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Gaps between primes in Beatty sequences

by

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1. Introduction. Let p_n denote the nth prime and t a natural number with $t \geq 2$. It has long been conjectured that

$$\liminf_{n\to\infty}(p_{n+t-1} - p_n) < \infty.$$

This was established recently for $t = 2$ by Y. Zhang [12], and shortly after for all t by J. Maynard [9]. Maynard showed that for $N > C(t)$, the interval $[N, 2N)$ contains a set $\mathcal{S}$ of t primes of diameter

$$D(\mathcal{S}) \ll t^3 \exp(4t),$$

where

$$D(\mathcal{S}) := \max\{n : n \in \mathcal{S}\} - \min\{n : n \in \mathcal{S}\}.$$

In the present paper, we adapt Maynard's method to prove a similar result where $\mathcal{S}$ is contained in a prescribed set $\mathcal{A}$ (see Theorem 1). We then work out applications (Theorems 2 and 3) to a section of a Beatty sequence, so that

$$\mathcal{A} = \{[\alpha m + \beta] : m \geq 1\} \cap [N, 2N).$$

The number α is assumed to be irrational with $\alpha > 1$, while β is a given real number. We require an auxiliary result (Theorem 4) for the estimation of errors of the form

$$\sum_{\substack{N \leq n < N' \\ \gamma n \in I \bmod 1 \\ n \equiv a \bmod q}} \Lambda(n) - \frac{(N - N')|I|}{\varphi(q)},$$

where I is an interval of length $|I| < 1$ and $\gamma = \alpha^{-1}$. Theorem 4 is of "Bombieri–Vinogradov type"; for completeness, we include a result of Barban–Davenport–Halberstam type for these errors (Theorem 5).

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We note that Chua, Park and Smith [5] have already used Maynard's method to prove the existence of infinitely many sets of k primes of diameter at most $C = C(\alpha, k)$ in a Beatty sequence $[\alpha n]$, where α is irrational and of finite type. However, no explicit bound for C is given.

Now we introduce some notation to be used throughout this paper. We suppose that $t \in \mathbb{N}$, $N \geq C(t)$ and write $\mathcal{L} = \log N$,

$$D_0 = \frac{\log \mathcal{L}}{\log\log \mathcal{L}}.$$

Moreover, (d, e) and $[d, e]$ stand for the greatest common divisor and the least common multiple of d and e, respectively; $\tau(q)$ and $\tau_k(q)$ are the usual divisor functions; and $\|x\|$ is the distance of between $x \in \mathbb{R}$ and the nearest integer. Set

$$P(z) = \prod_{p<z} p \quad \text{with } z \geq 2 \quad \text{and} \quad \psi(n, z) = \begin{cases} 1 & \text{if } (n, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$X(E; n)$ stands for the indicator function of a set E, and $\mathbb{P}$ for the set of primes. Let ε be a positive constant, sufficiently small in terms of t. The constant implied in "$\ll$", when it appears, may depend on ε and on A (if A appears in the statement of the result). "$F \asymp G$" means both $F \ll G$ and $G \ll F$ hold. As usual, $e(y) = \exp(2\pi i y)$, and $o(1)$ indicates a quantity tending to 0 as N tends to infinity. Furthermore,

$$\sum_{\chi \bmod q}, \quad \sum_{\chi \bmod q}{}', \quad \sum_{\chi \bmod q}{}^{\star}$$

denote, respectively, a sum over all Dirichlet characters modulo q, a sum over non-principal characters modulo q, and a sum restricted to primitive characters, other than $\chi = 1$, modulo q. We write $\hat{\chi}$ for the primitive character that induces χ. A set $\mathcal{H} = \{h_1, \ldots, h_k\}$ of distinct non-negative integers is *admissible* if for every prime p, there is an integer a_p such that $a_p \not\equiv h$ (mod p) for all $h \in \mathcal{H}$.

In Sections 1 and 2, let θ be a positive constant. Let $\mathcal{A}$ be a subset of $[N, 2N) \cap \mathbb{N}$. Suppose that $Y > 0$ and Y/q_0 is an approximation to $\#\mathcal{A}$, the cardinality of $\mathcal{A}$. Let q_0, q_1 be given natural numbers not exceeding N with $(q_1, q_0 P(D_0)) = 1$ and $\varphi(q_1) = q_1(1 + o(1))$. Suppose that $n \equiv a_0 \pmod{q_0}$ for all $n \in \mathcal{A}$ with $(a_0, q_0) = 1$. An admissible set $\mathcal{H}$ is given with

$$h \equiv 0 \pmod{q_0} \quad (h \in \mathcal{H})$$

and

$$p \mid h - h' \quad \text{with } h, h' \in \mathcal{H},\ h \neq h',\ p > D_0 \quad \text{implies} \quad p \mid q_0. \tag{1.1}$$

We now state "regularity conditions" on $\mathcal{A}$:

(I) We have

$$\sum_{\substack{q\le N^\theta\\(q,q_0q_1)=1}} \mu(q)^2\tau_{3k}(q)\Bigg|\sum_{n\equiv a_q \bmod qq_0} X(\mathcal{A};n)-\frac{Y}{qq_0}\Bigg| \ll \frac{Y}{q_0\mathcal{L}^{k+\varepsilon}} \tag{1.2}$$

(for any $a_q \equiv a_0 \pmod{q_0}$).

(II) There are non-negative functions $\varrho_1,\dots,\varrho_s$ defined on $[N,2N)$ (with s constant, $0<a\le s$) such that

$$X(\mathbb{P};n)\ge \varrho_1(n)+\dots+\varrho_a(n)-\big(\varrho_{a+1}(n)+\dots+\varrho_s(n)\big) \tag{1.3}$$

for $n\in[N,2N)$. There are positive $Y_{g,m}$ ($g=1,\dots,s$ and $m=1,\dots,k$) with

$$Y_{g,m}=Y(b_{g,m}+o(1))\mathcal{L}^{-1},$$

where the positive constants $b_{g,m}$ satisfy

$$b_{1,m}+\dots+b_{a,m}-(b_{a+1,m}+\dots+b_{s,m})\ge b>0 \tag{1.4}$$

for $m=1,\dots,k$. Moreover, for $m\le k$, $g\le s$ and any $a_q\equiv a_0 \pmod{q_0}$ with $(a_q,q)=1$ defined for $q\le x^\theta$, $(q,q_0q_1)=1$, we have

$$\sum_{\substack{q\le N^\theta\\(q,q_0q_1)=1}} \mu(q)^2\tau_{3k}(q)\Bigg|\sum_{n\equiv a_q \bmod qq_0} \varrho_g(n)X((\mathcal{A}+h_m)\cap\mathcal{A};n)-\frac{Y_{g,m}}{\varphi(q_0q)}\Bigg| \ll \frac{Y}{\varphi(q_0)\mathcal{L}^{k+\varepsilon}}. \tag{1.5}$$

Finally, $\varrho_g(n)=0$ unless $(n,P(N^{\theta/2}))=1$.

THEOREM 1. *Under the above hypotheses on $\mathcal{H}$ and $\mathcal{A}$, there is a set $\mathcal{S}$ of t primes in $\mathcal{A}$ with diameter not exceeding $D(\mathcal{H})$, provided $k\ge k_0(t,b,\theta)$ (k_0 is defined at the end of this section).*

In proving Theorem 2, we shall take $s=a=1$, $q_0=q_1=1$, $\rho_1(n)=X(\mathbb{P};n)$. A more complicated example with $s=5$, of the inequality (1.3), occurs in proving Theorem 3, but again $q_0=q_1=1$. We shall consider elsewhere a result in which q_0, q_1 are large. Maynard's Theorem 3.1 in [10] overlaps with our Theorem 1, but neither subsumes the other.

THEOREM 2. *Let $\alpha>1$, $\gamma=\alpha^{-1}$ and $\beta\in\mathbb{R}$. Suppose that*

$$\|\gamma r\|\gg r^{-3} \tag{1.6}$$

for all $r\in\mathbb{N}$. Then for any $N>c_1(t,\alpha,\beta)$, there is a set of t primes of the form $[\alpha m+\beta]$ in $[N,2N)$ having diameter

$$< C_2\alpha(\log\alpha+t)\exp(8t),$$

where C_2 is an absolute constant.

Theorem 3. *Let α be irrational with $\alpha > 1$ and $\beta \in \mathbb{R}$. Let $r \ge C_3(\alpha, \beta)$ and*

$$\left|\frac{1}{\alpha} - \frac{b}{r}\right| < \frac{1}{r^2}, \qquad b \in \mathbb{N},\ (b, r) = 1.$$

Let $N = r^2$. There is a set of t primes of the form $[\alpha n + \beta]$ in $[N, 2N)$ having diameter

$$< C_4 \alpha (\log \alpha + t) \exp(7.743t),$$

where C_4 is an absolute constant.

Theorem 3 improves Theorem 2 in that α can be any irrational number in $(1, \infty)$ and $7.743 < 8$, but we lose the arbitrary placement of N.

Turning our attention to our theorem of Bombieri–Vinogradov type, we write

$$E(N, N', \gamma, q, a) = \sup_I \Bigg| \sum_{\substack{N \le n < N' \\ \gamma n \in I \bmod 1 \\ n \equiv a \bmod q}} \Lambda(n) - \frac{(N' - N)|I|}{\varphi(q)} \Bigg|.$$

Here, I runs over intervals of length $|I| < 1$.

Theorem 4. *Let $A > 0$, γ be a real number and b/r a rational approximation to γ,*

$$\left|\gamma - \frac{b}{r}\right| \le \frac{1}{rN^{3/4}}, \qquad N^{\varepsilon} \le r \le N^{3/4},\ (b, r) = 1. \tag{1.7}$$

Then for $N < N' \le 2N$ and any $A > 0$, we have

$$\sum_{q \le \min(r, N^{1/4}) N^{-\varepsilon}} \max_{(a,q)=1} E(N, N', \gamma, q, a) \ll N\mathcal{L}^{-A}. \tag{1.8}$$

Our Barban–Davenport–Halberstam type result is the following.

Theorem 5. *Let $A > 0$ and γ be an irrational number. Suppose that for each $\eta > 0$ and sufficiently large $r \in \mathbb{N}$, we have*

$$\|\gamma r\| > \exp(-r^{\eta}). \tag{1.9}$$

Let $N\mathcal{L}^{-A} \le R \le N$. Then for $N < N' \le 2N$,

$$\sum_{q \le R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} E(N, N', \gamma, q, a)^2 \ll NR\mathcal{L}(\log \mathcal{L})^2. \tag{1.10}$$

There are weaker results overlapping with Theorems 4 and 5, due to W. D. Banks and I. E. Shparlinski [4].

Let γ be irrational, $\eta > 0$ and suppose that

$$\|\gamma r\| \le \exp(-r^{\eta})$$

for infinitely many $r \in \mathbb{N}$. Then (1.10) fails (so Theorem 5 is optimal in this sense). To see this, take $N = \exp(r^{\eta/2})$, $N' = 2N$ and $R = N\mathcal{L}^{-8/\eta}$. We have, for some $u \in \mathbb{Z}$,

$$\left|\gamma n - \frac{un}{r}\right| \le 2Nr^{-1}\exp(-r^{\eta}) < \frac{1}{4r} \qquad (n \le 2N).$$

From this, we infer that

$$\gamma n \notin \left(\frac{1}{4r}, \frac{3}{4r}\right) \pmod 1 \qquad (n \le 2N).$$

So

$$E(N, 2N, \gamma, q, a)^2 \ge \frac{N^2}{4r^2\varphi(q)} \qquad (q \le R,\ (a,q) = 1).$$

Therefore,

$$\sum_{q \le R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} E(N, 2N, \gamma, q, a)^2 \ge \frac{N^2}{4r^2} \sum_{q \le R} \frac{1}{\varphi(q)} > \frac{N^2}{r^2} = NR\mathcal{L}^{4/\eta}.$$

We now turn to the definition of $k_0(t, b, \theta)$. For a smooth function F supported on

$$\mathcal{R}_k = \Big\{(x_1, \ldots, x_k) \in [0,1]^k : \sum_{i=1}^{k} x_i \le 1\Big\},$$

set

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 \, dt_1 \cdots dt_k,$$

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \Big(\int_0^1 F(t_1, \ldots, t_k)\, dt_m\Big)^2 dt_1 \cdots dt_{m-1}\, dt_{m+1} \cdots dt_k$$

for $m = 1, \ldots, k$. Let

$$M_k = \sup_F \frac{\sum_{m=1}^{k} J_k^{(m)}(F)}{I_k(F)},$$

where the sup is taken over all functions F specified above and subject to the conditions $I_k(F) \ne 0$ and $J_k^{(m)}(F) \ne 0$ for each m. Sharpening a result of Maynard [9], D. H. J. Polymath [11] gives the lower bound

$$M_k \ge \log k + O(1). \tag{1.11}$$

Now let $k_0(t, b, \theta)$ be the least integer k for which

$$M_k > \frac{2t - 2}{b\theta}. \tag{1.12}$$

2. Deduction of Theorem 1 from two propositions. We first write down some lemmas that we shall need later.

LEMMA 1. *Let $\kappa, A_1, A_2, L > 0$. Suppose that γ is a multiplicative function satisfying*

$$0 \le \frac{\gamma(p)}{p} \le 1 - A_1$$

for all prime p, and

$$-L \le \sum_{w \le p \le z} \frac{\gamma(p)\log p}{p} - \kappa \log\frac{z}{w} \le A_2$$

for any w and z with $2 \le w \le z$. Let g be the totally multiplicative function defined by

$$g(p) = \frac{\gamma(p)}{p - \gamma(p)}.$$

Suppose that $G : [0,1] \to \mathbb{R}$ is a piecewise differentiable function with

$$|G(y)| + |G'(y)| \le B$$

for $0 \le y \le 1$ and

$$S = \prod_p \left(1 - \frac{\gamma(p)}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^{\kappa}. \tag{2.1}$$

Then for $z > 1$, we have

$$\sum_{d<z} \mu(d)^2 g(d) G\left(\frac{\log d}{\log z}\right) = \frac{S(\log z)^\kappa}{\Gamma(\kappa)} \int_0^1 t^{\kappa-1} G(t)\,dt + O(SLB(\log z)^{\kappa-1}).$$

The implied constant above depends on A_1, A_2, κ, but is independent of L.

Proof. This is [7, Lemma 4]. ∎

Throughout this section, we assume that the hypotheses of Theorem 1 hold. Moreover, we write

$$W_1 = \prod_{p \le D_0 \text{ or } p \mid q_0 q_1} p, \qquad W_2 = \prod_{\substack{p \le D_0 \\ p \nmid q_0}} p, \qquad R = N^{\theta/2-\varepsilon}.$$

Recalling the definition of admissible set, we pick a natural number ν_0 with

$$(\nu_0 + h_m, W_2) = 1 \qquad (m = 1, \dots, k).$$

LEMMA 2. *Suppose that $\gamma(p) = 1 + O(p^{-1})$ if $p \nmid W_1$, and $\gamma(p) = 0$ if $p \mid W_1$. Let $\kappa = 1$ and S be as defined in (2.1). Then*

$$S = \frac{\varphi(W_1)}{W_1}(1 + O(D_0^{-1})).$$

Proof. We have

$$S = \prod_{p\mid W_1}\Big(1-\frac{1}{p}\Big)\prod_{p\nmid W_1}\Big(1-\frac{1}{p}+O\Big(\frac{1}{p^2}\Big)\Big)^{-1}\Big(1-\frac{1}{p}\Big)$$
$$= \frac{\varphi(W_1)}{W_1}\prod_{\substack{p>D_0\\ p\nmid q_0q_1}}(1+O(p^{-2})),$$

from which the statement of the lemma can be readily obtained. ■

LEMMA 3. *Let $H>1$ and*

$$T_1 = \sum_{\substack{d\le R\\ (d,W_1)=1}}\frac{\mu(d)^2}{d}\sum_{a\mid d}\frac{4^{\omega(a)}}{a},\qquad T_2 = \sum_{H<d\le R}\frac{\mu(d)^2}{d^2}\sum_{a\mid d}a^{-1/2}.$$

Then

$$T_1 \ll \frac{\varphi(W_1)}{W_1}\mathcal{L}, \tag{2.2}$$

$$T_2 \ll H^{-1}. \tag{2.3}$$

Proof. Let $\gamma(p)=0$ if $p\mid W_1$, and

$$\gamma(p) = \frac{p^2+4p}{p^2+p+4}$$

if $p\nmid W_1$. Then $g(p)$, as defined in the statement of Lemma 1, is

$$g(p) = \frac{1}{p}+\frac{4}{p^2}$$

if $p\nmid W_1$. Therefore, if d is square-free and $(d,W_1)=1$, then

$$\frac{1}{d}\sum_{a\mid d}\frac{4^{\omega(a)}}{a} = \frac{1}{d}\prod_{p\mid d}\Big(1+\frac{4}{p}\Big) = g(d).$$

Otherwise, if $(d,W_1)\ne 1$, then $g(d)=0$. Using Lemma 1 with $G(y)=1$ and Lemma 2, we have

$$T_1 = \sum_{d\le R}\mu(d)^2 g(d) G\Big(\frac{\log d}{\log R}\Big)$$
$$= \frac{\varphi(W_1)}{W_1}(1+O(D_0^{-1}))\log R + O\Big(\frac{\varphi(W_1)}{W_1}L\Big),$$

where we can take

$$L = \sum_{p\mid W_1}\frac{\log p}{p} \ll \log D_0 + \log\omega(q_0) \ll \log\mathcal{L}.$$

Combining everything, we get (2.2).

To prove (2.3), we interchange the summations and get

$$T_2 \le \sum_{a\le R} a^{-5/2} \sum_{Ha^{-1}<k\le Ra^{-1}} k^{-2} \ll \sum_{a\le R} a^{-3/2} H^{-1} \ll H^{-1}. \ \blacksquare$$

LEMMA 4. *Let f_0, f_1 be multiplicative functions with $f_0(p) = f_1(p) + 1$. Then for square-free d and e,*

$$\frac{1}{f_0([d,e])} = \frac{1}{f_0(d)f_0(e)} \sum_{k\mid d,e} f_1(k).$$

Proof. We have

$$\frac{1}{f_0(d)f_0(e)} \sum_{k\mid d,e} f_1(k) = \frac{1}{f_0(d)f_0(e)} \prod_{p\mid(d,e)} (1+f_1(p))$$
$$= \frac{1}{f_0(d)f_0(e)} \prod_{p\mid(d,e)} f_0(p) = \prod_{p\mid[d,e]} (f_0(p))^{-1}.$$

The lemma follows from this. $\blacksquare$

We now present two propositions that readily yield Theorem 1 when combined. To state them, we define weights $y_{\mathbf{r}}$ and $\lambda_{\mathbf{r}}$ for tuples

$$\mathbf{r} = (r_1,\dots,r_k) \in \mathbb{N}^k$$

having the properties

$$\Big(\prod_{i=1}^k r_i, W_1\Big) = 1, \quad \mu\Big(\prod_{i=1}^k r_i\Big)^2 = 1. \tag{2.4}$$

We set $y_{\mathbf{r}} = \lambda_{\mathbf{r}} = 0$ for all other tuples. Let F be a smooth function with $|F| \le 1$ and the properties given at the end of Section 1. Let

$$y_{\mathbf{r}} = F\Big(\frac{\log r_1}{\log R},\dots,\frac{\log r_k}{\log R}\Big), \tag{2.5}$$

$$\lambda_{\mathbf{d}} = \prod_{i=1}^k \mu(d_i)d_i \sum_{\substack{\mathbf{r}\\ d_i\mid r_i\,\forall i}} \frac{y_{\mathbf{r}}}{\prod_{i=1}^k \varphi(r_i)}. \tag{2.6}$$

We have

$$\lambda_{\mathbf{r}} \ll \mathcal{L}^k \tag{2.7}$$

(see [9, (5.9)]). For $n \equiv \nu_0 \pmod{W_2}$, let

$$w_n = \Big(\sum_{d_i\mid n+h_i\,\forall i} \lambda_{\mathbf{d}}\Big)^2,$$

and $w_n = 0$ for all other natural numbers n.

Proposition 1. *Let*

$$S_1 = \sum_{N \le n < 2N} w_n X(\mathcal{A}; n).$$

Then

$$S_1 = \frac{(1+o(1))\varphi(W_1)^k Y (\log R)^k I_k(F)}{q_0 W_1^k W_2}.$$

Proposition 2. *Let*

$$S_2(g,m) = \sum_{\substack{N \le n < 2N \\ n \in \mathcal{A} \cap (\mathcal{A} - h_m)}} w_n \varrho_g(n + h_m).$$

Then for $1 \le g \le s$ *and* $1 \le m \le k$,

$$S_2(g,m) = \frac{b_{g,m}(1+o(1))\varphi(W_1)^{k+1} Y (\log R)^{k+1} J_k^{(m)}(F)}{\varphi(q_0)\varphi(W_2) W_1^{k+1} \mathcal{L}}.$$

Before proving the above propositions, we shall deduce Theorem 1 from them.

Proof of Theorem 1. Let

$$Z = \frac{Y\varphi(W_1)^k}{q_0 W_1^k W_2} (\log R)^k,$$

$$S(N) = \sum_{\substack{N \le n < 2N \\ n \in \mathcal{A}}} w_n \Big(\sum_{m=1}^{k} X(\mathbb{P} \cap \mathcal{A}; n + h_m) - (t-1) \Big).$$

Since $w_n \ge 0$, (1.3) gives

$$S(N) \ge \sum_{m=1}^{k} \Big(\sum_{g=1}^{a} S_2(g,m) - \sum_{g=a+1}^{s} S_2(g,m) \Big) - (t-1) S_1.$$

By Propositions 1 and 2, the right-hand side of the above is

$$(1+o(1)) Z \bigg(\sum_{m=1}^{k} \Big(\sum_{g=1}^{a} b_{g,m} - \sum_{g=a+1}^{s} b_{g,m} \Big) J_k^{(m)}(F) \Big(\frac{\theta}{2} - \varepsilon \Big) - (t-1) I_k(F) \bigg).$$

Here we have used

$$\frac{\varphi(q_0)\varphi(q_1)\varphi(W_2)}{q_0 q_1 W_2} \frac{W_1}{\varphi(W_1)} = 1 \quad \text{and} \quad \frac{\varphi(q_1)}{q_1} = 1 + o(1).$$

Therefore, from (1.4) we get

$$S(N) \ge (1+o(1)) Z \bigg(b \sum_{m=1}^{k} J_k^{(m)}(F) \Big(\frac{\theta}{2} - \varepsilon \Big) - (t-1) I_k(F) \bigg) > 0$$

for a suitable choice of F. The positivity of the above expression is a consequence of (1.12). Therefore, there must be at least one $n \in \mathcal{A}$ for which

$$\sum_{m=1}^{k} X(\mathbb{P} \cap \mathcal{A}; n + h_m) > t - 1.$$

For this n, there is a set of t primes $n + h_{m_1}, \ldots, n + h_{m_t}$ in $\mathcal{A}$. ∎

3. Proof of Propositions 1 and 2

Proof of Proposition 1. We first show that

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{r}} \frac{y_{\mathbf{r}}^2}{\prod_{i=1}^{k} \varphi(r_i)} + O\Big(\frac{Y\varphi(W_1)^k \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\Big). \tag{3.1}$$

From the definition of w_n, we get

$$S_1 = \sum_{\mathbf{d},\mathbf{e}} \lambda_{\mathbf{d}} \lambda_{\mathbf{e}} \sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 \bmod W_2 \\ [d_i, e_i] \mid n + h_i \,\forall i}} X(\mathcal{A}; n). \tag{3.2}$$

Recall that $n \equiv a_0 \pmod{q_0}$ for all $n \in \mathcal{A}$. The inner sum of the above takes the form

$$\sum_{\substack{N \le n < 2N \\ n \equiv a_q \bmod q q_0}} X(\mathcal{A}; n), \quad \text{where} \quad q = W_2 \prod_{i=1}^{k} [d_i, e_i],$$

provided that $W_2, [d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime. The latter restriction reduces to

$$(d_i, e_j) = 1 \tag{3.3}$$

for all $i \ne j$, and we exhibit this condition on the summation by writing

$$\sideset{}{'}\sum_{\mathbf{d},\mathbf{e}}.$$

Outside of $\sum'_{\mathbf{d},\mathbf{e}}$, the inner sum is empty. To see this, suppose that $p \mid d_i$, $p \mid e_j$ with $i \ne j$; then the conditions

$$[d_i, e_i] \mid n + h_i \quad \text{and} \quad [d_j, e_j] \mid n + h_j$$

imply that $p \mid h_i - h_j$. This means that either $p \le D_0$ or $p \mid q_0$, both contrary to $p \mid d_i$.

Counting the number of times a given q can arise, we get

$$S_1 - \frac{Y}{q_0 W_2} \sideset{}{'}\sum_{\mathbf{d},\mathbf{e}} \frac{\lambda_{\mathbf{d}}\lambda_{\mathbf{e}}}{\prod_{i=1}^k [d_i, e_i]} \ll \Big(\max_{\mathbf{d}} |\lambda_{\mathbf{d}}|\Big)^2 \sum_{\substack{q \le R^2 W_2 \\ (q,q_0)=1}} \mu(q)^2 \tau_{3k}(q) \Bigg| \sum_{n \equiv a_q \bmod qq_0} X(\mathcal{A}; n) - \frac{Y}{qq_0}\Bigg|. \tag{3.4}$$

Since $R^2 W_2 \le N^\theta$, we can appeal to (1.2) and (2.7) to majorize the right-hand side of (3.4) by

$$\ll \frac{Y}{q_0}\mathcal{L}^{2k-(k+\varepsilon)} \ll \frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}.$$

Applying Lemma 4 with $f_1 = \varphi$, we see that

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \varphi(u_i) \sideset{}{'}\sum_{\substack{\mathbf{d},\mathbf{e} \\ u_i | d_i, e_i \, \forall i}} \frac{\lambda_{\mathbf{d}}\lambda_{\mathbf{e}}}{\prod_{i=1}^k d_i e_i} + O\bigg(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\bigg).$$

Now we follow [9] verbatim to transform this equation into

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \varphi(u_i) \sideset{}{^*}\sum_{s_{1,2},\dots,s_{k,k-1}} \prod_{\substack{1 \le i,j \le k \\ i \ne j}} \mu(s_{i,j}) \times \sum_{\substack{\mathbf{d},\mathbf{e} \\ u_i | d_i, e_i \, \forall i \\ s_{i,j} | d_i, e_j \, \forall i \ne j}} \frac{\lambda_{\mathbf{d}}\lambda_{\mathbf{e}}}{\prod_{i=1}^k d_i e_i} + O\bigg(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\bigg). \tag{3.5}$$

Here $\sum^*$ indicates that $(s_{i,j}, u_i u_j) = 1$ and $(s_{i,j}, s_{i,c}) = 1 = (s_{i,j}, s_{d,j})$ for $c \ne j$ and $d \ne i$. Now define

$$a_j = u_j \prod_{i \ne j} s_{j,i}, \qquad b_j = u_j \prod_{i \ne j} s_{i,j}. \tag{3.6}$$

As in [9], we recast (3.5) as

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \sideset{}{^*}\sum_{s_{1,2},\dots,s_{k,k-1}} \prod_{\substack{1 \le i,j \le k \\ i \ne j}} \mu(s_{i,j}) \times \sum_{\substack{\mathbf{d},\mathbf{e} \\ u_i | d_i, e_i \, \forall i \\ s_{ij} | d_i, e_j \, \forall i \ne j}} \frac{\mu(s_{i,j})}{\varphi(s_{i,j})^2} y_{\mathbf{a}} y_{\mathbf{b}} + O\bigg(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\bigg). \tag{3.7}$$

For the non-zero terms on the right-hand side of (3.7), either $s_{i,j}=1$ or $s_{i,j}>D_0$. The terms of the latter kind (for given i, j with $i\neq j$) contribute

$$\text{(3.8)}\quad \ll \frac{Y}{q_0W_2}\Bigg(\sum_{\substack{u<R\\(u,W_1)=1}}\frac{\mu(u)^2}{\varphi(u)}\Bigg)^k\Bigg(\sum_{s_{i,j}>D_0}\frac{\mu(s_{i,j})^2}{\varphi(s_{i,j})^2}\Bigg)\Bigg(\sum_{s\geq 1}\frac{\mu(s)}{\varphi(s)^2}\Bigg)^{k^2-k-1}$$
$$= \frac{Y}{q_0W_2}U_1U_2U_3,$$

say. Clearly, $U_3\ll 1$. Now if u is square-free, we have

$$\frac{1}{\varphi(u)}=\frac{1}{u}\prod_{p\mid u}\left(1-\frac{1}{p}\right)^{-1}\ll\frac{1}{u}\sum_{a\mid u}\frac{1}{a}$$

and

$$\frac{1}{\varphi(u)^2}\ll\frac{1}{u^2}\prod_{p\mid u}\left(1+\frac{2}{p}\right)=\frac{1}{u^2}\sum_{a\mid u}\frac{2^{\omega(a)}}{a}\ll\frac{1}{u^2}\sum_{a\mid u}a^{-1/2}.$$

So (2.2) and (2.3) give, respectively,

$$U_1\ll\left(\frac{\varphi(W_1)}{W_1}\mathcal{L}\right)^k\quad\text{and}\quad U_2\ll\frac{1}{D_0}.$$

Hence, the right-hand side of (3.8) is

$$\ll\frac{\varphi(W_1)^kY\mathcal{L}^k}{q_0W_2W_1^kD_0}$$

and we arrive at (3.1).

Now, we shall deduce Proposition 1 from (3.1). Mindful of (2.6), we have

$$S_1=\frac{Y}{q_0W_2}\sum_{\substack{\mathbf{u}\\(u_l,u_j)=1\,\forall l\neq j\\(u_l,W_1)=1\,\forall l}}\prod_{i=1}^k\frac{\mu(u_i)^2}{\varphi(u_i)}F\left(\frac{\log u_1}{\log R},\ldots,\frac{\log u_k}{\log R}\right)^2$$
$$+O\Bigg(\frac{\varphi(W_1)^kY\mathcal{L}^k}{q_0W_2W_1^kD_0}\Bigg).$$

Note that the common prime factors of two integers both coprime to W_1 are strictly greater than D_0. Thus, we may drop the condition $(u_l,u_j)=1$ in the above expression at the cost of an error of size

$$\ll\frac{Y}{q_0W_2}\sum_{p>D_0}\sum_{\substack{u_1\cdots u_k<R\\p\mid u_l,u_j\\(u_l,W_1)=1\,\forall l}}\prod_{i=1}^k\frac{\mu(u_i)^2}{\varphi(u_i)}$$
$$\ll\frac{Y}{q_0W_2}\sum_{p>D_0}\frac{1}{(p-1)^2}\Bigg(\sum_{\substack{u<R\\(u,W_1)=1}}\frac{\mu(u)^2}{\varphi(u)}\Bigg)^k\ll\frac{\varphi(W_1)^kY\mathcal{L}^k}{q_0W_2W_1^kD_0},$$

by virtue of (2.2).

It remains to evaluate the sum

$$\sum_{\substack{\mathbf{u} \\ (u_l,W_1)=1\,\forall l}} \prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} F\biggl(\frac{\log u_1}{\log R}, \ldots, \frac{\log u_k}{\log R}\biggr)^2. \tag{3.9}$$

This requires applying Lemma 1 k times with

$$\gamma(p) = \begin{cases} 0, & p \mid W_1, \\ 1, & p \nmid W_1. \end{cases}$$

We take A_1 and A_2 to be suitable constants and

$$L \ll 1 + \sum_{p \mid W_1} \frac{\log p}{p} \ll \log \mathcal{L}$$

as noted earlier. In the jth application, we replace the summation over u_j by the integral over $[0,1]$. Ultimately, we express the sum in (3.9) in the form

$$\frac{\varphi(W_1)^k}{W_1^k} (\log R)^k I_k(F) + O\biggl(\frac{\varphi(W_1)(\log \mathcal{L})\mathcal{L}^{k-1}}{W_1^k}\biggr),$$

and Proposition 1 follows at once. ∎

We shall need the following lemma in the proof of Proposition 2.

Lemma 5. *Let $1 \le m \le k$ and suppose that $r_m = 1$. Let*

$$y_{\mathbf{r}}^{(m)} = \prod_{i=1}^{k} \mu(r_i) g(r_i) \sum_{\substack{\mathbf{d} \\ r_i \mid d_i\,\forall i \\ d_m = 1}} \frac{\lambda_{\mathbf{d}}}{\prod_{i=1}^{k} \varphi(d_i)}.$$

Then

$$y_{\mathbf{r}}^{(m)} = \sum_{a_m} \frac{y_{r_1,\ldots,r_{m-1},a_m,r_{m+1},\ldots,r_k}}{\varphi(a_m)} + O\biggl(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\biggr).$$

Proof. Following [9] verbatim, we have

$$y_{\mathbf{r}}^{(m)} = \prod_{i=1}^{k} \mu(r_i) g(r_i) \sum_{\substack{\mathbf{a} \\ r_i \mid a_i\,\forall i}} \frac{y_{\mathbf{a}}}{\prod_{i=1}^{k} \varphi(a_i)} \prod_{i \ne m} \frac{\mu(a_i) r_i}{\varphi(a_i)}. \tag{3.10}$$

Fix j with $1 \le j \le k$. In (3.10), the non-zero terms will have either $a_j = r_j$ or $a_j > D_0 r_j$. The contribution from the terms with $a_j \ne r_j$ is

(3.11)

$$\ll \prod_{i=1}^{k} g(r_i) r_i \biggl(\sum_{\substack{a_j > D_0 r_j \\ r_j \mid a_j}} \frac{\mu(a_j)^2}{\varphi(a_j)^2}\biggr) \biggl(\sum_{\substack{a_m < R \\ (a_m, W_1)=1}} \frac{\mu(a_m)^2}{\varphi(a_m)}\biggr) \prod_{\substack{1 \le i \le k \\ i \ne j,m}} \sum_{r_i \mid a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2}.$$

Now, as before, from (2.2) and (2.3), we have

$$\sum_{\substack{a_j > D_0 r_j \\ r_j \mid a_j}} \frac{\mu(a_j)^2}{\varphi(a_j)^2} \ll \frac{1}{D_0 \varphi(r_j)^2}, \qquad \sum_{\substack{a_m < R \\ (a_m, W_1)=1}} \frac{\mu(a_m)^2}{\varphi(a_m)} \ll \frac{\varphi(W_1)}{W_1}\mathcal{L}$$

and

$$\sum_{r_i \mid a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2} \le \frac{\mu(r_i)^2}{\varphi(r_i)^2} \sum_k \frac{\mu(k)}{\varphi(k)^2} \ll \frac{1}{\varphi(r_i)^2},$$

majorizing (3.11) by

$$\ll \prod_{i=1}^k \frac{g(r_i) r_i}{\varphi(r_i)^2} \frac{\varphi(W_1)}{W_1 D_0}\mathcal{L} \ll \frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}.$$

Hence (3.10) becomes

$$y_{\mathbf{r}}^{(m)} = \prod_{i=1}^k \frac{g(r_i) r_i}{\varphi(r_i)^2} \sum_{a_m} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\biggl(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\biggr),$$

and the proof is completed by applying Lemma 2. ∎

Proof of Proposition 2. Let

$$y_{\max}^{(m)} = \max_{\mathbf{r}} |y_{\mathbf{r}}^{(m)}|,$$

where $y_{\mathbf{r}}^{(m)}$ is defined in Lemma 5. We shall first show that

$$\begin{aligned}(3.12)\qquad & S_2(g,m) \\ &= \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)} \sum_{\mathbf{u}} \frac{(y_{\mathbf{u}}^{(m)})^2}{\prod_{i=1}^k g(u_i)} + O\biggl(\frac{Y\mathcal{L}^{k-2}\varphi^{k-1}(W_1)(y_{\max}^{(m)})^2}{\varphi(q_0)\varphi(W_2)W_1^{k-1}D_0} + \frac{Y\mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\biggr).\end{aligned}$$

From the definition of w_n, we have

$$(3.13)\qquad S_2(g,m) = \sum_{\mathbf{d},\mathbf{e}} \lambda_{\mathbf{d}}\lambda_{\mathbf{e}} \sum_{\substack{n \in \mathcal{A}\cap(\mathcal{A}-h_m) \\ N \le n < 2N,\, n \equiv \nu_0 \bmod W_2 \\ [d_i,e_i] \mid n+h_i\ \forall i}} \varrho_g(n+h_m).$$

As in the proof of Proposition 1, $\sum_{\mathbf{d},\mathbf{e}}$ reduces to $\sum'_{\mathbf{d},\mathbf{e}}$. Let $n' = n + h_m$. Since $n + h_m \equiv a_0 \pmod{q_0}$ for $n \in \mathcal{A}$, the inner sum of (3.13) reduces to

$$T(\mathbf{d},\mathbf{e}) := \sum_{\substack{n' \equiv \nu_0 + h_m \bmod W_2 \\ n' \equiv a_0 \bmod q_0 \\ n' \equiv h_m - h_i \bmod [d_i,e_i]\ \forall i}} X(\mathcal{A}\cap(\mathcal{A}+h_m), n')\varrho_g(n').$$

Recall that $\varrho_g(n') = 0$ if n' is divisible by a prime divisor of $[d_i,e_i]$. Since one condition of the summation is $[d_m,e_m] \mid n'$, we have $T(\mathbf{d},\mathbf{e}) = 0$ unless

$d_m = e_m = 1$. When $d_m = e_m = 1$,

$$T(\mathbf{d},\mathbf{e}) = \sum_{n\equiv a_q \bmod qq_0} X(\mathcal{A}\cap(\mathcal{A}+h_m),n)\varrho_g(n).$$

Here we have

$$q = W_2\prod_{i=1}^{k}[d_i,e_i], \qquad (a_q,q)=1, \qquad a_q \equiv a_0 \pmod{q_0}.$$

For $(a_q,q)=1$, we need $(h_m-h_i,[d_i,e_i])=1$ whenever $m\neq i$, which was noted earlier.

Arguing as in the proof of Proposition 1 and using (1.5) now gives

$$S_2(g,m) = \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)}\sideset{}{'}\sum_{\substack{\mathbf{d},\mathbf{e}\\ d_m=e_m=1}}\frac{\lambda_{\mathbf{d}}\lambda_{\mathbf{e}}}{\prod_{i=1}^{k}\varphi([d_i,e_i])} + O\biggl(\frac{Y\mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\biggr).$$

With a_j and b_j as in (3.6), we follow [9] to obtain

$$S_2(g,m) = \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)}\sum_{\mathbf{u}}\prod_{i=1}^{k}\frac{\mu(u_i)^2}{g(u_i)}\sideset{}{^*}\sum_{s_{1,2},\dots,s_{k,k-1}} \times\prod_{\substack{1\le i,j\le k\\ i\neq j}}\frac{\mu(s_{i,j})^2}{g(s_{i,j})^2}y_{\mathbf{a}}^{(m)}y_{\mathbf{b}}^{(m)} + O\biggl(\frac{Y\mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\biggr). \tag{3.14}$$

Here g is the totally multiplicative function with $g(p)=p-2$ for all p and we have used Lemma 4 with $f_1 = g$.

The contribution to the sum in (3.14) from $s_{i,j}\neq 1$ (for given i,j) is

$$\begin{aligned} &\ll \frac{Y(y_{\max}^{(m)})^2}{\varphi(q_0)\varphi(W_2)\mathcal{L}}\biggl(\sum_{\substack{u<R\\(u,W_1)=1}}\frac{\mu(u)^2}{g(u)}\biggr)^{k-1} \\ &\quad\times\biggl(\sum_{s}\frac{\mu(s)^2}{g(s)^2}\biggr)^{k(k-1)-1}\biggl(\sum_{s_{i,j}>D_0}\frac{\mu(s_{i,j})^2}{g(s_{i,j})^2}\biggr) \\ &= \frac{Y(y_{\max}^{(m)})^2}{\varphi(q_0)\varphi(W_2)\mathcal{L}}V_1V_2V_3, \end{aligned} \tag{3.15}$$

say. Clearly, $V_2\ll 1$. Using (2.2) while mindful of the estimate

$$\frac{1}{g(s)} \ll \frac{1}{s}\sum_{a\mid s}\frac{2^{\omega(a)}}{a}$$

yields

$$V_1 \ll \biggl(\frac{\varphi(W_1)}{W_1}\mathcal{L}\biggr)^{k-1}.$$

From (2.3) and the observation that, for s square-free,

$$\frac{1}{g^2(s)} \ll \frac{1}{s^2}\sum_{a|s}\frac{4^{\omega(a)}}{a} \ll \frac{1}{s^2}\sum_{a|s} a^{-1/2},$$

we get

$$V_3 \ll D_0^{-1}.$$

Note the bound in (3.15) is

$$\ll \frac{Y(y_{\max}^{(m)})^2\mathcal{L}^{k-2}}{\varphi(q_0)\varphi(W_2)}\left(\frac{\varphi(W_1)}{W_1}\right)^{k-1}\frac{1}{D_0},$$

and we have established (3.12).

Now we use Lemma 5 in (3.12), recalling (2.5). When $r_m = 1$, we have

$$(3.16)\qquad y_{\mathbf{r}}^{(m)} = \sum_{(u, W_1\prod_{i=1}^k r_i)=1}\frac{\mu(u)^2}{\varphi(u)} \times F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, \frac{\log u}{\log R}, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) + O\left(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\right).$$

From this, we find that

$$y_{\max}^{(m)} \ll \frac{\varphi(W_1)}{W_1}\mathcal{L}.$$

We shall apply Lemma 1 to (3.16) with $\kappa = 1$,

$$\gamma(p) = \begin{cases} 1, & p \nmid W_1\prod_{i=1}^k r_i,\\ 0, & \text{otherwise,}\end{cases}$$

A_1, A_2 suitably chosen and

$$L \ll \log \mathcal{L}$$

(similar to the proof of (2.2)). Define

$$F_{\mathbf{r}}^{(m)} = \int_0^1 F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, t_m, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) dt_m.$$

We obtain

$$y_{\mathbf{r}}^{(m)} = \log R\,\frac{\varphi(W_1)}{W_1}\left(\prod_{i=1}^k \frac{\varphi(r_i)}{r_i}\right)F_{\mathbf{r}}^{(m)} + O\left(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\right).$$

Inserted into (3.12), the above produces the main term

$$(3.17)\qquad \frac{(\log R)^2 Y_{g,m}\varphi(W_1)^2}{\varphi(q_0)\varphi(W_2)W_1^2}\sum_{\substack{\mathbf{r}\\ (r_i,W_1)=1\,\forall i\\ (r_i,r_j)=1\,\forall i\neq j\\ r_m=1}}\prod_{i=1}^{k}\frac{\varphi(r_i)\mu(r_i)^2}{g(r_i)r_i^2}(F_{\mathbf{r}}^{(m)})^2$$

and an error term of size

$$\ll \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)}\sum_{\substack{\mathbf{r}\\ r_m=1}}\frac{\varphi(W_1)^2\mathcal{L}^2}{W_1^2 D_0\prod_{i=1}^{k} g(r_i)}$$
$$\ll \frac{Y\varphi(W_1)^2\mathcal{L}^2}{\varphi(q_0)\varphi(W_2)W_1^2 D_0}\bigg(\sum_{\substack{r<R\\ (r,W_1)=1}}\frac{1}{g(r)}\bigg)^{k-1} \ll \frac{Y\varphi(W_1)^{k+1}\mathcal{L}^k}{\varphi(q_0)\varphi(W_2)W_1^{k+1}D_0}.$$

Recall that $Y_{g,m}\ll Y\mathcal{L}^{-1}$. Now we remove the condition $(r_i,r_j)=1$ from (3.17). As before, this introduces an error of size

$$\ll \frac{\mathcal{L}^2 Y\varphi(W_1)^2}{\varphi(q_0)\varphi(W_2)W_1^2}\bigg(\sum_{p>D_0}\frac{\varphi(p)^2}{g(p)^2p^2}\bigg)\bigg(\sum_{\substack{r<R\\ (r,W_1)=1}}\frac{\mu(r)^2\varphi(r)}{g(r)r}\bigg)^{k-1}$$
$$\ll \frac{Y\mathcal{L}^k\varphi(W_1)^{k+1}}{\varphi(q_0)\varphi(W_2)W_1^{k+1}D_0}$$

by an application of Lemma 3. Combining all our results, we get

$$S_2(g,m)=\frac{(\log R)^2 Y_{g,m}\varphi(W_1)^2}{\varphi(q_0)\varphi(W_2)W_1^2}\sum_{\substack{\mathbf{r}\\ (r_i,W_1)=1\,\forall i\\ r_m=1}}\prod_{i=1}^{k}\frac{\varphi(r_i)^2\mu(r_i)^2}{g(r_i)r_i^2}(F_{\mathbf{r}}^{(m)})^2$$
$$+O\bigg(\frac{Y\varphi(W_1)^{k+1}\mathcal{L}^k}{\varphi(q_0)\varphi(W_2)W_1^{k+1}D_0}\bigg).$$

The last sum is evaluated by applying Lemma 1 to each summation variable in turn, taking

$$\gamma(p)=\begin{cases}\dfrac{p^3-2p^2+p}{p^3-p^2-2p+1}, & p\nmid W_1,\\ 0, & p\mid W_1,\end{cases}$$

to produce the right value of $\gamma(p)/(p-\gamma(p))$. Of course,

$$S=\frac{\varphi(W_1)}{W_1}(1+O(D_0^{-1}))$$

by Lemma 2, while $L\ll\log\mathcal{L}$. Our final conclusion is that

$$S_2(g,m)=\frac{(\log R)^{k+1}Y_{g,m}\varphi(W_1)^{k+1}J_k^{(m)}}{\varphi(q_0)\varphi(W_2)W_1^{k+1}}(1+o(1)),$$

completing the proof of Proposition 2. ∎

4. Further lemmas. Let $\gamma = \alpha^{-1}$. As noted in [4], the set of $[\alpha m + \beta]$ in $[N, 2N)$ may be written as

$$\{n \in [N, 2N) : \gamma n \in (\gamma\beta - \gamma, \beta\gamma] \ (\mathrm{mod}\ 1)\}.$$

LEMMA 6. *Let $I = (a, b)$ be an interval of length l with $0 < l < 1$, and let h be a natural number satisfying*

$$0 < -h\gamma < 2\varepsilon \ (\mathrm{mod}\ 1),$$

where $2\varepsilon < l$. Let

$$\mathcal{A} = \{n \in [N, 2N) : \gamma n \in I \ (\mathrm{mod}\ 1)\}.$$

Then

$$\mathcal{A} \cap (\mathcal{A} + h) = \{n \in [N + h, 2N) : \gamma n \in J \ (\mathrm{mod}\ 1)\},$$

where J is an interval of length l' with

$$l - 2\varepsilon < l' < l.$$

Proof. Let $t \equiv -h\gamma \ (\mathrm{mod}\ 1)$, $0 < t < 2\varepsilon$. Clearly $\mathcal{A} \cap (\mathcal{A} + h)$ consists of the integers in $[N + h, 2N)$ for which

$$\gamma n \in (a, b) \ (\mathrm{mod}\ 1), \qquad \gamma n + t \in (a, b) \ (\mathrm{mod}\ 1).$$

The lemma follows with $J = (a, b - t)$. ∎

LEMMA 7. *Let I be an interval of length l, $0 < l < 1$. Let $x_1, \ldots, x_N$ be real. Then:*

(i) *There exists z such that*

$$\#\{j \le N : x_j \in z + I \ (\mathrm{mod}\ 1)\} \ge Nl.$$

(ii) *We have (for $a_j \ge 0$, $j = 1, \ldots, N$, and $L \ge 1$)*

$$\sum_{\substack{j=1 \\ x_j \in I \ \mathrm{mod}\ 1}}^{N} a_j - l \sum_{j=1}^{N} a_j \ll L^{-1} \sum_{j=1}^{N} a_j + \sum_{h=1}^{L} h^{-1} \Big| \sum_{j=1}^{N} a_j e(hx_j) \Big|.$$

Proof. We leave (i) as an exercise; (ii) is a slight variant of [1, Theorem 2.1]. ∎

LEMMA 8. *Let $1 \le Q \le N$ and F a non-negative function defined on Dirichlet characters. Then for some Q_1, $1 \le Q_1 \le Q$,*

$$\sum_{q \le Q} \sideset{}{'}\sum_{\chi \bmod q} F(\hat{\chi}) \ll \frac{\mathcal{L}Q}{Q_1} \sum_{Q_1 \le q_1 < 2Q_1} \sideset{}{^\star}\sum_{\psi \bmod q_1} F(\psi).$$

Proof. We recall that $\hat{\chi}$ is the primitive character that induces χ, so that $F(\hat{\chi})$ may be quite different from $F(\chi)$.

The left-hand side of the claimed inequality is

$$\sum_{q_1\le Q}\ \sideset{}{^\star}\sum_{\psi \bmod q_1} F(\psi) \sum_{\substack{\chi \bmod q\\ q\le Q,\, q_1\mid q\\ \psi \text{ induces } \chi}} 1 \le \sum_{q_1\le Q}\ \sideset{}{^\star}\sum_{\psi \bmod q_1} F(\psi)\frac{Q}{q_1}.$$

The lemma follows on applying a splitting-up argument to q_1. ■

Lemma 9. *Let $f(j)$ $(j\ge 1)$ be a periodic function with period q,*

$$S(f,n)=\sum_{j=1}^{n} f(j)e\Big(-\frac{nj}{q}\Big),$$

$F>0$, and $R\ge 1$. Let $H(y)$ be a real function with $H'(y)$ monotonic and

$$|H'(y))|\le Fy^{-1}$$

for $R\le y\le 2R$. Then for $J=[R,R']$ with $R<R'\le 2R$,

$$\sum_{m\in J} f(m)H(m)-q^{-1}\sum_{1\le|n|\le 2FqR^{-1}} S(f,n)\int_J e\Big(\frac{ny}{q}+H(y)\Big)\,dy$$
$$\ll \frac{R|S(f,0)|}{qF}+\sum_{|n|\in J'}\frac{|S(f,n)|}{n},$$

where

$$J'=[\min\{2FqR^{-1},q/2\},\max\{2FqR^{-1},q\}+q].$$

Proof. This is [2, Theorem 8]. ■

For a finite sequence $\{a_k : K\le k<K'\}$, set

$$\|a\|_2=\Big(\sum_{K\le k<K'}|a_k|^2\Big)^{1/2}.$$

Lemma 10. *Let $R,M,H\ge 1$. Let β be real and*

$$\Big|\beta-\frac{u_1}{r_1}\Big|\le\frac{H}{r_1^2}, \tag{4.1}$$

where $r_1\ge H$ and $(u_1,r_1)=1$. Then for $M_1\in\mathbb{N}$,

$$\sum_{m=M_1+1}^{M_1+M}\min\Big(R,\frac{1}{\|m\beta\|}\Big)\ll\Big(\frac{HM}{r_1}+1\Big)(R+r_1\log r_1). \tag{4.2}$$

If $M<r_1$ and

$$M\Big|\beta-\frac{u_1}{r_1}\Big|\le\frac{1}{2r_1},$$

then

$$\sum_{m=1}^{M} \frac{1}{\|m\beta\|} \ll r_1 \log 2r_1. \tag{4.3}$$

Proof. For (4.2), it suffices to show that a block of $[r_1/H]$ consecutive m's contributes

$$\ll R + \sum_{l=1}^{r_1} \frac{r_1}{l}.$$

Writing $m = m_0 + j$, $1 \le j \le [r_1/H]$, yields

$$\left|(m_0 + j)\beta - m_0\beta - \frac{ju_1}{r_1}\right| \le \frac{jH}{r_1^2} \le \frac{1}{r_1},$$

so there are $O(1)$ values of j for which the bound

$$\|(m_0 + j)\beta\| \ge \frac{1}{2}\left\|m_0\beta + \frac{ju_1}{r_1}\right\|$$

fails. Our block estimate follows immediately.

The argument for (4.3) is similar. In this case,

$$\left|m\beta - \frac{mu_1}{r_1}\right| \le \frac{1}{2r_1}$$

if $1 \le m \le M$. Therefore, the left-hand side of (4.3) can be estimated by $\sum_{l=1}^{r_1} r_1/l$. ∎

LEMMA 11. *Let $N < N' \le 2N$, $MK \asymp N$ and $N \ge K \ge M \ge 1$. Suppose that*

$$\left|\gamma - \frac{u}{r}\right| \le \frac{H}{r^2}, \qquad (u, r) = 1,\ H \le r \le N. \tag{4.4}$$

Let $(a_m)_{M \le m < 2M}$ and $(b_k)_{K \le k < 2K}$ be sequences of complex numbers. Then

$$S := \sum_{Q \le q < 2Q} \sum_{\chi \bmod q} \Big| \sum_{\substack{M \le m < 2M \\ N \le mk < N'}} \sum_{K \le k < 2K} a_m b_k \chi(mk) e(\gamma mk) \Big| \tag{4.5}$$

satisfies the bound

$$S \ll \|a\|_2 \|b\|_2 \mathcal{L}^{3/2} D^{1/2} \times \left(Q^2 M^{1/2} + \frac{Q^{3/2} H^{1/2} N^{1/2}}{r^{1/2}} + Q^{3/2} H^{1/2} K^{1/2} + Q^{3/2} r^{1/2}\right),$$

where

$$D = \max_{n < N} \#\{q \in [Q, 2Q) : n = lq\}.$$

Proof. Let S' be the sum obtained from S by removing the condition $N \le mk < N'$. It suffices to prove the same bound, with $\mathcal{L}^{1/2}$ in place of $\mathcal{L}^{3/2}$, for S', since the condition can be restored at the cost of a factor of $\mathcal{L}$. See [8, Section 3.2].

We have

$$S' \le \sum_{Q\le q<2Q} \sum_{\chi \bmod q} \sum_{M\le m<2M} |a_m| \Big| \sum_{K\le k<2K} b_k \chi(k) e(\gamma mk)\Big| = \sum_q S_q,$$

say. We may also assume that $b_k = 0$ if $(k,q) > 1$. By Cauchy's inequality, and with summations subject to the obvious restrictions on m, k_1 and k_2,

$$S_q^2 \le \varphi(q)\|a\|_2^2 \sum_{\chi \bmod q} \sum_m \sum_{k_1} \sum_{k_2} b_{k_1}\bar{b}_{k_2}\chi(k_1)\overline{\chi}(k_2)e(\gamma m(k_1-k_2)).$$

Bringing the sum over χ inside we see that the right-hand side of the above is

$$\varphi(q)^2\|a\|_2^2 \sum_{\substack{k_1,k_2\\ k_1\equiv k_2 \bmod q}} b_{k_1}\bar{b}_{k_2} \sum_m e(\gamma m(k_1-k_2))$$
$$\le \varphi(q)^2\|a\|_2^2 \sum_{k_1} |b_{k_1}|^2 \sum_{k_1\equiv k_2 \bmod q} \Big|\sum_m e(\gamma m(k_1-k_2))\Big|$$

upon using the parallelogram rule

$$|b_{k_1}b_{k_2}| \le \tfrac{1}{2}(|b_{k_1}|^2 + |b_{k_2}|^2).$$

Now summing the geometric sum over m and then summing over q, we see that

(4.6)
$$\sum_{Q\le q<2Q} S_q^2 \ll Q^3\|a\|_2^2\|b\|_2^2 M + Q^2\|a\|_2^2\|b\|_2^2 \sum_{Q\le q<2Q}\sum_{1\le l<K/q} \min\Big(M, \frac{1}{\|\gamma lq\|}\Big).$$

Now we combine the variables l and q and then apply (4.2), which leads to

$$\sum_{Q\le q<2Q} S_q^2 \ll Q^3\|a\|_2^2\|b\|_2^2 M + Q^2\|a\|_2^2\|b\|_2^2 D\Big(\frac{HK}{r}+1\Big)(M + r\log r)$$
$$\ll \|a\|_2^2\|b\|_2^2\Big(Q^3M + \mathcal{L}Q^2D\Big(\frac{HN}{r} + HK + M + r\Big)\Big).$$

The desired bound for S' follows by another application of Cauchy's inequality. ∎

LEMMA 12. *Under the hypotheses of Lemma* 11*, suppose that* $4MQ < N$*,* $b_k = 1$ *for* $K \le k < 2K$ *and* $|a_m| \le 1$ *for* $M \le m < 2M$*. Define* D *as in Lemma* 11*. Then:*

(i) *We have*

$$S \ll Q^{3/2}\mathcal{L}D\Big(\frac{QMH}{r}+1\Big)\Big(\frac{K}{Q}+r\Big).$$

(ii) *If* $4MQ < r$ *and*

$$4MQ\left|\gamma - \frac{u}{r}\right| \le \frac{1}{2r},$$

then

$$S \ll \mathcal{L}DQ^{3/2}r.$$

Proof. Let I_m (here and after) denote a subinterval of $[N/m, N'/m)$. We have

$$S \le QS^* + S^{**},$$

where, for a suitably chosen non-principal $\chi_q \pmod q$,

$$S^* = \sum_{Q\le q<2Q}\ \sum_{M\le m<2M}\left|\sum_{k\in I_m}\chi_q(k)e(\gamma mk)\right|,$$

$$S^{**} = \sum_{Q\le q<2Q}\ \sum_{M\le m<2M}\left|\sum_{k\in I_m}\chi_0(k)e(\gamma mk)\right|.$$

To prove part (1), it suffices to show that

$$S^* \ll Q^{1/2}\mathcal{L}D\left(\frac{QMH}{r}+1\right)\left(\frac{K}{Q}+r\right),$$

$$S^{**} \ll Q\mathcal{L}D\left(\frac{QMH}{r}+1\right)\left(\frac{K}{Q}+1\right).$$

We give the proof for S^*; the proof for S^{**} is similar.

Given q and m, Lemma 9 together with

$$|S(\chi, q)| \le \sqrt{q}$$

(see [6, Chapter 9]) gives

$$\sum_{k\in I_m}\chi_q(k)e(\gamma mk) - \frac{1}{q}\sum_{1\le|n|\ll Mq} S(\chi_q, n)\int_{I_m} e\left(\left(\frac{n}{q}+\gamma m\right)y\right)dy$$
$$\ll q^{-1/2}M^{-1} + q^{1/2}\sum_{1\le n\ll Mq} n^{-1} \ll q^{1/2}\mathcal{L}.$$

Therefore

$$\sum_{k\in I_m}\chi_q(k)e(\gamma mk) \ll q^{1/2}\mathcal{L} + q^{-1/2}\sum_{1\le|n|\ll Mq}\min\left(K, \frac{1}{|\gamma m - n/q|}\right).$$

Summing over m and q, we get

$$S^* \ll MQ^{3/2}\mathcal{L} + Q^{1/2}\sum_{Q\le q<2Q}\ \sum_{M\le m<2M}\ \sum_{1\le|n|\ll Mq}\min\left(\frac{K}{Q}, \frac{1}{|\gamma mq - n|}\right).$$

The contribution to the right-hand side from n's with $|n-\gamma mq| > 1/2$ is

$$\ll MQ^{3/2}\mathcal{L}.$$

Combining the variables m and q, we see that

$$S^* \ll MQ^{3/2}\mathcal{L} + Q^{1/2}D \sum_{MQ \le m' < 4MQ} \min\left(\frac{K}{Q}, \frac{1}{\|\gamma m'\|}\right). \tag{4.7}$$

We deduce the desired bound for S^* by applying (4.2).

Now for part (2), we note that (4.3) is applicable to the reciprocal sum in (4.7) with $4MQ$ and γ in place of M and β. Hence

$$S^* \ll MQ^{3/2}\mathcal{L} + Q^{1/2}Dr\log 2r \ll D\mathcal{L}Q^{1/2}r$$

since $4MQ < r$. Similarly $S^{**} \ll D\mathcal{L}Qr$, and part (2) follows. ∎

Lemma 13. *Suppose that*

$$\left|\gamma - \frac{u}{r}\right| \le \frac{\mathcal{L}^{A+1}}{r^2}$$

with $(u,r) = 1$, *and that* $r^2 \le N \le r^2\mathcal{L}^{2A+2}$. *Then:*

(i) *For* $Q < N^{2/7-\varepsilon}$, $N^{4/7} \ll K \ll N^{5/7}$ *and any* a_m, b_k *with* $|a_m| \le \tau(m)^B$, $|b_k| \le \tau(k)^B$, *where* B *is an absolute constant, the sum* S *in* (4.5) *satisfies the bound*

$$S \ll QN^{1-\varepsilon/4}. \tag{4.8}$$

(ii) *For* $Q \le N^{2/7-\varepsilon}$, $M \ll N^{4/7}$ *and* $b_k = 1$ *for* $K \le k < 2K$, $|a_m| \le 1$ *for* $M \le m < 2M$, *the sum* S *in* (4.5) *satisfies* (4.8).

Proof. In order to prove (i), we use Lemma 11. As $D \ll N^{\varepsilon/15}$,

$$\begin{aligned} SQ^{-1}N^{-1+\varepsilon/4} &\ll Q^{-1}N^{-1/2+\varepsilon/3}(Q^2N^{3/14} + Q^{3/2}N^{5/14}) \\ &\ll N^{-1/2+\varepsilon/2}(QN^{3/14} + Q^{1/2}N^{5/14}) \ll 1. \end{aligned}$$

To prove (ii), we consider two cases. If $K < N^{1-\varepsilon}$, then by Lemma 12(i),

$$\begin{aligned} SQ^{-1}N^{-1+\varepsilon/4} &\ll Q^{1/2}N^{-1+\varepsilon/2}\left(N^{1/2} + MQ + \frac{N^{1-\varepsilon}}{Q}\right) \\ &\ll N^{1/7-1/2+\varepsilon} + N^{3/7+4/7-1-\varepsilon} + N^{-\varepsilon/2} \ll 1. \end{aligned}$$

If $K \ge N^{1-\varepsilon}$, then $M \ll N^{\varepsilon}$ and Lemma 12(ii) is applicable since

$$4MQ\left|\gamma - \frac{u}{r}\right| \ll N^{-1+2/7+\varepsilon}.$$

Hence

$$SQ^{-1}N^{-1+\varepsilon/4} \ll Q^{1/2}N^{-1/2+\varepsilon} \ll 1,$$

giving the desired majorant. ∎

Lemma 14. *Let f be an arbitrary complex function on $[N, 2N)$. Let $N < N' \le 2N$. The sum*

$$S = \sum_{N \le n < N'} \Lambda(n) f(n)$$

can be decomposed into $O(\mathcal{L}^2)$ sums of the form

$$\sum_{M < m \le 2M} a_m \sum_{\substack{K \le k < 2K \\ N \le mk < N'}} f(mk) \quad \text{or} \quad \int_N^{N'} \sum_{M \le m < 2M} a_m \sum_{\substack{k \ge w \\ K \le k < 2K \\ N \le mk < N'}} f(mk) \frac{dw}{w}$$

with $M \le N^{1/4}$ and $|a_m| \le 1$, together with $O(\mathcal{L})$ sums of the form

$$\sum_{M < m \le 2M} a_m \sum_{\substack{K \le k < 2K \\ N \le mk < N'}} b_k f(mk)$$

with $N^{1/2} \le K \ll N^{3/4}$ and $\|a\|_2 \|b\|_2 \ll N^{1/2} \mathcal{L}^2$.

Proof. This follows from the arguments in [6, Chapter 24] by taking $U = V = N^{1/4}$. ∎

We record a special case of [3, Lemma 14]. For more background on the "Harman sieve", see [8].

Lemma 15. *Let $W(n)$ be a complex function with support in $(N, 2N] \cap \mathbb{Z}$, $|W(n)| \le N^{1/\varepsilon}$. For $r \in \mathbb{N}$, $z \ge 2$, let*

$$S^*(r, z) = \sum_{(n, P(z)) = 1} W(rn). \tag{4.9}$$

Suppose that for some constant $c > 0$, $0 \le d \le 1/2$, and for some $Y > 0$, we have, for any coefficients a_m, b_k with $|a_m| \le 1$, $|b_k| \le \tau(k)$,

$$\sum_{m \le 2N^c} a_m \sum_k W(mk) \ll Y \tag{4.10}$$

and

$$\sum_{N^c \le m \le 2N^{c+d}} a_m \sum_k b_k W(mk) \ll Y. \tag{4.11}$$

Let u_r $(r \le N^c)$ be complex numbers such that $|u_r| \le 1$ and $u_r = 0$ for $(r, P(N^\varepsilon)) > 1$. Then

$$\sum_{r \le (2N)^c} u_r S^*(r, (2N)^d) \ll Y \mathcal{L}^3.$$

The following application of Lemma 15 will be used in the proof of Theorem 3. We take

$$W(n)=\sum_{Q\le q<2Q}\sum_{\chi \bmod q}\eta_\chi\chi(n)e(\gamma n) \tag{4.12}$$

for $N\le n<N'$; otherwise, $W(n)=0$. Here η_χ is arbitrary with $|\eta_\chi|\le 1$.

LEMMA 16. *Suppose that*

$$\left|\gamma-\frac{u}{r}\right|\le\frac{\mathcal{L}^{A+1}}{r^2},\qquad (u,r)=1,\qquad N=r^2,\qquad 1\le Q\le N^{2/7-\varepsilon}.$$

Define $S^*(r,z)$ *as above with* W *defined in* (4.12). *Then*

$$\sum_{r\le(2N)^{4/7}}u_rS^*(r,(2N)^{1/7})\ll N\mathcal{L}^{-A}$$

for every $A>0$, *provided that* $|u_r|\le 1$ *and* $u_r=0$ *for* $(r,P(N^\varepsilon))>1$.

Proof. We need to verify (4.10) and (4.11) with $c=4/7$, $d=1/7$ and $Y=N\mathcal{L}^{-A-3}$. This is an application of Lemma 13. ∎

We now introduce some subsets of $\mathbb{R}^j$ needed in the proof of Theorem 3. Write E_j for the set of j-tuples $\boldsymbol{\alpha}_j=(\alpha_1,\dots,\alpha_j)$ satisfying

$$1/7\le\alpha_j<\alpha_{j-1}<\dots<\alpha_1\le 1/2\quad\text{and}\quad\alpha_1+\dots+\alpha_{j-1}+2\alpha_j\le 1.$$

A tuple $\boldsymbol{\alpha}_j$ is said to be *good* if some subsum of $\alpha_1+\dots+\alpha_j$ is in $[2/7,3/7]\cup[4/7,5/7]$, and *bad* otherwise.

We use the notation $p_j=(2N)^{\alpha_j}$. For instance, the sum

$$\sum_{\substack{p_1p_2n_3=k\\(2N)^{1/7}\le p_2<p_1<(2N)^{1/2}}}\psi(n_3,p_2)$$

will be written as

$$\sum_{\substack{p_1p_2n_3=k\\ \boldsymbol{\alpha}_2\in E_2}}\psi(n_3,p_2).$$

LEMMA 17. *Let* γ, u/r, N, Q *be as in Lemma* 16, *and* E *be a subset of* E_j *defined by a bounded number of inequalities of the form*

$$c_1\alpha_1+\dots+c_j\alpha_j<c_{j+1}\quad(\text{or }\le c_{j+1}). \tag{4.13}$$

Suppose that all points in E *are good and that throughout* E, z_j *is either the function* $z_j=(2N)^{\alpha_j}$ *or the constant* $z_j=(2N)^{1/7}$. *Then for arbitrary* η_χ *with* $|\eta_\chi|\le 1$,

$$\sum_{Q\le q<2Q}\sum_{\chi \bmod q}\eta_\chi\sum_{\substack{N\le p_1\cdots p_jn_{j+1}<N'\\ \boldsymbol{\alpha}_j\in E}}\chi(p_1\cdots p_jn_{j+1})e(\gamma p_1\cdots p_jn_{j+1})\psi(n_{j+1},z_j)\ll N\mathcal{L}^{-A}$$

for every $A>0$.

Proof. This is a consequence of Lemma 13(i). On grouping a subset of the variables as a product $m = \prod_{i\in\mathcal{S}} p_i$, with $S \subset \{1,\ldots,j\}$, we obtain a sum S of the form appearing in Lemma 13(i), except that a bounded number of inequalities of the form (4.13) are present. These inequalities may be removed at the cost of a log power, by the mechanism noted earlier. See [3, p. 184] for a few more details of a similar argument. The lemma follows at once. ∎

LEMMA 18. *Let* $D = \{(\alpha_1,\alpha_2) \in E_2 : (\alpha_1,\alpha_2)$ *is bad,* $\alpha_1 + 2\alpha_2 > 5/7\}$. *Then*

$$X(\mathbb{P};n) - \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in D}} \psi(n_3,p_2) = \varrho_1(n) + \varrho_2(n) + \varrho_3(n) - \varrho_4(n) - \varrho_5(n).$$

Here

$$\varrho_1(n) = \psi(n,(2N)^{1/7}),$$
$$\varrho_2(n) = \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in E_2\setminus D}} \psi(n_3,(2N)^{1/7}), \qquad \varrho_3(n) = \sum_{\substack{p_1p_2p_3p_4n_5=n\\ \boldsymbol{\alpha}_4\in E_4\\ (\alpha_1,\alpha_2)\in E_2\setminus D}} \psi(n_5,p_4),$$
$$\varrho_4(n) = \sum_{\substack{p_1n_2=n\\ \boldsymbol{\alpha}_1\in E_1}} \psi(n_2,(2N)^{1/7}), \qquad \varrho_5(n) = \sum_{\substack{p_1p_2p_3n_4=n\\ \boldsymbol{\alpha}_3\in E_3\\ (\alpha_1,\alpha_2)\in E_2\setminus D}} \psi(n_4,(2N)^{1/7}).$$

Proof. We repeatedly use Buchstab's identity in the form

$$\psi(m,z) = \psi(m,w) - \sum_{\substack{ph=m\\ w\le p<z}} \psi(h,p) \qquad (2 \le w < z).$$

Thus

$$\begin{aligned} X(\mathbb{P};n) &= \psi(n,(2N)^{1/2}) \\ &= \psi(n,(2N)^{1/7}) - \sum_{\substack{(2N)^{1/7}\le p_1<(2N)^{1/2}\\ p_1n_2=n}} \psi(n_2,p_1) \\ &= \varrho_1(n) - \varrho_4(n) + \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in E_2}} \psi(n_3,p_2), \end{aligned} \tag{4.14}$$

$$X(\mathbb{P};n) - \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in D}} \psi(n_3,p_2) = \varrho_1(n) - \varrho_4(n) + \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in E_2\setminus D}} \psi(n_3,p_2).$$

The last sum decomposes as

$$\sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in E_2\setminus D}} \psi(n_3,p_2) = \sum_{\substack{p_1p_2n_3=n\\ \boldsymbol{\alpha}_2\in E_2\setminus D}} \psi(n_3,(2N)^{1/7}) \tag{4.15}$$

$$- \sum_{\substack{p_1p_2p_3n_4=n\\ \boldsymbol{\alpha}_3\in E_3\\ (\alpha_1,\alpha_2)\in E_2\setminus D}} \psi(n_4,(2N)^{1/7}) + \sum_{\substack{p_1p_2p_3p_4n_5=n\\ \boldsymbol{\alpha}_4\in E_4\\ (\alpha_1,\alpha_2)\in E_2\setminus D}} \psi(n_5,p_4).$$

Combining (4.14) and (4.15), we complete the proof of the lemma. ∎

Lemma 19. *Let r, u/r, N and Q be as in Lemma 16 with $\varrho_1,\ldots,\varrho_5$ as in Lemma 18. Then*

$$\sum_{Q\le q<2Q}\ \sum_{\chi \bmod q} \eta_\chi \sum_{N\le n<N} \varrho_j(n)\chi(n)e(\gamma n) \ll QN\mathcal{L}^{-A}$$

for arbitrary η_χ with $|\eta_\chi|\le 1$ and any $A>0$.

Proof. This follows from Lemmas 16 and 17 for $j=1,2,4,5$ on noting that $\alpha_1+\alpha_2+\alpha_3\le\alpha_1+2\alpha_2\le 5/7$ for $j=5$, so that either $\boldsymbol{\alpha}_3$ is good or $\alpha_1+\alpha_2+\alpha_3<4/7$ (similarly for $j=2$). For $j=3$, we need to show that each $\boldsymbol{\alpha}_4$ counted is good. Suppose that some $\boldsymbol{\alpha}_4$ is bad. Then we have $\alpha_1+\alpha_2+\alpha_3+2\alpha_4\le 1$. Hence $\alpha_1+\alpha_2+\alpha_3\le 5/7$, from which we infer that $\alpha_1+\alpha_2+\alpha_3<4/7$. Therefore, $\alpha_1+\alpha_2<3/7$. But we know that $\alpha_1+\alpha_2>2/7$. This makes $\boldsymbol{\alpha}_4$ good, a contradiction. ∎

5. Proof of Theorems 4 and 5

Proof of Theorem 4. With a suitable choice of a_q with $(a_q,q)=1$, we have

$$\begin{aligned}
&\max_{(a,q)=1} E(N,N',\gamma,q,a)\\
&\quad\le \sup_I \Bigl| \sum_{\substack{N\le n<N'\\ \gamma n\in I \bmod 1\\ n\equiv a_q \bmod q}} \Lambda(n) - |I| \sum_{\substack{N\le n<N'\\ n\equiv a_q \bmod q}} \Lambda(n)\Bigr| + \Bigl| \sum_{\substack{N\le n<N'\\ n\equiv a_q \bmod q}} \Lambda(n) - \frac{N'-N}{\varphi(q)}\Bigr|\\
&\quad= T_1(q)+T_2(q),
\end{aligned}$$

say. In view of the Bombieri–Vinogradov theorem, we need only bound $\sum_q T_1(q)$, which is, by Lemma 7,

$$\begin{aligned}
&\ll \sum_{q\le N^{1/4-\varepsilon}} \mathcal{L}^{-A-1} \sum_{\substack{N\le n<N'\\ n\equiv a_q \bmod q}} \Lambda(n)\\
&\quad+ \sum_{q\le \min(r,N^{1/4})N^{-\varepsilon}}\ \sum_{h\le \mathcal{L}^{A+1}} \frac{1}{h} \Bigl| \sum_{\substack{N\le n<N'\\ n\equiv a_q \bmod q}} \Lambda(n)e(\gamma nh)\Bigr|.
\end{aligned}$$

Let $H=\mathcal{L}^{A+1}$. In view of the Brun–Titchmarsh inequality, it remains to show that for $1\le h\le H$,

$$\sum_{q\le \min(N^{1/4},r)N^{-\varepsilon}} \Bigl| \sum_{\substack{N\le n<N'\\ n\equiv a_q \bmod q}} \Lambda(n)e(\gamma nh)\Bigr| \ll N\mathcal{L}^{-A-1}.$$

Reducing hu/r to lowest terms, we need only show that

$$\sum_{q\le \min(N^{1/4},r)N^{-\varepsilon/2}} \eta_q \sum_{\substack{N\le n<N' \\ n\equiv a_q \bmod q}} \Lambda(n)e(\gamma n) \ll N\mathcal{L}^{-A-1}$$

under the modified hypothesis (4.4) on γ (with $H = \mathcal{L}^{A+1}$), whenever $|\eta_q| \le 1$.

Using Lemma 14, it suffices to show that

$$\sum_{q\le \min(N^{1/4},r)N^{-\varepsilon/2}} \eta_q \sum_{\substack{M\le m<2M \\ N\le mk<N' \\ mk\equiv a_q \bmod q}} \sum_{K\le k<2K} a_m b_k e(\gamma mk) \ll N\mathcal{L}^{-A-3} \tag{5.1}$$

under either of the following sets of conditions:

(a) $\|a\|_2\|b\|_2 \ll N^{1/2}\mathcal{L}^2$ and $N^{1/2} \le K \le N^{3/4}$;
(b) $|a_m| \le 1$, $b_k = 1$ for $k \in I_m \subset [K, 2K)$, $b_k = 0$ otherwise, $M \le N^{1/4}$.

We use Dirichlet characters to detect the congruence relation in (5.1), and we require the estimate

$$\sum_{q\le \min(N^{1/4},r)N^{-\varepsilon/2}} \frac{\eta_q}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a_q) \sum_{\substack{M\le m<2M \\ N\le mk<N'}} \sum_{K\le k<2K} a_m b_k \chi(mk) e(\gamma mk) \ll N\mathcal{L}^{-A-4}.$$

It suffices to show that

$$S := \sum_{Q\le q<2Q} \sum_{\chi \bmod q} \Big| \sum_{\substack{M\le m<2M \\ N\le mk<N'}} \sum_{K\le k<2K} a_m b_k \chi(mk) e(\gamma mk)\Big| \ll QN\mathcal{L}^{-A-6} \tag{5.2}$$

for $Q \le \min(N^{1/4}, r)N^{-\varepsilon/2}$.

In case (a), we apply Lemma 11, which gives

$$\begin{aligned} S &\ll N^{1/2+\varepsilon/6}\Big(Q^2M^{1/2} + \frac{Q^{3/2}N^{1/2}}{r^{1/2}} + Q^{3/2}K^{1/2} + Q^{3/2}r^{1/2}\Big) \\ &\ll N^{3/4+\varepsilon/6}Q^2 + \frac{N^{1+\varepsilon/6}Q^{3/2}}{r^{1/2}} + Q^{3/2}N^{7/8+\varepsilon/6}. \end{aligned}$$

Each one of these three terms is $\ll QN\mathcal{L}^{-A-6}$ as

$$\begin{gathered} N^{3/4+\varepsilon/6}Q^2(QN\mathcal{L}^{-A-6})^{-1} \ll QN^{-1/4+\varepsilon/5} \ll 1, \\ N^{1+\varepsilon/6}Q^{3/2}r^{-1/2}(QN\mathcal{L}^{-A-6})^{-1} \ll Q^{1/2}N^{\varepsilon/4}r^{-1/2} \ll 1, \end{gathered}$$

since $Q \le rN^{-\varepsilon/2}$, and

$$N^{7/8+\varepsilon/6}Q^{3/2}(QN\mathcal{L}^{-A-6})^{-1} \ll N^{-1/8+\varepsilon/5}Q^{1/2} \ll 1.$$

In case (b), we use Lemma 12. Suppose that $K < N^{1-\varepsilon/4}$; Lemma 12(i) gives

$$S \ll Q^{3/2} N^{\varepsilon/6}\left(\frac{N}{r} + QM + \frac{K}{Q} + r\right).$$

Each of the above four terms is $\ll QN\mathcal{L}^{-A-6}$, since

$$\begin{aligned}
\frac{Q^{3/2}N^{1+\varepsilon/6}}{r}(QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{1/2}r^{-1}N^{\varepsilon/5} \ll 1,\\
Q^{5/2}N^{\varepsilon/6}M(QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{3/2}N^{-3/4+\varepsilon/5} \ll 1,\\
Q^{1/2}N^{\varepsilon/6}K(QN\mathcal{L}^{-A-6})^{-1} &\ll KN^{-1+\varepsilon/4} \ll 1,\\
Q^{3/2}N^{\varepsilon/6}r(QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{1/2}N^{-1/4+\varepsilon/5} \ll 1.
\end{aligned}$$

Now suppose that $K \ge N^{1-\varepsilon/4}$. Then

$$4MQ \ll QN^{\varepsilon/4}, \quad \text{thus} \quad 4MQ < r$$

and

$$4MQr\left|\gamma - \frac{u}{r}\right| \ll MQN^{-3/4}, \quad \text{hence} \quad 4MQr\left|\gamma - \frac{u}{r}\right| \le \frac{1}{2}.$$

So of Lemma 12(ii) gives comfortably

$$S \ll N^{\varepsilon}Q^{3/2}r \ll QN\mathcal{L}^{-A-6},$$

completing the proof. ∎

Proof of Theorem 5. We first show that the contribution to the sum in (1.10) from $q \le \mathcal{L}^{A+1}$ is

$$\ll N^2\mathcal{L}^{-A} \ll NR.$$

Since, for some $Q \le \mathcal{L}^{A+1}$,

$$\begin{aligned}
\sum_{q\le\mathcal{L}^{A+1}} \sum_{\substack{a=1\\(a,q)=1}}^{q} E^2 &\ll N \sum_{q\le\mathcal{L}^{A+1}} \frac{1}{\varphi(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} E(N, N', \gamma, q, a)\\
&\ll \frac{N\mathcal{L}}{Q} \sum_{Q\le q<2Q} \max_{(a,q)=1} E(N, N', \gamma, q, a),
\end{aligned}$$

it suffices to show for this Q that

$$\sum_{Q\le q<2Q} \max_{(a,q)=1} E(N, N', \gamma, q, a) \ll QN\mathcal{L}^{-A-1}. \tag{5.3}$$

We may suppose that A is large. Arguing as in the proof of Theorem 4, we need only show that (5.2) follows from either (a) or (b). By Dirichlet's theorem, there is a rational approximation b/r to γ satisfying (1.7). For any $\eta > 0$,

$$N^{-3/4} \ge \|\gamma r\| \gg \exp(-r^{\eta}),$$

hence $r \gg \mathcal{L}^{5A}$. Now we apply Lemma 11 to prove the desired bound under (a). Since $D \le Q \le \mathcal{L}^{A+1}$, the term

$$\|a\|_2\|b\|_2\mathcal{L}^2 D^{1/2}Q^{3/2}H^{1/2}N^{1/2}r^{-1/2}$$

presents no difficulty; the other terms are clearly all small enough. For the bound under (b), a similar remark applies to Lemma 12 and the terms

$$\begin{array}{ll} Q^{3/2}\mathcal{L}DNHr^{-1} & \text{if } K < N^{1-\varepsilon/4}, \\ \mathcal{L}DQ^{3/2}r & \text{if } K \ge N^{1-\varepsilon/4}. \end{array}$$

This establishes (5.3).

It remains to examine the contribution to the sum in (1.10) from $q \in [Q, 2Q)$ with $\mathcal{L}^{A+1} \le Q \le R$. We have

$$\begin{aligned} &\sum_{Q\le q<2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} E(N,N',\gamma,q,a)^2 \\ &\quad \ll \sum_q \sum_a \sup_I \Big| \sum_{\substack{N<n\le N'\\ \{\gamma n\}\in I\\ n\equiv a \bmod q}} \Lambda(n) - |I| \sum_{\substack{N<n\le N'\\ n\equiv a \bmod q}} \Lambda(n)\Big|^2 \\ &\qquad + \sum_q \sum_a \Big(\sum_{\substack{N<n\le N'\\ n\equiv a \bmod q}} \Lambda(n) - \frac{N'-N}{\varphi(q)}\Big)^2 = T_1(Q) + T_2(Q), \end{aligned}$$

say. Since $T_2(Q)$ is covered by a slight variant of the discussion in [6, Chapter 29], we focus our attention on $T_1(Q)$. By Lemma 7,

$$\begin{aligned} T_1(Q) &\ll \sum_{Q\le q<2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \mathcal{L}^{-2A}\Big(\sum_{\substack{N<n\le N'\\ n\equiv a \bmod q}} \Lambda(n)\Big)^2 \\ &\quad + \sum_{Q\le q<2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \Big(\sum_{h\le \mathcal{L}^A} \frac{1}{h}\Big| \sum_{\substack{N<n\le N'\\ n\equiv a \bmod q}} \Lambda(n)e(\gamma n h)\Big|\Big)^2 \\ &= T_3(Q) + T_4(Q), \end{aligned}$$

say. The Brun–Titchmarsh theorem gives a satisfactory bound for $T_3(Q)$. Applying Cauchy's inequality to $T_4(Q)$, we get

$$\begin{aligned} T_4(Q) &\le \Big(\sum_{h\le \mathcal{L}^A} \frac{1}{h}\Big) \sum_{h\le \mathcal{L}^A} \frac{1}{h} \sum_{Q\le q<2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \Big| \sum_{\substack{N<n\le N'\\ n\equiv a \bmod q}} \Lambda(n)e(\gamma n h)\Big|^2 \\ &\ll (\log \mathcal{L})^2 \sum_{Q\le q<2Q} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \Big| \sum_{N<n\le N'} \Lambda(n)\chi(n)e(\gamma n h)\Big|^2 \end{aligned}$$

for some $h \le \mathcal{L}^A$. From this point, we can conclude the proof by following, with slight changes, the argument in [6, pp. 170–171]. ∎

6. Proof of Theorems 2 and 3

Proof of Theorem 2. Let $\gamma = \alpha^{-1}$ and $N \ge C_1(\alpha, t)$, $0 < \varepsilon < C_2(\alpha, t)$. By Dirichlet's theorem, there is a reduced fraction b/r satisfying (1.7). Our hypothesis on α implies that

$$N^{-3/4} \ge \|\gamma r\| \gg r^{-3}, \qquad r \gg N^{1/4}.$$

Let $h_1'', \dots, h_l''$ be the first l primes in (l, ∞). Any translate

$$\mathcal{H} = \{h_1', \dots, h_k'\} + h, \qquad h \in \mathbb{N},$$

with $\{h_1', \dots, h_k'\} \subset \{h_1'', \dots, h_l''\}$ is an admissible set. Using Lemma 7(i), we choose $h_1', \dots, h_k'$ so that

$$k \ge \varepsilon \gamma l, \tag{6.1}$$

and for some real η,

$$-\gamma h_m' \in (\eta, \eta + \varepsilon\gamma) \pmod 1$$

for every $m = 1, \dots, k$. Now choose $h \in \mathbb{N}$, $h \ll_\gamma 1$, so that

$$h\gamma \in (\eta - \varepsilon\gamma, \eta) \pmod 1.$$

Thus, writing $h_m = h_m' + h$, we have

$$-\gamma h_m = -\gamma h_m' - \gamma h \in (0, 2\varepsilon\gamma) \pmod 1.$$

We apply Theorem 1 to the set

$$\mathcal{A} = \{n \in [N, 2N) : \gamma m \in I \pmod 1\}$$

where $I = (\gamma\beta - \gamma, \gamma\beta)$, taking $q_0 = q_1 = 1$, $s = 1$, $\varrho(n) = X(\mathbb{P}; n)$, $\theta = 1/4 - \varepsilon$, $b = 1 - 2\varepsilon$,

$$Y = \gamma N, \qquad Y_{1,m} = l_m \int_N^{2N} \frac{dt}{\log t} = \frac{l_m Y}{\mathcal{L}\gamma}(1 + o(1)).$$

Here J_m, l_m are the interval J and its length l in Lemma 6 (with $\varepsilon\gamma$ in place of ε), so that

$$\gamma > l_m > \gamma(1 - 2\varepsilon).$$

Since (1.2) can be proved in a similar (but simpler) fashion to (1.5), we only show that (1.5) holds. We can rewrite this in the form

$$\sum_{q \le x^{1/4-\varepsilon}} \mu(q)^2 \tau_{3k}(q) \Bigg| \sum_{\substack{N + h_m \le p < 2N \\ p \equiv a_q \bmod q \\ \gamma p \in J_m \bmod 1}} 1 - \frac{l_m}{\varphi(q)} \int_N^{2N} \frac{dt}{\log t} \Bigg| \ll N\mathcal{L}^{-k-\varepsilon}. \tag{6.2}$$

The function $E(N, N', \gamma, q, a)$ appearing in Theorem 4 is not quite in the form that we need. However, discarding prime powers and using partial summation in the standard way, we readily deduce a variant of (6.2) from Theorem 4, in which $N\mathcal{L}^{-A}$ appears in place of $N\mathcal{L}^{-k-\varepsilon}$, and the weight

$\mu(q)^2\tau_{3k}(q)$ is absent. We then obtain (6.2) by using Cauchy's inequality; see [9, (5.20)] for a very similar computation.

We are now in a position to use Theorem 1, obtaining a set $\mathcal{S}$ of t primes in $\mathcal{A} \cap [N, 2N)$, which of course have the form $[\alpha n + \beta]$, with

$$D(\mathcal{S}) \le h_k - h_1 \le h_l''$$

provided that

$$M_k > \frac{2t-2}{(1-2\varepsilon)(1/4-\varepsilon)}. \tag{6.3}$$

We take l to be the least integer with

$$\log(\varepsilon\gamma l) \ge \frac{2t-2}{(1-2\varepsilon)(1/4-\varepsilon)} + C$$

for a suitable absolute constant C, so that (6.3) follows from (6.1) and (1.11). Therefore,

$$\gamma l \ll \exp(8t), \quad l \ll \alpha \exp(8t), \quad D(\mathcal{S}) \ll l \log l \ll \alpha(t + \log\alpha)\exp(8t). \ \blacksquare$$

In the proof of Theorem 3, we shall need the following.

LEMMA 20. *Let D be as in Lemma* 18 *and let $\omega_0(t)$ denote Buchstab's function.*

(i) *The points of D lie in two triangles A_1, A_2, where A_1 has vertices*

$$(5/21, 5/21), \ (2/7, 3/14), \ (2/7, 2/7),$$

and A_2 has vertices

$$(1/2, 3/14), \ (3/7, 2/7), \ (1/2, 1/4).$$

(ii) *For $j = 1, 2$, let*

$$I_j = \int_{A_j} \frac{1}{\alpha_1\alpha_2^2}\omega_0\left(\frac{1-\alpha_1-\alpha_2}{\alpha_2}\right) d\alpha_1\, d\alpha_2.$$

Then $I_1 < 0.03925889$ and $I_2 < 0.0566295$.

Proof. Let $(\alpha_1, \alpha_2) \in D$. If $\alpha_1 + \alpha_2 > 5/7$, then

$$\alpha_1 + \alpha_2 > 5/7, \quad \alpha_1 + 2\alpha_2 \le 1, \quad \alpha_1 \le 1/2.$$

This defines a triangle which is easily verified to be A_2. If $\alpha_1 + \alpha_2 \le 5/7$, then as $\boldsymbol{\alpha}_2$ is bad, we have in turn

$$\alpha_1 + \alpha_2 < 4/7, \quad \alpha_1 < 3/7, \quad \alpha_1 < 2/7.$$

Altogether, we have

$$\alpha_1 + 2\alpha_2 > 5/7, \quad \alpha_1 < 2/7, \quad \alpha_2 < \alpha_1.$$

This defines a triangle which we can verify to be A_1. This proves (1).

Part (2) requires a computer calculation, which was kindly carried out by Andreas Weingartner. $\blacksquare$

Proof of Theorem 3. With a different value of l, we choose $h_1'', \ldots, h_l''$ and $h_1, \ldots, h_k$ exactly as in the proof of Theorem 2. In applying Theorem 1, we also take I, $\mathcal{A}$, q_0, q_1, Y, J_m, l_m as in that proof, but now $\theta = 2/7 - \varepsilon$, $s = 5$, $a = 3$; the functions $\varrho_1(n), \dots, \varrho_5(n)$ are given in Lemma 18.

There is little difficulty in verifying (1.2) by a similar but simpler version of the proof of (1.5). So we concentrate on (1.5). We recall that this can be rewritten as

$$\sum_{q \le x^\theta} \mu(q)^2 \tau_{3k}(q) \Bigg| \sum_{\substack{n \equiv a_q \bmod q \\ \gamma n \in J_m \bmod 1 \\ N + h_m \le n < 2N}} \varrho_g(n) - \frac{Y_{g,m}}{\varphi(q)} \Bigg| \ll N\mathcal{L}^{-k-\varepsilon}. \tag{6.4}$$

We define $Y_{g,m}$ by

$$Y_{g,m} = l_m \sum_{N \le n < 2N} \varrho_g(n).$$

It is well known that

$$Y_{g,m} = \frac{l_m c_g N}{\mathcal{L}}(1 + o(1)), \tag{6.5}$$

where c_g is given by a multiple integral. In fact,

$$c_1 + c_2 + c_3 - c_4 - c_5 = 1 - \int_{\boldsymbol{\alpha}_2 \in D} \frac{1}{\alpha_1 \alpha_2^2} \omega_0\left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2}\right) d\alpha_1\, d\alpha_2.$$

Similar calculations are found in [8, Chapter 1].

Fix m and g. By analogy with the proof of Theorem 4, we can obtain (1.5) by showing

$$\sum_{q \le N^{2/7-\varepsilon}} \Bigg| \sum_{\substack{N \le n < N' \\ n \equiv a_q \bmod q}} \varrho_g(n) - \frac{1}{\varphi(q)} \sum_{N \le n < N'} \varrho_g(n) \Bigg| \ll N\mathcal{L}^{-A} \tag{6.6}$$

for every $A > 0$, and

$$\sum_{q \le N^{2/7-\varepsilon}} \Bigg| \sum_{\substack{N \le n < N' \\ n \equiv a_q \bmod q}} \varrho_g(n) e(\gamma n h) \Bigg| \ll N\mathcal{L}^{-A} \tag{6.7}$$

for $1 \le h \le \mathcal{L}^{A+1}$ and for every $A > 0$. Again adapting the argument of Theorem 4, we see that (6.7) is a consequence of Lemma 19.

For (6.6), it suffices to show, recalling Lemma 8, that for arbitrary $\eta_\chi \ll 1$ and $Q \le N^{2/7-\varepsilon}$,

$$\sum_{Q \le q < 2Q} \sideset{}{^\star}\sum_{\chi \bmod q} \eta_\chi \sum_{N \le n < N'} \varrho_g(n) \chi(n) \ll QN\mathcal{L}^{-A} \tag{6.8}$$

for every $A > 0$. This can be readily deduced from the Siegel–Walfisz theorem for $Q \le \mathcal{L}^{2A}$, so we assume that $Q > \mathcal{L}^{2A}$.

We apply Lemma 15 with

$$W(n) = \sum_{Q \le q < 2Q} \sideset{}{^\star}\sum_{\chi \bmod q} \eta_\chi \chi(n)$$

if $N \le n < N'$, and $W(n) = 0$ otherwise.

For example, when $g = 3$, the left-hand side of (6.8) is

$$\sum_{\substack{N \le p_1p_2p_3n_4 < N' \\ (n_4, P((2N)^{1/7}))=1 \\ \boldsymbol{\alpha}_3 \in E_3 \\ (\alpha_1,\alpha_2) \in E_2 \setminus D}} W(p_1p_2p_3n_4) = \sum_{\substack{\boldsymbol{\alpha}_3 \in E_3 \\ (\alpha_1,\alpha_2)\in E_2\setminus D}} S^*(p_1p_2p_3, (2N)^{1/7}).$$

We shall show that (4.10) and (4.11) hold with $Y = QN\mathcal{L}^{-A-3}$, $c = 4/7$ and $d = 1/7$. (We could reduce the constraints on c and d, but that would not be useful in the present context.) Once we have done this, we can follow the proof of Lemma 19 to deduce (6.8).

To prove (4.10), we use the Pólya–Vinogradov bound for character sums to obtain

$$\begin{aligned}\sum_{m \le 2N^{4/7}} \sum_k W(nk) &= \sum_{m \le 2N^{4/7}} a_m \sum_{Q \le q < 2Q} \sideset{}{^\star}\sum_{\substack{\chi \bmod q \\ N \le mk < N'}} \eta_\chi \chi(mk) \\ &\ll \mathcal{L} \sum_{m \le 2N^{4/7}} \sum_{Q \le q < 2Q} q^{1/2} \ll \mathcal{L}Q^{3/2}N^{4/7-\varepsilon} \ll QN\mathcal{L}^{-A-3}.\end{aligned}$$

Now to prove (4.11), we note that by the method of [8, Section 3.2] mentioned earlier, it suffices to show that

$$\sum_{M \le m < 2M} a_m \sum_{K \le k < 2K} b_k W(mk) \ll QN\mathcal{L}^{-A}$$

whenever $|a_m| \le 1$ and $|b_k| \le \tau(k)$, $N^{4/7} \ll M \ll N^{5/7}$, $MK \asymp N$. That is, it suffices to show that

$$\sum_{Q \le q < 2Q} \sideset{}{^\star}\sum_{\chi \bmod q} \Big| \sum_{M \le m < 2M} a_m \chi(m) \Big| \Big| \sum_{K \le k < 2K} b_k \chi(k) \Big| \ll QN\mathcal{L}^{-A}. \tag{6.9}$$

Following the proof of (6) in [6, Chapter 28], we find that the left-hand side of (6.9) is

$$\begin{aligned}&\ll \mathcal{L}(M+Q^2)^{1/2}(K+Q^2)^{1/2}\|a\|_2\|b\|_2 \ll \mathcal{L}^3(N^{1/2} + M^{1/2}Q + Q^2)N^{1/2} \\ &\ll QN\mathcal{L}^{-A},\end{aligned}$$

since $\mathcal{L}^3Q^{-1}N \ll \mathcal{L}^{3-A}N$, $\mathcal{L}^3M^{1/2}N^{1/2} \ll \mathcal{L}^3N^{6/7} \ll N\mathcal{L}^{-A}$ and $\mathcal{L}^3QN^{1/2} \ll \mathcal{L}^3N^{11/14} \ll N\mathcal{L}^{-A}$. This proves (1.5) with the present choice of $\mathcal{A}$, $Y_{g,m}$, etc.

Applying Theorem 1, we find that there is a set $\mathcal{S}$ of t primes in $\mathcal{A}$ (and thus of the form $[\alpha m + \beta]$) having diameter

$$D(\mathcal{S}) \le h_k - h_1 \ll l \log l$$

provided that

$$M_k > \frac{2t-2}{b(2/7-\varepsilon)}.$$

Here b must have the property

$$b_{1,m} + b_{2,m} + b_{3,m} - b_{4,m} - b_{5,m} \geq b > 0;$$

that is,

$$l_m(c_1 + c_2 + c_3 - c_4 - c_5) \geq b\gamma > 0.$$

We can choose

$$b = (1 - 2\varepsilon)\left(1 - \int\limits_{\boldsymbol{\alpha}_2 \in D} \frac{1}{\alpha_1 \alpha_2^2} \omega_0\left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2}\right) d\alpha_1 \, d\alpha_2\right).$$

Using Lemma 20, we see that

$$b > 0.90411.$$

Now we proceed just as in the proof of Theorem 2. We may choose any l for which

$$\log(\varepsilon\gamma l) \geq \frac{2t - 2}{0.90411(2/7 - \varepsilon)} + C$$

for a suitable constant C, and now it is a simple matter to deduce that

$$D(\mathcal{S}) < C_4 \alpha(\log \alpha + t) \exp(7.743t),$$

where C_4 is an absolute constant. ∎

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