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The minimal operator and the John–Nirenberg theorem for weighted grand Lebesgue spaces

by

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Abstract. We introduce the minimal operator on weighted grand Lebesgue spaces, discuss some weighted norm inequalities and characterize the conditions under which the inequalities hold. We also prove that the John–Nirenberg inequalities in the framework of weighted grand Lebesgue spaces are valid provided that the weight function belongs to the Muckenhoupt A_p class.

1. Introduction. The purpose of this paper is to investigate the minimal inequality and the John–Nirenberg theorem in the framework of weighted grand Lebesgue spaces. For a locally integrable function $f: \mathbb{R}^n \to \mathbb{C}$, define the minimal function of f by

$$mf(x) := \inf_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the infimum is taken over all cubes Q which contain x with sides parallel to the coordinate axes. The minimal operator was introduced by Cruz-Uribe and Neugebauer [CN], who used it to study the fine structure of functions which satisfy the reverse Hölder inequality. Furthermore, they proved the following weighted minimal inequality. Let w be a weight, that is, a positive and integrable function on $\mathbb{R}^n$. Then w belongs to the Muckenhoupt A_p class if and only if the weighted norm inequality

$$\|\log mf\|_{L_{p,w}} \le C_p \|\log f\|_{L_{p,w}} \tag{1.1}$$

is true for $p > 1$ and all f such that $0 \le f \le 1$ and $\log f$ is in $L_{p,w}$. The constant C_p depends only on the A_p constant of w and p. We refer to [C, CNO, ZL] for more information on the minimal operator.

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To state our main results, we need to describe the grand Lebesgue spaces $L_{p)}$. Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set with Lebesgue measure $|\Omega| < \infty$. For $1 < p < \infty$, the grand Lebesgue space $L_{p)}(\Omega)$ is defined as the set of all measurable functions f on Ω such that

$$\|f\|_{p)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon \frac{1}{|\Omega|} \int_{\Omega} |f|^{p-\varepsilon}\, dx\right)^{1/(p-\varepsilon)} < \infty.$$

Then $(L_{p)}, \|\cdot\|_{p)})$ is a Banach function space. Such Lebesgue spaces were introduced in 1992 by Iwaniec and Sbordone [IS] in the study of the integrability of the Jacobian under minimal hypotheses. Since then the structural properties of grand Lebesgue spaces have been studied in [CF, DF, F, KMR]. Grand Lebesgue spaces play an important role in PDE theory (see [DSS, FMR, FS, G2, GIS]) and in function space theory (see [CFG, FG, FGJ, K, KM]).

In 2008, Fiorenza, Gupta and Jain [FGJ, Theorem 4.1] investigated the maximal theorem for weighted grand Lebesgue spaces. Then it is natural to ask whether the minimal theorem is also true for them. The first goal of this paper is to extend (1.1) to this case. We consider more general grand Lebesgue spaces $L_{p),\varphi,w}(\Omega)$. The space $L_{p),\varphi,w}(\Omega)$, or simply $L_{p),\varphi,w}$, is called a weighted grand Lebesgue space and defined as the set of all measurable functions on Ω such that

$$\|f\|_{L_{p),\varphi,w}} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\left(\frac{1}{|\Omega|} \int_{\Omega} |f|^{p-\varepsilon} w\, dx\right)^{1/(p-\varepsilon)} < \infty,$$

where w is a weight on Ω and $\varphi : (0, p-1) \to \mathbb{R}_+$ is a finite non-decreasing function with $\lim_{t\to 0} \varphi(t) = 0$. If $w = 1$, then $L_{p),\varphi,1}(\Omega) = L_{p),\varphi}(\Omega)$; when in addition $\varphi(\varepsilon) = \varepsilon^{1/(p-\varepsilon)}$, the space $L_{p),\varepsilon^{1/(p-\varepsilon)}}(\Omega)$ reduces to the grand Lebesgue space $L_{p)}(\Omega)$. In Section 2, we study the weighted norm inequality for the minimal operator defined on $L_{p),\varphi,w}(\Omega)$. Let $p > 1$ and let w be a weight on Ω. We prove that w belongs to the Muckenhoupt class A_p if and only if there exists a positive constant $C_{p,\varphi,w}$ depending only on p, φ and w such that

$$\|\log mf\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w} \|\log f\|_{L_{p),\varphi,w}}$$

for every measurable function f satisfying $0 \le f \le 1$ and $\log f \in L_{p),\varphi,w}$ (see Theorem 2.5).

We now turn to the second objective of this paper, of discussing the John–Nirenberg inequalities in weighted grand Lebesgue spaces. Recall that a locally integrable function f belongs to BMO_p if

$$\|f\|_{\mathrm{BMO}_p} := \sup_Q \frac{\|(f - f_Q)\chi_Q\|_{L_p}}{\|\chi_Q\|_{L_p}}$$

is finite, where $f_Q = |Q|^{-1} \int_Q f(x)\, dx$ and the supremum is taken over all cubes Q contained in Ω. For convenience, we denote BMO_1 by BMO. The classical John–Nirenberg inequality (see e.g. [G1, Theorem 7.1.6]) says that for any $1 < p < \infty$ we have

$$\mathrm{BMO}_p = \mathrm{BMO}, \quad \text{with equivalent norms.}$$

We now define new BMO spaces corresponding to grand Lebesgue spaces and weighted grand Lebesgue spaces, respectively, by setting

$$\|f\|_{\mathrm{BMO}_{p),\varphi}} := \sup_Q \frac{\|(f - f_Q)\chi_Q\|_{L_{p),\varphi}}}{\|\chi_Q\|_{L_{p),\varphi}}}$$

and

$$\|f\|_{\mathrm{BMO}_{p),\varphi,w}} := \sup_Q \frac{\|(f - f_Q)\chi_Q\|_{L_{p),\varphi,w}}}{\|\chi_Q\|_{L_{p),\varphi,w}}},$$

where the supremum is taken over all cubes Q contained in Ω. In Section 3, we prove that

$$\mathrm{BMO}_{p),\varphi} = \mathrm{BMO} \quad \text{for every } 1 < p < \infty,$$

with equivalent norms. Furthermore, if w belongs to the Muckenhoupt class A_p, we have

$$\mathrm{BMO}_{p),\varphi,w} = \mathrm{BMO} \quad \text{for every } 1 < p < \infty,$$

with equivalent norms.

2. Weighted minimal inequalities. In this section, we study the minimal inequality on weighted grand Lebesgue spaces. Intuitively, the maximal operator controls the behavior of a function where it is large and the minimal operator controls the behavior of a function f where it is small, and therefore any norm inequality needs to reflect this fact. Similarly to [CN, Theorem 3.4], we obtain the minimal inequality in Theorem 2.5 by replacing f by $\log f$, which is large where f is small.

We first state the formal definition of grand Lebesgue spaces [KMR, Definition 3.1].

Definition 2.1. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $1 < p < \infty$ and $\varphi : (0, p-1) \to \mathbb{R}_+$ be a finite non-decreasing function with $\lim_{t\to 0} \varphi(t) = 0$. The (*generalized*) *grand Lebesgue space*, denoted by $L_{p),\varphi} := L_{p),\varphi}(\Omega, \mathcal{A}, \mu)$, is the set of all measurable functions for which

$$\|f\|_{L_{p),\varphi}} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\biggl(\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon}\,dx\biggr)^{1/(p-\varepsilon)}$$

is finite.

It is well known that $L_{p),\varphi}$ is complete [KMR, Theorem 3.6]. From now on, all of the above conditions will be tacitly assumed whenever we speak of grand Lebesgue spaces.

Let w be a *weight* on Ω, that is, a positive and integrable function on Ω. Recall that the weighted Lebesgue space, denoted by $L_{p,w}(\Omega,\mathcal{A},\mu)$ or simply $L_{p,w}$, is the set of all measurable functions f on Ω for which

$$\|f\|_{L_{p,w}} := \Big(\int_\Omega |f|^p w\,dx\Big)^{1/p} < \infty.$$

We now define *weighted grand Lebesgue spaces*.

Definition 2.2. Let $1 < p < \infty$. The space $L_{p),\varphi,w} := L_{p),\varphi,w}(\Omega)$ is the collection of all measurable functions f defined on $(\Omega,\mathcal{A},\mu)$ such that

$$\rho(f) := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon} w\,dx < \infty.$$

We equip this space with the (quasi-)norm

$$\|f\|_{L_{p),\varphi,w}} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\bigg(\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon} w\,dx\bigg)^{1/(p-\varepsilon)}.$$

It is obvious that the (quasi-)norm has the following equivalent expression:

$$\|f\|_{L_{p),\varphi,w}} \approx \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{L_{p-\varepsilon,w}},$$

where the equivalence constant depends only on p. Thus, for convenience, we sometimes write $\|f\|_{L_{p),\varphi,w}} = \sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|f\|_{L_{p-\varepsilon,w}}$.

The following definition is taken from [CN, Definition 1.1].

Definition 2.3. If f is an integrable function, define the *minimal function* of f, mf, by

$$mf(x) = \inf_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the infimum is taken over all cubes Q which contain x with sides parallel to the coordinate axes.

It is immediate that mf is a locally bounded function, and by the Lebesgue differentiation theorem, $mf(x) \le |f(x)| \le Mf(x)$ almost everywhere. Here, M is the classical Hardy–Littlewood maximal operator defined by the formula

$$Mf(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where Q runs over all non-degenerate cubes with sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

The Hardy–Littlewood maximal inequality is well known in harmonic analysis. We state it as follows.

Lemma 2.4. *Let $f \in L_p$ and $1 < p \le \infty$. Then*

$$\|Mf\|_{L_p} \le C(p')^{1/p}\|f\|_{L_p},$$

where $p' = (p-1)/p$ is the conjugate index of p.

Let $1 < p < \infty$. In the framework of standard weighted Lebesgue spaces, it is well known that

$$\int_{\Omega} (Mf(x))^p w(x)\,dx \le C \int_{\Omega} f(x)^p w(x)\,dx, \qquad f \in L_p,$$

if and only if w satisfies the A_p condition of Muckenhoupt [M]:

$$\sup \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx\right)^{p-1} =: A_p(w) < \infty.$$

We write by $w \in A_p$ if the weight w satisfies the inequality above.

We now state our main result in this section.

Theorem 2.5. *Let w be a weight and let $1 < p < \infty$. Then w is in A_p if and only if the weighted-norm inequality*

$$\|\log mf\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|\log f\|_{L_{p),\varphi,w}} \tag{2.1}$$

holds for all f such that $0 \le f \le 1$ and $\log f$ is in $L_{p),\varphi,w}$. The constant $C_{p,\varphi,w}$ depends only on the A_p constant of w, φ and p.

In order to prove the theorem above, we first state a lemma, which was proved in [M, Lemma 5].

Lemma 2.6. *If $1 < p < \infty$ and $w \in A_p$ on Ω with $A_p(w) = K$, then there exist constants $\theta > 0$ and $L > 0$ such that $w \in A_{p-\varepsilon}$ on Ω with $A_{p-\varepsilon}(w) \le L$ for all $0 < \varepsilon < \theta$.*

The lemma below plays a crucial role in the proof of Theorem 2.5, and also extends [FGJ, Theorem 4.1].

Lemma 2.7. *Let $1 < p < \infty$. Then $w \in A_p$ if and only if there is a constant $C_{p,\varphi,w} > 0$ such that for all bounded functions f,*

$$\|Mf\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|f\|_{L_{p),\varphi,w}}.$$

Proof. Let us first assume that $w \in A_p$. By Lemma 2.6, there exists $\theta \in (0, p-1)$ such that

$$\|Mf\|_{L_{p-\varepsilon,w}} \le C\|f\|_{L_{p-\varepsilon,w}}, \qquad \forall \varepsilon \in (0,\theta], \tag{2.2}$$

for some constant C independent of f and ε.

We now consider $\varepsilon > \theta$. In this case, we let $\varepsilon \in (\theta, p-1)$ be given. Applying Hölder's inequality, we have

$$\begin{aligned}
\|Mf\|^{p-\varepsilon}_{L_{p-\varepsilon,w}} &= \int_\Omega |Mf|^{p-\varepsilon} w\,dx = \int_\Omega |Mf|^{p-\varepsilon} w^{\frac{p-\varepsilon}{p-\theta}} w^{\frac{\varepsilon-\theta}{p-\theta}}\,dx \\
&\le \Big(\int_\Omega \big(|Mf|^{p-\varepsilon} w^{\frac{p-\varepsilon}{p-\theta}}\big)^{\frac{p-\theta}{p-\varepsilon}}\,dx\Big)^{\frac{p-\varepsilon}{p-\theta}} \Big(\int_\Omega \big(w^{\frac{\varepsilon-\theta}{p-\theta}}\big)^{\frac{p-\theta}{\varepsilon-\theta}}\,dx\Big)^{\frac{\varepsilon-\theta}{p-\theta}} \\
&= \Big(\int_\Omega |Mf|^{p-\theta} w\,dx\Big)^{\frac{p-\varepsilon}{p-\theta}} \Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{p-\theta}}.
\end{aligned}$$

This means

$$\begin{aligned}
\|Mf\|_{L_{p-\varepsilon,w}} &\le \Big(\int_\Omega |Mf|^{p-\theta} w\,dx\Big)^{\frac{1}{p-\theta}} \Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}} \\
&= \|Mf\|_{L_{p-\theta,w}} \Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|Mf\|_{L_{p),\varphi,w}} = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}} \\
&= \max\Big\{\sup_{0<\varepsilon<\theta} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}}, \sup_{\theta\le\varepsilon<p-1} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}}\Big\} \\
&\le \max\Big\{\sup_{0<\varepsilon<\theta} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}}, \sup_{\theta\le\varepsilon<p-1} \varphi(\varepsilon)\|Mf\|_{L_{p-\theta,w}}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\} \\
&= \max\Big\{\sup_{0<\varepsilon<\theta} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}}, \\
&\qquad\qquad \sup_{\theta\le\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1}\varphi(\theta)\|Mf\|_{L_{p-\theta,w}}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\} \\
&\le \max\Big\{1, \sup_{\theta<\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\} \sup_{0<\varepsilon\le\theta} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}}.
\end{aligned}$$

It follows from (3.1) that

$$\sup_{0<\varepsilon\le\theta} \varphi(\varepsilon)\|Mf\|_{L_{p-\varepsilon,w}} \le \sup_{0<\varepsilon<\theta} \varphi(\varepsilon)\|f\|_{L_{p-\varepsilon,w}} \le \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{L_{p-\varepsilon,w}}.$$

Thus, we get

$$\|Mf\|_{L_{p),\varphi,w}} \le \max\Big\{1, \sup_{\theta<\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\}\|f\|_{L_{p),\varphi,w}}.$$

It is not hard to see that $h(t) := \frac{t-\theta}{(p-\theta)(p-t)}$ is an increasing function on

$(\theta, p-1)$. So it is easy to see that

$$\max\Big\{1, \sup_{\theta<\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\}$$

$$\leq \max\Big\{1, \sup_{\theta<\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1} \sup_{\theta\leq\varepsilon<p-1}\Big(\int_\Omega w\,dx\Big)^{\frac{\varepsilon-\theta}{(p-\theta)(p-\varepsilon)}}\Big\}$$

$$\leq \sup_{\theta<\varepsilon<p-1} \varphi(\varepsilon)\varphi(\theta)^{-1}\Big(1+\int_\Omega w\,dx\Big)^{\frac{p-1-\theta}{p-\theta}}.$$

Thus we obtain

$$\|Mf\|_{L_{p),\varphi,w}} \leq C_{p,\varphi,w}\|f\|_{L_{p),\varphi,w}},$$

where

$$C_{p,\varphi,w} := \sup_{\theta<\varepsilon<p-1}\Big(1+\int_\Omega w\,dx\Big)^{\frac{p-1-\theta}{p-\theta}}\varphi(\varepsilon)\varphi(\theta)^{-1} < \infty.$$

To prove the converse, we assume that for all bounded functions f,

$$\|Mf\|_{L_{p),\varphi,w}} \leq C_{p,\varphi,w}\|f\|_{L_{p),\varphi,w}}. \tag{2.3}$$

Let $Q \subseteq \Omega$ be a non-degenerate cube with sides parallel to the coordinate axes. By the definition of maximal operator, we have

$$\Big(\frac{1}{|Q|}\int_Q |f|\Big)\chi_Q \leq M(f\chi_Q).$$

Using (2.3), we obtain

$$\|M(f\chi_Q)\|_{L_{p),\varphi,w}} \leq C_{p,\varphi,w}\|f\chi_Q\|_{L_{p),\varphi,w}}.$$

By Hölder's inequality, we have

$$\Big(\frac{1}{|Q|}\int_Q |f|\,dx\Big)\|\chi_Q\|_{L_{p),\varphi,w}} = \Big\|\Big(\frac{1}{|Q|}\int_Q |f|\,dx\Big)\chi_Q\Big\|_{L_{p),\varphi,w}}$$

$$\leq \|M(f\chi_Q)\|_{L_{p),\varphi,w}} \leq C_{p,\varphi,w}\|f\chi_Q\|_{L_{p),\varphi,w}}$$

$$= C_{p,\varphi,w}\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_Q |f|^{p-\varepsilon}w\,dx\Big)^{\frac{1}{p-\varepsilon}}$$

$$= C_{p,\varphi,w}\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_Q |f|^{p-\varepsilon}w^{\frac{p-\varepsilon}{p}}w^{\frac{\varepsilon}{p}}\,dx\Big)^{\frac{1}{p-\varepsilon}}$$

$$\leq C_{p,\varphi,w}\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_Q \big(|f|^{p-\varepsilon}w^{\frac{p-\varepsilon}{p}}\big)^{\frac{p}{p-\varepsilon}}\,dx\Big)^{1/p}\Big(\int_Q \big(w^{\frac{\varepsilon}{p}}\big)^{\frac{p}{\varepsilon}}\,dx\Big)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= C_{p,\varphi,w}\Big(\int_Q |f|^p w\,dx\Big)^{1/p}\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)w(Q)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= C_{p,\varphi,w}\Big(\int_Q |f|^p w\,dx\Big)^{1/p} \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) w(Q)^{\frac{1}{p-\varepsilon}} w(Q)^{-1/p}$$
$$= C_{p,\varphi,w}\, w(Q)^{-1/p}\Big(\int_Q |f|^p w\,dx\Big)^{1/p} \|\chi_Q\|_{L_{p),\varphi,w}},$$

where $w(Q) = \int_Q w\,dx$. This means that for all bounded functions f,

$$\frac{1}{|Q|}\int_Q |f|\,dx \le C_{p,\varphi,w}\, w(Q)^{-1/p}\Big(\int_Q |f|^p w\,dx\Big)^{1/p}. \tag{2.4}$$

Let g be an arbitrary integrable function and let $g_n = g\chi_{\{|g|\le n\}}$. Then (2.4) implies that

$$\frac{1}{|Q|}\int_Q |g_n|\,dx \le C_{p,\varphi,w}\, w(Q)^{-1/p}\Big(\int_Q |g_n|^p w\,dx\Big)^{1/p}.$$

Letting $n\to\infty$, we get for every integrable function g,

$$\frac{1}{|Q|}\int_Q |g|\,dx \le C_{p,\varphi,w}\, w(Q)^{-1/p}\Big(\int_Q |g|^p w\,dx\Big)^{1/p}.$$

Let now $g = w^{-1/(p-1)}$. Then we have $g^p w = w^{-1/(p-1)}$ and also

$$\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx \le C_{p,\varphi,w}\, w(Q)^{-1/p}\Big(\int_Q w^{-1/(p-1)}\,dx\Big)^{1/p}.$$

Thus we get

$$\sup\Big(\frac{1}{|Q|}\int_Q w\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx\Big)^{p-1} < \infty.$$

The proof is complete.

We are now in a position to prove Theorem 2.5.

Proof of Theorem 2.5. Suppose first that w is in A_p. Let f be as in the hypotheses. Then by Jensen's inequality, at each point x,

$$\log\Big(\frac{1}{|Q|}\int_Q f(y)\,dy\Big) \ge \frac{1}{|Q|}\int_Q \log f(y)\,dy$$

for all cubes Q which contain x. Since $0\le f\le 1$, $0\le mf(x)\le 1$, we have

$$|\log mf(x)| \le \Big|\frac{1}{|Q|}\int_Q \log f(y)\,dy\Big| \le \sup \frac{1}{|Q|}\int_Q |\log f(y)|\,dy = M(\log f)(x).$$

Since w is in A_p, by Lemma 2.7, the maximal operator is bounded on $L_{p),\varphi,w}$, that is,

$$\|M(\log f)\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|\log f\|_{L_{p),\varphi,w}}.$$

Then we get

$$\|\log mf\|_{L_{p),\varphi,w}} \le \|M(\log f)\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|\log f\|_{L_{p),\varphi,w}}.$$

To prove the converse, suppose w is such that (2.1) holds for all f. Fix $0 < a < 1$. Define $\Phi_a(t) = (\log(1/t))^a$ for $t \in (0,1)$. A simple calculation shows that Φ_a is decreasing and concave on $(e^{a-1}, 1)$. Now let f be any function on Ω such that $\alpha < f(x) < 1$ for some $\alpha > 0$. Then there exists $k > 0$ such that $f(x)^k > e^{a-1}$. Therefore, by Jensen's inequality, for any cube Q, we have $|Q|^{-1}\int_Q \Phi_a(f(x)^k)\,dx \le \Phi_a(|Q|^{-1}\int_Q f(x)^k\,dx)$, or equivalently,

$$\left(\frac{1}{|Q|}\int\limits_Q |\log(f(x)^k)|^a\,dx\right)^{1/a} \le \left|\log\left(\frac{1}{|Q|}\int\limits_Q f(x)^k\,dx\right)\right|.$$

Fix x and take the supremum over all Q containing x; this yiels

$$M(|\log(f^k)|^a)^{1/a} \le |\log m(f^k)|.$$

Since the maximal operator and the minimal operator are positive homogeneous, by (2.1) we see that

$$\|M(|\log f|^a)^{1/a}\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|\log f\|_{L_{p),\varphi,w}}. \tag{2.5}$$

However, the constant in (2.4) is independent of a, so we get

$$\|M(\log f)\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|\log f\|_{L_{p),\varphi,w}}. \tag{2.6}$$

Indeed, since $|\log f|$ is a bounded function, it follows that $|\log(f)|^a \to |\log(f)|$ in L_∞ when $a \to 1$, and therefore it is easy to see that

$$\lim_{a\to 1}\|M(|\log f|^a)^{1/a} - M(\log f)\|_\infty = 0.$$

Noting that $L_\infty \hookrightarrow L_{p),\varphi,w}$, we get (2.6). Thus, for all bounded functions g we have

$$\|M(g)\|_{L_{p),\varphi,w}} \le C_{p,\varphi,w}\|g\|_{L_{p),\varphi,w}}.$$

By Lemma 2.7, w is in A_p. The proof of the theorem is complete.

3. The new John–Nirenberg theorem. For f an integrable function on Ω, set

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|}\int\limits_Q |f(x) - f_Q|\,dx,$$

where $f_Q = |Q|^{-1}\int_Q f(x)\,dx$ and the supremum is taken over all cubes Q in Ω. The function f is called *of bounded mean oscillation* if $\|f\|_{\mathrm{BMO}} < \infty$, and $\mathrm{BMO}(\Omega)$ is the set of all integrable functions f on Ω with $\|f\|_{\mathrm{BMO}} < \infty$. It is well known that BMO is a linear space; moreover, $\mathrm{BMO}/\mathbb{C}$ is a Banach space. Obviously,

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{\|(f - f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}}.$$

The classical John–Nirenberg theorem (see e.g. [G1, Theorem 7.1.6]) says that there exist constants $C_1, C_2 > 0$ such that for any $\alpha > 0$ and all cubes $Q \subseteq \Omega$,

$$\big|\{x \in Q : |f(x) - f_Q| > \alpha\}\big| \le C_1 \exp\Big[-\frac{C_2\alpha}{\|f\|_{\mathrm{BMO}}}\Big]|Q|.$$

This statement is equivalent to the following: There exists an absolute constant C such that for all $1 \le p < \infty$,

$$\|f\|_{\mathrm{BMO}} \le \sup_{Q\subseteq\Omega} \frac{\|(f - f_Q)\chi_Q\|_{L_p}}{\|\chi_Q\|_{L_p}} \le Cp\|f\|_{\mathrm{BMO}}. \tag{3.1}$$

We define $\mathrm{BMO}_p(\Omega)$ to be the set of all integrable functions f on Ω such that

$$\|f\|_{\mathrm{BMO}_p} = \sup_{Q\subseteq\Omega} \frac{\|(f - f_Q)\chi_Q\|_{L_p}}{\|\chi_Q\|_{L_p}}$$

is finite. By (3.1), we get $\mathrm{BMO} = \mathrm{BMO}_p$ with equivalent norms.

We now define new bounded mean oscillation norms on grand Lebesgue spaces and weighted grand Lebesgue spaces.

Definition 3.1. For f an integrable function on Ω, set

$$\|f\|_{\mathrm{BMO}_{p),\varphi}} = \sup_{Q} \frac{\|(f - f_Q)\chi_Q\|_{L_{p),\varphi}}}{\|\chi_Q\|_{L_{p),\varphi}}},$$

where the supremum is taken over all cubes Q in Ω. $\mathrm{BMO}_{p),\varphi}(\Omega)$ is the set of all integrable functions f on Ω with $\|f\|_{\mathrm{BMO}_{p),\varphi}} < \infty$.

Definition 3.2. Let w be a weight and let f be an integrable function on Ω. Set

$$\|f\|_{\mathrm{BMO}_{p),\varphi,w}} = \sup_{Q} \frac{\|(f - f_Q)\chi_Q\|_{L_{p),\varphi,w}}}{\|\chi_Q\|_{L_{p),\varphi,w}}},$$

where the supremum is taken over all cubes Q in Ω. $\mathrm{BMO}_{p),\varphi,w}(\Omega)$ is the set of all integrable functions f on Ω with $\|f\|_{\mathrm{BMO}_{p),\varphi,w}} < \infty$.

When $w \equiv 1$ the space $\mathrm{BMO}_{p),\varphi,1}(\Omega)$ reduces to $\mathrm{BMO}_{p),\varphi}(\Omega)$, and when $\varphi \equiv 1$ the space $\mathrm{BMO}_{p),1,w}(\Omega)$ reduces to the classical weighted bounded mean oscillation space $\mathrm{BMO}_{p,w}(\Omega)$, i.e.,

$$\mathrm{BMO}_{p,w} = \{f \in L_1(\Omega) : \|f\|_{\mathrm{BMO}_{p,w}} < \infty\},$$

where

$$\|f\|_{\mathrm{BMO}_{p,w}} = \sup_{Q\subseteq\Omega} \frac{\|(f - f_Q)\chi_Q\|_{L_{p,w}}}{\|\chi_Q\|_{L_{p,w}}}.$$

When $p = 1$ the space $\mathrm{BMO}_{1,w}(\Omega)$ reduces to BMO_w.

Theorems 3.3 and 3.4 below are the main results in this section.

Theorem 3.3. *Let $1 < p < \infty$. Then*

$$\mathrm{BMO}_{p),\varphi} = \mathrm{BMO}, \tag{3.2}$$

with equivalent norms.

Proof. By Hölder's inequality, we have

$$\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} \le \frac{\|(f-f_Q)\chi_Q\|_{L_p}}{\|\chi_Q\|_{L_p}} \quad \text{for any } 1<p<\infty.$$

Given $1<p<\infty$, we obtain

$$\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} \le \frac{\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}}{\|\chi_Q\|_{L_{p-\varepsilon}}}$$

for any $0<\varepsilon<p-1$. Thus

$$\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} \cdot \|\chi_Q\|_{L_{p-\varepsilon}} \le \|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}.$$

Hence

$$\begin{aligned}\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} &= \frac{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}}\cdot\|\chi_Q\|_{L_{p-\varepsilon}}}{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon}}}\\ &\le \frac{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}}{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon}}} = \frac{\|(f-f_Q)\chi_Q\|_{L_{p),\varphi}}}{\|\chi_Q\|_{L_{p),\varphi}}}.\end{aligned}$$

Taking the supremum over all cubes $Q\subseteq\Omega$, we obtain

$$\|f\|_{\mathrm{BMO}_1} \le \|f\|_{\mathrm{BMO}_{p),\varphi}}.$$

Conversely, from (3.1), we get

$$\begin{aligned}\frac{\|(f-f_Q)\chi_Q\|_{L_{p),\varphi}}}{\|\chi_Q\|_{L_{p),\varphi}}} &= \frac{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}}{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon}}}\\ &\le \sup_{0<\varepsilon<p-1}\left\{\frac{\varphi(\varepsilon)\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}}{\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon}}}\right\}\\ &= \sup_{0<\varepsilon<p-1}\left\{\frac{\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon}}}{\|\chi_Q\|_{L_{p-\varepsilon}}}\right\}\\ &\le C\sup_{0<\varepsilon<p-1}(p-\varepsilon)\|f\|_{\mathrm{BMO}} \le Cp\|f\|_{\mathrm{BMO}}.\end{aligned}$$

Taking the supremum over all cubes $Q\subseteq\Omega$, we obtain

$$\|f\|_{\mathrm{BMO}_{p),\varphi}} \le Cp\|f\|_{\mathrm{BMO}}.$$

To sum up, we have

$$\|f\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}_{p),\varphi}} \le Cp\|f\|_{\mathrm{BMO}}.$$

The proof of the theorem is complete.

THEOREM 3.4. *Let $1<p<\infty$ and $w\in A_p$. Then*

$$\mathrm{BMO}_{p),\varphi,w} = \mathrm{BMO},$$

with equivalent norms.

LEMMA 3.5 ([G1]). *If an integrable function $w : \Omega \to [0,\infty)$ belongs to A_p, then there exist a $\sigma > 0$ and a constant $C > 0$ such that for any cube $Q \subseteq \Omega$ and all measurable subsets E of Q, we have*

$$\frac{w(E)}{w(Q)} \le C\left(\frac{|E|}{|Q|}\right)^{\sigma}. \tag{3.3}$$

Proof of Theorem 3.4. Let us first prove $\mathrm{BMO} \hookrightarrow \mathrm{BMO}_{p),\varphi,w}$. By the John–Nirenberg inequality, we have

$$\big|\{x \in Q : |f(x) - f_Q| > \alpha\}\big| \le C_1 \exp\left[-\frac{C_2\alpha}{\|f\|_{\mathrm{BMO}}}\right]|Q|.$$

According to (3.3), we get

$$w(\{x \in Q : |f(x) - f_Q| > \alpha\}) \le CC_1^{\sigma} \exp\left[-\frac{C_2\sigma\alpha}{\|f\|_{\mathrm{BMO}}}\right]w(Q)$$

for some $C > 0$. Thus there exists a constant $C_3 > 0$ such that for any $Q \subseteq \Omega$,

$$\begin{aligned}
\frac{\|(f - f_Q)\chi_Q\|_{L_{p),\varphi,w}}}{\|\chi_Q\|_{L_{p),\varphi,w}}}\ & \frac{\sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|(f - f_Q)\chi_Q\|_{L_{p-\varepsilon,w}}}{\sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon,w}}} \\
&\le \sup_{0<\varepsilon<p-1} \left\{\frac{\|(f - f_Q)\chi_Q\|^{p-\varepsilon}_{L_{p-\varepsilon,w}}}{w(Q)}\right\}^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \left\{\frac{(p-\varepsilon)\int_0^\infty \alpha^{p-\varepsilon-1} w\big(\{x \in Q : |f(x) - f_Q| > \alpha\}\big)\,d\alpha}{w(Q)}\right\}^{\frac{1}{p-\varepsilon}} \\
&\le \sup_{0<\varepsilon<p-1} \left\{CC_1^{\sigma}(p-\varepsilon)\int_0^\infty \alpha^{p-\varepsilon-1} \exp\left[-\frac{C_2\sigma\alpha}{\|f\|_{\mathrm{BMO}}}\right]\right\}^{\frac{1}{p-\varepsilon}} \\
&\le C_3\|f\|_{\mathrm{BMO}},
\end{aligned}$$

where $w(Q) = \int_Q w(x)\,dx$.

We now prove $\mathrm{BMO}_{p),\varphi,w} \hookrightarrow \mathrm{BMO}$. By Lemma 2.6, there exists θ in $(0, p-1)$ such that $w \in A_{p-\varepsilon}$ for all $\varepsilon \in (0,\theta]$. Applying Hölder's inequality, we have

$$\begin{aligned}
\int_Q |f(x) - f_Q|\,dx &= \int_Q |f(x) - f_Q| w(x)^{\frac{1}{p-\varepsilon}} w(x)^{-\frac{1}{p-\varepsilon}}\,dx \\
&\le \Big(\int_Q |f(x) - f_Q|^{p-\varepsilon} w(x)\,dx\Big)^{\frac{1}{p-\varepsilon}} \Big(\int_Q w(x)^{-\frac{1}{p-\varepsilon-1}}\,dx\Big)^{\frac{p-\varepsilon-1}{p-\varepsilon}} \\
&\le \Big(\int_Q |f(x) - f_Q|^{p-\varepsilon} w(x)\,dx\Big)^{\frac{1}{p-\varepsilon}} \frac{|Q|}{w(Q)^{1/(p-\varepsilon)}}.
\end{aligned}$$

That is,

$$\frac{\|(f - f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} \cdot \|\chi_Q\|_{L_{p-\varepsilon,w}} \le \|(f - f_Q)\chi_Q\|_{L_{p-\varepsilon,w}}.$$

Thus we obtain

$$
\begin{aligned}
\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}} &= \frac{\sup_{0<\varepsilon\le\theta}\varphi(\varepsilon)\frac{\|(f-f_Q)\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}}\cdot\|\chi_Q\|_{L_{p-\varepsilon,w}}}{\sup_{0<\varepsilon\le\theta}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon,w}}} \\
&\le \frac{\sup_{0<\varepsilon\le\theta}\varphi(\varepsilon)\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon,w}}}{\sup_{0<\varepsilon\le\theta}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon,w}}} \\
&\le C_{p,\theta}\frac{\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|(f-f_Q)\chi_Q\|_{L_{p-\varepsilon,w}}}{\sup_{\theta\le\varepsilon<p-1}\frac{\varphi(\theta)}{\varphi(\varepsilon)}\cdot\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|\chi_Q\|_{L_{p-\varepsilon,w}}} \\
&= C_{p,\theta}\sup_{\theta\le\varepsilon<p-1}\frac{\varphi(\varepsilon)}{\varphi(\theta)}\,\frac{\|(f-f_Q)\chi_Q\|_{L_{p),\varphi,w}}}{\|\chi_Q\|_{L_{p),\varphi,w}}}. \\
&\le C_{p,\theta}\frac{\varphi(p-1)}{\varphi(\theta)}\,\frac{\|(f-f_Q)\chi_Q\|_{L_{p),\varphi,w}}}{\|\chi_Q\|_{L_{p),\varphi,w}}}.
\end{aligned}
$$

Taking the supremum over all cubes $Q\subseteq\Omega$, we get

$$\|f\|_{\mathrm{BMO}}\le C_{p,\theta,\phi}\|f\|_{\mathrm{BMO}_{p),\varphi,w}},$$

where $\theta\in(0,p-1)$ is a fixed constant depending only on w. The proof of the theorem is complete.

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