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Maximality of dual coactions on sectional C^*-algebras of Fell bundles and applications

by

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Abstract. We give a simple proof of the maximality of dual coactions on full cross-sectional C^*-algebras of Fell bundles over locally compact groups. This result was only known for discrete groups or for saturated (separable) Fell bundles over locally compact groups. Our proof, which is derived as an application of the theory of universal generalised fixed-point algebras for weakly proper actions, is different from these previously known cases and works for general Fell bundles over locally compact groups. As applications, we extend certain exotic crossed-product functors in the sense of Baum, Guentner and Willett to the category of Fell bundles and the category of partial actions, and we obtain results about the K-theory of (exotic) cross-sectional algebras of Fell bundles over K-amenable groups. As a bonus, we give a characterisation of maximal coactions of discrete groups in terms of maximal tensor products.

1. Introduction. The theory of Fell bundles $\mathcal{B}$ over a locally compact group G (also called C^*-algebraic bundles in [15]) and their cross-sectional algebras give far reaching generalisations of the theory of crossed products by strongly continuous actions $\alpha : G \to \mathrm{Aut}(A)$ of G on C^*-algebras A. Important examples of Fell bundles come from (twisted) partial actions (see [21]) of G on C^*-algebras A and in this case the crossed products for such actions are by definition given as the cross-sectional C^*-algebras of the associated Fell bundles.

Recall from [14, 15] that a *Fell bundle* $\mathcal{B}$ over G consists of a topological space $\mathcal{B}$ together with a continuous open surjection $p : \mathcal{B} \twoheadrightarrow G$ such that the fibres $B_s := p^{-1}(\{s\})$ are Banach spaces for all $s \in G$ and such that all operations like multiplication with scalars, fibre-wise addition, and norm are continuous on $\mathcal{B}$. Moreover, $\mathcal{B}$ comes equipped with an associative continuous multiplication function

$$\cdot : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, \qquad (a, b) \mapsto a \cdot b,$$

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which is bilinear when restricted to $B_s \times B_t$ for all $s,t \in G$ and such that $B_s \cdot B_t \subseteq B_{st}$. In addition, $\mathcal{B}$ is equipped with a continuous involution $* : \mathcal{B} \to \mathcal{B}$, $b \mapsto b^*$, which sends B_t to $B_{t^{-1}}$ for all $t \in G$ and which is compatible with multiplication and addition on $\mathcal{B}$ in a sense extending the usual properties for involutions on C^*-algebras. In particular, the C^*-condition $\|b^*b\| = \|b\|^2$ and the positivity condition $b^*b \geq 0$ in B_e are required to hold for all $b \in \mathcal{B}$. Note that the unit fibre B_e in a Fell bundle $\mathcal{B}$ is always a C^*-algebra. A Fell bundle is called *saturated* if $\operatorname{span}(B_t^* B_t) = B_e$ for all $t \in G$.

Given a Fell bundle $\mathcal{B}$, let $C_c(\mathcal{B})$ denote the space of continuous sections with compact support. It carries multiplication and involution given by the formulas

$$f * g(s) = \int_G f(t)g(t^{-1}s)\,dt \quad \text{and} \quad f^*(s) = \Delta_G(s^{-1})f(s^{-1})^*. \tag{1.1}$$

In general there might exist many possible C^*-completions of $C_c(\mathcal{B})$. The largest (L^1-bounded) C^*-norm on $C_c(\mathcal{B})$ is the *universal* (or *maximal*) *cross-sectional algebra* $C^*(\mathcal{B})$ whose representations are in one-to-one correspondence with the continuous representations of the bundle $\mathcal{B}$. At the other extreme there is the *reduced* cross-sectional algebra $C_r^*(\mathcal{B})$, which is defined as the image of $C^*(\mathcal{B})$ under the regular representation $\Lambda : \mathcal{B} \to \mathcal{L}(L^2(\mathcal{B}))$.

If $\mathcal{B}$ is a Fell bundle and if $u_G : G \to U\mathcal{M}(C^*(G))$ denotes the universal representation of G, then the integrated form of the representation

$$\delta_{\mathcal{B}} = \iota_{\mathcal{B}} \otimes u_G : \mathcal{B} \to \mathcal{M}(C^*(\mathcal{B}) \otimes C^*(G)), \qquad b_t \mapsto b_t \otimes u_G(t),$$

where b_t denotes an element in the fibre B_t of $\mathcal{B}$, defines a *dual coaction* of G (or rather of the Hopf-C^*-algebra $C^*(G)$) on $C^*(\mathcal{B})$ (see [24]). It is the main purpose of our paper to show that this coaction always satisfies Katayama duality for the maximal bidual crossed product in the sense that a certain canonical surjective homomorphism

$$\Phi_{\mathcal{B}} : C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \rtimes_{\widehat{\delta_{\mathcal{B}}}} G \twoheadrightarrow C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G))$$

is actually an isomorphism. Coactions with this property have been called *maximal* in [17], where it has first been shown that every coaction (B,δ) admits a *maximalisation* (B_m, δ_m). If G is discrete and (B,δ) is any coaction of G on some C^*-algebra B, then it follows from results of Ng and Quigg [33, 36] (see also [19]) that B is isomorphic to a C^*-completion $C_\mu^*(\mathcal{B})$ of $C_c(\mathcal{B})$ for some Fell bundle $\mathcal{B}$ with respect to some norm $\|\cdot\|_\mu$ lying between the universal norm $\|\cdot\|_u$ and the reduced norm $\|\cdot\|_r$ such that the coaction δ is the natural dual coaction of this algebra. It has then been shown in [19] (see also [17, §4]) that δ is maximal if and only if $B = C^*(\mathcal{B})$, the universal cross-sectional algebra. This gives the desired result in the discrete case.

For second countable locally compact groups the result that $(C^*(\mathcal{B}), \delta_{\mathcal{B}})$ is a maximal coaction has been obtained in the special case of *separable saturated* Fell bundles by Kaliszewski, Muhly, Quigg and Williams [29]. Separability is not a strong assumption, but note that Fell bundles arising from (twisted) partial actions are saturated if and only if the action is actually a global (twisted) action, so that there are many important examples of Fell bundles which are not saturated. Moreover, the proof given in [29] relies on some heavy machinery involving Fell bundles on groupoids while our proof depends on the notion of generalised fixed-point algebras for weakly proper actions as introduced recently in [9]. Both proofs depend on non-trivial results, but we believe that our proof is much shorter and technically easier than the proof given for the special case of saturated bundles in [29]. We should point out, however, that for the special case of Fell bundles associated to partial actions, the maximality result can also be deduced from the paper [2] by Abadie and Martí Pérez. Indeed, as we will see in Section 4.2, the maximality of the dual coaction on the C^*-algebra $C^*(\mathcal{B})$ of a Fell bundle associated to a partial action (A, α) is essentially equivalent to the fact (proven in [2]) that the full crossed product of the Morita enveloping action of (A, α) is Morita equivalent to the original full crossed product by the partial action. Hence our main result will provide an alternative proof of one of the main results in [2].

The paper is organised as follows. After a short preliminary section (§2) on cross-sectional algebras, coactions, and generalised fixed-point algebras for weakly proper actions we shall give the proof of our main result (Theorem 3.1) in §3. We will then have a number of interesting applications, starting from extensions of crossed-product functors from ordinary actions to Fell bundle categories, K-amenability for cross-sectional C^*-algebras and some applications to partial actions. Our results imply, in particular, that one can extend Morita compatible exotic crossed-product functors from ordinary actions to partial actions. In other words, given a Morita compatible crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ defined only for (global) G-actions (A, α), we can extend this functor and define $A \rtimes_{\alpha,\mu} G$ for every given partial G-action (A, α). More generally, we can extend the functor to the category of Fell bundles over G, and define exotic versions $C^*_\mu(\mathcal{B})$ of cross-sectional C^*-algebras of Fell bundles $\mathcal{B}$ over G. These include partial actions, or more generally twisted partial actions. We give an alternative proof and recover some of the main results of [2] on enveloping actions and amenability for partial actions. More generally, we prove that all exotic cross-sectional C^*-algebras $C^*_\mu(\mathcal{B})$ associated to so called correspondence crossed-product functors $\rtimes_\mu$ have the same K-theory if the underlying group is K-amenable.

In the final section, we give a simple characterisation of maximal coactions of discrete groups (which is not available for general locally compact

groups), proving that a coaction $\delta : B \to B \otimes C^*(G)$ of a discrete group G is maximal exactly when it admits a lift $\delta_{\max} : B \to B \otimes_{\max} C^*(G)$, where $\otimes_{\max}$ denotes the maximal tensor product. (And, as usual, all the unlabelled tensor products $\otimes$ between C^*-algebras mean the minimal tensor product.)

2. Preliminaries

2.1. Fell bundles and their cross-sectional algebras. Suppose that $p : \mathcal{B} \to G$ is a Fell bundle over the locally compact group G as defined in the introduction. Main references on Fell bundles and their cross-sectional algebras are the books by Doran and Fell [14, 15], to which we refer for more details (see also the book by Exel [23]). Let $C_c(\mathcal{B})$ be the set of all continuous sections of $\mathcal{B}$ with compact supports. Then, equipped with convolution and involution as in (1.1), $C_c(\mathcal{B})$ becomes a *-algebra. Let $L^1(\mathcal{B})$ denote the completion of $C_c(\mathcal{B})$ with respect to $\|f\|_1 = \int_G \|f(t)\|\, dt$. Then the universal cross-sectional algebra $C^*(\mathcal{B})$ is defined as the enveloping C^*-algebra of the Banach-* algebra $L^1(\mathcal{B})$, i.e., it is the completion of $L^1(\mathcal{B})$ with respect to

$$\|f\|_u := \sup_\pi \|\pi(f)\|$$

where π runs through all representations of $L^1(\mathcal{B})$ on Hilbert space.

It has been shown by Fell [25, §§15–16] that $C^*(\mathcal{B})$ is universal for (continuous) representations of $\mathcal{B}$, and since we shall need it later, let us explain (a modern version of) this result in some detail: By a *representation* $\pi : \mathcal{B} \to \mathcal{M}(D)$ of $\mathcal{B}$ in the multiplier algebra of some C^*-algebra D we understand a strictly continuous map $b \mapsto \pi(b)$ which is linear on each fibre B_t and preserves multiplication and involution on $\mathcal{B}$. Such a representation is called *nondegenerate* if its restriction $\pi_e : B_e \to \mathcal{M}(D)$ to the unit fibre B_e of $\mathcal{B}$ is nondegenerate in the usual sense that $\operatorname{span}(\pi_e(B_e)D)$ is dense in D (then Cohen's factorisation theorem implies that $\pi_e(B_e)D = D$). There is a canonical nondegenerate representation $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}(C^*(\mathcal{B}))$ which is determined by the formulas

$$(\iota_{\mathcal{B}}(b_t)f)(s) = b_t f(t^{-1}s) \quad \text{and} \quad (f\iota_{\mathcal{B}}(b_t))(s) = \Delta_G(t^{-1})f(st^{-1})b_t$$

for $b_t \in B_t$, $f \in C_c(\mathcal{B})$ and $t, s \in G$ (see e.g. [25, p. 138]). We have the following well-known result:

PROPOSITION 2.1. *There are one-to-one correspondences between*

(1) *nondegenerate representations* $\pi : \mathcal{B} \to \mathcal{M}(D)$,
(2) *nondegenerate representations* $\tilde{\pi} : C_c(\mathcal{B}) \to \mathcal{M}(D)$ *which are continuous with respect to the inductive limit topology on* $C_c(\mathcal{B})$ *and the norm topology on* $\mathcal{M}(D)$,
(3) *nondegenerate representations* $\tilde{\pi} : L^1(\mathcal{B}) \to \mathcal{M}(D)$, *and*
(4) *nondegenerate representations* $\tilde{\pi} : C^*(\mathcal{B}) \to \mathcal{M}(D)$.

The correspondences in the proposition are as follows: If $\pi : \mathcal{B} \to \mathcal{M}(D)$ is as in (1), then the corresponding representation $\tilde{\pi} : C_c(\mathcal{B}) \to \mathcal{M}(D)$ is given by integration:

$$\tilde{\pi}(f)c := \int_G \pi(f(t))c\,dt \quad \text{for all } f \in C_c(\mathcal{B}) \text{ and } c \in D.$$

We call it the *integrated form* of π. It is straightforward to check that it is continuous in the inductive limit topology (given by uniform convergence with controlled compact supports) and with respect to $\|\cdot\|_1$. This shows how objects in (2), (3) correspond to objects in (1). By construction of the enveloping C^*-algebra, every representation of $L^1(\mathcal{B})$ uniquely extends to $C^*(\mathcal{B})$, which shows how objects in (4) correspond to objects in (3). Conversely, if $\tilde{\pi} : C^*(\mathcal{B}) \to \mathcal{M}(D)$ is as in (4), then $\pi = \tilde{\pi} \circ \iota_{\mathcal{B}}$ is the corresponding representation as in (1). The only missing link is how objects in (1) correspond to objects in (2). But this link is given by [25, Theorem 16.1] by representing D faithfully on a Hilbert space.

In what follows, we shall make no notational difference between a representation π of $\mathcal{B}$ and the corresponding representations of $C_c(\mathcal{B}), L^1(\mathcal{B})$, and $C^*(\mathcal{B})$.

Let $L^2(\mathcal{B})$ denote the Hilbert module over B_e given as the completion of $C_c(\mathcal{B})$ with respect to the B_e-valued inner product

$$\langle \xi, \eta \rangle_{B_e} := (\xi^* * \eta)(e) = \int_G \xi(t)^* \eta(t)\,dt.$$

Then the action of $C_c(\mathcal{B})$ on itself given by convolution extends to the *regular representation* $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{L}(L^2(\mathcal{B}))$ by adjointable operators on the B_e-Hilbert module $L^2(\mathcal{B})$, and the image $C^*_r(\mathcal{B}) := \Lambda_{\mathcal{B}}(C^*(\mathcal{B})) \subseteq \mathcal{L}(L^2(\mathcal{B}))$ is called the *reduced* cross-sectional C^*-algebra of $\mathcal{B}$.

2.2. The Fell bundle of an action. We now want to give a brief outline of the well-known fact that the construction of maximal cross-sectional C^*-algebras of Fell bundles extends the construction of full and reduced crossed products of a G-C^*-algebra (A, α) (see [14, 15]). Indeed, if $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of a locally compact group G on a C^*-algebra A, then we can form an associated Fell bundle $\mathcal{B} = A \rtimes^\alpha G$ in which $A \rtimes^\alpha G$ consists of the trivial bundle $A \times G$ over G equipped with multiplication and involution given for all $(a, s), (b, t) \in A \rtimes^\alpha G$ by

$$(a, s)(b, t) = (a\alpha_s(b), st) \quad \text{and} \quad (a, t)^* = (\alpha_t(a)^*, t^{-1}).$$

The *-algebra $C_c(A \rtimes^\alpha G)$ then identifies with $C_c(G, A)$ equipped with the usual convolution and involution given for $f, g \in C_c(G, A)$ by

$$f * g(s) = \int_G f(t)\alpha_{t^{-1}}(g(t^{-1}s))\,dt \quad \text{and} \quad f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*).$$

Moreover, there is a one-to-one correspondence between nondegenerate covariant representations (ρ, v) of (A, G, α) in $\mathcal{M}(D)$ and nondegenerate strictly continuous representations $\pi : A \rtimes^\alpha G \to \mathcal{M}(D)$ for any C^*-algebra D. To see how this works, recall that a nondegenerate covariant representation (ρ, v) of (A, G, α) in $\mathcal{M}(D)$ consists of a nondegenerate *-homomorphism $\rho : A \to \mathcal{M}(D)$ and a strictly continuous homomorphism $v : G \to U\mathcal{M}(D)$ such that $\rho(\alpha_s(a)) = v_s \rho(a) v_s^*$ for all $a \in A$, $s \in G$. Given such a covariant representation (ρ, v) the corresponding representation $\pi : A \rtimes^\alpha G \to \mathcal{M}(D)$ is given by $\pi(a, s) = \rho(a)v(s)$ for all $(a, s) \in A \rtimes^\alpha G$. It is then an easy exercise to check that modulo the identification $C_c(A \rtimes^\alpha G) = C_c(G, A)$ the integrated form $\tilde{\pi} : C_c(A \rtimes^\alpha G) \to \mathcal{M}(D)$ coincides with the integrated form $\rho \rtimes v : C_c(G, A) \to \mathcal{M}(D)$ given for $f \in C_c(G, A)$ by

$$\rho \rtimes v(f) = \int_G \rho(f(s)) v_s \, ds.$$

This implies that the full cross-sectional C^*-algebra $C^*(A \rtimes^\alpha G)$ coincides with the full crossed product $A \rtimes_\alpha G$ which is defined as the completion of $C_c(G, A)$ under the norm $\|f\|_u := \sup_{(\rho,v)} \|\rho \rtimes v(f)\|$, where (ρ, v) runs through all possible (nondegenerate) covariant homomorphisms of (A, G, α).

Notice that under the correspondence between representations of $A \rtimes^\alpha G$ and covariant representations of (A, G, α), the universal representation $\iota_{A \rtimes^\alpha G} : A \rtimes^\alpha G \to \mathcal{M}(C^*(A \rtimes^\alpha G))$ corresponds to the universal covariant representation (i_A, i_G) of (A, G, α) in $\mathcal{M}(A \rtimes_\alpha G)$ which is determined by the equations

$$(i_A(a)f)(t) = af(t) \quad \text{and} \quad (i_G(s)f)(t) = \alpha_s(f(s^{-1}t))$$

for all $a \in A, s, t \in G$ and $f \in C_c(G, A)$.

For the reduced crossed product one checks that identifying sections of $A \rtimes^\alpha G$ with A-valued functions on G identifies the Hilbert A-module $L^2(A \rtimes^\alpha G)$ constructed from the Fell bundle $A \rtimes^\alpha G$ with the Hilbert A-module $L^2(G, A)$, and the left action $\Lambda_{A \rtimes^\alpha G} : C_c(A \rtimes^\alpha G) \to \mathcal{L}(L^2(A \rtimes^\alpha G))$ then identifies with the regular representation of $C_c(G, A) \subseteq A \rtimes_\alpha G$. Hence, the reduced cross-sectional algebra $C_r^*(A \rtimes^\alpha G)$ identifies with the reduced crossed product $A \rtimes_{\alpha,r} G$ as well.

2.3. Coactions and their crossed products. A *coaction* of a locally compact group on a C^*-algebra B is an injective nondegenerate *-homomorphism $\delta : B \to \mathcal{M}(B \otimes C^*(G))$ which satisfies the identity

$$(\mathrm{id}_B \otimes \delta_G) \circ \delta = (\delta \otimes \mathrm{id}_G) \circ \delta, \tag{2.1}$$

where $\delta_G : C^*(G) \to \mathcal{M}(C^*(G) \otimes C^*(G))$ is the comultiplication on $C^*(G)$ which is given by the integrated form of the representation $s \mapsto u_G(s) \otimes u_G(s) \in U\mathcal{M}(C^*(G) \otimes C^*(G))$, where in turn $u_G : G \to U\mathcal{M}(C^*(G))$

denotes the universal representation of G. Note that in (2.1) (and in many other places) we make no notational difference between a nondegenerate *-homomorphism and its unique extension to the multiplier algebra. We shall assume that our coactions δ always satisfy the following (strong) nondegeneracy condition:

$$\overline{\operatorname{span}}(\delta(B)(1\otimes C^*(G))) = B\otimes C^*(G).$$

If $\delta : B \to \mathcal{M}(B\otimes C^*(G))$ is a coaction of G, we let $\delta^r := (1\otimes\lambda)\circ\delta : B\to\mathcal{M}(B\otimes C^*_r(G))$ denote the *reduction* of δ, where $\lambda : G\to U(L^2(G))$ is the regular representation of G, and we let $M : C_0(G)\to\mathbb{B}(L^2(G))$ be the representation by multiplication operators. We may then represent the crossed product $B\rtimes_\delta\widehat{G}$ faithfully in $\mathcal{M}(B\otimes\mathcal{K}(L^2(G)))$ via the regular representation $\Lambda^{\widehat{G}}_B := \delta^r\rtimes(1\otimes M)$. Hence, via this representation, we may define

$$B\rtimes_\delta\widehat{G} := \overline{\operatorname{span}}\big\{\delta^r(B)\big(1\otimes M(C_0(G))\big)\big\}\subseteq\mathcal{M}\big(B\otimes\mathcal{K}(L^2(G))\big).$$

Since locally compact groups are always "co-amenable", this "reduced crossed product" coincides with the "universal crossed product", which is universal for covariant representations of the co-system (B,δ). Note that in the notation of crossed products by coactions we use the symbol $\widehat{G}$ to indicate that this construction is *dual* to the construction of crossed products by actions of G. We refer to [18, Appendix A] for an extensive survey on crossed products by actions and coactions of groups on C^*-algebras.

The *dual action* $\widehat{\delta} : G\to\operatorname{Aut}(B\rtimes_\delta\widehat{G})$ is determined by the equation

$$\widehat{\delta}_s\big(\delta^r(b)(1\otimes M(\varphi))\big) = \delta^r(b)\big(1\otimes M(\sigma_s(\varphi))\big),\qquad \forall b\in B,\ \varphi\in C_0(G),$$

where $\sigma : G\to\operatorname{Aut}(C_0(G))$ denotes the *right translation action*. One checks that $(\Lambda^{\widehat{G}}_B, 1\otimes\rho)$ is a covariant representation of the dual system $(B\rtimes_\delta\widehat{G}, G, \widehat{\delta})$ on $\mathcal{M}(B\otimes\mathcal{K}(L^2(G)))$, and it follows from [32, Corollary 2.6] that the integrated form of this representation gives a surjective *-homomorphism

$$\Phi_B := \big((\mathrm{id}\otimes\lambda)\circ\delta\rtimes(1\otimes M)\big)\rtimes(1\otimes\rho) : B\rtimes_\delta\widehat{G}\rtimes_{\widehat{\delta}}G\twoheadrightarrow B\otimes\mathcal{K}(L^2(G)).$$

The map Φ_B might be called the *Katayama-duality map*. Now, following [17] a coaction (B,δ) is called *maximal* if the homomorphism Φ_B is an isomorphism.

At the other extreme, a coaction (B,δ) of G is called *normal* if the surjection Φ_B factors through an isomorphism

$$B\rtimes_\delta\widehat{G}\rtimes_{\widehat{\delta},r}G\cong B\otimes\mathcal{K}(L^2(G)).$$

It has been shown by Quigg [36] that every coaction (B,δ) has a *normalisation* (B_n,δ_n), which can be constructed by passing from B to the quotient $B_n := B/\ker\delta^r$. In particular, it follows that (B,δ) is normal if and only if its reduction δ^r is injective. On the other hand, it has been shown in

[17] that every coaction also has a *maximalisation* (B_m, δ_m) such that there exist $\widehat{G}$-equivariant surjections $B_m \twoheadrightarrow B \twoheadrightarrow B_n$ which induce isomorphisms between the respective coaction-crossed products.

For later use we need the construction of the maximalisation and normalisation of (B, δ) as given in [9], using the notion of generalised fixed-point algebras for weakly proper actions. In what follows let us write $(j_B, j_{C_0(G)})$ for the covariant representation $(\delta^r, 1 \otimes M)$ when viewed as a representation in $\mathcal{M}(B \rtimes_\delta \widehat{G})$. It is then clear that $j_{C_0(G)} : C_0(G) \to \mathcal{M}(B \rtimes_\delta \widehat{G})$ is a nondegenerate σ-$\widehat{\delta}$-equivariant *-homomorphism which gives $(B \rtimes_\delta \widehat{G}, \widehat{\delta})$ the structure of a weakly proper $G \rtimes G$-algebra in the sense of [9]. For simpler notation let us write $A := B \rtimes_\delta \widehat{G}$ and $\varphi \cdot m := j_B(\varphi)m$ and $m \cdot \varphi := m j_B(\varphi)$ for all $m \in \mathcal{M}(A)$, $\varphi \in C_0(G)$. Moreover, let $A_c := C_c(G) \cdot A \cdot C_c(G)$, which is a dense *-subalgebra of A, and let

$$A_c^G := \{m \in \mathcal{M}(A)^G : m \cdot \varphi, \varphi \cdot m \in A_c \ \forall \varphi \in C_c(G)\}, \tag{2.2}$$

where $\mathcal{M}(A)^G$ denotes the set of fixed points in $\mathcal{M}(A)$ for the extended action $\widehat{\delta}$. We call A_c^G the *generalised fixed-point algebra with compact supports*. Following ideas of Rieffel [38, 39], it has then been shown in [9, Proposition 2.2] that $\mathcal{F}_c(A) := C_c(G) \cdot A$ can be made into a pre-Hilbert $C_c(G, A)$-module by defining a $C_c(G, A)$-valued inner product on $\mathcal{F}_c(A)$ and a right action of $C_c(G, A)$ on $\mathcal{F}_c(A)$ by

$$\langle \xi, \eta \rangle_{C_c(G,A)} = [s \mapsto \Delta_G(s)^{-1/2} \xi^* \widehat{\delta}_s(\eta)], \qquad \xi \cdot \varphi = \int_G \Delta(t)^{-1/2} \alpha_t(\xi \varphi(t^{-1}))\, dt$$

for $\xi, \eta \in \mathcal{F}_c(A)$ and $\varphi \in C_c(G, A)$.

Let $A \rtimes_{\widehat{\delta},\mu} G$ be any C^*-completion of $C_c(G, A)$ with respect to a C^*-norm $\|\cdot\|_\mu$ on $C_c(G, A)$ such that $\|\cdot\|_u \geq \|\cdot\|_\mu \geq \|\cdot\|_r$, where $\|\cdot\|_u$ and $\|\cdot\|_r$ denote the universal (i.e., maximal) and the reduced norm on $C_c(G, A)$, respectively. Then the above defined inner product takes values in $A \rtimes_{\widehat{\delta},\mu} G$, and the completion $\mathcal{F}_\mu(A)$ of $\mathcal{F}_c(A)$ with respect to this inner product becomes a full $A \rtimes_{\widehat{\delta},\mu} G$-Hilbert module (the module is full since the translation action of G on itself is free and proper). Now, if we define a left action of A_c^G on $\mathcal{F}_c(A)$ by taking products inside $\mathcal{M}(A)$ (where both spaces are located), this action extends to a faithful *-homomorphism $\Psi_\mu : A_c^G \to \mathcal{K}(\mathcal{F}_\mu(A))$ with dense image. Hence $A_\mu^G := \mathcal{K}(\mathcal{F}_\mu(A))$ can be viewed as the completion of A_c^G with respect to the operator norm for the left action of A_c^G on $\mathcal{F}_\mu(A)$. In particular, $\mathcal{F}_\mu(A)$ becomes a A_μ^G-$A \rtimes_{\widehat{\delta},\mu} G$-equivalence bimodule.

Moreover, if the dual coaction on $A \rtimes_{\widehat{\delta}} G$ factors through a coaction on $A \rtimes_{\widehat{\delta},\mu} G$ (a property which depends on the norm $\|\cdot\|_\mu$), it is shown in [9, Theorem 4.6] that there are canonical coactions $\delta_{A_\mu^G}$ and $\delta_{\mathcal{F}_\mu(A)}$ of G on A_μ^G and $\mathcal{F}_\mu(A)$, respectively, such that $(\mathcal{F}_\mu(A), \delta_{\mathcal{F}_\mu(A)})$ becomes a

$\widehat{G}$-equivariant Morita equivalence between $(A^G_\mu, \delta_{A^G_\mu})$ and $(A \rtimes_{\widehat{\delta},\mu} G, \widehat{\widehat{\delta}})$. It is also shown in [9, Lemma 4.8] that there exists a unique crossed-product norm $\|\cdot\|_\mu$ on $C_c(G,A)$ such that $(A^G_\mu, \delta_{A^G_\mu})$ is isomorphic to the original coaction (B,δ). Moreover, if $\|\cdot\|_\mu = \|\cdot\|_u$ is the universal norm on $C_c(G,A)$, then the corresponding system $(B_m,\delta_m) := (A^G_u, \delta_{A^G_u})$ is a maximalisation for (B,δ), and if $\|\cdot\|_\mu = \|\cdot\|_r$ is the reduced norm, then $(B_n,\delta_n) := (A^G_r, \delta_{A^G_r})$ is a normalisation of (B,δ). Identifying (B,δ) with $(A^G_\mu, \delta_{A^G_\mu})$ as above, we see that the identity map on A^G_c induces the $\widehat{G}$-equivariant surjections $B_m \twoheadrightarrow B \twoheadrightarrow B_n$ which induce isomorphisms of crossed products

$$B_m \rtimes_{\delta_m} \widehat{G} \cong B \rtimes_\delta \widehat{G} \cong B_n \rtimes_{\delta_n} \widehat{G}.$$

3. The main result. Assume that $p : \mathcal{B} \to G$ is a Fell bundle over a locally compact group G. Then there is a canonical coaction

$$\delta_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{M}(C^*(\mathcal{B}) \otimes C^*(G)),$$

called the *dual coaction* of G on $C^*(\mathcal{B})$, given as the integrated form of the *-representation $\delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}(C^*(\mathcal{B}) \otimes C^*(G))$ which sends $b_t \in B_t$ to $\iota_{\mathcal{B}}(b_t) \otimes u_G(t)$, where $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}(C^*(\mathcal{B}))$ is the universal representation of $\mathcal{B}$ and $u_G : G \to U\mathcal{M}(C^*(G))$ is the universal representation of G.

The name "dual coaction" for the coaction $\delta_{\mathcal{B}}$ is motivated by the fact that it extends the well-known construction of dual coactions $\widehat{\alpha} : A \rtimes_\alpha G \to \mathcal{M}(A \rtimes_\alpha G \otimes C^*(G))$ which is given as the integrated form of the covariant homomorphism $(i_A \otimes 1, i_G \otimes u_G)$, where (i_A, i_G) denotes the universal covariant representation of (A, G, α) into $\mathcal{M}(A \rtimes_\alpha G)$. Indeed, as explained in Section 2.2, if $\mathcal{B} = A \rtimes^\alpha G$ is the Fell bundle associated to the G-C^*-algebra (A,α), this follows from the correspondence between the universal representation $\iota_{A \rtimes^\alpha G}$ of $A \rtimes^\alpha G$ and the universal representation (i_A, i_G) of (A, G, α).

THEOREM 3.1. *Let $\mathcal{B}$ be a Fell bundle over a locally compact group G. Then the dual coaction $\delta_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{M}(C^*(\mathcal{B}) \otimes C^*(G))$ is maximal.*

Let $(B,\delta) := (C^*(\mathcal{B}), \delta_{\mathcal{B}})$ and let (B_m,δ_m) be the maximalisation of (B,δ) as constructed from (B,δ) in the previous section. We will show that there exists a δ-δ_m-equivariant surjection $\Psi : B \twoheadrightarrow B_m$ which induces an isomorphism of crossed products $B \rtimes_\delta \widehat{G} \cong B_m \rtimes_{\delta_m} \widehat{G}$. The result will then follow from the following easy lemma, which should be well known to experts:

LEMMA 3.2. *Let (B,δ) and (B_m,δ_m) be coactions of G with (B_m,δ_m) maximal. Suppose that $\Psi : B \twoheadrightarrow B_m$ is a δ-δ_m-equivariant surjection which induces an isomorphism of crossed products. Then Ψ is an isomorphism and (B,δ) is maximal as well.*

Proof. Since $\Psi : B \twoheadrightarrow B_m$ is δ-δ_m-equivariant, we obtain a commutative diagram

$$\begin{array}{ccc} B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} G & \xrightarrow{\Phi_B} & B \otimes \mathcal{K}(L^2(G)) \\ {\scriptstyle \Psi \rtimes \widehat{G} \rtimes G}\downarrow & & \downarrow{\scriptstyle \Psi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}} \\ B_m \rtimes_{\delta_m} \widehat{G} \rtimes_{\widehat{\delta}_m} G & \xrightarrow{\Phi_{B_m}} & B_m \otimes \mathcal{K}(L^2(G)) \end{array}$$

By our assumptions, the left vertical and the lower horizontal arrows are isomorphisms. It then follows that the upper horizontal arrow has to be injective. Since it is always surjective, it is an isomorphism. Hence (B, δ) is maximal. Moreover, it follows that the right vertical arrow is an isomorphism, which then implies that $\Psi : B \to B_m$ is an isomorphism as well. ∎

Proof of Theorem 3.1. Let $(B, \delta) := (C^*(\mathcal{B}), \delta_{\mathcal{B}})$ and let

$$A := B \rtimes_\delta \widehat{G} = \overline{\operatorname{span}}\big\{\delta^r(B)\big(1 \otimes M(C_0(G))\big)\big\}.$$

As explained in the previous section, we view A as a weakly proper $G \rtimes G$-algebra. Then, as explained above, the maximalisation of (B, δ) is given by the coaction $(B_m, \delta_m) = (A_u^G, \delta_{A_u^G})$ where A_u^G denotes the universal generalised fixed-point algebra of A. We will show that the restriction of δ^r to $C_c(\mathcal{B})$ defines a *-homomorphism $\Psi : C_c(\mathcal{B}) \to A_c^G \subseteq \mathcal{M}(A)$ which extends to the desired δ-δ_m-equivariant surjective *-homomorphism $\Psi : C^*(\mathcal{B}) \twoheadrightarrow A_u^G$. First of all, it follows directly from the definition of the dual action $\widehat{\delta}$ that $\delta^r(B)$ lies in the fixed-point algebra $\mathcal{M}(A)^G$. To see that it sends $C_c(\mathcal{B})$ into the generalised fixed-point algebra A_c^G with compact supports it suffices to show that all elements of the form $\delta^r(b)(1 \otimes M(f)), (1 \otimes M(f))\delta^r(b)$ lie in $A_c = C_c(G) \cdot A \cdot C_c(G)$ for all $b \in C_c(\mathcal{B})$ and $f \in C_c(G)$. For this we first note that $\delta^r = (1 \otimes \lambda) \circ \delta_{\mathcal{B}}$ is the integrated form of the representation $\delta^r : \mathcal{B} \to \mathcal{M}(C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G)))$, $b_t \mapsto \iota_{\mathcal{B}}(b_t) \otimes \lambda(t)$. Suppose now that $b \in C_c(\mathcal{B})$ is a continuous section with compact support $K = \operatorname{supp}(b)$. Then, if $f \in C_c(G)$ is fixed, we may choose a function $g \in C_c(G)$ such that $g \equiv 1$ on $K \cdot \operatorname{supp}(f) \cup \operatorname{supp}(f)$. Hence for $a = \delta^r(b)(1 \otimes M(f))$ we clearly have $a \cdot g = \delta^r(b)(1 \otimes M(fg)) = \delta^r(b)(1 \otimes M(f)) = a$. On the other hand, using $\lambda(t)M(g)\lambda(t)^{-1} = M(\tau_t(g))$, where $\tau : G \to \operatorname{Aut}(C_0(G))$ denotes the left translation action of G on itself, we compute

$$\begin{aligned} g \cdot a &= (1 \otimes M(g))\delta^r(b)(1 \otimes M(f)) \\ &= \int_K (1 \otimes M(g))(\iota_{\mathcal{B}}(b_t) \otimes \lambda(t))(1 \otimes M(f))\, dt \\ &= \int_K (\iota_{\mathcal{B}}(b_t) \otimes \lambda(t))(1 \otimes M(\tau_{t^{-1}}(g)f))\, dt \\ &= \int_K (\iota_{\mathcal{B}}(b_t) \otimes \lambda(t))(1 \otimes M(f))\, dt = a \end{aligned} \tag{3.1}$$

since for $t \in K$ and $s \in \operatorname{supp}(f)$ we have $\tau_{t^{-1}}(g)(s) = g(ts) = 1$ because $ts \in K \cdot \operatorname{supp}(f)$. This proves that $\delta^r(C_c(\mathcal{B}))(1 \otimes M(f))$ lies in A_c, and a similar argument gives $(1 \otimes M(f))\delta^r(b) \in A_c$.

Now we need to show that $\delta^r : C_c(\mathcal{B}) \to A_c^G$ extends to an equivariant surjective *-homomorphism $\Psi : C^*(\mathcal{B}) \to A_u^G = \mathcal{K}(\mathcal{F}_u(A))$. For this we need to recall from [9, Definition 2.6] the notion of convergence in the inductive limit topology on the spaces $A_c = C_c(G) \cdot A \cdot C_c(G)$, $\mathcal{F}_c(A) = C_c(G) \cdot A$ and A_c^G, respectively. First of all, a net $(\xi_j)_{j\in J}$ in $\mathcal{F}_c(A)$ (resp. A_c) converges to $\xi \in \mathcal{F}_c(A)$ (resp. $\xi \in A_c$) in the inductive limit topology if $\xi_j \to \xi$ in the norm topology of A and there exists a function $g \in C_c(G)$ such that $\xi = g \cdot \xi$, $\xi_j = g \cdot \xi_j$ (resp. $\xi = g \cdot \xi \cdot g$, $\xi_j = g \cdot \xi_j \cdot g$) for all $j \in J$. For A_c^G, a net $(m_j)_{j\in J}$ in A_c^G converges to $m \in A_c^G$ in the inductive limit topology if for all $f \in C_c(G)$ we have $f \cdot m_j \to f \cdot m$ and $m_j \cdot f \to m \cdot f$ in the inductive limit topology of A_c (the fact that G/G is a one-point set implies that this definition coincides with the one given in [9, Definition 2.6]). Now it is shown in [9, Lemma 2.7] that all pairings in the A_c^G-$C_c(G, A)$-pre-imprimitivity bimodule $\mathcal{F}_c(A)$ are jointly continuous with respect to the inductive limit topologies, where on $C_c(G, A)$ we use the usual notion of inductive limit convergence. Since inductive limit convergence in $C_c(G, A)$ is stronger than norm convergence with respect to any given L^1-bounded C^*-norm $\|\cdot\|_\mu$ on $C_c(G, A)$, it follows from this that the inductive limit topology on A_c^G is stronger than any norm topology induced on A_c^G via the left action on the $A \rtimes_{\widehat{\delta},\mu} G$-Hilbert module $\mathcal{F}_\mu(A)$. In particular, inductive limit convergence in A_c^G implies norm convergence in A_u^G.

Assume now that $(b_j)_{j\in J}$ is a net in $C_c(\mathcal{B})$ which converges to some $b \in C_c(\mathcal{B})$ in the inductive limit topology of $C_c(\mathcal{B})$ in the sense that $b_j \to b$ uniformly on G and that all b_j have supports in a fixed compact subset K of G. Then the computation in (3.1) can easily be modified to show that $\delta^r(b_j) \to \delta^r(b)$ in the inductive limit topology of A_c^G. Thus $\delta^r(b_j) \to \delta^r(b)$ in the universal completion A_u^G. We therefore obtain a representation $\Psi : C_c(\mathcal{B}) \to A_u^G$ which is continuous for the inductive limit topology on $C_c(\mathcal{B})$. But then Proposition 2.1 implies that Ψ extends to a *-homomorphism $\Psi : C^*(\mathcal{B}) \to A_u^G$.

To see that the image is dense, we first show that

$$\mathcal{E}_c := \operatorname{span}\Big(\delta^r(C_c(\mathcal{B}))\big(1 \otimes M(C_c(G))\big)\Big) \subseteq A_c$$

is inductive limit dense in A_c. Since $\mathcal{E}_c$ is norm dense in A, it is clear that

$$\operatorname{span}\Big(\big(1 \otimes M(C_c(G))\big)\delta^r(C_c(\mathcal{B}))\big(1 \otimes M(C_c(G))\big)\Big)$$

is inductive limit dense in A_c. Hence it suffices to show that every element of the form $(1 \otimes M(g))\delta^r(b)(1 \otimes M(f))$ with $f, g \in C_c(G)$ and $b \in C_c(\mathcal{B})$ can

be inductive limit approximated by elements in $\mathcal{E}_c$. By (3.1) we know that

$$(1 \otimes M(g))\delta^r(b)(1 \otimes M(f)) = \int_K (\iota_{\mathcal{B}}(b_t) \otimes \lambda(t))\big(1 \otimes M(\tau_{t^{-1}}(g)f)\big)\, dt,$$

where $K = \operatorname{supp}(b)$. Now, for each $\varepsilon > 0$ and $t \in K$ we find a neighbourhood V_t of t in G such that $\|\tau_{s^{-1}}(g)f - \tau_{t^{-1}}(g)f\|_\infty < \varepsilon$ for all $s \in V_t$. Let $t_1, \dots, t_n$ be given such that $K \subseteq \bigcup_{l=1}^n V_{t_l}$, let $\varphi_1, \dots, \varphi_n$ be a partition of unity for K with $\operatorname{supp}\varphi_l \subseteq V_{t_l}$ for $1 \le l \le n$, and let $b_l := \varphi_l \cdot b$ (pointwise product). Then $a := \sum_{l=1}^n \delta^r(b_l)(1 \otimes M(\tau_{t_l^{-1}}(g)f)) \in \mathcal{E}_c$ satisfies

$$\begin{aligned}
&\Big\|(1 \otimes M(g))\delta^r(b)(1 \otimes M(f)) - \sum_{l=1}^n \delta^r(b_l)\big(1 \otimes M(\tau_{t_l^{-1}}(g)f)\big)\Big\| \\
&\qquad = \Big\| \int_K (\iota_{\mathcal{B}}(b_s) \otimes \lambda(s))\big(1 \otimes M(\tau_{s^{-1}}(g)f)\big) \\
&\qquad\qquad - \sum_{l=1}^n \big(\iota_{\mathcal{B}}(\varphi_l(s)b_s) \otimes \lambda(s)\big)\big(1 \otimes M(\tau_{t_l^{-1}}(g)f)\big)\, dt \Big\| \\
&\qquad \le \int_K \sum_{i=1}^n \varphi_l(s)\|\iota_{\mathcal{B}}(b_s)\|\, \|\tau_{s^{-1}}(g)f - \tau_{t_l^{-1}}(g)f\|_\infty\, dt \le \varepsilon\mu(K)\|b\|_\infty,
\end{aligned}$$

where $\mu(K)$ denotes the Haar measure of K. One checks as before that for any function $\varphi \in C_c(G)$ with $\varphi \equiv 1$ on $\operatorname{supp}(f) \cup K \cdot \operatorname{supp}(f)$ we have $\varphi \cdot a = a \cdot \varphi = a$, which this time shows that $\mathcal{E}_c$ is inductive limit dense in A_c.

Recall now from [9, Lemma 2.3] that there is a surjective linear map $\mathbb{E} : A_c \to A_c^G$ given by the equation $\mathbb{E}(a)c = \int_G \widehat{\delta}_t(a)c\, dt$ for all $a, c \in A_c$, such that for all $m \in A_c^G$ and $f \in C_c(G)$ we have $\mathbb{E}(m \cdot f) = \mathbb{E}(m)\mathbb{E}_\tau(f)$ with $\mathbb{E}_\tau(f) := \int_G f(t)\, dt$. For $m = \delta^r(b)$ with $b \in C_c(\mathcal{B})$ we get $m \cdot f = \delta^r(b)(1 \otimes M(f))$ and it follows that $\mathbb{E}(\mathcal{E}_c) = \delta^r(C_c(\mathcal{B}))$. A slight adaptation of the last part of the proof of [9, Lemma 2.7] shows that $\mathbb{E} : A_c \to A_c^G$ is continuous for the inductive limit topologies. Therefore, since $\mathcal{E}_c$ is inductive limit dense in A_c it now follows that $\delta^r(C_c(\mathcal{B})) = \mathbb{E}(\mathcal{E}_c)$ is inductive limit dense in A_c^G, hence norm dense in A_u^G. Therefore $\delta^r : C_c(\mathcal{B}) \to A_c^G$ extends to a surjective *-homomorphism $\Psi : C^*(\mathcal{B}) \to A_u^G$.

We now check that Ψ is equivariant with respect to the dual coaction on $C^*(\mathcal{B})$ and the coaction $\delta_{A_u^G}$ on A_u^G as defined in [9] on the dense subspace A_c^G by the formula

$$\delta_{A_u^G}(m) = (\phi \otimes \mathrm{id})(w_G)(m \otimes 1)(\phi \otimes \mathrm{id})(w_G)^*$$

where $w_G \in \mathcal{M}(C_0(G) \otimes C^*(G))$ is the unitary given by the function $t \mapsto u_G(t)$ and $\phi = 1 \otimes M : C_0(G) \to \mathcal{M}(A)$. Recall that the equivari-

ance of Ψ means that

$$\delta_{A_u^G}(\Psi(b)) = (\Psi \otimes \mathrm{id})(\delta_{\mathcal{B}}(b)) \quad \forall b \in C_c(\mathcal{B}). \tag{3.2}$$

Using $\Psi = \delta^r$ on $C_c(\mathcal{B})$, we see that the right-hand side is given by

$$(\Psi \otimes \mathrm{id})(\delta_{\mathcal{B}}(b)) = (\Psi \otimes \mathrm{id})\Big(\int_G \iota_{\mathcal{B}}(b_t) \otimes u_G(t)\, dt\Big) = \int_G \iota_{\mathcal{B}}(b_t) \otimes \lambda(t) \otimes u_G(t)\, dt.$$

To compare this with the left-hand side, observe that since $\phi = 1 \otimes M$, we have $(\phi \otimes \mathrm{id})(w_G) = 1 \otimes \tilde{w}_G$, where $\tilde{w}_G := (M \otimes \mathrm{id})(w_G) \in \mathcal{M}(\mathcal{K}(L^2(G)) \otimes C^*(G)) = \mathcal{L}(L^2(G, C^*(G)))$ is the unitary given by the formula $\tilde{w}_G\zeta(t) = u_G(t)\zeta(t)$ for all $\zeta \in C_c(G, C^*(G)) \subseteq L^2(G, C^*(G))$ (we view $L^2(G, C^*(G)) = L^2(G) \otimes C^*(G)$ as a Hilbert module over $C^*(G)$ and write $\mathcal{L}(L^2(G, C^*(G)))$ for the C^*-algebra of adjointable operators on it). It follows that

$$\begin{aligned}
\delta_{A_u^G}(\Psi(b)) &= (\phi \otimes \mathrm{id})(w_G)(\Psi(b) \otimes 1)(\phi \otimes \mathrm{id})(w_G)^* \\
&= (1 \otimes \tilde{w}_G)\Big(\int_G \iota_{\mathcal{B}}(b_t) \otimes \lambda(t) \otimes 1\, dt\Big)(1 \otimes \tilde{w}_G)^* \\
&= \int_G \iota_{\mathcal{B}}(b_t) \otimes \tilde{w}_G(\lambda(t) \otimes 1)\tilde{w}_G^*\, dt.
\end{aligned}$$

Now a simple computation shows that $\tilde{w}_G(\lambda(t) \otimes 1)\tilde{w}_G^* = \lambda(t) \otimes u_G(t)$, which implies (3.2).

To finish the proof we only need to check that Ψ induces an isomorphism $\Psi \rtimes \widehat{G} : C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \xrightarrow{\sim} A_u^G \rtimes_{\delta_{A_u^G}} \widehat{G}$. But for every coaction $\delta : B \to \mathcal{M}(B \otimes C^*(G))$ it is known that the image of $\delta^r : B \to \mathcal{M}(B \rtimes_\delta \widehat{G})$ is the reduced generalised fixed-point algebra A_r^G for the weakly proper $G \rtimes G$-algebra $A = B \rtimes_\delta \widehat{G}$ (endowed with the dual action and the canonical embedding $C_0(G) \to \mathcal{M}(A)$). A first reference for this fact is Quigg's original version of Landstad duality for coactions (see [37]). We have shown in [9] that A_r^G carries a coaction $\delta_{A_r^G}$ given on A_c^G by the same formula as $\delta_{A_u^G}$ and that $(A_r^G, \delta_{A_r^G})$ is the normalisation of (B, δ) where $\delta^r : B \to A_r^G$ serves as the normalisation map. This in particular means that δ^r induces an isomorphism $\delta^r \rtimes \widehat{G} : B \rtimes_\delta \widehat{G} \xrightarrow{\sim} A_r^G \rtimes_{\delta_{A_r^G}} \widehat{G}$. Now it is clear that the map $\Psi : C^*(\mathcal{B}) \to A_u^G$ constructed above composed with the normalisation map $\nu : A_u^G \to A_r^G$ (given by the identity map on A_c^G) is the canonical map $\delta^r : C^*(\mathcal{B}) \to A_r^G$. Hence the composition

$$C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \xrightarrow{\Psi \rtimes \widehat{G}} A_u^G \rtimes_{\delta_{A_u^G}} \widehat{G} \xrightarrow{\nu \rtimes \widehat{G}} A_r^G \rtimes_{\delta_{A_r^G}} \widehat{G}$$

is an isomorphism. Since $(A_r^G, \delta_{A_r^G})$ is also a normalisation for $(A_u^G, \delta_{A_u^G})$ and hence $\nu \rtimes \widehat{G} : A_u^G \rtimes_{\delta_{A_u^G}} \widehat{G} \to A_r^G \rtimes_{\delta_{A_r^G}} \widehat{G}$ is also an isomorphism, this implies the desired isomorphism $\Psi \rtimes \widehat{G} : C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \xrightarrow{\sim} A_u^G \rtimes_{\delta_{A_u^G}} \widehat{G}$. ∎

REMARK 3.3. The normalisation of $(C^*(\mathcal{B}), \delta_{\mathcal{B}})$ can be realised concretely as the *dual coaction* $\delta_{\mathcal{B},r} : C^*_r(\mathcal{B}) \to \mathcal{M}(C^*_r(\mathcal{B}) \otimes C^*(G))$ of G on $C^*_r(\mathcal{B})$, which is constructed as follows. Consider the regular representation $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ and view it as a representation $\Lambda_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}(C^*_r(\mathcal{B}))$ of $\mathcal{B}$. Now consider the tensor product representation $\Lambda_{\mathcal{B}} \otimes \lambda : \mathcal{B} \to \mathcal{M}(C^*_r(\mathcal{B}) \otimes C^*_r(G))$. By Fell's absorbtion theorem [24, Corollary 2.15], the integrated form of this representation factors faithfully through $C^*_r(\mathcal{B})$ and hence yields a faithful *-homomorphism $\Lambda_{\mathcal{B}} \otimes \lambda : C^*_r(\mathcal{B}) \to \mathcal{M}(C^*_r(\mathcal{B}) \otimes C^*_r(G))$. It is not difficult to check directly (see [24, Proposition 2.10] for details) that this is a reduced coaction (that is, an injective coaction of the Hopf-C^*-algebra $C^*_r(G)$), and therefore lifts to a normal coaction $\delta_{\mathcal{B},r} : C^*_r(\mathcal{B}) \to \mathcal{M}(C^*_r(\mathcal{B}) \otimes C^*(G))$. This is the desired normalisation of the dual coaction $\delta_{\mathcal{B}} : C^*(\mathcal{B}) \to \mathcal{M}(C^*(\mathcal{B}) \otimes C^*(G))$, with the regular representation $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ serving as the normalisation map (see [8, Proposition 6.9.8]).

4. Some applications. In this section we want to give some simple applications of our main Theorem 3.1.

4.1. Extension of exotic crossed-product functors. Recall from [10, 5] that an *exotic crossed-product functor* is a functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ from the category of G-C^*-algebras with G-equivariant *-homomorphisms to the category of C^*-algebras such that $A \rtimes_{\alpha,\mu} G$ is a C^*-completion of the convolution *-algebra $C_c(G, A)$ in such a way that the identity map $C_c(G, A) \to C_c(G, A)$ extends to surjective *-homomorphisms

$$A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,r} G.$$

Theorem 3.1 allows us to extend every Morita compatible G-crossed-product functor $\rtimes_\mu$ to the category of Fell bundles over G, that is, we can extend the definition of $\rtimes_\mu$ to the realm of Fell bundles over G in a natural and functorial way. Recall from [10] that a crossed product functor is called *Morita compatible* ([1]) if Morita equivalent actions are sent to (canonically) Morita equivalent crossed products. We refer to [10] for a detailed discussion of this property and for the stronger notion of a correspondence functor. As shown there, many crossed-product functors do have this property, and it follows from work of Okayasu [34] together with the papers [28, 10] that there are uncountably many different correspondence functors for any discrete group which contains the free group on two generators.

We shall show that if we start with a crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ for G, then for every Fell bundle $\mathcal{B}$ over G we can complete $C_c(\mathcal{B})$

([1]) Also called "strongly Morita compatible" in [10] to differentiate it from the formally weaker (but essentially equivalent) notion of Morita compatibility introduced in [5].

to a C^*-algebra $C^*_\mu(\mathcal{B})$ lying between $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$ in the sense that the identity map on $C_c(\mathcal{B})$ extends to surjections

$$C^*(\mathcal{B}) \twoheadrightarrow C^*_\mu(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$$

and so that the assignment $\mathcal{B} \to C^*_\mu(\mathcal{B})$ is a functor from the category of Fell bundles over G (with appropriate morphisms) to the category of C^*-algebras with *-homomorphisms as morphisms. We make this precise in what follows.

Definition 4.1. Given a crossed-product functor $\rtimes_\mu$ for a locally compact group G and a Fell bundle $\mathcal{B}$ over G, we define $C^*_\mu(\mathcal{B})$ as the unique quotient of $C^*(\mathcal{B})$ such that Katayama's duality map

$$\Phi_{\mathcal{B}} : C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \rtimes_{\widehat{\delta_{\mathcal{B}}}} G \xrightarrow{\sim} C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G))$$

factors through an isomorphism

$$C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \rtimes_{\widehat{\delta_{\mathcal{B}}},\mu} G \cong C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)).$$

Although the above construction makes sense for every crossed-product functor, as we will see, it will only give a completion $C^*_\mu(\mathcal{B})$ with good properties if we assume that the given functor $\rtimes_\mu$ has extra properties (for instance, Morita compatibility). We are specially interested in correspondence functors, where essentially all good properties are present (see [10]).

To make the construction $\mathcal{B} \mapsto C^*_\mu(\mathcal{B})$ into a functor, we need to introduce morphisms and turn Fell bundles over G into a category. As with C^*-algebras, there are several types of morphisms we can consider between Fell bundles, but the most basic one is defined as follows.

Definition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over G. By a morphism $\mathcal{A} \to \mathcal{B}$ we mean a continuous map $\pi : \mathcal{A} \to \mathcal{B}$ that maps each fibre A_t linearly into the fibre B_t and which is compatible with multiplication and involution in the sense that

$$\pi(a \cdot b) = \pi(a) \cdot \pi(b) \quad \text{and} \quad \pi(a)^* = \pi(a)^*$$

for all $a, b \in \mathcal{A}$.

A morphism $\pi : \mathcal{A} \to \mathcal{B}$ induces a map $\tilde{\pi} : C_c(\mathcal{A}) \to C_c(\mathcal{B})$, $\xi \mapsto \tilde{\pi}(\xi)(t) := \pi(\xi(t))$, which is clearly continuous with respect to the inductive limit topologies and hence extends to a *-homomorphism $\tilde{\pi}_u : C^*(\mathcal{A}) \to C^*(\mathcal{B})$. This shows that the construction $\mathcal{B} \mapsto C^*(\mathcal{B})$ is a functor. The following result shows that this remains true for the assignment $\mathcal{B} \mapsto C^*_\mu(\mathcal{B})$ as in Definition 4.1:

Proposition 4.3. *Let $\rtimes_\mu$ be any crossed-product functor. Then $\mathcal{B} \mapsto C^*_\mu(\mathcal{B})$ is a functor from the category of Fell bundles over G with morphisms as defined in Definition* 4.2 *in the sense that the canonical map $\tilde{\pi} : C_c(\mathcal{A}) \to C_c(\mathcal{B})$ induced from any morphism $\pi : \mathcal{A} \to \mathcal{B}$ extends to a *-homomorphism $\tilde{\pi} : C^*_\mu(\mathcal{A}) \to C^*_\mu(\mathcal{B})$.*

Proof. Consider the diagram

$$\begin{array}{ccc} C^*(\mathcal{A}) \rtimes_{\delta_\mathcal{A}} \widehat{G} \rtimes_{\widehat{\delta_\mathcal{A}}} G & \xrightarrow[\cong]{\Phi_\mathcal{A}} & C^*(\mathcal{A}) \otimes \mathcal{K}(L^2(G)) \\ \downarrow{\scriptstyle \tilde{\pi}_u \rtimes \widehat{G} \rtimes G} & & \downarrow{\scriptstyle \tilde{\pi}_u \otimes \mathrm{id}_\mathcal{K}} \\ C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta_\mathcal{B}}} G & \xrightarrow[\cong]{\Phi_\mathcal{B}} & C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G)) \end{array}$$

It follows easily from the definition of the dual coactions on $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$, respectively, that the morphism $\tilde{\pi}_u : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ is $\delta_\mathcal{A}$-$\delta_\mathcal{B}$-equivariant, which implies that the left vertical arrow exists. Moreover, using the fact that $\Phi_\mathcal{A}$ is given by the covariant homomorphism $((\mathrm{id} \otimes \lambda) \circ \delta_\mathcal{A} \rtimes (1 \otimes M)) \rtimes (1 \otimes \rho)$ (and similarly for $\Phi_\mathcal{B}$), the $\delta_\mathcal{A}$-$\delta_\mathcal{B}$-equivariance of $\tilde{\pi}_u$ also implies that the diagram commutes. Now, since $\rtimes_\mu$ is a crossed-product functor, the vertical arrow on the left factors through a *-homomorphism

$$\tilde{\pi}_u \rtimes \widehat{G} \rtimes_\mu G : C^*(\mathcal{A}) \rtimes_{\delta_\mathcal{A}} \widehat{G} \rtimes_{\widehat{\delta_\mathcal{A}},\mu} G \to C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta_\mathcal{B}},\mu} G$$

and hence the vertical arrow $\tilde{\pi}_u \otimes \mathrm{id}_\mathcal{K}$ on the right-hand side of the diagram must also factor through a well-defined homomorphism

$$(\tilde{\pi}_u \otimes \mathrm{id}_\mathcal{K})_\mu : C^*_\mu(\mathcal{A}) \otimes \mathcal{K}(L^2(G)) \to C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)).$$

But this is only possible if $\tilde{\pi}_u : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ factors through a homomorphism $\tilde{\pi}_\mu : C^*_\mu(\mathcal{A}) \to C^*_\mu(\mathcal{B})$, whence the result. ∎

The above proposition shows that given any crossed-product functor, the procedure given in Definition 4.1 determines a functor on the category of Fell bundles. But does it always extend the given functor if we apply the new functor to the semidirect product Fell bundle $A \rtimes^\alpha G$ associated to a given action $\alpha : G \to \mathrm{Aut}(A)$? Recall that the underlying topological space of $A \rtimes^\alpha G$ is the trivial bundle $A \times G$ with multiplication and involution defined by

$$(a,t)(b,s) = (a\alpha_t(b), ts) \quad \text{and} \quad (a,t)^* = (\alpha_{t^{-1}}(a)^*, t^{-1})$$

for $(a,t),(b,s) \in A \times G$. The notation $A \rtimes^\alpha G$ for this Fell bundle should not be mistaken with the notation for the universal crossed product $A \rtimes_\alpha G$. The following example shows that the answer to the above question is negative in general:

Example 4.4. Let G be any non-amenable group. We define a crossed-product functor $(A, G, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ by letting $A \rtimes_{\alpha,\mu} G$ be the completion of the convolution algebra $C_c(G, A)$ with respect to the C^*-norm

$$\|f\|_\mu = \sup\{\|\pi \rtimes U(f)\| : U \prec \lambda\},$$

where (π, U) runs through all covariant representations such that U *is weakly contained in* λ, which just means that the kernel of U in $C^*(G)$ contains the

kernel of λ in $C^*(G)$. The functor we get in this way is just the Brown–Guentner functor associated to the reduced group algebra $C_r^*(G)$ as discussed in [6, 5, 10].

We now consider the case of the trivial action of G on $\mathbb{C}$. Then the μ-crossed product $\mathbb{C} \rtimes_\mu G$ will just be the reduced group algebra $C_r^*(G)$ of G. On the other hand, the corresponding Fell bundle will just be the trivial bundle $\mathbb{C} \times G$, and the full cross-sectional algebra will be the full group algebra $C^*(G)$. The crossed product $C^*(G) \rtimes_{\delta_G} \widehat{G}$ by the dual coaction is isomorphic to the algebra of compact operators $\mathcal{K}(L^2(G))$ with faithful representation $\lambda \rtimes M : C^*(G) \rtimes_{\delta_G} \widehat{G} \to \mathbb{B}(L^2(G))$ (see e.g. [18, Example A.62]). A straightforward computation shows that in this picture the dual action $\widehat{\delta}_G : G \to \mathrm{Aut}(\mathcal{K}(L^2(G)))$ is given by $\widehat{\delta}_G(s) = \mathrm{Ad}\,\rho(s)$, where $\rho : G \to U(L^2(G))$ is the right regular representation. Hence, this action is implemented by a unitary representation and there is a *-isomorphism

$$\Phi : \mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho} G \xrightarrow{\cong} \mathcal{K}(L^2(G)) \rtimes_{\mathrm{id}} G \cong \mathcal{K}(L^2(G)) \otimes C^*(G)$$

which sends an element f in the dense subalgebra $C_c(G, \mathcal{K}(L^2(G)))$ to the function $[s \mapsto f(s)\rho(s)] \in C_c(G, \mathcal{K}(L^2(G)))$. The representations of $\mathcal{K}(L^2(G)) \otimes C^*(G)$ are all of the form $\mathrm{id}_\mathcal{K} \otimes V$, where $V : G \to U(\mathcal{H})$ is a unitary representation of G viewed as a representation of $C^*(G)$ via integration. The corresponding covariant representation of $(\mathcal{K}(L^2(G)), G, \mathrm{id})$ is given by the pair $(\mathrm{id}_\mathcal{K} \otimes 1_\mathcal{H}, 1_{L^2(G)} \otimes V)$ and it is easy to check that the representation of $\mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho} G$ corresponding to $(\mathrm{id}_\mathcal{K} \otimes 1_\mathcal{H}, 1_{L^2(G)} \otimes V)$ via the isomorphism Φ is given by the covariant pair $(\mathrm{id}_\mathcal{K} \otimes 1, \rho \otimes V)$. By Fell's trick we know that $\rho \otimes V$ is a multiple of ρ. Since ρ is unitarily equivalent to λ, we see that the unitary part of any representation of $\mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho} G$ is weakly equivalent to λ. But this implies that

$$\begin{aligned} C^*(G) \rtimes_{\delta_G} \widehat{G} \rtimes_{\widehat{\delta_G},\mu} G &\cong \mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho,\mu} G = \mathcal{K}(L^2(G)) \rtimes_{\mathrm{Ad}\,\rho} G \\ &\cong \mathcal{K}(L^2(G)) \otimes C^*(G). \end{aligned}$$

As a consequence we see that

$$C_\mu^*(\mathbb{C} \times G) \cong C^*(G) \ncong C_r^*(G) = \mathbb{C} \rtimes_\mu G.$$

Thus, in general, our procedure does not reproduce the given functor for actions when applied to semidirect product bundles $A \rtimes^\alpha G$.

The problem in the above example comes from the fact that the Brown–Guentner crossed-product functor associated to the reduced group algebra $C_r^*(G)$ is not Morita compatible in the sense discussed in [10]. Recall from [10] that a crossed-product functor is called *Morita compatible* if it preserves Morita equivalences in the following sense: If (X, γ) is a G-equivariant equivalence bimodule between two systems (A, G, α) and (B, G, β), and if

we make $C_c(G,X)$ into a $C_c(G,A)$-$C_c(G,B)$ pre-equivalence bimodule in the usual way, then there is a completion $X \rtimes_{\gamma,\mu} G$ of $C_c(G,X)$ which becomes an $A \rtimes_{\alpha,\mu} G$-$B \rtimes_{\beta,\mu} G$ equivalence bimodule. Note that the results of [10] show that Morita compatibility—or the even stronger assumption that $\rtimes_\mu$ is a correspondence functor—are quite reasonable to assume for a "well-behaved" crossed-product functor. Recall also from the discussions in [10] that for many non-amenable groups there exist uncountably many different correspondence (and hence Morita compatible) functors for G.

THEOREM 4.5. *Assume that $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is a Morita compatible crossed-product functor for G. Then there is a canonical isomorphism $C^*_\mu(A \rtimes^\alpha G) \cong A \rtimes_{\alpha,\mu} G$ for any G-algebra (A,α) given on the dense subalgebra $C_c(A \rtimes^\alpha G)$ by the canonical identification $C_c(A \rtimes^\alpha G) = C_c(G,A) \subseteq A \rtimes_{\alpha,\mu} G$. Thus, for Morita compatible crossed-product functors, the procedure of Definition* 4.1 *defines an extension of the functor $\rtimes_\mu$ to the category of Fell bundles.*

Proof. The result is a consequence of the Imai–Takai duality theorem for actions: Assume that $\alpha : G \to \mathrm{Aut}(A)$ is an action. Then the full crossed product $A \rtimes_\alpha G$ coincides with the full cross-sectional algebra $C^*(A \rtimes^\alpha G)$ since both are universal for covariant representations (see Section 2.2). Moreover, the dual coactions of G on $A \rtimes_\alpha G$ and $C^*(A \rtimes^\alpha G)$ coincide (see beginning of Section 3). Hence the Imai–Takai duality theorem shows that

$$C^*(A \rtimes^\alpha G) \rtimes_{\widehat\alpha} \widehat G = A \rtimes_\alpha G \rtimes_{\widehat\alpha} \widehat G \cong A \otimes \mathcal{K}(L^2(G)),$$

which is equivariant for the bi-dual action $\widehat{\widehat\alpha}$ on the left and the action $\alpha \otimes \mathrm{Ad}\,\rho$ on the right (see e.g. [26] or [18, Theorem A.67]). As already observed in the previous example, the action $\alpha \otimes \mathrm{Ad}\,\rho$ is Morita equivalent to $\alpha \otimes \mathrm{id}$ with respect to the equivariant bimodule $(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \rho)$. By [10, Corollary 5.4], it follows that the integrated form of the covariant homomorphism $(i_A^\mu \otimes \mathrm{id}_{\mathcal{K}}, i_G^\mu \otimes \rho)$ factors through an isomorphism

$$\Psi_\mu : \big(A \otimes \mathcal{K}(L^2(G))\big) \rtimes_{\alpha\otimes\mathrm{Ad}\,\rho,\mu} G \xrightarrow{\cong} (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}(L^2(G)),$$

where (i_A^μ, i_G^μ) denotes the canonical representation of (A,G,α) into the multiplier algebra $\mathcal{M}(A \rtimes_{\alpha,\mu} G)$. Altogether, we obtain an isomorphism

$$C^*(A \rtimes^\alpha G) \rtimes_{\widehat\alpha} \widehat G \rtimes_{\widehat{\widehat\alpha},\mu} G \cong (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}(L^2(G))$$

which fits into a commutative diagram

$$\begin{array}{ccc} C^*(A \rtimes^\alpha G) \rtimes_{\widehat\alpha} \widehat G \rtimes_{\widehat{\widehat\alpha}} G & \xrightarrow{\Psi_u} & (A \rtimes_\alpha G) \otimes \mathcal{K}(L^2(G)) \\ \big\downarrow & & \big\downarrow \\ C^*(A \rtimes^\alpha G) \rtimes_{\widehat\alpha} \widehat G \rtimes_{\widehat{\widehat\alpha},\mu} G & \xrightarrow{\Psi_\mu} & (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}(L^2(G)) \end{array}$$

where both vertical arrows are induced by the natural inclusion

$$C_c(A \rtimes^{\alpha} G) = C_c(G, A) \subseteq A \rtimes_{\alpha,\mu} G. \ \blacksquare$$

In particular the above result applies to *correspondence* crossed-product functors as introduced in [10]. This is a class of crossed-product functors which extend (in a suitable way) to the category of G-algebras with equivariant correspondences as their morphisms. The equivariant Morita equivalences are the isomorphisms in this category, so it is no surprise that these functors are Morita compatible. In [10, Theorem 4.9] a list of equivalent conditions is given in order to check whether a crossed-product functor is a correspondence functor. It is shown in [10] that correspondence functors have very nice properties. For instance, they behave very well with respect to K-theory, and we will explore this point in the next section. Here we want to use the fact, proven in [10, Theorem 5.6], that correspondence functors always admit dual coactions for ordinary crossed products, and deduce the following consequence:

COROLLARY 4.6. *Let $\rtimes_\mu$ be a correspondence crossed-product functor for G and let $\mathcal{B}$ be a Fell bundle over G. Then the dual coaction $\delta_{\mathcal{B}}$ on $C^*(\mathcal{B})$ factors through a coaction $\delta_{\mathcal{B},\mu} : C^*_\mu(\mathcal{B}) \to \mathcal{M}(C^*_\mu(\mathcal{B}) \otimes C^*(G))$, which we call the* dual μ-coaction. *The quotient maps $C^*(\mathcal{B}) \twoheadrightarrow C^*_\mu(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$ induce isomorphisms*

$$C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \xrightarrow{\sim} C^*_\mu(\mathcal{B}) \rtimes_{\delta_{\mathcal{B},\mu}} \widehat{G} \xrightarrow{\sim} C^*_r(\mathcal{B}) \rtimes_{\delta^r_{\mathcal{B}}} \widehat{G}. \tag{4.1}$$

Hence the dual μ-coaction satisfies μ-duality *in the sense that Katayama's map is an isomorphism*

$$C^*_\mu(\mathcal{B}) \rtimes_{\delta_{\mathcal{B},\mu}} \widehat{G} \rtimes_{\widehat{\delta}_{\mathcal{B},\mu}} G \xrightarrow{\sim} C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)). \tag{4.2}$$

This isomorphism sends the bidual coaction $\widehat{\widehat{\delta}}_{\mathcal{B},\mu}$ to the coaction $\mathrm{Ad}_W \circ (\delta_{\mathcal{B},\mu} \otimes_* \mathrm{id})$, *where $W = 1 \otimes w^*_G$, $w_G \in \mathcal{M}(C_0(G) \otimes C^*(G))$ is the fundamental unitary (which can be seen as the universal representation $t \mapsto u_G(t)$ of G), and $\delta_{\mathcal{B},\mu} \otimes_* \mathrm{id}$ denotes the obvious coaction $C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)) \to \mathcal{M}(C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)) \otimes C^*(G))$.*

Proof. By Theorem 3.1, Katayama's homomorphism is an isomorphism

$$C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \rtimes_{\widehat{\delta}_{\mathcal{B}}} G \xrightarrow{\sim} C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G)). \tag{4.3}$$

It is well known (see e.g. [17]) that the bidual coaction $\widehat{\widehat{\delta}}_{\mathcal{B}}$ on the left-hand side corresponds to the coaction $\mathrm{Ad}_W \circ (\delta_{\mathcal{B}} \otimes_* \mathrm{id})$ as in the statement. By definition, $C^*_\mu(\mathcal{B})$ is the quotient of $C^*(\mathcal{B})$ that turns (4.3) into an isomorphism

$$C^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}}} \widehat{G} \rtimes_{\widehat{\delta}_{\mathcal{B},\mu}} G \xrightarrow{\sim} C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)). \tag{4.4}$$

Since $\rtimes_\mu$ is a correspondence functor, the left-hand side carries a (bi)dual

coaction $\widehat{\widehat{\delta}}_{\mathcal{B},\mu}$ (by [10, Theorem 5.6]). More precisely, the bidual coaction on the full crossed product $C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta}_\mathcal{B}} G$ factors through the coaction $\widehat{\widehat{\delta}}_{\mathcal{B},\mu}$. It follows that the coaction $\mathrm{Ad}_W \circ (\delta_\mathcal{B} \otimes_* \mathrm{id})$ also factors through a coaction on $C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G))$ of the form $\mathrm{Ad}_W \circ (\delta_{\mathcal{B},\mu} \otimes_* \mathrm{id})$, where $\delta_{\mathcal{B},\mu}$ is a coaction of $C^*_\mu(\mathcal{B})$ which factors the dual coaction $\delta_\mathcal{B}$ on $C^*(\mathcal{B})$. This holds in particular for the reduced cross-sectional algebra $C^*_r(\mathcal{B})$ and, as already observed in Remark 3.3, in this case the coaction $\delta_{\mathcal{B},r}$ is a normalisation of $\delta_\mathcal{B}$. In particular the quotient homomorphism $C^*(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$ (which is the regular representation of $\mathcal{B}$) induces an isomorphism $C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \xrightarrow{\sim} C^*_r(\mathcal{B}) \rtimes_{\delta_{\mathcal{B},r}} \widehat{G}$. It follows that the same is true for every other exotic quotient $C^*_\mu(\mathcal{B})$ because the quotient map $C^*(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$ (and hence also the induced map on crossed products) factors as a composition $C^*(\mathcal{B}) \twoheadrightarrow C^*_\mu(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$. This implies the isomorphism (4.1), and the isomorphism (4.2) is then just a reinterpretation of the defining isomorphism (4.4). ∎

4.2. Partial actions. The notion of partial actions of the group of integers has been introduced by Exel [20] and subsequently generalised to arbitrary discrete groups by McClanahan in [31]. In [21] Exel generalises both notions and defines twisted partial actions of locally compact groups. Every twisted partial action gives rise to a Fell bundle via a construction similar to the semidirect Fell bundle associated to an ordinary (global, untwisted) action. Moreover, the main result, Theorem 7.3, of [21] shows that for second countable G, after stabilisation, every separable Fell bundle is isomorphic to one of this form, that is, a Fell bundle associated to a twisted partial action (and for discrete groups or saturated Fell bundles the twist can be removed; see [23, 40, 35, 16]). In this section, we will focus only on partial actions, but essentially all results go through with essentially no change (except that the notation becomes slightly more complicated) for general twisted partial actions.

Let α be a partial action of a locally compact group G on a C^*-algebra A. This consists of partial automorphisms $\alpha_t : D_{t^{-1}} \to D_t$ between certain (closed, two-sided) ideals $D_t \subseteq A$ with $D_e = A$, $\alpha_e = \mathrm{id}_A$ and such that α_{st} extends $\alpha_s \circ \alpha_t$ for all $s, t \in G$. An appropriate continuity condition for the family of maps α_t is also required to hold. We refer the reader to [21] for details. Given such a partial action, the associated Fell bundle is the bundle $A \rtimes^\alpha G := \{(a,t) \in A \times G : a \in D_t\}$ with algebraic operations defined by

$$(a,t) \cdot (b,s) = (\alpha_t(\alpha_{t^{-1}}(a)b), ts) \quad \text{and} \quad (a,t)^* = (\alpha_{t^{-1}}(a)^*, t^{-1}).$$

The full (resp. reduced) crossed product of (A,α) can defined as the full (resp. reduced) cross-sectional C^*-algebra of $A \rtimes^\alpha G$. More generally, we can introduce:

DEFINITION 4.7. Given a partial action (A,α) of G and a Morita compatible G-crossed-product functor $\rtimes_\mu$, we define the *μ-crossed product* $A\rtimes_{\alpha,\mu}G$ as the μ-cross-sectional C^*-algebra $C^*_\mu(A\rtimes^\alpha G)$ (as in Definition 4.1).

Notice that by Theorem 4.5 (and our assumption of Morita compatibility) the above definition recovers the original μ-crossed product for global actions. Also, Proposition 4.3 allows us to extend the original crossed-product functor to a functor $(A,\alpha)\mapsto A\rtimes_{\alpha,\mu}G$ from the category of partial G-actions to the category of C^*-algebras, where a morphism between two partial G-actions (A,α) and (B,β) is defined as a G-equivariant *-homomorphism $\pi:A\to B$, meaning that $\pi(D^\alpha_t)\subseteq D^\beta_t$ and $\beta_t(\pi(a))=\pi(\alpha_t(a))$ for all $t\in G$ and $a\in D^\alpha_{t^{-1}}$. These are exactly the morphisms between the associated semidirect Fell bundles $A\rtimes^\alpha G$ and $B\rtimes^\beta G$.

Given a partial action (A,α), we view the dual coaction on the full cross-sectional algebra $C^*(A\rtimes^\alpha G)$ as the *dual coaction* of $A\rtimes_\alpha G$ and denote it by $\widehat{\alpha}$. Our main result (Theorem 3.1) says that we have a natural isomorphism

$$(A\rtimes_\alpha G)\rtimes_{\widehat{\alpha}}\widehat{G}\rtimes_{\widehat{\widehat{\alpha}}}G\cong(A\rtimes_\alpha G)\otimes\mathcal{K}(L^2(G)). \tag{4.5}$$

In particular this implies that, after tensoring with $\mathcal{K}(L^2(G))$, every partial crossed product is isomorphic to a global crossed product. More precisely, the tensor product $(A\rtimes_\alpha G)\otimes\mathcal{K}(L^2(G))$ is naturally isomorphic to $\tilde{A}\rtimes_{\tilde{\alpha}}G$, where $\tilde{A}:=(A\rtimes_\alpha G)\rtimes_{\widehat{\alpha}}\widehat{G}$ and $\tilde{\alpha}:=\widehat{\widehat{\alpha}}$. The G-algebra $(\tilde{A},\tilde{\alpha})$ has a natural interpretation in terms of the original partial action: it follows from [1, Proposition 8.1] that $(\tilde{A},\tilde{\alpha})$ is a *Morita enveloping action* for the partial action (A,α); we call $(\tilde{A},\tilde{\alpha})$ the *canonical Morita enveloping action* of (A,α). It is shown in [1, Proposition 6.3] that all Morita enveloping actions are Morita equivalent, so it is unique up to Morita equivalence. We refer to [1] for the relevant notion of Morita equivalence for partial actions.

Let us recall that the assertion that $(\tilde{A},\tilde{\alpha})$ is a Morita enveloping action of (A,α) means that $\tilde{A}$ contains a (closed, two-sided) ideal $I\subseteq\tilde{A}$ such that the orbit $\{\tilde{\alpha}_t(I):t\in G\}$ of I generates a dense subspace of $\tilde{A}$ and such that the partial action on I given by restriction of $\tilde{\alpha}$ (as described in [1]) is Morita equivalent to the original partial action (A,α). For the canonical Morita enveloping action, the ideal I and also $\tilde{A}$ (together with their actions) can be described directly in terms of the Fell bundle $A\rtimes^\alpha G$ associated to (A,α) as certain algebras of "kernels", but this description does not concern us here. We refer to [1] for further details.

Thus the natural isomorphism (4.5) (that is, our main Theorem 3.1 applied for partial actions) can be seen as the statement that every partial crossed product $A\rtimes_\alpha G$ is naturally Morita equivalent to its canonical Morita

enveloping crossed product. As already mentioned above, it is proven in [1] that all Morita enveloping actions of a given partial action (A,α) are Morita equivalent. It follows that the full crossed product $A \rtimes_\alpha G$ is Morita equivalent to the full crossed product of any of the Morita enveloping actions of (A,α). Note that the paper [1] by Abadie only contained a version of this result for the reduced crossed products (see [1, Proposition 4.6]). The version for full crossed products was obtained in a second paper [2, Corollary 1.3] by Abadie and Martí Pérez. We now use our approach to generalise these results to all exotic crossed products related to Morita compatible crossed-product functors:

COROLLARY 4.8. *For every partial action (A,α) of G and for every Morita compatible crossed-product functor $\rtimes_\mu$, the partial crossed product $A \rtimes_{\alpha,\mu} G$ is Morita equivalent to its canonical Morita enveloping crossed product $\tilde{A} \rtimes_{\tilde{\alpha},\mu} G$. More precisely, the canonical isomorphism* (4.5) *factors through an isomorphism*

$$\tilde{A} \rtimes_{\tilde{\alpha},\mu} G \cong (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}(L^2(G)). \tag{4.6}$$

More generally, every other Morita enveloping action (B,β) of (A,α) has crossed product $B \rtimes_{\beta,\mu} G \sim_M A \rtimes_{\alpha,\mu} G$.

Proof. The first statement is an immediate consequence of our definitions. Indeed, by definition, the exotic partial crossed product $A \rtimes_{\alpha,\mu} G$ is exactly the quotient of $A \rtimes_\alpha G$ that turns (4.5) into the isomorphism (4.6). And the final assertion follows from the already mentioned fact that all Morita enveloping actions are Morita equivalent and the assumption that our functor $\rtimes_\mu$ preserves Morita equivalence. Indeed, if (B,β) is a Morita enveloping action for (A,α), then (B,β) is Morita equivalent to $(\tilde{A},\tilde{\alpha})$ by [1, Proposition 6.3]. Since our crossed-product functor $\rtimes_\mu$ preserves Morita equivalences, we conclude that $B \rtimes_{\beta,\mu} G \sim_M \tilde{A} \rtimes_{\tilde{\alpha},\mu} G \sim_M A \rtimes_{\alpha,\mu} G$. ∎

We also obtain one of the main results on amenability of partial actions shown in [2]. Following the terminology from [2], we say that a partial action (A,α) is *amenable* if its associated Fell bundle is amenable (in the sense of Exel [22]). Hence, by definition, a partial action (A,α) is amenable if and only if its full and reduced crossed products coincide.

COROLLARY 4.9. *A partial action (A,α) is amenable if and only if its canonical Morita enveloping action $(\tilde{A},\tilde{\alpha})$ is amenable, if and only if all Morita enveloping actions of (A,α) are amenable. In this case all exotic (partial) crossed products involving these algebras coincide. More generally, given Morita compatible G-crossed product functors $\rtimes_\mu$ and $\rtimes_\nu$, we have $A \rtimes_{\alpha,\mu} G = A \rtimes_{\alpha,\nu} G$ if and only if $\tilde{A} \rtimes_{\tilde{\alpha},\mu} G = \tilde{A} \rtimes_{\tilde{\alpha},\nu} G$, if and only if $B \rtimes_{\beta,\mu} G = B \rtimes_{\beta,\nu} G$ for every Morita enveloping action (B,β) of (A,α).*

Proof. The first assertion will follow from the last assertion by taking the full and reduced crossed products for $\rtimes_\mu$ and $\rtimes_\nu$. To prove the last assertion notice that (by definition) the equality $A \rtimes_{\alpha,\mu} G = A \rtimes_{\alpha,\nu} G$ means that the ideals in the full crossed product $A \rtimes_\alpha G$ corresponding to the quotient maps $A \rtimes_\alpha G \to A \rtimes_{\alpha,\mu} G, A \rtimes_{\alpha,\nu} G$ coincide, and of course the same meaning has to be given to the equality $B \rtimes_{\beta,\mu} G = B \rtimes_{\beta,\nu} G$. But then the last assertion in the statement follows from the Rieffel correspondence between ideals induced by the Morita equivalences $A \rtimes_{\alpha,\mu} G \sim_M B \rtimes_{\beta,\mu} G$ and $A \rtimes_{\alpha,\nu} G \sim_M B \rtimes_{\beta,\nu} G$ and the fact that both are quotients of the Morita equivalence for full crossed products: $A \rtimes_\alpha G \sim_M B \rtimes_\beta G$. ∎

4.3. K-amenability. The concept of K-amenable groups was first introduced for discrete groups by Cuntz [13] and then extended to locally compact groups by Julg and Valette [27]. It follows from the results of Cuntz that a countable discrete group G is K-amenable if and only if the regular representation $\lambda : C^*(G) \to C^*_r(G)$ is a KK-equivalence, which then implies that for all actions $\alpha : G \to \mathrm{Aut}(A)$ on a separable C^*-algebra A the regular representation $\Lambda^G_A : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is a KK-equivalence as well. The definition of K-amenability for general second countable locally compact groups is slightly more technical, but as a consequence regular representations of crossed products induce KK-equivalences between the full and reduced crossed products. More generally, it is shown in [10, Theorem 6.6] that, if G is K-amenable, then for every separable G-C^*-algebra (A, α) and for every correspondence functor $\rtimes_\mu$ for G the canonical quotient maps

$$A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,r} G$$

are KK-equivalences. Cuntz [13] has shown that all free groups are K-amenable and that K-amenability enjoys some nice permanence properties. Moreover, a more recent result of Tu [41] shows that all a-T-menable groups are K-amenable as well.

The results of the previous section now allow us to extend [10, Theorem 6.6] to cross-sectional algebras of Fell bundles. For general Fell bundles, the result seems to be new even for the quotient map $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$, but this special case is known for Fell bundles associated to partial actions of discrete groups (see [31] and the discussion following Corollary 4.12 below).

Corollary 4.10. *Let G be a second countable K-amenable locally compact group and let $\rtimes_\mu$ be a correspondence crossed-product functor for G. Then, for every separable Fell bundle over G, both *-homomorphisms in the sequence*

$$C^*(\mathcal{B}) \twoheadrightarrow C^*_\mu(\mathcal{B}) \twoheadrightarrow C^*_r(\mathcal{B})$$

given by the identity map on $C_c(\mathcal{B})$ are KK-equivalences. In particular, they induce isomorphisms $K_*(C^*(\mathcal{B})) \cong K_*(C^*_\mu(\mathcal{B})) \cong K_*(C^*_r(\mathcal{B}))$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta_G}} G & \xrightarrow[\cong]{\Psi_\mathcal{B}} & C^*(\mathcal{B}) \otimes \mathcal{K}(L^2(G)) \\
\downarrow & & \downarrow \\
C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta_G},\mu} G & \xrightarrow[\cong]{\Psi_\mu} & C^*_\mu(\mathcal{B}) \otimes \mathcal{K}(L^2(G)) \\
\downarrow & & \downarrow \\
C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \rtimes_{\widehat{\delta_G},r} G & \xrightarrow[\cong]{\Psi_r} & C^*_r(\mathcal{B}) \otimes \mathcal{K}(L^2(G))
\end{array}$$

Since G is K-amenable, the vertical arrows on the left-hand side are all KK-equivalences by [10, Theorem 6.6]. Hence the right vertical arrows are KK-equivalences as well. Since being KK-equivalent is stable under stabilisation by compact operators, the result follows. ∎

Remark 4.11. The above result can be generalised as follows. Let $\rtimes_\mu$ be any crossed-product functor. Following [9] we say that a given coaction $\delta : B \to \mathcal{M}(B \otimes C^*(G))$ is a μ-coaction if Katayama's duality surjection $B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} G \twoheadrightarrow B \otimes \mathcal{K}(L^2(G))$ factors through an isomorphism $B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta},\mu} G \cong B \otimes \mathcal{K}(L^2(G))$. In particular, if $\rtimes_\mu$ is a correspondence functor and $\mathcal{B}$ is a Fell bundle over G, then the dual coaction $\delta_{\mathcal{B},\mu}$ on $C^*_\mu(\mathcal{B})$ is a μ-coaction.

Suppose now that G is K-amenable and second countable, $\rtimes_\mu$ is a correspondence functor for G and (B, δ) is a μ-coaction on a separable C^*-algebra B. Then, if (B_m, δ_m) and (B_n, δ_n) are the maximalisation and normalisation of (B, δ), respectively, the corresponding quotient maps

$$B_m \overset{q_m}{\twoheadrightarrow} B \overset{q_n}{\twoheadrightarrow} B_n$$

are KK-equivalences. This follows directly from the commutative diagram

$$\begin{array}{ccc}
B_m \rtimes_{\delta_m} \widehat{G} \rtimes_{\widehat{\delta},\mu} G & \xrightarrow{\cong} & B_m \otimes \mathcal{K}(L^2(G)) \\
{\scriptstyle q_m \rtimes \widehat{G} \rtimes G}\downarrow & & \downarrow{\scriptstyle q_m \otimes \mathrm{id}_\mathcal{K}} \\
B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta},\mu} G & \xrightarrow{\cong} & B \otimes \mathcal{K}(L^2(G)) \\
{\scriptstyle q_n \rtimes \widehat{G} \rtimes G}\downarrow & & \downarrow{\scriptstyle q_n \otimes \mathrm{id}_\mathcal{K}} \\
B_n \rtimes_{\delta_n} \widehat{G} \rtimes_{\widehat{\delta},\mu} G & \xrightarrow{\cong} & B_n \otimes \mathcal{K}(L^2(G))
\end{array}$$

and the fact that both morphisms in the sequence

$$B_m \rtimes_{\delta_m} \widehat{G} \xrightarrow{q_m \rtimes \widehat{G}} B \rtimes_\delta \widehat{G} \xrightarrow{q_n \rtimes \widehat{G}} B_n \rtimes_{\delta_n} \widehat{G}$$

are G-equivariant isomorphisms. In particular, if G is K-amenable, all C^*-algebras B_m, B and B_n have the same K-theory and K-homology groups.

As a direct consequence of Corollary 4.10, together with the discussion on partial actions given in Section 4.2, we also obtain the following result:

COROLLARY 4.12. *Let (A,α) be a partial action of a second countable locally compact K-amenable group G on a separable C^*-algebra A. If $\rtimes_\mu$ is a correspondence G-crossed-product functor, then the quotient homomorphisms*

$$A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,r} G$$

are KK-equivalences. In particular, $A \rtimes_{\alpha,\mu} G$ has the same K-theory and K-homology as $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$.

For partial actions of *discrete* groups, the above result was proven by McClanahan in [31] for the special case of the quotient map $A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ linking the full and reduced crossed products by partial actions. Notice that the result of McClanahan does not imply the result for general exotic crossed products for discrete groups. Indeed, as shown in [10], there are examples of crossed-product functors that are not correspondence functors for which the above result fails even for crossed products by ordinary actions. In the recent paper [3] by Ara and Exel (see in particular Corollary 6.9) the authors have applied the result by McClanahan to some interesting partial actions of free groups associated to separated graphs in order to deduce that certain full and reduced crossed products have the same K-theory (and the K-theory is effectively computed in [3]). By the above result these computations extend to the respective exotic crossed product related to any given correspondence crossed-product functor.

5. Maximal coactions of discrete groups. As a bonus, we derive in this section a characterisation of maximal coactions of discrete groups. Recall from [33, 36] that every coaction $\delta : B \to B \otimes C^*(G)$ of a discrete group G determines a Fell bundle $\mathcal{B}$ over G with fibres

$$B_t = \{b \in B : \delta(b) = b \otimes u_G(t)\}$$

where $u : G \to U(C^*(G))$ denotes the inclusion map. There is a canonical embedding

$$C_c(\mathcal{B}) = \operatorname{span}\Big(\bigcup_{t\in G} B_t\Big) \hookrightarrow B,$$

which then extends to a surjective $\delta_{\mathcal{B}}$-δ-equivariant *-homomorphism $\kappa : C^*(\mathcal{B}) \twoheadrightarrow B$. On the other hand, it has also been observed by Quigg that the dual coaction δ_n of G on $C_r^*(\mathcal{B})$ is the normalisation of (B,δ), so that there is also a δ-δ_n-equivariant *-homomorphism $\Lambda_B : B \mapsto B_r := C_r^*(\mathcal{B})$. This shows that, in a sense, we may view B as an exotic completion of $C_c(\mathcal{B})$. We know from [17] (and now also from our main Theorem 3.1) that $(C^*(\mathcal{B}),\delta_{\mathcal{B}})$ is the maximalisation of (B,δ), so that the quotient maps $C^*(\mathcal{B}) \xrightarrow{\kappa} B \xrightarrow{\Lambda_B} C_r^*(\mathcal{B})$

induce isomorphisms

$$C^*(\mathcal{B}) \rtimes_{\delta_\mathcal{B}} \widehat{G} \xrightarrow{\sim} B \rtimes_\delta \widehat{G} \xrightarrow{\sim} C^*_r(\mathcal{B}) \rtimes_{\delta_n} \widehat{G}.$$

THEOREM 5.1. *Let G be a discrete group and let $\delta : B \to B \otimes C^*(G)$ be a coaction. Let $\mathcal{B}$ be the Fell bundle over G corresponding to δ as explained above. Then the following assertions are equivalent:*

(1) *(B, δ) is maximal;*
(2) *the canonical *-homomorphism $C^*(\mathcal{B}) \twoheadrightarrow B$ is an isomorphism;*
(3) *δ can be lifted to a *-homomorphism $\delta_{\max} : B \to B \otimes_{\max} C^*(G)$;*
(4) *the reduction $\delta^r := (\mathrm{id} \otimes \lambda) \circ \delta : B \to B \otimes C^*_r(G)$ can be lifted to a *-homomorphism $\delta^r_{\max} : B \to B \otimes_{\max} C^*_r(G)$;*
(5) *the *-homomorphism $(\Lambda_B \otimes \mathrm{id}) \circ \delta^r = (\Lambda_B \otimes \lambda) \circ \delta : B \to B_r \otimes C^*_r(G)$ can be lifted to a homomorphism $(\Lambda_B \otimes_{\max} \mathrm{id}) \circ \delta^r_{\max} : B \to B_r \otimes_{\max} C^*_r(G)$.*

Moreover, if they exist, the lifted homomorphisms in (3)–(5) *are all faithful.*

Proof. The equivalence (1)⇔(2) follows from the fact that $(C^*(\mathcal{B}), \delta_\mathcal{B})$ is the maximisation of (B, δ) by [17, Proposition 4.2]. Notice that (3) holds for $(B, \delta) = (C^*(\mathcal{B}), \delta_\mathcal{B})$ because of the universal property of $C^*(\mathcal{B})$, so we also get (2)⇒(3), and it is obvious that (3)⇒(4). We will finish with the proof of the implications (4)⇒(5)⇒(2). For this assume that (4) holds. It is shown in [4, Theorem 6.2] that, for every Fell bundle $\mathcal{B}$ over a discrete group G, the representation of $\mathcal{B}$ in $C^*_r(\mathcal{B}) \otimes_{\max} C^*_r(G)$ given by $b_t \mapsto b_t \otimes_{\max} \lambda(t)$ extends to a faithful *-homomorphism $(\Lambda_\mathcal{B} \otimes_{\max} \lambda) \circ \delta_\mathcal{B} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B}) \otimes_{\max} C^*_r(G)$ (and this is exactly the lift homomorphism in (5) for the coaction $\delta_\mathcal{B} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(G)$). Now notice that the homomorphism $\kappa : C^*(\mathcal{B}) \to B$ fits into the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad\delta^r_{\max}\quad} & B \otimes_{\max} C^*_r(G) \\ {\scriptstyle\kappa}\uparrow & & \downarrow{\scriptstyle \Lambda_B \otimes_{\max} \mathrm{id}} \\ C^*(\mathcal{B}) & \xrightarrow{(\Lambda_\mathcal{B} \otimes_{\max} \lambda) \circ \delta_\mathcal{B}} & C^*_r(\mathcal{B}) \otimes_{\max} C^*_r(G) \end{array} \tag{5.1}$$

Since $(\Lambda_\mathcal{B} \otimes_{\max} \lambda) \circ \delta_\mathcal{B}$ is injective, it follows that κ is also injective and therefore an isomorphism (so we just proved (4)⇒(2)). But then $(\Lambda_B \otimes_{\max} \mathrm{id}) \circ \delta^r_{\max} = (\Lambda_\mathcal{B} \otimes_{\max} \lambda) \circ \delta_\mathcal{B}$, hence $(\Lambda_B \otimes_{\max} \mathrm{id}) \circ \delta^r_{\max}$ is injective. So (4)⇒(5). Conversely, if (5) holds, then diagram (5.1) implies that κ is injective and therefore an isomorphism $C^*(\mathcal{B}) \xrightarrow{\sim} B$. Hence (5)⇒(2). We saw above that $(\Lambda_B \otimes_{\max} \mathrm{id}) \circ \delta^r_{\max}$, if exists, is faithful. But then $\delta^r_{\max}$ and δ^r are faithful as well. ∎

REMARK 5.2. If $\delta : B \to \mathcal{M}(B \otimes C^*(G))$ is a maximal coaction of a locally compact group G, then it is Morita equivalent to a dual coaction on

a maximal crossed product. Using this it is not difficult to see that such a coaction lifts to $\delta_{\max} : B \to \mathcal{M}(B \otimes_{\max} C^*(G))$. But the converse is not true in general and the above theorem does not extend to general locally compact groups G. The problem is that by the fundamental results of Choi–Effros and Connes [11, 12], the full and reduced group algebras of (almost) connected second countable groups are always nuclear—even if the groups are not amenable (like $G = \mathrm{SL}_2(\mathbb{R})$). But then it is clear that the dual coaction $\delta_{G,r} : C_r^*(G) \to \mathcal{M}(C_r^*(G) \otimes C^*(G))$, which is not maximal if G is not amenable, extends to a faithful map $\delta_{G,r}^{\max} : C_r^*(G) \to \mathcal{M}(C_r^*(G) \otimes_{\max} C^*(G))$.

Notice that the main ingredient in the proof of Theorem 5.1 is [4, Theorem 6.2], which, as indicated by the authors, is based on an idea of Kirchberg. Part (2) of the following corollary has already been observed in [4]. Recall from [22] that a Fell bundle $\mathcal{B}$ is said to be *amenable* if the regular representation $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C_r^*(\mathcal{B})$ is faithful.

COROLLARY 5.3. *Let G be a discrete group.*

(1) *A Fell bundle $\mathcal{B}$ over G is amenable if and only if the dual coaction $\delta_{\mathcal{B}}$ on $C^*(\mathcal{B})$ is normal, if and only if the the dual coaction $\delta_{\mathcal{B},r}$ on $C_r^*(\mathcal{B})$ is maximal.*
(2) *If the full or reduced cross-sectional C^*-algebra of a Fell bundle $\mathcal{B}$ over G is nuclear, then $\mathcal{B}$ is amenable.*
(3) *If G is amenable (that is, if $C_r^*(G)$ or $C^*(G)$ is nuclear), then every Fell bundle over G is amenable.*

We point out that (3) has already been shown by Exel [22, Theorem 4.7].

Proof of Corollary 5.3. The first statement follows directly from the fact that the regular representation $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C_r^*(\mathcal{B})$ can be thought of as either the normalisation map for $(C^*(\mathcal{B}), \delta_{\mathcal{B}})$, or the maximalisation map for $(C_r^*(\mathcal{B}), \delta_{\mathcal{B},r})$. The second item follows directly from the combination of (1) and Theorem 5.1. And the third item also follows from (1) and the fact that every G-coaction is maximal and normal for amenable G. ∎

Observe that the converse of (2) above does not hold, that is, there are amenable Fell bundles (over discrete groups) for which $C^*(\mathcal{B}) \cong C_r^*(\mathcal{B})$ is not nuclear. To see an example, let A be any non-nuclear C^*-algebra and let an amenable group G act trivially on A. Then $A \rtimes_{\mathrm{id}} G = A \rtimes_{\mathrm{id},r} G$, hence the corresponding Fell bundle $A \rtimes^{\mathrm{id}} G$ is amenable. But $A \rtimes_{\mathrm{id}} G \cong A \otimes C^*(G)$ is not nuclear, since the tensor product of a non-nuclear C^*-algebra with a nuclear C^*-algebra is never nuclear. A partial converse of Corollary 5.3(2) holds under the assumption that the unit fiber $\mathcal{B}_e$ is nuclear (see [23, Proposition 25.10]).

If $\mathcal{B}$ is not amenable, we know from the above corollary that $C^*(\mathcal{B})$ and $C_r^*(\mathcal{B})$ are non-nuclear C^*-algebras, so there exist C^*-algebras D, E

such that the algebraic tensor products $C^*(\mathcal{B}) \odot D$ and $C^*_r(\mathcal{B}) \odot E$ do not admit unique C^*-norms. Indeed, the methods above allow us to make explicit choices for D and E:

COROLLARY 5.4. *If a Fell bundle $\mathcal{B}$ over a discrete group G is not amenable, then all the algebraic tensor products $C^*(\mathcal{B}) \odot C^*_r(G)$, $C^*_r(\mathcal{B}) \odot C^*(G)$ and $C^*_r(\mathcal{B}) \odot C^*_r(G)$ do not admit unique C^*-norms, that is,*

$$C^*(\mathcal{B}) \otimes_{\max} C^*_r(G) \neq C^*(\mathcal{B}) \otimes_{\min} C^*_r(G),$$
$$C^*_r(\mathcal{B}) \otimes_{\max} C^*_r(G) \neq C^*_r(\mathcal{B}) \otimes_{\min} C^*_r(G),$$
$$C^*_r(\mathcal{B}) \otimes_{\max} C^*(G) \neq C^*_r(\mathcal{B}) \otimes_{\min} C^*(G).$$

Proof. The dual coaction $\delta_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(G)$ is maximal and hence its reduction $\delta^r_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*_r(G)$ lifts to an injective homomorphism $\delta^r_{\mathcal{B},\max} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes_{\max} C^*_r(G)$. But the reduction $\delta^r_{\mathcal{B}}$ is weakly equivalent to $\Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$, hence if $C^*(\mathcal{B}) \otimes_{\max} C^*_r(G) = C^*(\mathcal{B}) \otimes_{\min} C^*_r(G)$, then $\mathcal{B}$ is amenable. This gives the statement for $C^*(\mathcal{B}) \odot C^*_r(G)$, and the other cases are treated similarly. ∎

Taking $\mathcal{B}$ to be the trivial Fell bundle $\mathcal{B} = \mathbb{C} \times G$, we deduce from the above result the non-uniqueness of C^*-norms on mixed tensor products of the form $C^*(G) \odot C^*_r(G)$ and $C^*_r(G) \odot C^*_r(G)$. This special case is well known and follows, for instance, from [7, Proposition 6.4.1] ([2]). Observe that we do not say anything about the tensor product $C^*(\mathcal{B}) \odot C^*(G)$. Indeed, as shown in [30, Proposition 8.1], uniqueness of the C^*-norm on $C^*(\mathcal{B}) \odot C^*(G)$ in the case of $\mathcal{B} = \mathbb{C} \times G$, $G = \mathbb{F}_\infty$ is equivalent to the solution of Connes' embedding conjecture!

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