mathcal{U}+\mathcal{V}$ says that if $B \in \mathcal{U}+\mathcal{V}$ then for some k, in fact for $\mathcal{U}$-almost all k, we have $B - k \in \mathcal{V}$. So $B - k$ is a set in $\mathcal{V}$ with a rightward translate, by k, included in B. ∎

We remark that (1) in Theorem 12 above is a strictly stronger property than $\mathcal{V} \leq_{\mathrm{fe}} \mathcal{W}$.

Corollary 13. *The ordering $\leq_{\mathrm{fe}}$ of ultrafilters on $\mathbb{N}$ is upward directed.*

Proof. Any $\mathcal{U}$ and $\mathcal{V}$ have $\mathcal{U} + \mathcal{V}$ as an upper bound. ∎

In view of the characterization of the upward cone of $\mathcal{U}$ as the (topological) closure of $\{\mathcal{U} + \mathcal{W} : \mathcal{W} \in \beta\mathbb{N}\}$, it is tempting to consider the closure of $\{\mathcal{W} + \mathcal{U} : \mathcal{W} \in \beta\mathbb{N}\}$ also, but this leads to nothing interesting. Indeed, $\{\mathcal{W}+\mathcal{U} : \mathcal{W} \in \beta\mathbb{N}\}$ is already closed, being the image of the compact set $\beta\mathbb{N}$ under the continuous function $\mathcal{W} \to \mathcal{W}+\mathcal{U}$. (Recall that addition in $\beta\mathbb{N}$ is a continuous function of the left summand when the right summand is fixed, but not vice versa.)

Let us summarize the preceding results, in the language of ideals of the semigroup $(\beta\mathbb{N}, +)$. Recall that $\{\mathcal{U} + \mathcal{W} : \mathcal{W} \in \beta\mathbb{N}\}$ and $\{\mathcal{W}+\mathcal{U} : \mathcal{W} \in \beta\mathbb{N}\}$ are, respectively, the right and left ideals generated by $\mathcal{U}$.

Corollary 14. *For any $\mathcal{U} \in \beta\mathbb{N}$, the upward cone $\{\mathcal{V} : \mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}\}$ is a closed, two-sided ideal in $\beta\mathbb{N}$. It is the smallest closed right ideal containing $\mathcal{U}$, and therefore it is also the smallest closed two-sided ideal containing $\mathcal{U}$.*

The fact that the closure of a right ideal is a two-sided ideal is a general property of Stone–Čech compactifications of discrete commutative semigroups. In fact, it holds in even greater generality (see [5, Theorem 2.19(a)]).

For completeness, we mention that we cannot replace "right" by "left" in this corollary. The smallest left ideal containing $\mathcal{U}$, namely $\beta\mathbb{N} + \mathcal{U}$, is already closed and, as the following example shows, it need not contain the right ideal generated by $\mathcal{U}$.

Example 15. Recall that the sets $X \subseteq \mathbb{N}$ whose density (i.e., the limit of $|X \cap n|/n$ as $n \to \infty$) exists and is zero constitute a proper ideal, so there are ultrafilters $\mathcal{U}$ that contain no such X. The set of all such ultrafilters $\mathcal{U}$ is easily seen to be a left ideal Δ (see [5, Theorem 6.79]). So, if we fix one such ultrafilter $\mathcal{U}$, then the left ideal it generates is included in Δ.

We claim that, in contrast, the right ideal generated by $\mathcal{U}$ contains an ultrafilter $\mathcal{U}+\mathcal{V}$ that contains a set of density zero. To see this, let $\mathcal{V}$ be any nonprincipal ultrafilter on $\mathbb{N}$ that contains the set $P = \{2^m : m \in \mathbb{N}\}$, and consider the set

$$Q = \{2^m + k : m, k \in \mathbb{N} \text{ and } k < m\}.$$

Observe that, for every fixed k, the set $Q-k$ contains all but finitely many elements of P and is therefore in $\mathcal{V}$. Thus, $Q \in \mathcal{U}+\mathcal{V}$. (Indeed, $Q \in \mathcal{U}'+\mathcal{V}$ for all ultrafilters $\mathcal{U}'$ on $\mathbb{N}$.) Finally, to see that Q has density 0, notice that the function $n \mapsto |A \cap n|/n$ achieves its local maxima at the numbers of the form $n = 2^m + m$, where its values are $(m^2+m)/(2^{m+1}+2m)$. These values tend to zero as $m \to \infty$, so Q has density zero. (The essential properties of Q used here are merely that it has density zero but is thick. Recall that thickness means that every finite set has a rightward translate included in Q, or equivalently that there is an ultrafilter $\mathcal{V}$ that contains all the sets $Q-k$.)

4. Nonstandard models. In this section we assume the reader to be familiar with the basics of nonstandard analysis, and in particular with the notions of *hyper-extension* ${}^*A$ (or nonstandard extension) of a set A, and with the *transfer principle*. Most notably, we will focus on the set of *hypernatural numbers* ${}^*\mathbb{N}$, and we will consider the nonstandard characterization of topology. We will work in a nonstandard framework where the $\mathfrak{c}^+$-*enlarging property* holds, namely the property that if a family $\mathfrak{X}$ of cardinality at most the continuum has the finite intersection property, then the intersection $\bigcap_{X\in\mathfrak{X}} {}^*X$ is nonempty. (This is a weaker property than $\mathfrak{c}^+$-saturation.) We refer to [2, §4.4] for the foundations, and to [3] for all nonstandard notions and results used in this section.

Finite embeddability has a suggestive nonstandard characterization, which was in fact the original motivation for this research. Precisely, $A \le_{\mathrm{fe}} B$ means that a (possibly infinite) rightward translation of A is included in the hyper-extension of B.

Proposition 16 ([4, §4]). *$A \le_{\mathrm{fe}} B$ if and only if $\mu + A \subseteq {}^*B$ for some $\mu \in {}^*\mathbb{N}$.*

Proof. Directly by the definition of $A \le_{\mathrm{fe}} B$, the family $\{B-a : a \in A\}$ has the finite intersection property. Then, by the enlarging property, we can pick $\mu \in \bigcap_{a\in A} {}^*(B-a)$. Since ${}^*(B-a) = {}^*B - a$, this means that $\mu + A \subseteq {}^*B$.

Conversely, given a finite subset $\{a_1,\dots,a_k\} \subseteq A$, the element μ witnesses that the following property holds:

$$\exists x \in {}^*\mathbb{N} \;\; x + a_1 \in {}^*B \wedge \cdots \wedge x + a_k \in {}^*B.$$

Then, by transfer, we obtain the existence of an element $x \in \mathbb{N}$ such that $x + a_i \in B$ for all $i = 1,\dots,k$. ■

There is a canonical way of associating an ultrafilter on $\mathbb{N}$ to each hypernatural number.

Definition 17. The *ultrafilter generated by* $\alpha \in {}^*\mathbb{N}$ is the family

$$\mathcal{U}_\alpha = \{A \subseteq \mathbb{N} : \alpha \in {}^*A\}.$$

It is readily checked that $\mathcal{U}_\alpha$ is actually an ultrafilter, and that $\mathcal{U}_\alpha$ is nonprincipal if and only if $\alpha \notin \mathbb{N}$, i.e., α is infinite. Notice that every ultrafilter on $\mathbb{N}$ is a family with the cardinality of the continuum that has the finite intersection property. So, by the $\mathfrak{c}^+$-enlarging property, each ultrafilter is generated by some (actually, by $\mathfrak{c}^+$ many) hypernatural numbers; in consequence,

$$\beta\mathbb{N} = \{\mathcal{U}_\mu : \mu \in {}^*\mathbb{N}\}.$$

For $B \subseteq \mathbb{N}$ and $\mu \in {}^*\mathbb{N}$, we consider the set of elements in ${}^*B$ that are placed at finite distance from μ on the right side:

$$B_\mu = \{n \in \mathbb{N} : \mu + n \in {}^*B\} = ({}^*B - \mu) \cap \mathbb{N}.$$

Trivially $B_\mu \leq_{\mathrm{fe}} B$ because $\mu + B_\mu \subseteq {}^*B$. Notice that B_μ is the leftward $\mathcal{U}_\mu$-shift of B; indeed, $n \in B - \mathcal{U}_\mu \Leftrightarrow B - n \in \mathcal{U}_\mu \Leftrightarrow \mu \in {}^*(B-n) \Leftrightarrow \mu + n \in {}^*B$. In consequence,

$$A \in \mathcal{U}_\alpha + \mathcal{U}_\beta \Leftrightarrow A_\beta = A - \mathcal{U}_\beta \in \mathcal{U}_\alpha \Leftrightarrow \alpha \in {}^*A_\beta.$$

A nice nonstandard characterization can also be given of the topology on $\mathcal{P}(\mathbb{N})$ considered in Section 2.

PROPOSITION 18. *A is in the topological closure, in the power set $\mathcal{P}(\mathbb{N})$, of the set $\{B - k : k \in \mathbb{N}\}$ of leftward shifts of B if and only if $A = B_\mu$ for some $\mu \in {}^*\mathbb{N}$.*

Proof. Recall that if X is a topological space of character κ, then in any nonstandard model with the κ^+-enlarging property, the following characterization holds: "An element x belongs to the closure of a set $Y \subseteq X$ if and only if $m(x) \cap {}^*Y \neq \emptyset$," where

$$m(x) = \bigcap \{{}^*I : I \text{ a neighborhood of } x\}$$

is the monad of x. (See e.g. [3, §3.1].) In our topological space, a base of neighborhoods of a set $A \subseteq \mathbb{N}$ is obtained by taking all finite intersections of sets of the form $I_{A,n} = \{B \in \mathcal{P}(\mathbb{N}) : n \in A \Leftrightarrow n \in B\}$. So, we have

$$m(A) = \bigcap_{n\in\mathbb{N}} {}^*I_{A,n} = \{X \in {}^*\mathcal{P}(\mathbb{N}) : X \cap \mathbb{N} = A\}.$$

By the above nonstandard characterization, A is in the closure of the set $\{B - k : k \in \mathbb{N}\}$ if and only if there exists $\mu \in {}^*\mathbb{N}$ such that ${}^*B - \mu \in m(A)$, i.e. $B_\mu = ({}^*B - \mu) \cap \mathbb{N} = A$. ∎

The above notions and characterizations can be used to give nonstandard proofs of all the results presented in this paper. Below, we consider in detail the main results (named here Theorems A and B) and leave the other proofs as exercises for the interested reader.

THEOREM A (Theorem 4). *For any $A, B \subseteq \mathbb{N}$, the following are equivalent:*

(1) $A \leq_{\mathrm{fe}} B$.
(2) *The family* $\{B - a : a \in A\}$ *has the finite intersection property.*
(3) *There exists an ultrafilter* $\mathcal{V}$ *on* $\mathbb{N}$ *such that* A *is a subset of the "leftward* $\mathcal{V}$*-shift" of* B*, namely*

$$B - \mathcal{V} = \{x \in \mathbb{N} : B - x \in \mathcal{V}\}.$$

(4) *There exists an ultrafilter* $\mathcal{V}$ *on* $\mathbb{N}$ *such that* $A = B' - \mathcal{V}$ *for some subset* B' *of* B.
(5) *The basic open sets* $\bar{A}$ *and* $\bar{B}$ *in the Stone–Čech compactification* $\beta\mathbb{N}$ *satisfy* $\bar{A} + \mathcal{V} \subseteq \bar{B}$ *for some ultrafilter* $\mathcal{V} \in \beta\mathbb{N}$.
(6) *Some superset of* A *is in the topological closure, in the power set* $\mathcal{P}(\mathbb{N})$*, of the set* $\{B - k : k \in \mathbb{N}\}$ *of leftward shifts of* B.
(7) A *is in the topological closure of the set of leftward shifts of some subset* B' *of* B.
(8) *There exists* $\mu \in {}^*\mathbb{N}$ *such that* $A \subseteq B_\mu$.
(9) $A' = B_\mu$ *for some superset* A' *of* A *and some* $\mu \in {}^*\mathbb{N}$.
(10) $A = B'_\mu$ *for some subset* B' *of* B *and some* $\mu \in {}^*\mathbb{N}$.

Nonstandard proof. We first reduce to the "nonstandard" items (8), (9) and (10).

(1)$\Leftrightarrow$(8). It is Proposition 16.

(2)$\Leftrightarrow$(8). Recall that a family has the finite intersection property if and only if it is included in an ultrafilter. So, item (2) is equivalent to the existence of a point $\mu \in {}^*\mathbb{N}$ such that $\{B - a : a \in A\} \subseteq \mathcal{U}_\mu$. Now, $B - a \in \mathcal{U}_\mu \Leftrightarrow \mu \in {}^*(B - a) = {}^*B - a \Leftrightarrow a \in B_\mu$, and therefore $\{B - a : a \in A\} \subseteq \mathcal{U}_\mu$ is equivalent to $A \subseteq B_\mu$.

(3)$\Leftrightarrow$(8). Item (3) says that $A \subseteq B - \mathcal{U}_\mu$ for some $\mu \in {}^*\mathbb{N}$. Then recall that $B - \mathcal{U}_\mu = B_\mu$.

(4)$\Leftrightarrow$(10). Condition (4) says that $A = B' - \mathcal{U}_\mu = B'_\mu$ for some $B' \subseteq B$ and some $\mu \in {}^*\mathbb{N}$.

(5)$\Leftrightarrow$(8). Notice that $\bar{A} = \{\mathcal{U}_\alpha : \alpha \in {}^*A\}$ and $\bar{B} = \{\mathcal{U}_\beta : \beta \in {}^*B\}$. So, item (5) says that there exists $\mu \in {}^*\mathbb{N}$ such that $B \in \mathcal{U}_\alpha + \mathcal{U}_\mu$ for all $\alpha \in {}^*A$. Now recall that $B \in \mathcal{U}_\alpha + \mathcal{U}_\mu$ if and only if $\alpha \in {}^*B_\mu$. Then the above property is equivalent to ${}^*A \subseteq {}^*B_\mu$, which in turn is equivalent to $A \subseteq B_\mu$, by transfer.

(6)$\Leftrightarrow$(9) and (7)$\Leftrightarrow$(10). They both directly follow from Proposition 18.

We are left to prove the equivalence of the three "nonstandard" conditions.

(8)$\Leftrightarrow$(9). Trivial.

(10)$\Rightarrow$(8). By transfer, $B \subseteq B' \Leftrightarrow {}^*B \subseteq {}^*B'$, and hence $B \subseteq B' \Rightarrow B_\mu \subseteq B'_\mu$ for all $\mu \in {}^*\mathbb{N}$.

(8)⇒(10). Assume first that $\mu = k \in \mathbb{N}$ is finite. By the hypothesis, $A \subseteq B_k = B - k$. If we let $B' = A + k$ then $B' \subseteq B$ and $A = B'_k$.

Now let $\mu \in {}^*\mathbb{N}$ be infinite. Denote by $A_n = \{a_1 < \cdots < a_n\}$ the set of the first n elements of A. Notice that every set $\Lambda_n = \{x \in \mathbb{N} : x + A_n \subseteq B\}$ is infinite, since the infinite number μ is in ${}^*\Lambda_n$. We now inductively define a sequence $\{x_n\}$ of numbers and a sequence $\{B_n\}$ of finite subsets of B as follows:

- Let $x_1 \in \Lambda_1$ and let $B_1 = \{x_1 + a_1\} = x_1 + A_1$.
- At the inductive step $n > 1$, pick $x_n \in \Lambda_n$ such that $x_n > x_{n-1} + a_{n-1}$ and let $B_n = x_n + A_n \subseteq [x_n, x_n + a_n]$.
- Define $B' = \bigcup_{n\in\mathbb{N}} B_n \subseteq B$.

The sets B_n are pairwise disjoint and so, for every n,

$$x_n + A_n = x_n + (A \cap [0, a_n]) = B' \cap [x_n, x_n + a_n],$$

i.e. $A \cap [0, a_n] = (B' - x_n) \cap [0, a_n]$. Then, by transfer, for all $N \in {}^*\mathbb{N}$,

$${}^*A \cap [0, a_N] = ({}^*B' - x_N) \cap [0, a_N].$$

If we pick $\nu = x_N$ for an infinite N, then $\mathbb{N} \subseteq [0, a_N]$, and hence $A = {}^*A \cap \mathbb{N} = ({}^*B' - \nu) \cap \mathbb{N} = B'_\nu$. ∎

Theorem B (Theorem 11). *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. Then $\mathcal{U} \leq_{\text{fe}} \mathcal{V}$ if and only if $\mathcal{V}$ is in the closure, in $\beta\mathbb{N}$, of the set $\{\mathcal{U} + \mathcal{W} : \mathcal{W} \in \beta\mathbb{N}\}$.*

Nonstandard proof. Pick $\alpha, \beta \in {}^*\mathbb{N}$ such that $\mathcal{U} = \mathcal{U}_\alpha$ and $\mathcal{V} = \mathcal{U}_\beta$. Assume first that $\mathcal{U}_\alpha \leq_{\text{fe}} \mathcal{U}_\beta$. We want to show that for every $B \in \mathcal{U}_\beta$ there exists $\mu \in {}^*\mathbb{N}$ such that $B \in \mathcal{U}_\alpha + \mathcal{U}_\mu$. By the hypothesis we can pick $A \in \mathcal{U}_\alpha$ with $A \leq_{\text{fe}} B$. By the nonstandard characterization, this means that there exists $\mu \in {}^*\mathbb{N}$ such that $A \subseteq B_\mu$. But then $B_\mu \in \mathcal{U}_\alpha$, i.e. $\alpha \in {}^*B_\mu$, and we conclude that $B \in \mathcal{U}_\alpha + \mathcal{U}_\mu$.

Conversely, if $\mathcal{U}_\beta$ is in the closure of $\{\mathcal{U}_\alpha + \mathcal{U}_\mu : \mu \in {}^*\mathbb{N}\}$, then for every $B \in \mathcal{U}_\beta$ there exists μ such that $B \in \mathcal{U}_\alpha + \mathcal{U}_\mu$, i.e. $\alpha \in {}^*B_\mu$. But then we have found a set $B_\mu \in \mathcal{U}_\alpha$ with $B_\mu \leq_{\text{fe}} B$, as desired. ∎

5. Questions

- Under which conditions does the following implication hold?

$$A \leq_{\text{fe}} B \text{ and } A' \leq_{\text{fe}} B' \Rightarrow (A - A') \leq_{\text{fe}} (B - B').$$

- For $A \subseteq \mathbb{N}$, is there a neat combinatorial description of the equivalence classes $[A] = \{B : A \leq_{\text{fe}} B \wedge B \leq_{\text{fe}} A\}$? And, for $\mathcal{U} \in \beta\mathbb{N}$, of the equivalence classes $[\mathcal{U}] = \{\mathcal{V} : \mathcal{U} \leq_{\text{fe}} \mathcal{V} \wedge \mathcal{V} \leq_{\text{fe}} \mathcal{U}\}$?
- If we modify the definition of $\leq_{\text{fe}}$ so that trivial right translations by 0 are not permitted, can we find a neat combinatorial description of the sets A such that $A \leq_{\text{fe}} A$? And of the ultrafilters $\mathcal{U}$ such that $\mathcal{U} \leq_{\text{fe}} \mathcal{U}$?

Remark 19. Proposition 4.2(6) of [4] looks as if it answers the first of these questions, but its proof relies on the definition in [4] of finite embeddability, which allows both leftward and rightward shifts. Isaac Goldbring has pointed out, in a private communication, that the same proof works in our setting, using only rightward shifts, under the additional hypothesis that, for each finite $F \subseteq A$, there are infinitely many $k \in \mathbb{N}$ such that $F + k \subseteq B$. Goldbring suggests calling this hypothesis *proper finite embeddability* of A in B, and notes that it is an intermediate property between our finite embeddability and dense embeddability as defined in [4]. Note that, as shown in the proof of (1)⇒(7) in Theorem 4 above, the only way A can be finitely embeddable but not properly finitely embeddable in B is that there is some $k \in \mathbb{N}$ with $A + k \subseteq B$.

It remains open whether proper finite embeddability is the exact condition for the first question or whether some weaker condition might suffice.

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