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ON TWISTED GROUP ALGEBRAS OF OTP REPRESENTATION TYPE OVER THE RING OF p-ADIC INTEGERS

BY

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Abstract. Let $\hat{\mathbb{Z}}_p$ be the ring of p-adic integers, $U(\hat{\mathbb{Z}}_p)$ the unit group of $\hat{\mathbb{Z}}_p$ and $G = G_p \times B$ a finite group, where G_p is a p-group and B is a p'-group. Denote by $\hat{\mathbb{Z}}_p^\lambda G$ the twisted group algebra of G over $\hat{\mathbb{Z}}_p$ with a 2-cocycle $\lambda \in Z^2(G, U(\hat{\mathbb{Z}}_p))$. We give necessary and sufficient conditions for $\hat{\mathbb{Z}}_p^\lambda G$ to be of OTP representation type, in the sense that every indecomposable $\hat{\mathbb{Z}}_p^\lambda G$-module is isomorphic to the outer tensor product $V \# W$ of an indecomposable $\hat{\mathbb{Z}}_p^\lambda G_p$-module V and an irreducible $\hat{\mathbb{Z}}_p^\lambda B$-module W.

1. Introduction. Assume that $p \geq 2$ is a prime, S is either a field of characteristic p, or a commutative discrete valuation domain, $U(S)$ is the unit group of S, and G is a finite group of order $|G|$. Denote by $Z^2(G, U(S))$ the group of all $U(S)$-valued normalized 2-cocycles $\lambda = (\lambda_{a,b})_{a,b \in G} : G \times G \to U(S)$ of the group G that acts trivially on $U(S)$. We recall that λ is defined to be *normalized* if $\lambda_{a,e} = \lambda_{e,a} = 1$ for all $a \in G$, where e is the identity element of G. By the *twisted group algebra* of G over S with a 2-cocycle $\lambda \in Z^2(G, U(S))$ we mean the free S-algebra $S^\lambda G$ with an S-basis $\{u_g : g \in G\}$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$. Such a basis is called *canonical* (corresponding to λ). We remark that $S^\lambda G$ is isomorphic to the group algebra SG if and only if λ is a 2-coboundary (see [29, pp. 67–68]).

Assume now that $G = G_p \times B$, where G_p is a p-group, B is a p'-group and $|G_p| > 1$, $|B| > 1$. This means that the Sylow p-subgroup G_p of G is a direct summand of G. We recall from [17, p. 9] that a finite group whose order is not divisible by p is called a *p'-group.* Given $\mu \in Z^2(G_p, U(S))$ and $\nu \in Z^2(B, U(S))$, the map $\mu \times \nu : G \times G \to U(S)$ defined by the formula

$$(\mu \times \nu)_{x_1 b_1, x_2 b_2} = \mu_{x_1, x_2} \cdot \nu_{b_1, b_2} \tag{1.1}$$

for all $x_1, x_2 \in G_p$, $b_1, b_2 \in B$ is a 2-cocycle in $Z^2(G, U(S))$. Every 2-cocycle

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$\lambda \in Z^2(G, U(S))$ is cohomologous to $\mu \times \nu$ in the second cohomology group

$$\mathrm{H}^2(G, U(S)) = Z^2(G, U(S))/B^2(G, U(S)),$$

where μ is the restriction of λ to $G_p \times G_p$, ν is the restriction of λ to $B \times B$ and $B^2(G, U(S))$ is the subgroup of all 2-coboundaries of $Z^2(G, U(S))$. If $\nu_{b_1,b_2} = 1$ for all $b_1, b_2 \in B$, we write $\lambda = \mu \times 1$. Similarly, $\lambda = 1 \times \nu$ if $\mu_{x_1,x_2} = 1$ for all $x_1, x_2 \in G_p$.

Henceforth, we suppose that every cocycle $\lambda \in Z^2(G, U(S))$ under consideration satisfies the condition $\lambda = \mu \times \nu$, and all $S^\lambda G$-modules are assumed to be finitely generated left $S^\lambda G$-modules which are S-free. Recall that the study of these $S^\lambda G$-modules is essentially equivalent to the study of projective S-representations of G with the 2-cocycle λ.

Let $\lambda = \mu \times \nu \in Z^2(G, U(S))$ and $\{u_g \colon g \in G\}$ be a canonical S-basis of $S^\lambda G$. Then $\{u_h \colon h \in G_p\}$ is a canonical S-basis of $S^\mu G_p$ and $\{u_b \colon b \in B\}$ is a canonical S-basis of $S^\nu B$. Moreover, if $g = hb$, where $g \in G$, $h \in G_p$, $b \in B$, then $u_g = u_h u_b = u_b u_h$. It follows that $S^\lambda G \cong S^\mu G_p \otimes_S S^\nu B$.

Given an $S^\mu G_p$-module V and an $S^\nu B$-module W, we denote by $V \# W$ the $S^\lambda G$-module whose underlying S-module is $V \otimes_S W$, the $S^\lambda G$-module structure is given by

$$u_{hb}(v \otimes w) = u_h v \otimes u_b w$$

for all $h \in G_p$, $b \in B$, $v \in V$, $w \in W$, and it is extended to $S^\lambda G$ and $V \otimes_S W$ by S-linearity. Following [29, p. 122], we call the module $V \# W$ the *outer tensor product* of V and W.

We next recall from [7, p. 10] the following definitions.

Definition 1.1. Assume that S, G are as above and $\lambda = \mu \times \nu \in Z^2(G, U(S))$ is a 2-cocycle as in (1.1).

(a) The algebra $S^\lambda G$ is defined to be of *OTP representation type* if every indecomposable $S^\lambda G$-module is isomorphic to the outer tensor product $V \# W$, where V is an indecomposable $S^\mu G_p$-module and W is an irreducible $S^\nu B$-module.

(b) The group $G = G_p \times B$ is said to be of *OTP projective S-representation type* if there is a cocycle $\lambda \in Z^2(G, U(S))$ for which the algebra $S^\lambda G$ is of OTP representation type.

(c) The group $G = G_p \times B$ is defined to be of *purely OTP projective S-representation type* if $S^\lambda G$ is of OTP representation type for any $\lambda \in Z^2(G, U(S))$.

In [14], Brauer and Feit proved that the group algebra SG is always of OTP representation type in case when S is an algebraically closed field of characteristic p.

Blau [12] and Gudyvok [23], [24] independently show that if S is an arbitrary field of characteristic p, then SG is of OTP representation type if and only if G_p is cyclic or S is a splitting field for SB. In [24]–[26], Gudyvok considers an analogous problem for the group algebra SG, where S is a commutative complete discrete valuation domain. In particular, he proved that the algebra $\hat{\mathbb{Z}}_pG$ is of OTP representation type if and only if the p-adic number field $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_pB$, or G_p is cyclic of order p^n, $n \leq 2$.

In [2], [4], [5], [7]–[9], the twisted group algebras $S^\lambda G$ of OTP representation type are described, where $G = G_p \times B$ and S is either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p. For this case, necessary and sufficient conditions on G and S were given, in [5], [9], for G to be of OTP projective S-representation type and of purely OTP projective S-representation type.

In the present paper we determine the twisted group algebras $\hat{\mathbb{Z}}_p^\lambda G$ of OTP representation type, where $G = G_p \times B$ and $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers. Moreover, we describe the groups $G_p \times B$ of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type.

The main results of the paper are the following three theorems proved as Theorems 3.6, 4.7 and 5.4.

THEOREM A. *Let $p \neq 2$, G_p be a cyclic p-group, $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$ be as in* (1.1). *Denote by d the number of simple blocks of the algebra $\hat{\mathbb{Q}}_p^\mu G_p$. The algebra $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *if $|G_p| > p^2$, then $d \leq 2$;*
(ii) *$\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$.*

We also prove that if G_p is non-cyclic then, under some assumption, the algebra $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type if and only if $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$.

THEOREM B. *Let $p = 2$, G_2 be a cyclic group of order 2^n, $G = G_2 \times B$, $\mu \in Z^2(G_2, U(\hat{\mathbb{Z}}_2))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times \nu$ be as in* (1.1). *The algebra $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *$\hat{\mathbb{Q}}_2^\mu G_2$ is a totally ramified field extension of $\hat{\mathbb{Q}}_2$;*
(ii) *$\hat{\mathbb{Q}}_2^\mu G_2$ is a field and the center of the algebra $\hat{\mathbb{Q}}_2 B$ is 2-irreducible* (*see Definition* 2.11)*;*
(iii) *$n \leq 2$ and $\hat{\mathbb{Z}}_2^\mu G_2$ is the group algebra of G_2 over $\hat{\mathbb{Z}}_2$;*
(iv) *$n = 2$, the number of simple blocks of $\hat{\mathbb{Q}}_2^\mu G_2$ is 2 and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible;*
(v) *$\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

Theorem C. *The group $G = G_p \times B$ is of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if one of the following conditions is satisfied:*

(i) *$p \neq 2$ and G_p is a cyclic group of order p or p^2;*
(ii) *$p = 2$, G_2 is a cyclic group of order 2 or 4 and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible;*
(iii) *$p \neq 2$ and there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that any projective $\hat{\mathbb{Q}}_p$-representation of B with a 2-cocycle in $Z^2(B, U(\hat{\mathbb{Z}}_p))$ lifts projectively to an ordinary $\hat{\mathbb{Q}}_p$-representation of $\widehat{B}$ and $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p\widehat{B}$;*
(iv) *$p = 2$ and $\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

We remark that conditions (iii) and (iv) of Theorem C do not hold for B if $B' \neq B$. Here $B' = [B, B]$ is the commutator subgroup of B.

Throughout the paper, we use the standard group representation theory notation and terminology introduced in the monographs by Curtis and Reiner [16]–[18], and Karpilovsky [29]. A systematic account of the projective representation theory can be found in [29]. For problems of the representation theory of orders in finite-dimensional algebras and of Cohen–Macaulay algebras, we refer to the books [16]–[18], [35] and to the articles [21] and [31]. A background of the modern representation theory of finite-dimensional algebras can be found in the monographs by Assem, Simson and Skowroński [1], Drozd and Kirichenko [22], Simson [30], and Simson and Skowroński [34], where among other things the representation types (finite, tame, wild) of finite groups and algebras are discussed. Various aspects of the representation types are also considered by Dowbor and Simson [19], [20], Simson [32], and Simson and Skowroński [33].

In particular, we use the following notation: $p \geq 2$ is a prime; $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers; $\hat{\mathbb{Q}}_p$ is the field of p-adic numbers; $U(\hat{\mathbb{Z}}_p)$ is the unit group of $\hat{\mathbb{Z}}_p$; $\Phi_{p^n}(X)$ is the cyclotomic polynomial of order p^n; $\mathrm{GF}(q)$ is the finite field of q-elements; $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is the residue class field of $\hat{\mathbb{Z}}_p$; $\operatorname{rad} A$ is the Jacobson radical of a ring A and $\overline{A} = A/\operatorname{rad} A$ is the factor ring of A by $\operatorname{rad} A$; $G = G_p \times B$ is a finite group, where G_p is a p-group, B is a p'-group, $|G_p| > 1$ and $|B| > 1$; $H' = [H, H]$ is the commutator subgroup of a group H, e is the identity element of H, $|h|$ is the order of $h \in H$; $\operatorname{soc} H$ is the socle of an abelian group H. If D is a subgroup of H, then the restriction of $\lambda \in Z^2(H, U(\hat{\mathbb{Z}}_p))$ to $D \times D$ will also be denoted by λ. We assume that in this case $\hat{\mathbb{Z}}_p^\lambda D$ is the $\hat{\mathbb{Z}}_p$-subalgebra of $\hat{\mathbb{Z}}_p^\lambda H$ consisting of all $\hat{\mathbb{Z}}_p$-linear combinations of elements $\{u_d \colon d \in D\}$, where $\{u_h \colon h \in H\}$ is a canonical $\hat{\mathbb{Z}}_p$-basis of $\hat{\mathbb{Z}}_p^\lambda H$ corresponding to λ. Given a $\hat{\mathbb{Z}}_p^\lambda H$-module, we write $\operatorname{End}_{\hat{\mathbb{Z}}_p^\lambda H}(M)$ for the ring of all $\hat{\mathbb{Z}}_p^\lambda H$-endomorphisms of M. Denote by

$A_1 \dot{\times} A_2$ the Kronecker (or tensor) product of the matrices A_1 and A_2 (see [16, p. 69]), and by E_m the identity matrix of order m.

2. Preliminaries. We start with some information on the structure of the units of $\hat{\mathbb{Z}}_p$ that we need in the paper (see [27, p. 236]).

If $p \neq 2$, then any unit η in $U(\hat{\mathbb{Z}}_p)$ can be represented uniquely in the form

$$\eta = \omega^r (1+p)^\alpha,$$

where ω is a primitive $(p-1)$th root of 1 and $\alpha \in \hat{\mathbb{Z}}_p$. Any unit η in $U(\hat{\mathbb{Z}}_2)$ can be represented uniquely in the form

$$\eta = \pm 5^\alpha, \qquad \alpha \in \hat{\mathbb{Z}}_2.$$

Denote by $U_t(\hat{\mathbb{Z}}_p)$ the maximal torsion subgroup of $U(\hat{\mathbb{Z}}_p)$. Hence

$$U_t(\hat{\mathbb{Z}}_p) = \begin{cases} \langle \omega \rangle & \text{if } p \neq 2, \\ \langle -1 \rangle & \text{if } p = 2. \end{cases}$$

Let

$$U_f(\hat{\mathbb{Z}}_p) = \begin{cases} \{(1+p)^\alpha \colon \alpha \in \hat{\mathbb{Z}}_p\} & \text{if } p \neq 2, \\ \{5^\alpha \colon \alpha \in \hat{\mathbb{Z}}_2\} & \text{if } p = 2. \end{cases}$$

We have $U(\hat{\mathbb{Z}}_p) = U_t(\hat{\mathbb{Z}}_p) \times U_f(\hat{\mathbb{Z}}_p)$.

LEMMA 2.1. *Let $p \neq 2$, D be a finite p-group and T a finite p'-group.*

(i) *For every 2-cocycle $\lambda \in Z^2(D, U(\hat{\mathbb{Z}}_p))$ there exists a 2-cocycle μ in $Z^2(D, U_f(\hat{\mathbb{Z}}_p))$ such that λ and μ are cohomologous in $\mathrm{H}^2(D, U(\hat{\mathbb{Z}}_p))$.*
(ii) *The restriction of any 2-cocycle $\lambda \in Z^2(D, U(\hat{\mathbb{Z}}_p))$ to $D' \times D'$ is a 2-coboundary.*
(iii) *For every 2-cocycle $\lambda \in Z^2(T, U(\hat{\mathbb{Z}}_p))$ there exists a 2-cocycle ν in $Z^2(T, U_t(\hat{\mathbb{Z}}_p))$ such that λ and ν are cohomologous in $\mathrm{H}^2(T, U(\hat{\mathbb{Z}}_p))$.*

Proof. Apply [29, Theorem 1.7, p. 11, and Corollary 4.10, p. 42]. ■

By Lemma 2.1, without loss of generality we may assume that if $G = G_p \times B$ and $p \neq 2$, then every 2-cocycle $\lambda \in Z^2(G, U(\hat{\mathbb{Z}}_p))$ satisfies the condition $\lambda = \mu \times \nu$, where $\mu \in Z^2(G_p, U_f(\hat{\mathbb{Z}}_p))$ and $\nu \in Z^2(B, U_t(\hat{\mathbb{Z}}_p))$.

LEMMA 2.2. *Let D be a finite 2-group and T a finite $2'$-group.*

(i) *The restriction of any 2-cocycle $\lambda \in Z^2(D, U_f(\hat{\mathbb{Z}}_2))$ to $D' \times D'$ is a 2-coboundary.*
(ii) *Every 2-cocycle $\lambda \in Z^2(T, U(\hat{\mathbb{Z}}_2))$ is a 2-coboundary.*

Proof. Again apply [29, Theorem 1.7, p. 11, and Corollary 4.10, p. 42]. ■

In view of Lemma 2.2, we may assume that if $p = 2$ and $G = G_2 \times B$, then every 2-cocycle $\lambda \in Z^2(G, U(\hat{\mathbb{Z}}_2))$ satisfies the condition $\lambda = \mu \times 1$, where μ is the restriction of λ to $G_2 \times G_2$.

Let $H = \langle a_1 \rangle \times \dots \times \langle a_m \rangle$ be an abelian p-group of type $(p^{n_1}, \dots, p^{n_m})$, $\mu \in Z^2(H, U(\hat{\mathbb{Z}}_p))$, $r_i = p^{n_i} - 1$ and $\gamma_i = \mu_{a_i,a_i}\mu_{a_i,a_i^2} \dots \mu_{a_i,a_i^{r_i}}$ for i in $\{1, \dots, m\}$. The algebra $\hat{\mathbb{Z}}_p^\mu H$ has a canonical $\hat{\mathbb{Z}}_p$-basis $\{u_h : h \in H\}$ satisfying the following conditions:

(1) if $h = a_1^{k_1} \dots a_m^{k_m}$ and $0 \le k_i < p^{n_i}$ for each $i \in \{1, \dots, m\}$, then
$$u_h = u_{a_1}^{k_1} \dots u_{a_m}^{k_m};$$

(2) $u_{a_i}^{p^{n_i}} = \gamma_i u_e$ for every $i \in \{1, \dots, m\}$.

We also denote $\hat{\mathbb{Z}}_p^\mu H$ by $[H, \hat{\mathbb{Z}}_p, \gamma_1, \dots, \gamma_m]$.

Recall that $u_{a_i}u_{a_j} = \varepsilon_{ij}u_{a_j}u_{a_i}$, where $i \ne j$ and $\varepsilon_{ij} = \mu_{a_i,a_j}\mu_{a_j,a_i}^{-1}$. It follows that $\varepsilon_{ij}^{|a_i|} = 1$. Hence, $\varepsilon_{ij} = 1$ for $p \ne 2$, and $\varepsilon_{ij} \in \{1, -1\}$ for $p = 2$. Consequently, if $p \ne 2$ then $\hat{\mathbb{Z}}_p^\mu H$ is a commutative algebra.

Now we collect several facts we apply later.

LEMMA 2.3. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$. The algebra $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type if and only if the outer tensor product $V \# W$ of any indecomposable $\hat{\mathbb{Z}}_p^\mu G_p$-module V and any irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W is an indecomposable $\hat{\mathbb{Z}}_p^\lambda G$-module.*

The proof is similar to that of the corresponding fact for the group algebra $\hat{\mathbb{Z}}_p G$ (see [12, p. 41], [26, p. 68] and [28, p. 658]).

LEMMA 2.4. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$. If V is an indecomposable $\hat{\mathbb{Z}}_p^\mu G_p$-module and W is an irreducible $\hat{\mathbb{Z}}_p^\nu B$-module, then*
$$\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\lambda G}(V \# W)} \cong \overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu G_p}(V)} \otimes_{\mathbb{Z}_p} \overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\nu B}(W)}.$$

Proof. See [7, p. 15] and [28, p. 657]. ∎

LEMMA 2.5. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$. If $\hat{\mathbb{Q}}_p$ is a splitting field for the $\hat{\mathbb{Q}}_p$-algebra $\hat{\mathbb{Q}}_p^\nu B$, then $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type.*

Proof. Again see [7, p. 15] and [28, p. 657]. ∎

LEMMA 2.6. *Let R be a commutative complete discrete valuation domain, H a finite group, $\lambda \in Z^2(H, U(R))$ and V an $R^\lambda H$-module. Then V is indecomposable if and only if $\overline{\mathrm{End}_{R^\lambda H}(V)}$ is a skew field.*

Proof. Apply [17, Proposition 6.10, p. 125]. ∎

LEMMA 2.7. *Let G_p be a finite p-group, H a subgroup of G_p, $\lambda \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$ and V an indecomposable $\hat{\mathbb{Z}}_p^\lambda H$-module. Assume that $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\lambda H}(V)}$ is isomorphic to the finite field $\mathrm{GF}(p^m)$ and one of the following conditions is satisfied:*

(i) *$G_p = H \cdot T$, where T is a subgroup of the center of G_p;*
(ii) *p does not divide m.*

Then $V^{G_p} := \hat{\mathbb{Z}}_p^\lambda G_p \otimes_{\hat{\mathbb{Z}}_p^\lambda H} V$ is an indecomposable $\hat{\mathbb{Z}}_p^\lambda G_p$-module, and the quotient algebra $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\lambda G_p}(V^{G_p})}$ is isomorphic to $\mathrm{GF}(p^m)$.

Proof. Apply [10, Theorem 2.6, p. 4138]. ∎

LEMMA 2.8. *Let K be a finite field extension of $\hat{\mathbb{Q}}_p$, R the ring of all integral elements of K, $\overline{R}$ the residue class field of R, and H either a cyclic group of order p^3, or an abelian group of type (p,p). Then, for any finite field extension F of $\overline{R}$, there exists an indecomposable RH-module M such that $\overline{\mathrm{End}_{RH}(M)} \cong F$.*

Proof. See [26, pp. 72–74]. ∎

LEMMA 2.9. *Let K be a finite ramified extension of $\hat{\mathbb{Q}}_p$, $K \neq \hat{\mathbb{Q}}_p$, R the ring of all integral elements of K, and H a cyclic group of order p^2. Then, for any finite field extension F of $\overline{R}$, there is an indecomposable RH-module M such that $\overline{\mathrm{End}_{RH}(M)} \cong F$.*

Proof. See [26, pp. 73–74]. ∎

LEMMA 2.10. *Let $G = G_p \times B$. The group algebra $\hat{\mathbb{Z}}_p G$ is of OTP representation type if and only if either $\hat{\mathbb{Q}}_p$ is a splitting field for the group algebra $\hat{\mathbb{Q}}_p B$, or G_p is a cyclic group of order p^r, $r \leq 2$.*

Proof. See [24, p. 583]. ∎

Assume that

(2.1)
η is a primitive $(p^m - 1)$th root of 1,
$f(X) \in \hat{\mathbb{Z}}_p[X]$ is the minimal monic polynomial of η,
A_f is the companion matrix of the polynomial f
in the sense of [15, p. 345].

It is well known (see [27, pp. 190, 211–212]) that:

(i) the polynomial f is irreducible modulo p and the degree of f is m;
(ii) $\hat{\mathbb{Q}}_p(\eta)$ is an unramified extension of $\hat{\mathbb{Q}}_p$ of degree m;
(iii) $\hat{\mathbb{Z}}_p[\eta]$ is the ring of all integral elements of $\hat{\mathbb{Q}}_p(\eta)$;
(iv) $\hat{\mathbb{Z}}_p[\eta]/p\hat{\mathbb{Z}}_p[\eta] \cong \mathrm{GF}(p^m)$.

Let $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$. Then $\hat{\mathbb{Q}}_p^{\nu} B$ is the quotient algebra of $\hat{\mathbb{Q}}_p \widehat{B}$, where $|\widehat{B}| = (p-1) \cdot |B|$ (see [29, pp. 136–137]). Denote by ξ a primitive $|\widehat{B}|$th root of 1. The field $\hat{\mathbb{Q}}_p(\xi)$ is a splitting field for $\hat{\mathbb{Q}}_p \widehat{B}$ (see [17, p. 386]) and hence $\hat{\mathbb{Q}}_p(\xi)$ is a splitting field for $\hat{\mathbb{Q}}_p^{\nu} B$. By [27, p. 211], $\hat{\mathbb{Q}}_p(\xi)$ is an unramified extension of $\hat{\mathbb{Q}}_p$. Since the index of every simple block of $\hat{\mathbb{Q}}_p^{\nu} B$ is 1 and $\hat{\mathbb{Q}}_p(\xi) \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathbb{Q}}_p^{\nu} B$ is a direct product of matrix algebras over $\hat{\mathbb{Q}}_p(\xi)$, we have

$$\hat{\mathbb{Q}}_p^{\nu} B \cong \mathbb{M}_{n_1}(F_1) \times \cdots \times \mathbb{M}_{n_r}(F_r), \tag{2.2}$$

where $F_1, \ldots, F_r$ are unramified extensions of $\hat{\mathbb{Q}}_p$. We recall that the algebras $\mathbb{M}_{n_1}(F_1), \ldots, \mathbb{M}_{n_r}(F_r)$ are called the *simple blocks* of $\hat{\mathbb{Q}}_p B$.

Let W_j be an irreducible $\hat{\mathbb{Z}}_p^{\nu} B$-module such that $\widetilde{W_j} := \hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} W_j$ is a direct summand of $\mathbb{M}_{n_j}(F_j)$, where $j \in \{1, \ldots, r\}$. Denote by Γ_j an irreducible matrix $\hat{\mathbb{Z}}_p$-representation of the algebra $\hat{\mathbb{Z}}_p^{\nu} B$ afforded by the module W_j. Let $\deg \Gamma_j = k_j$. Assume that

$$\begin{aligned} L_j &:= \{A \in \mathbb{M}_{k_j}(\hat{\mathbb{Q}}_p) \colon A\Gamma_j(x) = \Gamma_j(x)A \text{ for every } x \in \hat{\mathbb{Z}}_p^{\nu} B\}, \\ S_j &:= \{C \in \mathbb{M}_{k_j}(\hat{\mathbb{Z}}_p) \colon C\Gamma_j(x) = \Gamma_j(x)C \text{ for every } x \in \hat{\mathbb{Z}}_p^{\nu} B\}. \end{aligned} \tag{2.3}$$

Then L_j is a $\hat{\mathbb{Q}}_p$-algebra and S_j is a $\hat{\mathbb{Z}}_p$-algebra. Moreover

$$L_j \cong \operatorname{End}_{\hat{\mathbb{Q}}_p^{\nu} B}(\widetilde{W_j}) \cong F_j, \qquad S_j \cong \operatorname{End}_{\hat{\mathbb{Z}}_p^{\nu} B}(W_j).$$

We identify $\alpha \in \hat{\mathbb{Q}}_p$ with the scalar matrix αE_{k_j}. Then $\hat{\mathbb{Q}}_p \subset L_j$ and $\hat{\mathbb{Z}}_p \subset S_j$. Suppose that $A \in L_j$ and $A \neq 0$. Then by [16, Corollary 76.16, p. 536], $A = p^l C$, where $l \in \mathbb{Z}$, $C \in S_j$ and C is invertible over $\hat{\mathbb{Z}}_p$. Since C is a root of the characteristic polynomial $\det(XE - C) \in \hat{\mathbb{Z}}_p[X]$ of C, the matrix C is integral over $\hat{\mathbb{Z}}_p$. If A is integral over $\hat{\mathbb{Z}}_p$, then so is AC^{-1}. It follows that $l \geq 0$, hence $A \in S_j$. Consequently, S_j is the integral closure of $\hat{\mathbb{Z}}_p$ in L_j.

Definition 2.11. Let B be a finite p'-group and $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$. We say that the center of the algebra $\hat{\mathbb{Q}}_p^{\nu} B$ is *p-irreducible* if $[F : \hat{\mathbb{Q}}_p]$ is not divisible by p for every simple block $\mathbb{M}_n(F)$ of $\hat{\mathbb{Q}}_p^{\nu} B$.

Denote by l_B the product of all pairwise distinct prime divisors of $|B|$. Let ξ be a primitive l_Bth root of 1. If $[\hat{\mathbb{Q}}_p(\xi) : \hat{\mathbb{Q}}_p]$ is not divisible by p, then for any $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ the center of $\hat{\mathbb{Q}}_p^{\nu} B$ is p-irreducible.

Proposition 2.12. *Let W_j be an irreducible $\hat{\mathbb{Z}}_p^{\nu} B$-module such that $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} W_j$ is a direct summand of $\mathbb{M}_{n_j}(F_j)$ (see (2.2)). Then:*

(i) $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^{\nu} B}(W_j)} \cong \mathrm{GF}(p^{k_j})$, *where* $k_j = [F_j : \hat{\mathbb{Q}}_p]$.

(ii) *$\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$ if and only if $\mathbb{Z}_p$ is a splitting field for $\mathbb{Z}_p^{\overline{\nu}} B := \hat{\mathbb{Z}}_p^\nu B / p\hat{\mathbb{Z}}_p^\nu B$.*

Proof. (i) By [17, Proposition 5.22, p. 112],

$$\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W_j)} \cong \operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W_j)/p \operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W_j) \cong S_j/pS_j = \mathrm{GF}(p^{k_j}),$$

where $k_j = [F_j : \hat{\mathbb{Q}}_p]$ (see the notation (2.3)).

(ii) By [17, Theorem 6.8, p. 124], for every simple $\mathbb{Z}_p^{\overline{\nu}} B$-module $\overline{W}$ there exists an irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W such that $W/pW \cong \overline{W}$. By [16, Theorem 76.8, p. 532 and Corollary 76.16, p. 536],

$$\operatorname{End}_{\mathbb{Z}_p^{\overline{\nu}} B}(\overline{W}) \cong \operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W)/p \operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W).$$

Moreover, by [16, Corollary 76.15, p. 536], W/pW is a simple $\mathbb{Z}_p^{\overline{\nu}} B$-module for any irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W.

Furthermore, $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$ if and only if

$$\operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W)/p \operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W) \cong \mathbb{Z}_p$$

for every irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W. It follows that $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$ if and only if $\operatorname{End}_{\mathbb{Z}_p^{\overline{\nu}} B}(\overline{W}) \cong \mathbb{Z}_p$ for any simple $\mathbb{Z}_p^{\overline{\nu}} B$-module $\overline{W}$, i.e. if and only if $\mathbb{Z}_p$ is a splitting field for $\mathbb{Z}_p^{\overline{\nu}} B$. ∎

PROPOSITION 2.13. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$. Assume that if G_p is a non-abelian group, then the center of the algebra $\hat{\mathbb{Q}}_p^\nu B$ is p-irreducible. Moreover, let T be a subgroup of G_p, $|T| > 1$ and $H = T \times B$. If $\hat{\mathbb{Z}}_p^\lambda H$ is not of OTP representation type, then neither is $\hat{\mathbb{Z}}_p^\lambda G$.*

Proof. Suppose that $\hat{\mathbb{Z}}_p^\lambda H$ is not of OTP representation type. Then, in view of Lemma 2.3, there exist an indecomposable $\hat{\mathbb{Z}}_p^\mu T$-module V and an irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W such that $V \# W$ is a decomposable $\hat{\mathbb{Z}}_p^\lambda H$-module. By Lemmas 2.4 and 2.6, $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu T}(V)} \otimes_{\mathbb{Z}_p} \overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\nu B}(W)}$ is not a skew field. In view of Lemma 2.7, the $\hat{\mathbb{Z}}_p^\mu G_p$-module $V^{G_p} := \hat{\mathbb{Z}}_p^\mu G_p \otimes_{\hat{\mathbb{Z}}_p^\mu T} V$ is indecomposable and the quotient algebra $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu G_p}(V^{G_p})}$ is isomorphic to the field $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu T}(V)}$. Hence, again by Lemmas 2.4 and 2.6, the $\hat{\mathbb{Z}}_p^\lambda G$-module $V^{G_p} \# W$ is decomposable. Applying Lemma 2.3, we conclude that the algebra $\hat{\mathbb{Z}}_p^\lambda G$ is not of OTP representation type. ∎

3. Twisted group algebras $\hat{\mathbb{Z}}_p^\lambda G$ of OTP representation type for $p \neq 2$. Let R be a commutative ring with 1, and t a root of the monic irreducible polynomial $f(X) \in R[X]$. Denote by

$$\widetilde{z} \in \mathbb{M}_{m+1}(R) \tag{3.1}$$

the matrix of multiplication by $z \in R[t]$ in the R-basis $1, t, \dots, t^m$ of the ring $R[t]$.

Throughout this section, we assume that $p \neq 2$.

Let δ, θ and ρ be roots of the irreducible polynomials

$$X^{p^n} - (1+p),\ X^{p^{n-1}} - (1+p),\ \Phi_p\left(\frac{X^{p^{n-1}}}{1+p}\right) \in \hat{\mathbb{Z}}_p[X],$$

respectively.

LEMMA 3.1. *Let $H = \langle a \rangle$ be a cyclic group of order p^n $(n \geq 1)$ and $\hat{\mathbb{Z}}_p^\mu H = [H, \hat{\mathbb{Z}}_p, (1+p)^{p^l}]$, where $l \in \{0, 1\}$ and $n \geq 2$ for $l = 1$.*

(i) *If $l = 0$ then, up to equivalence, the algebra $\hat{\mathbb{Z}}_p^\mu H$ has only one indecomposable matrix $\hat{\mathbb{Z}}_p$-representation $\Gamma\colon u_a \mapsto \widetilde{\delta}$.*

(ii) *If $l = 1$ then, up to equivalence, the indecomposable matrix $\hat{\mathbb{Z}}_p$-representations of the algebra $\hat{\mathbb{Z}}_p^\mu H$ are the following:*

$$\Gamma_1\colon u_a \mapsto \widetilde{\theta}, \quad \Gamma_2\colon u_a \mapsto \widetilde{\rho}, \quad \Gamma_{3j}\colon u_a \mapsto \begin{pmatrix} \widetilde{\theta} & \langle \pi^j \rangle \\ 0 & \widetilde{\rho} \end{pmatrix},\ j = 0, 1, \dots, p^{n-1}-1,$$

where $\pi = 1 - \theta$ is a prime element of $\hat{\mathbb{Z}}_p[\theta]$ and $\langle \pi^j \rangle$ is the matrix in which all columns but the last one are zero, and the last column consists of the coordinates of π^j in the $\hat{\mathbb{Z}}_p$-basis $1, \theta, \dots, \theta^{p^{n-1}-1}$ of the ring $\hat{\mathbb{Z}}_p[\theta]$.

Proof. (i) If $l = 0$ then $\hat{\mathbb{Z}}_p^\mu H \cong \hat{\mathbb{Z}}_p[\delta]$. Each $\hat{\mathbb{Z}}_p^\mu H$-module M can be considered as a torsionfree module over the principal ideal domain $\hat{\mathbb{Z}}_p[\delta]$, therefore if $M \neq 0$ then $M \cong \hat{\mathbb{Z}}_p[\delta] \oplus \dots \oplus \hat{\mathbb{Z}}_p[\delta]$. Hence, up to equivalence, the algebra $\hat{\mathbb{Z}}_p^\mu H$ has only one indecomposable matrix $\hat{\mathbb{Z}}_p$-representation $u_a \mapsto \widetilde{\delta}$.

(ii) Let $l = 1$, M be an arbitrary non-zero $\hat{\mathbb{Z}}_p^\mu H$-module and

$$N := \{v \in M\colon (u_a^{p^{n-1}} - (1+p)u_e)v = 0\}.$$

Then N is a $\hat{\mathbb{Z}}_p^\mu H$-submodule of M. Since M is a $\hat{\mathbb{Z}}_p$-torsionfree module, $\alpha m \in N$ implies $m \in N$ for all $m \in M$ and for all non-zero $\alpha \in \hat{\mathbb{Z}}_p$. One can view the $\hat{\mathbb{Z}}_p^\mu H$-module N as a module over the algebra

$$\hat{\mathbb{Z}}_p^\mu H / (u_a^{p^{n-1}} - (1+p)u_e)\hat{\mathbb{Z}}_p^\mu H \cong \hat{\mathbb{Z}}_p[\theta].$$

Since $\hat{\mathbb{Z}}_p[\theta]$ is a principal ideal domain and N is a $\hat{\mathbb{Z}}_p[\theta]$-torsionfree module, there is a decomposition $N \cong \hat{\mathbb{Z}}_p[\theta] \oplus \cdots \oplus \hat{\mathbb{Z}}_p[\theta]$. Moreover, we have

$$\hat{\mathbb{Z}}_p^\mu H/\Phi_p\Big(\frac{u_a^{p^{n-1}}}{1+p}\Big)\hat{\mathbb{Z}}_p^\mu H \cong \hat{\mathbb{Z}}_p[\rho],$$

where $\hat{\mathbb{Z}}_p[\rho]$ is a principal ideal domain. The $\hat{\mathbb{Z}}_p^\mu H$-module M/N can be viewed as a $\hat{\mathbb{Z}}_p[\rho]$-module. If $z \in \hat{\mathbb{Z}}_p[\rho]$ and $z \neq 0$, then the equality $z(v+N) = N$ yields $v \in N$. This means that M/N is a torsionfree module over $\hat{\mathbb{Z}}_p[\rho]$. Hence in the case $N \neq M$ we have $M/N \cong \hat{\mathbb{Z}}_p[\rho] \oplus \cdots \oplus \hat{\mathbb{Z}}_p[\rho]$.

Every $\hat{\mathbb{Z}}_p$-basis of N can be extended to an $\hat{\mathbb{Z}}_p$-basis of M (see [16, p. 100]), and hence up to equivalence, any matrix $\hat{\mathbb{Z}}_p$-representation Γ of the algebra $\hat{\mathbb{Z}}_p^\mu H$ afforded by the $\hat{\mathbb{Z}}_p^\mu H$-module M can be written in the form

$$\Gamma(u_a) = \begin{pmatrix} \widetilde{\theta} \dot{\times} E_s & * \\ 0 & \widetilde{\rho} \dot{\times} E_t \end{pmatrix},$$

where $\widetilde{\theta} \dot{\times} E_s$ is the Kronecker product of the matrices $\widetilde{\theta}$ and E_s. Using the technique of [11, pp. 880–888], we conclude that indecomposable matrix $\hat{\mathbb{Z}}_p$-representations of the algebra $\hat{\mathbb{Z}}_p^\mu H$ are $\Gamma_1, \Gamma_2, \Gamma_{3j}$, as asserted. ∎

LEMMA 3.2. *Let $H = \langle a \rangle$ be a cyclic group of order p^n and let μ be in $Z^2(H, U(\hat{\mathbb{Z}}_p))$. If the algebra $\hat{\mathbb{Q}}_p^\mu H$ has at most two simple blocks, then $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu H}(W)} \cong \mathbb{Z}_p$ for each indecomposable $\hat{\mathbb{Z}}_p^\mu H$-module W.*

Proof. Keeping the notation of Lemma 3.1, assume that $\hat{\mathbb{Z}}_p^\mu H$ is not the group algebra $\hat{\mathbb{Z}}_p H$ and $\hat{\mathbb{Q}}_p^\mu H$ is not a field. Then $n \geq 2$ and $\hat{\mathbb{Z}}_p^\mu H = [H, \hat{\mathbb{Z}}_p, (1+p)^p]$. If W_1 is an underlying $\hat{\mathbb{Z}}_p^\mu H$-module of the representation Γ_1, then $\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu H}(W_1) \cong \hat{\mathbb{Z}}_p[\theta]$, and consequently

$$\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu H}(W_1)} \cong \hat{\mathbb{Z}}_p[\theta]/(1-\theta)\hat{\mathbb{Z}}_p[\theta] \cong \mathbb{Z}_p.$$

Let W_{3j} be an underlying $\hat{\mathbb{Z}}_p^\mu H$-module of the representation Γ_{3j},

$$S := \{C \in \mathbb{M}_{p^n}(\hat{\mathbb{Z}}_p) \colon C\Gamma_{3j}(u_a) = \Gamma_{3j}(u_a)C\},$$
$$S_1 := \{C_1 \in \mathbb{M}_{p^{n-1}}(\hat{\mathbb{Z}}_p) \colon C_1\widetilde{\theta} = \widetilde{\theta}C_1\}.$$

The ring S is isomorphic to $\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu H}(W_{3j})$, and the ring S_1 is isomorphic to $\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu H}(W_1)$. If $C \in S$ then

$$C = \begin{pmatrix} C_1 & D \\ 0 & C_2 \end{pmatrix},$$

where $C_1 \in S_1$ and $C_2\widetilde{\rho} = \widetilde{\rho}C_2$. Since $S_1/\mathrm{rad}\, S_1 \cong \mathbb{Z}_p$, we have $C_1 = \alpha E' + T_1$, where $\alpha \in \hat{\mathbb{Z}}_p$, E' is the identity matrix of order p^{n-1} and $T_1 \in \mathrm{rad}\, S_1$, i.e. T_1 is a non-invertible matrix over $\hat{\mathbb{Z}}_p$. It follows that $C = \alpha E + T$, where E

is the identity matrix of order p^n and $T \in S$. Because S is a local ring and T is a non-invertible matrix over $\hat{\mathbb{Z}}_p$, we conclude that $T \in \operatorname{rad} S$. It follows that $S/\operatorname{rad} S \cong \mathbb{Z}_p$.

The case when $\hat{\mathbb{Q}}_p^\mu H$ is a field and the case when $|H| = p$ and $\hat{\mathbb{Z}}_p^\mu H$ is the group algebra can be treated similarly. ■

LEMMA 3.3. *Let $H = \langle a \rangle$ be a cyclic p-group and $\mu \in Z^2(H, U(\hat{\mathbb{Z}}_p))$. Assume that the algebra $\hat{\mathbb{Q}}_p^\mu H$ has three simple blocks.*

(i) *If $\hat{\mathbb{Z}}_p^\mu H$ is the group algebra $\hat{\mathbb{Z}}_p H$, then $|H| = p^2$ and $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu H}(W)} \cong \mathbb{Z}_p$ for each indecomposable $\hat{\mathbb{Z}}_p^\mu H$-module W.*

(ii) *If μ is not a 2-coboundary, then, for any positive integer m, there is an indecomposable $\hat{\mathbb{Z}}_p^\mu H$-module M such that*

$$\overline{\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu H}(M)} \cong \mathrm{GF}(p^m).$$

Proof. Statement (i) was proved in [24, p. 583]. Now we prove (ii). In view of Lemma 2.7, we may assume that $|H| = p^3$ and

$$\hat{\mathbb{Z}}_p^\mu H = [H, \hat{\mathbb{Z}}_p, (1+p)^{p^2}].$$

Denote by $\theta_1, \theta_2, \theta_3$ roots of the irreducible polynomials

$$X^p - (1+p), \Phi_p\left(\frac{X^p}{1+p}\right), \Phi_{p^2}\left(\frac{X^p}{1+p}\right) \in \hat{\mathbb{Z}}_p[X],$$

respectively, and by s_j the $\hat{\mathbb{Z}}_p$-rank of $\hat{\mathbb{Z}}_p[\theta_j]$ for $j = 1, 2, 3$. Let $\pi_j = 1 - \theta_j$ for $j = 1, 2$, A_f be the companion matrix of the polynomial f as in (2.1) and Γ be the matrix $\hat{\mathbb{Z}}_p$-representation of the algebra $\hat{\mathbb{Z}}_p^\mu H$ defined by

$$\Gamma(u_a) = \begin{pmatrix} \widetilde{\theta}_1 \dot{\times} E_m & \langle \pi_1 \rangle \dot{\times} E_m & \langle 1 \rangle \dot{\times} A_f \\ 0 & \widetilde{\theta}_2 \dot{\times} E_m & \langle \pi_2 \rangle \dot{\times} E_m \\ 0 & 0 & \widetilde{\theta}_3 \dot{\times} E_m \end{pmatrix},$$

where m is the order of A_f, and $\langle \delta_j \rangle$ is the matrix all of whose columns except the last one are zero, whereas the last column consists of the coordinates of the element $\delta_j \in \hat{\mathbb{Z}}_p[\theta_j]$ in the $\hat{\mathbb{Z}}_p$-basis $1, \theta_j, \dots, \theta_j^{s_j-1}$ of the ring $\hat{\mathbb{Z}}_p[\theta_j]$, $1 \le j \le 2$.

By the same arguments as in [11, pp. 889–894], we can prove that the representation Γ is indecomposable. Denote by M the underlying $\hat{\mathbb{Z}}_p^\mu H$-module of Γ. The algebra $\operatorname{End}_{\hat{\mathbb{Z}}_p^\mu H}(M)$ is isomorphic to the algebra

$$S = \{C \in \mathbb{M}_{mp^3}(\hat{\mathbb{Z}}_p) \colon C\Gamma(u_a) = \Gamma(u_a)C\}.$$

For a matrix $\Omega = (x_{kl}) \in \mathrm{GL}(m, \hat{\mathbb{Z}}_p[\theta_j])$, we set $\widetilde{\Omega} = (\widetilde{x}_{kl})$ (see the notation (3.1)).

By Lemma 2.6, S is a local ring. If $C \in S$ and C is a non-invertible matrix, then $C \in \operatorname{rad} S$. Let $C \in S$ be an invertible matrix. Arguing as in [11, pp. 890–892], we conclude that C is of the form

$$C = \begin{pmatrix} \widetilde{\Omega}_1 & C_1 & C_2 \\ 0 & \widetilde{\Omega}_2 & C_3 \\ 0 & 0 & \widetilde{\Omega}_3 \end{pmatrix},$$

where $\Omega_j \in \mathrm{GL}(m, \hat{\mathbb{Z}}_p[\theta_j])$ for $j = 1, 2, 3$ and $\Omega_1^{-1} A_f \Omega_1 \equiv A_f \pmod{\pi_1}$. The matrix Ω_1 can be written as $\Omega_1 = T_1 + \pi_1 \Omega_1'$, where $T_1 \in \mathrm{GL}(m, \hat{\mathbb{Z}}_p)$, $\Omega_1' \in \mathbb{M}_m(\hat{\mathbb{Z}}_p[\theta_1])$ and $T_1^{-1} A_f T_1 \equiv A_f \pmod{p}$. By [16, Theorem 76.8, p. 532], there is a matrix $D_1 \in \mathrm{GL}(m, \hat{\mathbb{Z}}_p)$ such that $D_1 \equiv T_1 \pmod{p}$ and $D_1^{-1} A_f D_1 = A_f$. Let $D := \mathrm{diag}[E_{s_1} \dot{\times} D_1, E_{s_2} \dot{\times} D_1, E_{s_3} \dot{\times} D_1]$. Then $D \in S$, hence $C - D \in S$. Since $\Omega_1 - D_1 \equiv 0 \pmod{\pi_1}$, the matrix $\widetilde{\Omega}_1 - \widetilde{D}_1$ is non-invertible over $\hat{\mathbb{Z}}_p$. Hence so is $C - D$, and therefore $C - D \in \operatorname{rad} S$.

Let $R = \{D_1 \in \mathbb{M}_m(\hat{\mathbb{Z}}_p) \colon D_1 A_f = A_f D_1\}$. The ring R is local, $\operatorname{rad} R = pR$ and $R/\operatorname{rad} R \cong \mathrm{GF}(p^m)$. The map $\varphi \colon S/\operatorname{rad} S \to R/\operatorname{rad} R$ defined by $\varphi(C + \operatorname{rad} S) = D_1 + \operatorname{rad} R$ is an algebra isomorphism. Consequently,

$$\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^{\mu} H}(M)} \cong \mathrm{GF}(p^m)$$

and the proof is complete. ∎

Lemma 3.4. *Let $H = \langle a \rangle \times \langle b \rangle$ be an abelian group of type (p^n, p^2), $\mu \in Z^2(H, U(\hat{\mathbb{Z}}_p))$ and $\hat{\mathbb{Z}}_p^{\mu} H = [H, \hat{\mathbb{Z}}_p, 1 + p, 1]$. Then, for any finite field F of characteristic p, there is an indecomposable $\hat{\mathbb{Z}}_p^{\mu} H$-module M such that $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^{\mu} H}(M)} \cong F$.*

Proof. Let $D := \langle a \rangle$ and $T := \langle b \rangle$. The algebra $\hat{\mathbb{Z}}_p^{\mu} D$ is isomorphic to the $\hat{\mathbb{Z}}_p$-algebra $R := \hat{\mathbb{Z}}_p[\rho]$, where $\rho^{p^n} = 1 + p$. The field $\hat{\mathbb{Q}}_p(\rho)$ is a totally ramified extension of $\hat{\mathbb{Q}}_p$ of degree p^n, R is the ring of all integral elements of $\hat{\mathbb{Q}}_p(\rho)$, $\pi = 1 - \rho$ is a prime element of R and $R/\pi R \cong \mathbb{Z}_p$. One can view $\hat{\mathbb{Z}}_p^{\mu} H$ as the group algebra RT. By Lemma 2.9, for any finite field F of characteristic p, there is an indecomposable RT-module M for which $\overline{\mathrm{End}_{RT}(M)} \cong F$. One can view M as an indecomposable $\hat{\mathbb{Z}}_p^{\mu} H$-module. Moreover $\mathrm{End}_{\hat{\mathbb{Z}}_p^{\mu} H}(M) \cong \mathrm{End}_{RT}(M)$. ∎

We are now able to prove the first main result of this paper.

Theorem 3.5. *Let $p \neq 2$, G_p be a cyclic p-group, $G = G_p \times B$, $\mu \in Z^2(G_p, U_f(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U_t(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$ be as in* (1.1). *The algebra $\hat{\mathbb{Z}}_p^{\lambda} G$ is of OTP representation type if and only if one the following conditions is satisfied:*

(i) *if* $|G_p| > p^2$, *then* $\hat{\mathbb{Z}}_p^\mu G_p = [G_p, \hat{\mathbb{Z}}_p, \alpha]$, *where* $\alpha \equiv 1 \pmod p$ *and* $\alpha \not\equiv 1 \pmod{p^3}$*;*
(ii) $\hat{\mathbb{Q}}_p$ *is a splitting field for* $\hat{\mathbb{Q}}_p^\nu B$.

Proof. We have $\hat{\mathbb{Z}}_p^\mu G_p = [G_p, \hat{\mathbb{Z}}_p, \alpha]$, where $\alpha \in U_f(\hat{\mathbb{Z}}_p)$. It is easy to show that $\hat{\mathbb{Z}}_p^\mu G_p = [G_p, \hat{\mathbb{Z}}_p, (1+p)^{p^k}]$, where $k = 0$ if $\alpha \not\equiv 1 \pmod{p^2}$; $k = 1$ if $\alpha \equiv 1 \pmod{p^2}$ and $\alpha \not\equiv 1 \pmod{p^3}$; $k \geq 2$ if $\alpha \equiv 1 \pmod{p^3}$.

If one of conditions (i)–(ii) is satisfied, then $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type, by Lemmas 2.3–2.6 and 3.1–3.3.

Let us prove the necessity. Assume that $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_p^\nu B$. In view of Proposition 2.12, there is an irreducible $\hat{\mathbb{Z}}_p^\nu B$-module W such that $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\nu B}(W)} \cong \mathrm{GF}(p^m)$, where $m > 1$. If $|G_p| > p^2$ and $\hat{\mathbb{Z}}_p^\mu G_p = [G_p, \hat{\mathbb{Z}}_p, \alpha]$, where $\alpha \equiv 1 \pmod{p^3}$, then, by Lemmas 2.7–2.8 and 3.3, there exists an indecomposable $\hat{\mathbb{Z}}_p^\mu G_p$-module V such that $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_p^\mu G_p}(V)} \cong \mathrm{GF}(p^m)$. Since $\mathrm{GF}(p^m) \otimes_{\mathbb{Z}_p} \mathrm{GF}(p^m)$ is not a field, the $\hat{\mathbb{Z}}_p^\lambda G$-module $V \# W$ is decomposable, by Lemmas 2.4 and 2.6. Consequently, in view of Lemma 2.3, the algebra $\hat{\mathbb{Z}}_p^\lambda G$ is not of OTP representation type. ∎

The previous theorem can be reformulated in the following way.

Theorem 3.6. *Let* $p \neq 2$, G_p *be a cyclic* p*-group,* $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ *and* $\lambda = \mu \times \nu$. *Denote by* d *the number of simple blocks of the algebra* $\hat{\mathbb{Q}}_p^\mu G_p$. *Then the algebra* $\hat{\mathbb{Z}}_p^\lambda G$ *is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *if* $|G_p| > p^2$, *then* $d \leq 2$*;*
(ii) $\hat{\mathbb{Q}}_p$ *is a splitting field for* $\hat{\mathbb{Q}}_p^\nu B$.

We remark that if $|G_p| \leq p^2$ then $d \leq 3$; moreover, if $d = 3$ then $|G_p| = p^2$ and $\hat{\mathbb{Q}}_p^\mu G_p = \hat{\mathbb{Q}}_p G_p$.

Suppose now that G_p is an abelian group of type (p^n, p) and μ is in $Z^2(G_p, U(\hat{\mathbb{Z}}_p))$. In this case $d \geq 2$. If $d = 2$ then there exists a direct decomposition $G_p = \langle a \rangle \times \langle b \rangle$, where $|a| = p^n$ and $|b| = p$, such that $\hat{\mathbb{Z}}_p^\mu G_p = [G_p, \hat{\mathbb{Z}}_p, 1+p, 1]$.

Proposition 3.7. *Let* $p \neq 2$, G_p *be an abelian group od type* (p^n, p), $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ *and* $\lambda = \mu \times \nu$. *If the number of simple blocks of* $\hat{\mathbb{Q}}_p^\mu G_p$ *is different from* 2, *then* $\hat{\mathbb{Z}}_p^\lambda G$ *is of OTP representation type if and only if* $\hat{\mathbb{Q}}_p$ *is a splitting field for* $\hat{\mathbb{Q}}_p^\nu B$.

Proof. Let $D = \mathrm{soc}\, G_p$. If $\hat{\mathbb{Z}}_p^\mu D = \hat{\mathbb{Z}}_p D$, then the assertion follows from Lemmas 2.5, 2.10 and Proposition 2.13. Assume now that $\hat{\mathbb{Z}}_p^\mu D$ is not $\hat{\mathbb{Z}}_p D$. Then there is a subgroup $T = \langle a \rangle \times \langle b \rangle$ of type (p^2, p) of G_p such that $\hat{\mathbb{Z}}_p^\mu T = [T, \hat{\mathbb{Z}}_p, 1, 1+p]$. Let $H = T \times B$. If $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_p^\nu B$

then, by Lemmas 2.3, 2.4, 2.6 and 3.4, $\hat{\mathbb{Z}}_p^\lambda H$ is not of OTP representation type. Applying Proposition 2.13, we conclude that neither is $\hat{\mathbb{Z}}_p^\lambda G$. ∎

THEOREM 3.8. *Let $p \neq 2$, G_p be a non-cyclic p-group, $G = G_p \times B$, $\mu \in Z^2(G_p, U(\hat{\mathbb{Z}}_p))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = \mu \times \nu$. Assume that if G_p/G_p' is of type (p^n, p), then G_p is non-abelian and the following conditions are satisfied:*

(i) *if μ is not a 2-coboundary, then the center of $\hat{\mathbb{Q}}_p^\nu B$ is p-irreducible;*
(ii) *if $|G_p| = p^3$ then $\exp G_p = p$.*

The algebra $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type if and only if $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$.

Proof. Assume that G_p/G_p' is not of type (p^n, p). In view of Lemma 2.1, we may assume that G_p is abelian. Let $D = \operatorname{soc} G_p$. If $|D| \geq p^3$ then $\hat{\mathbb{Z}}_p^\mu D$ contains a group algebra $\hat{\mathbb{Z}}_p^\mu H = \hat{\mathbb{Z}}_p H$, where H is a group of type (p,p). In this case the theorem follows from Lemmas 2.5, 2.10 and Proposition 2.13. The case when $|D| = p^2$ and $\hat{\mathbb{Z}}_p^\mu D = \hat{\mathbb{Z}}_p D$ is treated similarly. Suppose now that $|D| = p^2$ and the restriction of μ to $D \times D$ is not a 2-coboundary. Then $\hat{\mathbb{Z}}_p^\mu G_p$ contains an algebra $\hat{\mathbb{Z}}_p^\mu H$ as in Lemma 3.4. Next apply Lemmas 2.3–2.6, 3.4 and Proposition 2.13.

Assume that G_p/G_p' is of type (p^n, p). If G_p' is not cyclic, then the assertion follows from Lemmas 2.1, 2.10 and Proposition 2.13. Assume that $G_p' = \langle c \rangle$, $|c| = p^s$ and $G_p/G_p' = \langle xG_p' \rangle \times \langle yG_p' \rangle$, where $|xG_p'| = p^n$, $|yG_p'| = p$. Let $T = \langle c^p \rangle$. Denote by D the subgroup of G_p such that $G_p' \subset D$ and $D/G_p' = \operatorname{soc}(G_p/G_p')$. By [3, Lemma 1.12, p. 288], $|D'| \leq p$. First, we examine the case when $x^{p^n} \in T$ and $y^p \in T$. If $s \geq 2$ then $D' \subset T$ and $D/T = \langle aT \rangle \times \langle bT \rangle \times \langle cT \rangle$, where $a = x^{p^{n-1}}$ and $b = y$. Arguing further as in the first part of the proof, we establish the desired conclusion. If $s = 1$ then $|D| = p^3$ and $\exp D = p$. The algebra $\hat{\mathbb{Z}}_p^\mu D$ contains a group algebra $\hat{\mathbb{Z}}_p^\mu H = \hat{\mathbb{Z}}_p H$, where H is an abelian group of type (p,p). Next we argue as previously.

We now consider the case in which $x^{p^n} \notin T$. Let $\{u_g \colon g \in G_p\}$ be a canonical $\hat{\mathbb{Z}}_p$-basis of $\hat{\mathbb{Z}}_p^\mu G_p$. We may assume that

$$u_x^{p^n} = (1+p)^j u_c, \quad u_y^p = (1+p)^k u_e, \quad \text{where } k \in \{0,1\}.$$

By Proposition 2.13 and Theorem 3.6, $|c| = p$, hence $n \geq 2$. If $k = 0$ then the $\hat{\mathbb{Z}}_p$-algebra generated by u_c and u_y is the group algebra $\hat{\mathbb{Z}}_p H$, where $H = \langle c \rangle \times \langle y \rangle$. If $k = 1$ then

$$(u_y^{-j} u_x^{p^{n-1}})^p = u_c.$$

Consequently, $\hat{\mathbb{Z}}_p^\mu G_p$ contains the twisted group algebra $\hat{\mathbb{Z}}_p^\mu H$ as in Lemma

3.4, where $H = \langle y\rangle \times \langle y^{-j}x^{p^{n-1}}\rangle$ is of type (p, p^2). Next apply Lemmas 2.3–2.6, 3.4 and Proposition 2.13.

If $x^{p^n} \in T$ and $y^p \notin T$, then $|c| = p$, $n \geq 2$ and

$$u_x^{p^n} = (1+p)^i u_e, \quad u_y^p = (1+p)^j u_c.$$

Let $i = pk$ and $v = (1+p)^{-k}u_x^{p^{n-1}}$. Then $v^p = u_e$, hence the $\hat{\mathbb{Z}}_p$-algebra generated by v and u_c is a group algebra of an abelian group of type (p, p). If p does not divide i, we may assume that $i = 1$. For $v = u_x^{-jp^{n-1}}u_y$ we have $v^p = u_c$. Therefore $\hat{\mathbb{Z}}_p^\mu G_p$ contains $\hat{\mathbb{Z}}_p^\mu H$ as in Lemma 3.4, where $H = \langle x^p\rangle \times \langle yx^{-jp^{n-1}}\rangle$ is of type (p^{n-1}, p^2). Applying Lemmas 2.3–2.6, 3.4 and Proposition 2.13, we finish the proof. ∎

PROPOSITION 3.9. *Let p be an arbitrary prime, $G = G_p \times B$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$ and $\lambda = 1 \times \nu \in Z^2(G, U(\hat{\mathbb{Z}}_p))$. The algebra $\hat{\mathbb{Z}}_p^\lambda G$ is of OTP representation type if and only if either $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p^\nu B$, or G_p is a cyclic group of order p^r, $r \leq 2$.*

Proof. Apply Lemmas 2.3–2.6, 2.8, 3.2 and 3.3. ∎

4. Twisted group algebras $\hat{\mathbb{Z}}_2^\lambda G$ of OTP representation type. In this section $\hat{\mathbb{Q}}_2$ is the field of 2-adic numbers, $\hat{\mathbb{Z}}_2$ is the ring of 2-adic integers, $G = G_2 \times B$ is a finite group, where G_2 is a 2-group, B is a $2'$-group and $|G_2|, |B| > 1$. In view of Lemma 2.2, the algebra $\hat{\mathbb{Z}}_2^\nu B$ is the group algebra $\hat{\mathbb{Z}}_2 B$ for any $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_2))$. Therefore every cocycle $\lambda \in Z^2(G, U(\hat{\mathbb{Z}}_2))$ satisfies the condition $\lambda = \mu \times 1$, where μ is the restriction of λ to $G_2 \times G_2$.

Let

$$\rho = \frac{1+\sqrt{5}}{2}, \qquad R = \hat{\mathbb{Z}}_2[\rho]. \tag{4.1}$$

We recall from [13, p. 277] that the field $\hat{\mathbb{Q}}_2(\sqrt{5})$ is an unramified extension of $\hat{\mathbb{Q}}_2$ of degree 2 and R is the ring of all integral elements of $\hat{\mathbb{Q}}_2(\sqrt{5})$.

Assume that $\mu \in Z^2(G_2, U(\hat{\mathbb{Z}}_2))$, $\Lambda = \hat{\mathbb{Z}}_2^\mu G_2$ and $\Lambda' = R \otimes_{\hat{\mathbb{Z}}_2} \Lambda$. If N is a Λ'-module, we denote by N_Λ the module N viewed as a Λ-module. By a result due to Jacobinski (see [17, pp. 697–698]), for any indecomposable Λ-module M there is an indecomposable Λ'-module U such that M is a direct summand of the module U_Λ. Moreover, if N is an indecomposable Λ'-module, then $R \otimes_{\hat{\mathbb{Z}}_2} N_\Lambda \cong N \oplus V$, where V is also an indecomposable Λ'-module and the R-rank of V is equal to the R-rank of N.

LEMMA 4.1. *Let $G = G_2 \times B$, $\mu \in Z^2(G_2, U_f(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times 1 \in Z^2(G, U_f(\hat{\mathbb{Z}}_2))$. If μ is not a 2-coboundary and $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type, then the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible.*

Proof. By Lemma 2.2, the restriction of μ to $G_2' \times G_2'$ is a 2-coboundary. Hence we may assume that $\mu_{x,y} = 1$ for all $x, y \in G_2'$. Let $\{u_g : g \in G_2\}$ be a canonical $\hat{\mathbb{Z}}_2$-basis of $\hat{\mathbb{Z}}_2^\mu G_2$. Then $u_g^{-1} u_h u_g = u_{g^{-1}hg}$ for all $g \in G_2$, $h \in G_2'$. Suppose that $F = G_2/G_2'$ and $I(G_2')$ is the augmentation ideal of $\hat{\mathbb{Z}}_2 G_2'$. Arguing as in the proof of [29, Lemma 5.5, p. 91], we may show that $\hat{\mathbb{Z}}_2^\mu G_2 \cdot I(G_2')$ is a two-sided ideal of $\hat{\mathbb{Z}}_2^\mu G_2$ and $\hat{\mathbb{Z}}_2^\mu G_2 / \hat{\mathbb{Z}}_2^\mu G_2 \cdot I(G_2') \cong \hat{\mathbb{Z}}_2^\tau F$ for some $\tau \in Z^2(F, U_f(\hat{\mathbb{Z}}_2))$ such that μ is cohomologous to $\inf(\tau) \in Z^2(G_2, U_f(\hat{\mathbb{Z}}_2))$, where $\inf(\tau)_{a,b} = \tau_{aG_2', bG_2'}$ for all $a, b \in G_2$. Since μ is not a 2-coboundary, neither is τ. Consequently, without loss of generality we may suppose that G_2 is abelian.

Up to cohomology, there is an element $x \in G_2$ of order 2^n such that

$$u_x^{2^n} = 5^{2^m} u_e, \qquad m < n.$$

Let $H = \langle x \rangle$, $D = \langle y \rangle$ be a cyclic group of order 2^{n-m}, $z = y^{2^{n-m-1}}$ and $T = \langle z \rangle$. There exists an algebra homomorphism of $\hat{\mathbb{Z}}_2^\mu H$ onto the twisted group algebra

$$\hat{\mathbb{Z}}_2^\sigma D = \bigoplus_{i=0}^{2^{n-m}-1} \hat{\mathbb{Z}}_2 v_y^i, \qquad v_y^{2^{n-m}} = 5 v_e.$$

Denote by M the underlying $\hat{\mathbb{Z}}_2^\sigma T$-module of the matrix representation Δ of $\hat{\mathbb{Z}}_2^\sigma T$ defined by

$$\Delta(v_z) = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \qquad \text{where } v_z = v_y^{2^{n-m-1}}.$$

The algebra $\operatorname{End}_{\hat{\mathbb{Z}}_2^\sigma T}(M)$ is isomorphic to the algebra

$$R = \{C \in \mathbb{M}_2(\hat{\mathbb{Z}}_2) : C\Delta(v_z) = \Delta(v_z)C\}.$$

We have

$$R = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha - \beta \end{pmatrix} : \alpha, \beta \in \hat{\mathbb{Z}}_2 \right\}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

we conclude that $R \cong \hat{\mathbb{Z}}_2[\rho]$, where $\rho = (1 + \sqrt{5})/2$. It follows that $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_2^\sigma T}(M)} \cong \mathrm{GF}(4)$. In view of Lemma 2.7, the induced module $N := M^D = \hat{\mathbb{Z}}_2^\sigma D \otimes_{\hat{\mathbb{Z}}_2^\sigma T} M$ is indecomposable and $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_2^\sigma D}(N)} \cong \mathrm{GF}(4)$. One can view the $\hat{\mathbb{Z}}_2^\sigma D$-module N as a $\hat{\mathbb{Z}}_2^\mu H$-module. By Lemma 2.7, the $\hat{\mathbb{Z}}_2^\mu G_2$-module $N^{G_2} := \hat{\mathbb{Z}}_2^\mu G_2 \otimes_{\hat{\mathbb{Z}}_2^\mu H} N$ is indecomposable and $\overline{\operatorname{End}_{\hat{\mathbb{Z}}_2^\mu G_2}(N^{G_2})}$ is isomorphic to GF(4). By applying Lemmas 2.3, 2.4, 2.6 and Proposition 2.12, one shows that the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible. ■

LEMMA 4.2. *Let H be an abelian group of type* $(2,2)$ *and* $\Lambda = [H, \hat{\mathbb{Z}}_2, 5, 1]$. *Then, for any odd number m, there is an indecomposable Λ-module M such that $\overline{\mathrm{End}_\Lambda(M)}$ contains a subfield which is isomorphic to* $\mathrm{GF}(2^m)$.

Proof. Let $\Lambda' = R \otimes_{\hat{\mathbb{Z}}_2} \Lambda$ (see the notation (4.1)). The algebra Λ' is the group algebra RH. Therefore, by Lemma 2.8, there is an indecomposable Λ'-module N for which $\overline{\mathrm{End}_{\Lambda'}(N)} \cong \mathrm{GF}(2^{2m})$. Assume that N_Λ is an indecomposable Λ-module. We have $\mathrm{End}_{\Lambda'}(N) \subset \mathrm{End}_\Lambda(N_\Lambda)$. Because the rings $\mathrm{End}_{\Lambda'}(N)$ and $\mathrm{End}_\Lambda(N_\Lambda)$ are local, $\mathrm{rad}\,\mathrm{End}_{\Lambda'}(N) \subset \mathrm{rad}\,\mathrm{End}_\Lambda(N_\Lambda)$. It follows that $\overline{\mathrm{End}_{\Lambda'}(N)}$ is isomorphic to a subfield of $\overline{\mathrm{End}_\Lambda(N_\Lambda)}$. Consequently, $\overline{\mathrm{End}_\Lambda(N_\Lambda)}$ contains a subfield which is isomorphic to $\mathrm{GF}(2^m)$.

We now consider the case when N_Λ is a decomposable Λ-module. Let d be the R-rank of N. Then $N_\Lambda = M \oplus V$, where M and V are indecomposable Λ-modules of $\hat{\mathbb{Z}}_2$-rank d and N is isomorphic to $R \otimes_{\hat{\mathbb{Z}}_2} M$. Denote by Δ a matrix $\hat{\mathbb{Z}}_2$-representation of the algebra Λ afforded by the Λ-module M. Let $\{u_h : h \in H\}$ be a canonical $\hat{\mathbb{Z}}_2$-basis of Λ, and

$$S := \{C \in \mathbb{M}_d(\hat{\mathbb{Z}}_2) : C\Delta(u_h) = \Delta(u_h)C \text{ for every } h \in H\},$$
$$S' := \{C' \in \mathbb{M}_d(R) : C'\Delta(u_h) = \Delta(u_h)C' \text{ for every } h \in H\}.$$

The ring S is isomorphic to $\mathrm{End}_\Lambda(M)$, and the ring S' is isomorphic to $\mathrm{End}_{\Lambda'}(N)$. Assume $C' = C_1 + \rho C_2$, where $\rho = (1+\sqrt{5})/2$ and $C_1, C_2 \in \mathbb{M}_d(\hat{\mathbb{Z}}_2)$. Because $\{1, \rho\}$ is a $\hat{\mathbb{Z}}_2$-basis of R, we conclude that $C' \in S'$ if and only if $C_1, C_2 \in S$. Hence $S' = S + \rho S$. By [17, Proposition 5.22 and Theorem 7.9], we may write $\overline{S'} \cong \mathrm{GF}(4) \otimes_{\mathbb{Z}_2} \overline{S}$, consequently $\overline{\mathrm{End}_\Lambda(M)} \cong \mathrm{GF}(2^m)$. ■

LEMMA 4.3. *Let $H = \langle a \rangle$ be a cyclic group of order 2^n and $\Lambda = [H, \hat{\mathbb{Z}}_2, 5^{2^k}]$, where $n \geq 3$ and $k \geq 1$. Then, for any odd number m, there is an indecomposable Λ-module M such that $\overline{\mathrm{End}_\Lambda(M)}$ contains a subfield isomorphic to* $\mathrm{GF}(2^m)$.

Proof. In view of Lemmas 2.7 and 2.8, we may assume that $n = 3$ and $k \in \{1, 2\}$. Keeping the notation (4.1), suppose that $k = 2$. The algebra $\Lambda' = R \otimes_{\hat{\mathbb{Z}}_2} \Lambda$ is the group algebra of H over R. By Lemma 2.8, there is an indecomposable Λ'-module N such that $\overline{\mathrm{End}_{\Lambda'}(N)}$ is isomorphic to $\mathrm{GF}(2^{2m})$. Arguing as in the proof of Lemma 4.2, we deduce the assertion.

Now consider the case when $k = 1$. Denote by θ_1, θ_2 and θ_3 roots of the irreducible polynomials $X^2 - \sqrt{5}$, $X^2 + \sqrt{5}$ and $X^4 + 5 \in R[X]$, respectively. Let $\Lambda' = R \otimes_{\hat{\mathbb{Z}}_2} \Lambda$ and N be an underlying Λ'-module of the matrix representation Γ of Λ' defined by the formula

$$\Gamma(u_a) = \begin{pmatrix} \widetilde{\theta}_1 \dot{\times} E_m & \langle \pi_1 \rangle \dot{\times} E_m & \langle 1 \rangle \dot{\times} A_f \\ 0 & \widetilde{\theta}_2 \dot{\times} E_m & \langle \pi_2 \rangle \dot{\times} E_m \\ 0 & 0 & \widetilde{\theta}_3 \dot{\times} E_m \end{pmatrix},$$

where $\pi_i = 1 - \theta_i$ for $i = 1, 2$ and A_f is the matrix as in (2.1); see also the notation in the proof of Lemma 3.3. Arguing as in the latter proof, we show that N is an indecomposable module and $\overline{\mathrm{End}_{\Lambda'}(N)} \cong \mathrm{GF}(2^{2m})$. By applying the same type of arguments as in the proof of Lemma 4.2, we finish the proof in this case. ∎

LEMMA 4.4. *Let $H = \langle a \rangle$ be a cyclic group of order 2^n and $\Lambda = [H, \hat{\mathbb{Z}}_2, 5]$. Then:*

(i) $\overline{\mathrm{End}_{\Lambda}(M)}$ *is isomorphic to a subfield of the field* $\mathrm{GF}(4)$ *for any indecomposable Λ-module M.*
(ii) *There exists an indecomposable Λ-module M_0 such that* $\overline{\mathrm{End}_{\Lambda}(M_0)} \cong \mathrm{GF}(4)$.

Proof. Let R be the ring as in (4.1). Denote by θ and σ roots of the polynomials $X^{2^{n-1}} - \sqrt{5}$ and $X^{2^{n-1}} + \sqrt{5}$, respectively. The fields $\hat{\mathbb{Q}}_2(\theta)$ and $\hat{\mathbb{Q}}_2(\sigma)$ are totally ramified extensions of $\hat{\mathbb{Q}}_2(\sqrt{5})$ of degree 2^{n-1}, and $R[\theta]$, $R[\sigma]$ are the rings of all integral elements of $\hat{\mathbb{Q}}_2(\theta)$ and $\hat{\mathbb{Q}}_2(\sigma)$, respectively. Clearly, $\theta^{2^n} = 5$ and $\Lambda \cong \hat{\mathbb{Z}}_2[\theta]$. Since $R[\theta] = \hat{\mathbb{Z}}_2[\theta] + \rho\hat{\mathbb{Z}}_2[\theta]$, the $\hat{\mathbb{Z}}_2$-order $\hat{\mathbb{Z}}_2[\theta]$ is of cyclic index in the maximal $\hat{\mathbb{Z}}_2$-order $R[\theta]$ in the $\hat{\mathbb{Q}}_2$-algebra $\hat{\mathbb{Q}}_2(\theta)$. By a result of Borevich–Faddeev (see [17, p. 789]), every Λ-module is isomorphic to a direct sum of ideals of Λ. It follows that the $\hat{\mathbb{Z}}_2$-rank of any indecomposable Λ-module is 2^n.

Write $\Lambda' = R \otimes_{\hat{\mathbb{Z}}_2} \Lambda$. Applying the arguments used in the proof of Lemma 3.1, we can prove that, up to equivalence, the indecomposable matrix R-representations of Λ' are the following:

$$\Gamma_1 \colon u_a \mapsto \widetilde{\theta}, \quad \Gamma_2 \colon u_a \mapsto \widetilde{\sigma}, \quad \Gamma_{3+k} \colon u_a \mapsto \begin{pmatrix} \widetilde{\theta} & \langle t^k \rangle \\ 0 & \widetilde{\sigma} \end{pmatrix}, \quad \text{where } t = 1 - \theta,$$

$k = 0, 1, \ldots, 2^{n-1} - 1$ (see the notation in (3.1) and in Lemma 3.1). Arguing as in the proof of Lemma 3.2, we can show that $\overline{\mathrm{End}_{\Lambda'}(U)} \cong \overline{R} = \mathrm{GF}(4)$ for every indecomposable Λ'-module U.

Assume that N is an underlying Λ'-module of the representation Γ_j, where $j \in \{1, 2\}$. Then N_Λ is an indecomposable Λ-module. The Λ'-module $V := R \otimes_{\hat{\mathbb{Z}}_2} N_\Lambda$ decomposes into a direct sum of two mutually non-isomorphic indecomposable Λ'-modules of R-rank 2^{n-1}. It follows that $\overline{\mathrm{End}_{\Lambda'}(V)} \cong \mathrm{GF}(4) \times \mathrm{GF}(4)$. The argument given in the proof of Lemma 4.2 shows that $\overline{\mathrm{End}_{\Lambda'}(V)} \cong \overline{R} \otimes_{\mathbb{Z}_2} \overline{\mathrm{End}_{\Lambda}(N_\Lambda)}$. Consequently, $\overline{\mathrm{End}_{\Lambda}(N_\Lambda)} \cong \mathrm{GF}(4)$.

Now, assume that N is an underlying Λ'-module of the representation Γ_j, where $j \in \{3, \dots, 2 + 2^{n-1}\}$. Then the $\hat{\mathbb{Z}}_2$-rank of N_Λ is equal to 2^{n+1}, and therefore $N_\Lambda = M \oplus V$, where M and V are indecomposable Λ-modules of $\hat{\mathbb{Z}}_2$-rank 2^n. By a result of Jacobinski (see [17, pp. 697–698]), the Λ'-module $R \otimes_{\hat{\mathbb{Z}}_2} M$ is indecomposable, hence $\overline{\mathrm{End}_{\Lambda'}(R \otimes_{\hat{\mathbb{Z}}_2} M)} \cong \overline{R}$. Since

$$\overline{\mathrm{End}_{\Lambda'}(R \otimes_{\hat{\mathbb{Z}}_2} M)} \cong \overline{R} \otimes_{\mathbb{Z}_2} \overline{\mathrm{End}_\Lambda(M)},$$

we conclude that $\overline{\mathrm{End}_\Lambda(M)} \cong \mathbb{Z}_2$. ∎

LEMMA 4.5. *Let $H = \langle a \rangle$ be a cyclic group of order 4 and $\Lambda = [H, \hat{\mathbb{Z}}_2, 5^2]$. Then:*

(i) *$\overline{\mathrm{End}_\Lambda(M)}$ is isomorphic to a subfield of* GF(4) *for every indecomposable Λ-module M.*

(ii) *There is an indecomposable Λ-module M_0 such that*

$$\overline{\mathrm{End}_\Lambda(M_0)} \cong \mathrm{GF}(4).$$

Proof. Denote by η_1, η_2 some roots of the polynomials $X^2 - 5$ and $X^2 + 5$, respectively. Let

$$\widetilde{\eta}_1 = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\eta}_2 = \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

By [6, Lemma 3.9], up to equivalence, the indecomposable matrix $\hat{\mathbb{Z}}_2$-representations of the algebra Λ are the following:

$$\Gamma_i \colon u_a \mapsto \widetilde{\eta}_i \; (i = 1, 2), \quad \Gamma_3 \colon u_a \mapsto \Delta, \quad \Gamma_4 \colon u_a \mapsto \begin{pmatrix} \widetilde{\eta}_1 & D \\ 0 & \widetilde{\eta}_2 \end{pmatrix},$$

$$\Gamma_5 \colon u_a \mapsto \begin{pmatrix} \widetilde{\eta}_1 & S \\ 0 & \widetilde{\eta}_2 \end{pmatrix}, \quad \Gamma_6 \colon u_a \mapsto \begin{pmatrix} \Delta & S \\ 0 & \widetilde{\eta}_2 \end{pmatrix}, \quad \Gamma_7 \colon u_a \mapsto \begin{pmatrix} \Delta & S & T \\ 0 & \widetilde{\eta}_2 & 0 \\ 0 & 0 & \widetilde{\eta}_2 \end{pmatrix}.$$

Let M_i be the underlying Λ-module of the representation Γ_i and $d_i = \mathrm{rank}_{\hat{\mathbb{Z}}_2} M_i$. Denote by R_i the set of all matrices $C \in \mathbb{M}_{d_i}(\hat{\mathbb{Z}}_2)$ such that $C\Gamma_i(u_a) = \Gamma_i(u_a)C$. Then R_i is a free $\hat{\mathbb{Z}}_2$-algebra and $R_i \cong \mathrm{End}_\Lambda(M_i)$. By Lemma 2.6, R_i is a local algebra.

We have shown in the proof of Lemma 4.1 that $\overline{R}_3 \cong \mathrm{GF}(4)$.

If $C \in R_6$, then

$$C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}, \quad \text{where } C_3 = \begin{pmatrix} x & -5y \\ y & x \end{pmatrix} \text{ with } x, y \in \hat{\mathbb{Z}}_2.$$

Let $A = xE_4 + y\Gamma_6(u_a)$. Then $A \in R_6$ and $C - A \in \operatorname{rad} R_6$. Since $\Gamma_6(u_a)^4 \equiv E_4 \pmod 2$, it follows that $C + \operatorname{rad} R_6 = (x+y)E_4 + \operatorname{rad} R_6$. Consequently, $\overline{R}_6 \cong \mathbb{Z}_2$.

If $C \in R_7$, then

$$C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}, \quad \text{where } C_1 = \begin{pmatrix} x & y \\ y & x-y \end{pmatrix} \text{ with } x, y \in \hat{\mathbb{Z}}_2.$$

Let $A = xE_6 + yL$, where

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \quad \text{with } L_1 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},\ L_2 = E_2 \dot{\times} L_1.$$

Then $A \in R_7$ and $C - A \in \operatorname{rad} R_7$. By (4.2), $L^2 + L = E_6$. Therefore $\overline{R}_7 \cong \mathrm{GF}(4)$. Similarly we can show that $\overline{R}_i \cong \mathbb{Z}_2$ for each $i \in \{1, 2, 4, 5\}$. ∎

Our second main result of this paper is the following theorem.

THEOREM 4.6. *Let $G_2 = \langle a \rangle$ be a cyclic group of order 2^n, $G = G_2 \times B$, $\mu \in Z^2(G_2, U(\hat{\mathbb{Z}}_2))$, $\lambda = \mu \times 1 \in Z^2(G, U(\hat{\mathbb{Z}}_2))$ and $\hat{\mathbb{Z}}_2^\mu G_2 = [G_2, \hat{\mathbb{Z}}_2, \alpha]$. The algebra $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) $\alpha \not\equiv 1 \pmod 4$;
(ii) $\alpha \equiv 1 \pmod 4$, $\alpha \not\equiv 1 \pmod 8$ *and the center of* $\hat{\mathbb{Q}}_2 B$ *is* 2*-irreducible;*
(iii) $n \le 2$ *and* $\hat{\mathbb{Z}}_2^\mu G_2 = \hat{\mathbb{Z}}_2 G_2$;
(iv) $n = 2$, $\alpha \equiv 1 \pmod 8$, $\alpha \not\equiv 1 \pmod{16}$ *and the center of* $\hat{\mathbb{Q}}_2 B$ *is* 2*-irreducible;*
(v) $\hat{\mathbb{Q}}_2$ *is a splitting field for* $\hat{\mathbb{Q}}_2 B$.

Proof. Assume that $\alpha \not\equiv 1 \pmod 4$. Denote by θ a root of the irreducible polynomial $X^{2^n} - \alpha \in \hat{\mathbb{Z}}_2[X]$. Then $\hat{\mathbb{Q}}_2(\theta)$ is a totally ramified field extension of $\hat{\mathbb{Q}}_2$ and $\hat{\mathbb{Z}}_2[\theta]$ is the ring of all integral elements of $\hat{\mathbb{Q}}_2(\theta)$ (see [27, p. 192]). Because $\hat{\mathbb{Z}}_2^\mu G_2 \cong \hat{\mathbb{Z}}_2[\theta]$, and $\hat{\mathbb{Z}}_2[\theta]$ is a principal ideal domain, every indecomposable $\hat{\mathbb{Z}}_2^\mu G_2$-module is isomorphic to the regular module. Since $\mathrm{End}_{\hat{\mathbb{Z}}_2^\mu G_2}(\hat{\mathbb{Z}}_2^\mu G_2) \cong \mathbb{Z}_2$, the algebra $\hat{\mathbb{Z}}_2^\lambda G_2$ is of OTP representation type, by Lemmas 2.3, 2.4, 2.6.

Assume now that $\alpha \equiv 1 \pmod 4$, i.e. $\mu \in Z^2(G_2, U_f(\hat{\mathbb{Z}}_2))$. It is easy to show that $\hat{\mathbb{Z}}_2^\mu G_2 = [G_2, \hat{\mathbb{Z}}_2, 5^{2^k}]$, where $k = 0$ if $\alpha \not\equiv 1 \pmod 8$; $k = 1$ if $\alpha \equiv 1 \pmod 8$ and $\alpha \not\equiv 1 \pmod{16}$; $k \ge 2$ if $\alpha \equiv 1 \pmod{16}$. If one of the conditions (ii)–(v) is satisfied, then $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type, by Lemmas 2.3–2.6, 2.10 and 4.4, 4.5. Conversely, let $\hat{\mathbb{Z}}_2^\lambda G$ be of OTP representation type. If μ is a 2-coboundary, then $\hat{\mathbb{Z}}_2^\mu G_2 = \hat{\mathbb{Z}}_2 G_2$, and in view of Lemma 2.10, one of conditions (iii), (v) is satisfied. Suppose μ is not a 2-coboundary. By Lemma 4.1, the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible. Suppose that $\alpha \equiv 1 \pmod 8$. If $n = 2$ then $\alpha \not\equiv 1 \pmod{16}$. If $n \geq 3$ then, by Lemmas 2.3–2.6, 4.3 and Proposition 2.12, condition (v) is satisfied. ∎

Under the identification of the field $\hat{\mathbb{Q}}_2$ with the field $\{\alpha u_e \colon \alpha \in \hat{\mathbb{Q}}_2\}$, we can reformulate Theorem 4.6 as follows.

THEOREM 4.7. *Let G_2 be a cyclic group of order 2^n, $G = G_2 \times B$, $\mu \in Z^2(G_2, U(\hat{\mathbb{Z}}_2))$, $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times \nu$. The algebra $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *$\hat{\mathbb{Q}}_2^\mu G_2$ is a totally ramified field extension of $\hat{\mathbb{Q}}_2$;*
(ii) *$\hat{\mathbb{Q}}_2^\mu G_2$ is a field and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible;*
(iii) *$n \leq 2$ and $\hat{\mathbb{Z}}_2^\mu G_2$ is the group algebra of G_2 over $\hat{\mathbb{Z}}_2$;*
(iv) *$n = 2$, the number of simple blocks of $\hat{\mathbb{Q}}_2^\mu G_2$ is 2 and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible;*
(v) *$\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

PROPOSITION 4.8. *Let G_2 be a non-cyclic 2-group, $G = G_2 \times B$, $\mu \in Z^2(G_2, U_f(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times 1 \in Z^2(G, U_f(\hat{\mathbb{Z}}_2))$. The algebra $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if $\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

Proof. By Lemmas 2.2 and 2.10, we may assume that G_2 is abelian and $\hat{\mathbb{Z}}_2^\mu G_2$ is not the group algebra $\hat{\mathbb{Z}}_2 G_2$. Denote by H the socle of G_2. If H is of type $(2, 2)$ and $\hat{\mathbb{Z}}_2^\mu H$ is not $\hat{\mathbb{Z}}_2 H$, the assertion follows from Lemmas 2.3–2.6, 4.1, 4.2 and Proposition 2.13. Let $|H| > 4$. There exists a non-cyclic subgroup D of H such that $\hat{\mathbb{Z}}_2^\mu D$ is $\hat{\mathbb{Z}}_2 D$. By applying Lemmas 2.5, 2.10 and Proposition 2.13, the proof follows in this case. ∎

PROPOSITION 4.9. *Let G_2 be an abelian 2-group, $G = G_2 \times B$, $\mu \in Z^2(G_2, U_t(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times 1 \in Z^2(G, U_t(\mathbb{Z}_2))$. Assume that $\hat{\mathbb{Z}}_2^\mu G_2$ is a commutative algebra. Then $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *G_2 is cyclic and μ is not a 2-coboundary;*
(ii) *G_2 is cyclic of order 2 or 4;*
(iii) *G_2 is of type $(2^n, 2)$ and the number of simple blocks of $\hat{\mathbb{Q}}_2^\mu G_2$ is 2;*
(iv) *$\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

Proof. If G_2 has at least three invariants, there is a non-cyclic subgroup H of G_2 such that $\hat{\mathbb{Z}}_2^\mu H = \hat{\mathbb{Z}}_2 H$. Applying Lemma 2.10 and Proposition 2.13, we deduce the proposition.

Assume that G_2 has two invariants and μ is not a 2-coboundary. Then $G_2 = \langle a \rangle \times \langle b \rangle$ and $\hat{\mathbb{Z}}_2^\mu G_2 = [G_2, \hat{\mathbb{Z}}_2, -1, 1]$. Let $|a| = 2^n$ and $|b| = 2^m$. Arguing as in the proof of Lemma 3.4, we conclude that if $m \geq 2$ then, for any finite field F of characteristic 2, there is an indecomposable $\hat{\mathbb{Z}}_2^\mu G_2$-module M such that $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_2^\mu G_2}(M)} \cong F$. In view of Lemmas 2.3–2.6, $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if $\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.

Let $m = 1$. Denote by ξ a root of the polynomial $X^{2^n} + 1$. The field $\hat{\mathbb{Q}}_2(\xi)$ is a totally ramified extension of $\hat{\mathbb{Q}}_2$ of degree 2^n, and $R := \hat{\mathbb{Z}}_2[\xi]$ is the ring of all integral elements of $\hat{\mathbb{Q}}_2(\xi)$. One can view $\hat{\mathbb{Z}}_2^\mu G_2$ as the group algebra RH of the group $H = \langle b\rangle$ of order 2 over R. Up to equivalence, the indecomposable matrix R-representations of RH are the following:

$$\Gamma_1 \colon u_b \mapsto 1, \qquad \Gamma_2 \colon u_b \mapsto -1, \qquad \Gamma_{j+3} \colon u_b \mapsto \begin{pmatrix} 1 & \pi^j \\ 0 & -1 \end{pmatrix},$$

where $\pi = 1 - \xi$ and $j = 0, 1, \ldots, 2^n - 1$. Denote by M_i the underlying RH-module of the representation Γ_i for $i \in \{1, \ldots, 2^n + 2\}$. Since $\overline{\mathrm{End}_{RH}(M_i)} \cong \overline{R} = \mathbb{Z}_2$ for every i, we see that $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type, by Lemmas 2.3–2.6. Note that in this case the number of simple blocks of $\hat{\mathbb{Q}}_2^\mu G_2$ equals 2.

In the case when G_2 is a cyclic group of order 2^n and μ is not a 2-coboundary we have $\hat{\mathbb{Z}}_2^\mu G_2 \cong \hat{\mathbb{Z}}_2[\xi]$, where $\xi^{2^n} = -1$. Because $\hat{\mathbb{Z}}_2[\xi]$ is a principal ideal domain, each indecomposable $\hat{\mathbb{Z}}_2^\mu G_2$-module is isomorphic to the regular module. Moreover, $\overline{\mathrm{End}_{\hat{\mathbb{Z}}_2^\mu G_2}(\hat{\mathbb{Z}}_2^\mu G_2)} \cong \mathbb{Z}_2$. By Lemmas 2.3, 2.4 and 2.6, $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type. ∎

PROPOSITION 4.10. *Let G_2 be an abelian 2-group, $G = G_2 \times B$, $\mu \in Z^2(G_2, U(\hat{\mathbb{Z}}_2))$ and $\lambda = \mu \times 1 \in Z^2(G, U(\hat{\mathbb{Z}}_2))$. Assume that the algebra $\hat{\mathbb{Z}}_2^\mu G_2$ is commutative and the number of invariants of G_2 is at least 3. Then $\hat{\mathbb{Z}}_2^\lambda G$ is of OTP representation type if and only if $\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

Proof. Let $D = \operatorname{soc} G_2$. There is a subgroup T of type $(2,2)$ in D such that $\hat{\mathbb{Z}}_2^\mu T$ is either the group algebra, or the algebra as in Lemma 4.2. Now we may apply Lemmas 2.3–2.6 and Proposition 2.13. ∎

5. Finite groups of OTP projective representation type. First we remark that, in view of (2.2), Propositions 2.2–2.9 in [5] relating to splitting fields for a twisted group algebra $K^\nu B$, where K is a field of characteristic p and B is a finite p'-group, remain valid also in the case when $K = \hat{\mathbb{Q}}_p$ and $\nu \in Z^2(B, U(\hat{\mathbb{Z}}_p))$.

PROPOSITION 5.1. *Let $p \neq 2$ and $G = G_p \times B$ with G_p/G_p' not of type (p^n, p). The group G is of OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if one of the following conditions is satisfied:*

(i) *G_p is cyclic;*
(ii) *$\hat{\mathbb{Q}}_p$ is a splitting field of $\hat{\mathbb{Q}}_p^\nu B$ for certain $\nu \in Z^2(B, U_t(\hat{\mathbb{Z}}_p))$.*

Proof. Apply Theorems 3.5 and 3.8. ∎

PROPOSITION 5.2. *Let $p \neq 2$, $G = G_p \times B$ be an abelian group with G_p not of type (p^n, p). The group G is of OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if one of the following conditions is satisfied:*

(i) *G_p is cyclic;*
(ii) *B has a subgroup H such that B/H is of symmetric type, i.e. $B/H \cong D \times D$, and $p-1$ is divisible by $m := \max\{\exp H, \exp(B/H)\}$.*

Proof. Apply Theorems 3.5, 3.8 and [5, Proposition 2.5]. ∎

PROPOSITION 5.3. *Let $p \neq 2$, G_p be an abelian p-group, B be a nilpotent p'-group and $G = G_p \times B$. Assume that G_p is not of type (p^n, p) and $p-1$ is not divisible by q for some prime q dividing $|B|$. The group G is of OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if G_p is cyclic.*

Proof. Apply Theorems 3.5, 3.8 and [5, Proposition 2.7]. ∎

Our final main result of this paper is the following theorem.

THEOREM 5.4. *The group $G = G_p \times B$ is of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if one of the following conditions is satisfied:*

(i) *$p \neq 2$ and G_p is a cyclic group of order p or p^2;*
(ii) *$p = 2$, G_2 is a cyclic group of order 2 or 4 and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible;*
(iii) *$p \neq 2$ and there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that any projective $\hat{\mathbb{Q}}_p$-representation of B with a 2-cocycle in $Z^2(B, U(\hat{\mathbb{Z}}_p))$ lifts projectively to an ordinary $\hat{\mathbb{Q}}_p$-representation of $\widehat{B}$ and $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_p\widehat{B}$;*
(iv) *$p = 2$ and $\hat{\mathbb{Q}}_2$ is a splitting field for $\hat{\mathbb{Q}}_2 B$.*

Proof. Apply Lemma 2.10, Theorems 3.5, 4.6 and [5, Proposition 2.9]. ∎

COROLLARY 5.5. *Let $G = G_p \times B$ and $B' \neq B$. The group G is of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if one of the following conditions is satisfied:*

(i) *$p \neq 2$ and G_p is a cyclic group of order p or p^2;*
(ii) *$p = 2$, G_2 is a cyclic group of order 2 or 4 and the center of $\hat{\mathbb{Q}}_2 B$ is 2-irreducible.*

Proof. Let $p \neq 2$. There is a normal subgroup H of B such that $\overline{B} := B/H$ is a cyclic group of order q, where q is a prime divisor of $|B|$. Let $p - 1 = q^m k$, where $m \geq 1$ and $(q, k) = 1$. Denote by ξ a primitive q^mth root of 1 and by $\hat{\mathbb{Z}}_p^{\nu}\overline{B}$ the algebra

$$\bigoplus_{i=0}^{q-1} \hat{\mathbb{Z}}_p u^i, \qquad u^q = \xi.$$

Since $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_p^{\nu}\overline{B} = \hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Z}}_p^{\nu}\overline{B}$, there is a twisted group algebra $\hat{\mathbb{Z}}_p^{\nu}B$ such that $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_p^{\nu}B$. If q does not divide $p-1$, then $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_p\overline{B}$. It follows that $\hat{\mathbb{Q}}_p$ is not a splitting field for $\hat{\mathbb{Q}}_pB$. Applying Theorem 5.4, we conclude that G is of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if G_p is a cyclic group of order p or p^2. In the case when $p = 2$ the corollary follows in a similar way. ■

Corollary 5.6. *Let $p \neq 2$ and $G = G_p \times B$. Assume that $p-1$ is not divisible by every prime q dividing $|B|$. Then $\mathrm{H}^2(B, U(\hat{\mathbb{Z}}_p)) = 1$ and G is of purely OTP projective $\hat{\mathbb{Z}}_p$-representation type if and only if either $\hat{\mathbb{Q}}_p$ is a splitting field for $\hat{\mathbb{Q}}_pB$, or G_p is a cyclic group of order p^r, $r \leq 2$.*

Proof. Apply [29, Theorem 1.7, p. 11] and Theorem 5.4. ■

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