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Heights, regulators and Schinzel's determinant inequality

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1. Introduction. Let k be an algebraic number field, $k^\times$ its multiplicative group of nonzero elements, and $h : k^\times \to [0,\infty)$ the absolute, logarithmic, Weil height. If α belongs to $k^\times$ and ζ is a root of unity in $k^\times$, then the identity $h(\zeta\alpha) = h(\alpha)$ is well known. It follows that h is constant on cosets in the quotient group

$$\mathcal{G}_k = k^\times/\mathrm{Tor}(k^\times).$$

Therefore the height is well defined as a map $h : \mathcal{G}_k \to [0,\infty)$.

Let S be a finite set of places of k such that S contains all the archimedean places. Then

$$O_S = \{\gamma \in k : |\gamma|_v \le 1 \text{ for all places } v \notin S\}$$

is the ring of S-integers in k, and

$$O_S^\times = \{\gamma \in k^\times : |\gamma|_v = 1 \text{ for all places } v \notin S\} \tag{1.1}$$

is the multiplicative group of S-units in the ring O_S. We write

$$\mathrm{Tor}(O_S^\times) = \mathrm{Tor}(k^\times) \tag{1.2}$$

for the torsion subgroup of $O_S^\times$, which is also the torsion subgroup of the multiplicative group $k^\times$. As is well known, (1.2) is a finite, cyclic group of even order, and

$$\mathfrak{U}_S(k) = O_S^\times/\mathrm{Tor}(O_S^\times) \subseteq \mathcal{G}_k \tag{1.3}$$

is a free abelian group of finite rank r, where $|S| = r + 1$.

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In this paper we establish simple inequalities between the S-regulator $\mathrm{Reg}_S(k)$ and products of the form

$$\prod_{j=1}^{r}([k:\mathbb{Q}]h(\alpha_j)),$$

where $\alpha_1,\dots,\alpha_r$ are multiplicatively independent elements in $\mathfrak{U}_S(k)$.

THEOREM 1.1. *Assume that $O_S^\times$ has positive rank r, and let $\alpha_1,\dots,\alpha_r$ be multiplicatively independent elements in the free group $\mathfrak{U}_S(k)$. If $\mathfrak{A}\subseteq\mathfrak{U}_S(k)$ is the multiplicative subgroup generated by $\alpha_1,\dots,\alpha_r$, then*

$$\mathrm{Reg}_S(k)[\mathfrak{U}_S(k):\mathfrak{A}]\le\prod_{j=1}^{r}([k:\mathbb{Q}]h(\alpha_j)). \tag{1.4}$$

A special case of (1.4) occurs when S is the collection of all archimedean places of k. We write O_k for the ring of algebraic integers in k, and $O_k^\times$ for the multiplicative group of units in O_k. If k is not $\mathbb{Q}$, and k is not an imaginary quadratic extension of $\mathbb{Q}$, then the quotient group

$$\mathfrak{U}(k)=O_k^\times/\mathrm{Tor}(O_k^\times)\subseteq\mathcal{G}_k$$

is a free abelian group of positive rank r, where $r+1$ is the number of archimedean places of k. It is known from work of Remak [22], [23], and Zimmert [28] that the regulator $\mathrm{Reg}(k)$ is bounded from below by an absolute constant. Further, Friedman [12] has shown that $\mathrm{Reg}(k)$ takes its minimum value at the unique number field k_0 having degree 6 over $\mathbb{Q}$, and having discriminant equal to -10051. Thus by Friedman's result we have

$$0.2052\ldots=\mathrm{Reg}(k_0)\le\mathrm{Reg}(k) \tag{1.5}$$

for all algebraic number fields k. Combining (1.4) and (1.5) leads to the following explicit lower bound.

COROLLARY 1.1. *Assume that k is not $\mathbb{Q}$, and k is not an imaginary quadratic extension of $\mathbb{Q}$, so that $\mathfrak{U}(k)$ has positive rank r. Let $\alpha_1,\dots,\alpha_r$ be multiplicatively independent elements in $\mathfrak{U}(k)$. If $\mathfrak{A}\subseteq\mathfrak{U}(k)$ is the subgroup generated by $\alpha_1,\dots,\alpha_r$, then*

$$(0.2052\ldots)[\mathfrak{U}(k):\mathfrak{A}]\le\prod_{j=1}^{r}([k:\mathbb{Q}]h(\alpha_j)). \tag{1.6}$$

Let k be an algebraic number field such that $O_k^\times$ has positive rank r. The inequality (1.6) implies that each collection $\alpha_1,\dots,\alpha_r$ of multiplicatively independent units must contain a unit, say α_1, that satisfies

$$(0.2052\ldots)\le[k:\mathbb{Q}]h(\alpha_1). \tag{1.7}$$

A result of this sort was proposed by Bertrand [5, comment (iii), p. 210], who observed that it would follow from an unproved hypothesis related to Lehmer's problem.

In a well known paper Lehmer [16] asked, in the language and notation developed here, whether there exists a positive constant c_0 such that

$$c_0 \leq [k:\mathbb{Q}]h(\gamma) \tag{1.8}$$

for all γ in $k^\times$ which are not in $\mathrm{Tor}(k^\times)$. If $\gamma \neq 0$ is not a unit, then it is easy to show that

$$\log 2 \leq [k:\mathbb{Q}]h(\gamma).$$

Hence (1.8) is of interest for nontorsion elements γ in $O_k^\times$, or equivalently, for a nontrivial coset representative γ in $\mathfrak{U}(k)$. The inequality (1.6) provides a solution to a form of Lehmer's problem on average. Further information about Lehmer's problem is given in [6, Section 1.6.15] and in [25].

In Section 3 we give an analogous upper bound for the relative regulator associated to an extension l/k of algebraic number fields.

We will show that the inequality (1.4) is sharp up to a constant that depends only on the rank r of the group $\mathfrak{U}_S(k)$, but *not* on the underlying field k. Related results have been proved by Brindza [7], Bugeaud and Győry [8], Hajdu [14], and Matveev [18], [19]. More general inequalities that apply to arbitrary finitely generated subgroups of $\overline{\mathbb{Q}}^\times$ were obtained in [26, Theorems 1 and 2]. The inequality (1.9) below is sharper but less general, as it applies only to subgroups of a group of S-units having maximum rank.

Theorem 1.2. *Assume that $O_S^\times$ has positive rank r, and let $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$ be a subgroup of rank r. Then there exist multiplicatively independent elements $\beta_1, \ldots, \beta_r$ in $\mathfrak{A}$ such that*

$$\prod_{j=1}^{r}([k:\mathbb{Q}]h(\beta_j)) \leq \frac{2^r(r!)^3}{(2r)!}\mathrm{Reg}_S(k)[\mathfrak{U}_S(k):\mathfrak{A}]. \tag{1.9}$$

We note that if $r = 2$ then (1.4) and (1.9) imply that the multiplicatively independent elements β_1 and β_2 contained in $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$ satisfy

$$\mathrm{Reg}_S(k)[\mathfrak{U}_S(k):\mathfrak{A}] \leq ([k:\mathbb{Q}]h(\beta_1))([k:\mathbb{Q}]h(\beta_2)) \leq \frac{4}{3}\mathrm{Reg}_S(k)[\mathfrak{U}_S(k):\mathfrak{A}].$$

It follows that β_1 and β_2 form a basis for the group $\mathfrak{A}$. More generally, by using a well known lemma proved by Mahler [17] and Weyl [27] (see also [9, Chapter V, Lemma 8]), we obtain the following bound on the product of the heights of a basis for $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$.

Corollary 1.2. *Assume that $O_S^\times$ has positive rank r, and let $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$ be a subgroup of rank r. Then there exists a basis $\gamma_1, \ldots, \gamma_r$ for the free* *group* $\mathfrak{A}$ *such that*

$$\prod_{j=1}^{r}([k:\mathbb{Q}]h(\gamma_j)) \le \frac{2(r!)^4}{(2r)!}\operatorname{Reg}_S(k)[\mathfrak{U}_S(k):\mathfrak{A}]. \tag{1.10}$$

Bounds for S-regulators play an important role in the effective theory of Diophantine equations (see, for example, [8] and the references given there).

In this article we use the height h on specific free groups, such as $\mathcal{G}_k$ and $\mathfrak{U}(k)$. The use of the height on free groups is discussed in [1].

2. Preliminary results. At each place v of k we write k_v for the completion of k at v, so that k_v is a local field. We select two absolute values $\|\ \|_v$ and $|\ |_v$ from the place v. The absolute value $\|\ \|_v$ extends the usual archimedean or nonarchimedean absolute value on the subfield $\mathbb{Q}$. Then $|\ |_v$ must be a power of $\|\ \|_v$, and we set

$$|\ |_v = \|\ \|_v^{d_v/d}, \tag{2.1}$$

where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree of the extension, and $d = [k : \mathbb{Q}]$ is the global degree. With these normalizations the *height* of an algebraic number $\alpha \neq 0$ that belongs to k is given by

$$h(\alpha) = \sum_v \log^+ |\alpha|_v = \frac{1}{2}\sum_v \big|\log|\alpha|_v\big|, \tag{2.2}$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$. Each sum in (2.2) is over the set of all places v of k, and the equality between the two sums follows from the product formula. Then $h(\alpha)$ depends on $\alpha \neq 0$, but not on the number field k that contains α. We have already noted that the height is well defined as a map

$$h : \mathcal{G}_k \to [0, \infty).$$

Elementary properties of the height show that $(\alpha, \beta) \mapsto h(\alpha\beta^{-1})$ defines a metric on the group $\mathcal{G}_k$.

Let $\eta_1, \ldots, \eta_r$ be multiplicatively independent elements in $\mathfrak{U}_S(k)$ that form a basis for $\mathfrak{U}_S(k)$ as a free abelian group of rank r. Then let

$$M = (d_v \log \|\eta_j\|_v)$$

denote the $(r+1)\times r$ real matrix where $v \in S$ indexes rows and $j = 1, \ldots, r$ indexes columns. At each place $\widehat{v}$ in S we write

$$M^{(\widehat{v})} = (d_v \log \|\eta_j\|_v) \tag{2.3}$$

for the $r \times r$ submatrix of M obtained by removing the row indexed by $\widehat{v}$. Then the S-regulator of $O_S^\times$ (or of $\mathfrak{U}_S(k)$) is the positive number

$$\operatorname{Reg}_S(k) = |\det M^{(\widehat{v})}|, \tag{2.4}$$

which is independent of the choice of $\widehat{v}$ in S. Using an inequality proved by A. Schinzel [24] that bounds the determinant of a real matrix, we will prove that

$$\mathrm{Reg}_S(k) \le \prod_{j=1}^{r}([k:\mathbb{Q}]h(\eta_j)). \tag{2.5}$$

If the better known inequality of Hadamard is used to estimate the determinant that defines the S-regulator on the right of (2.4), we obtain an upper bound that is larger than (2.5) by a factor of 2^r.

Assume more generally that $\alpha_1, \ldots, \alpha_r$ are multiplicatively independent elements in $\mathfrak{U}_S(k)$, but not necessarily a basis for the free group $\mathfrak{U}_S(k)$. It follows that there exists an $r \times r$ nonsingular matrix $B = (b_{ij})$ with entries in $\mathbb{Z}$ such that

$$\log \|\alpha_j\|_v = \sum_{i=1}^{r} b_{ij} \log \|\eta_i\|_v \tag{2.6}$$

for each place v in S and each $j = 1, \ldots, r$. Alternatively, (2.6) can be written as the matrix identity

$$(d_v \log \|\alpha_j\|_v) = (d_v \log \|\eta_j\|_v)B. \tag{2.7}$$

If

$$\mathfrak{A} = \langle \alpha_1, \ldots, \alpha_r \rangle \subseteq \mathfrak{U}_S(k) \tag{2.8}$$

is the multiplicative subgroup generated by $\alpha_1, \ldots, \alpha_r$, we find that

$$[\mathfrak{U}_S(k) : \mathfrak{A}] = |\det B|. \tag{2.9}$$

This will lead to the more general inequality (1.4).

3. Relative regulators. Throughout this section we suppose that k and l are algebraic number fields with $k \subseteq l$. We write $r(k)$ for the rank of the unit group $O_k^{\times}$, and similarly for $r(l)$. Then k has $r(k)+1$ archimedean places, and l has $r(l)+1$ archimedean places. In general $r(k) \le r(l)$, and we recall (see [21, Proposition 3.20]) that $r(k) = r(l)$ if and only if l is a CM-field, and k is the maximal totally real subfield of l.

The norm is a homomorphism of multiplicative groups

$$\mathrm{Norm}_{l/k} : l^{\times} \to k^{\times}.$$

If v is a place of k, then each element α in $l^{\times}$ satisfies

$$[l:k] \sum_{w|v} \log |\alpha|_w = \log |\mathrm{Norm}_{l/k}(\alpha)|_v, \tag{3.1}$$

where $|\ |_v$ and $|\ |_w$ are normalized as in (2.1). It follows from (3.1) that the norm restricted to $O_l^\times$ is a homomorphism

$$\mathrm{Norm}_{l/k} : O_l^\times \to O_k^\times,$$

and the norm restricted to the torsion subgroup in $O_l^\times$ is also a homomorphism

$$\mathrm{Norm}_{l/k} : \mathrm{Tor}(O_l^\times) \to \mathrm{Tor}(O_k^\times).$$

Therefore we get a homomorphism

$$\mathrm{norm}_{l/k} : O_l^\times/\mathrm{Tor}(O_l^\times) \to O_k^\times/\,\mathrm{Tor}(O_k^\times),$$

well defined by

$$\mathrm{norm}_{l/k}(\alpha\,\mathrm{Tor}(O_l^\times)) = \mathrm{Norm}_{l/k}(\alpha)\,\mathrm{Tor}(O_k^\times).$$

To simplify notation we write

$$F_k = O_k^\times/\mathrm{Tor}(O_k^\times) \quad \text{and} \quad F_l = O_l^\times/\mathrm{Tor}(O_l^\times),$$

and we write the elements of F_k and F_l as coset representatives rather than cosets. Obviously F_k and F_l are free abelian groups of rank $r(k)$ and $r(l)$, respectively.

Following Costa and Friedman [10], we define the subgroup of relative units in $O_l^\times$ as

$$\{\alpha \in O_l^\times : \mathrm{Norm}_{l/k}(\alpha) \in \mathrm{Tor}(O_k^\times)\}.$$

Alternatively, we work in the free group F_l, where the image of the subgroup of relative units is the kernel of $\mathrm{norm}_{l/k}$. That is, we define the subgroup of *relative units* in F_l to be

$$E_{l/k} = \{\alpha \in F_l : \mathrm{norm}_{l/k}(\alpha) = 1\}. \tag{3.2}$$

We also write

$$I_{l/k} = \{\mathrm{norm}_{l/k}(\alpha) : \alpha \in F_l\} \subseteq F_k$$

for the image of $\mathrm{norm}_{l/k}$. If β in F_l represents a coset in F_k, then

$$\mathrm{norm}_{l/k}(\beta) = \beta^{[l:k]}.$$

Therefore $I_{l/k} \subseteq F_k$ is a subgroup of rank $r(k)$, and

$$[F_k : I_{l/k}] < \infty. \tag{3.3}$$

It follows that $E_{l/k} \subseteq F_l$ is a subgroup of rank $r(l/k) = r(l) - r(k)$, and we restrict our attention to extensions l/k such that $r(l/k)$ is positive.

Let $\eta_1, \ldots, \eta_{r(l/k)}$ be a collection of multiplicatively independent relative units that form a basis for the subgroup $E_{l/k}$. At each archimedean place v of k we select a place $\widehat{w}_v$ of l such that $\widehat{w}_v \mid v$. Then we define an $r(l/k) \times r(l/k)$ real matrix

$$M_{l/k} = ([l_w : \mathbb{Q}_w] \log \|\eta_j\|_w), \tag{3.4}$$

where w is an archimedean place of l, but $w \neq \widehat{w}_v$ for each $v \mid \infty$, and $j = 1, \ldots, r(l/k)$. We write l_w for the completion of l at the place w, $\mathbb{Q}_w$ for the completion of $\mathbb{Q}$ at the place w, and we write $[l_w : \mathbb{Q}_w]$ for the local degree. Of course $\mathbb{Q}_w$ is isomorphic to $\mathbb{R}$ in the situation considered here. As in [10], we define the *relative regulator* of l/k to be the positive number

$$\mathrm{Reg}(E_{l/k}) = |\det M_{l/k}|. \tag{3.5}$$

It follows, as in the proof of [10, Theorem 1] (see also [11]), that the determinant on the right of (3.5) does not depend on the choice of places $\widehat{w}_v$ for each archimedean place v of k.

THEOREM 3.1. *Let $k \subseteq l$ be algebraic number fields such that the group $E_{l/k}$ has positive rank $r(l/k) = r(l) - r(k)$. Let $\varepsilon_1, \ldots, \varepsilon_{r(l/k)}$ be a collection of multiplicatively independent relative units in $E_{l/k}$. If $\mathfrak{E} \subseteq E_{l/k}$ is the multiplicative subgroup generated by $\varepsilon_1, \ldots, \varepsilon_{r(l/k)}$, then*

$$\mathrm{Reg}(E_{l/k})[E_{l/k} : \mathfrak{E}] \leq \prod_{j=1}^{r(l/k)} ([l : \mathbb{Q}]h(\varepsilon_j)). \tag{3.6}$$

The relative regulator can also be expressed as a ratio of the (ordinary) regulators $\mathrm{Reg}(k)$ and $\mathrm{Reg}(l)$ by using the basic identity

$$[F_k : I_{l/k}]\,\mathrm{Reg}(k)\,\mathrm{Reg}(E_{l/k}) = \mathrm{Reg}(l), \tag{3.7}$$

established in [10, Theorem 1]. A slightly different definition of a relative regulator was considered by Bergé and Martinet [2]–[4]. We have used the definition proposed by Costa and Friedman [10], [11], as it leads more naturally to (3.6). Further lower bounds for the product on the right of (3.6) follow from inequalities for the relative regulator obtained by Friedman and Skoruppa [13].

4. Schinzel's norm. For a real number x we write

$$x^+ = \max\{0, x\} \quad \text{and} \quad x^- = \max\{0, -x\},$$

so that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. If $\boldsymbol{x} = (x_n)$ is a (column) vector in $\mathbb{R}^N$, we define $\delta : \mathbb{R}^N \to [0, \infty)$ by

$$\delta(\boldsymbol{x}) = \max\Big\{\sum_{m=1}^{N} x_m^+, \sum_{n=1}^{N} x_n^-\Big\}. \tag{4.1}$$

The following inequality was proved by A. Schinzel [24].

THEOREM 4.1. *If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are column vectors in $\mathbb{R}^N$, then*

$$|\det(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N)| \leq \delta(\boldsymbol{x}_1) \cdots \delta(\boldsymbol{x}_N). \tag{4.2}$$

A slightly sharper upper bound was established by C. R. Johnson and M. Newman [15]. However, their bound does not lead to a significant improvement of the results we obtain here.

If a and b are nonnegative real numbers then

$$2\max\{a, b\} = |a+b| + |a-b|.$$

This leads to the identity

$$\delta(\boldsymbol{x}) = \max\Big\{\sum_{m=1}^{N} x_m^+, \sum_{n=1}^{N} x_n^-\Big\} = \frac{1}{2}\Big|\sum_{n=1}^{N} x_n\Big| + \frac{1}{2}\sum_{n=1}^{N} |x_n|. \tag{4.3}$$

It follows easily from (4.3) that $\boldsymbol{x} \mapsto \delta(\boldsymbol{x})$ is a continuous, symmetric distance function, or norm, on $\mathbb{R}^N$. Let

$$K_N = \{\boldsymbol{x} \in \mathbb{R}^N : \delta(\boldsymbol{x}) \le 1\} \tag{4.4}$$

be the associated unit ball. Then K_N is a compact, convex, symmetric subset of $\mathbb{R}^N$ with nonempty interior.

Lemma 4.1. *Let $\delta : \mathbb{R}^N \to [0, \infty)$ be the continuous distance function defined by* (4.3), *and let K_N be the unit ball defined by* (4.4). *Then*

$$\mathrm{Vol}_N(K_N) = \frac{(2N)!}{(N!)^3}. \tag{4.5}$$

Proof. We write J for the $(N+1) \times N$ matrix

$$J = \frac{1}{2}\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \\ -1 & \cdots & -1 \end{pmatrix}.$$

Then it is obvious that J has rank N. Let

$$\mathcal{D}_N = \{\boldsymbol{y} \in \mathbb{R}^{N+1} : y_0 + y_1 + \cdots + y_N = 0\}$$

be the N-dimensional subspace of $\mathbb{R}^{N+1}$ spanned by the columns of J. Further, let

$$\mathcal{B}_{N+1} = \{\boldsymbol{y} \in \mathbb{R}^{N+1} : \|\boldsymbol{y}\|_1 = |y_0| + |y_1| + \cdots + |y_N| \le 1\}$$

denote the unit ball in $\mathbb{R}^{N+1}$ with respect to the $\| \ \|_1$-norm. If $\boldsymbol{x}$ is a (column) vector in $\mathbb{R}^N$, we find that $\delta(\boldsymbol{x}) = \|J\boldsymbol{x}\|_1$, and therefore

$$K_N = \{\boldsymbol{x} \in \mathbb{R}^N : \|J\boldsymbol{x}\|_1 \le 1\}.$$

It follows that

$$\mathrm{Vol}_N(K_N) = \int_{\mathbb{R}^N} \chi_{\mathcal{B}_{N+1}}(J\boldsymbol{x})\, d\boldsymbol{x} = |\det U| \int_{\mathbb{R}^N} \chi_{\mathcal{B}_{N+1}}(JU\boldsymbol{x})\, d\boldsymbol{x}, \tag{4.6}$$

where $\boldsymbol{y} \mapsto \chi_{\mathcal{B}_{N+1}}(\boldsymbol{y})$ is the characteristic function of $\mathcal{B}_{N+1}$, and U is an arbitrary $N \times N$ nonsingular real matrix.

We select U so that the columns of JU form an orthonormal basis for $\mathcal{D}_N$. With this choice we find that

$$\int_{\mathbb{R}^N} \chi_{\mathcal{B}_{N+1}}(JU\boldsymbol{x})\,d\boldsymbol{x} = \mathrm{Vol}_N(\mathcal{D}_N \cap \mathcal{B}_{N+1}) = \frac{\sqrt{N+1}(2N)!}{2^N(N!)^3}, \tag{4.7}$$

where the second equality follows from a result of Meyer and Pajor [20, Proposition II.7]. Because the columns of JU are orthonormal, we get

$$\mathbf{1}_N = (JU)^T(JU). \tag{4.8}$$

For each $m = 1, \ldots, N+1$ let $J^{(m)}$ be the $N \times N$ submatrix of J obtained by removing the mth row. From (4.8) and the Cauchy–Binet formula,

$$1 = \det((JU)^T(JU)) = (\det U)^2 \det(J^TJ) = (\det U)^2 \sum_{m=1}^{N+1} (\det J^{(m)})^2$$
$$= (\det U)^2 4^{-N}(N+1),$$

and therefore

$$|\det U| = \frac{2^N}{\sqrt{N+1}}. \tag{4.9}$$

The identity (4.5) follows by combining (4.6), (4.7) and (4.9). ■

Next let

$$A = (\boldsymbol{a}_1 \ \cdots \ \boldsymbol{a}_N)$$

be an $N \times N$ nonsingular matrix with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$. Obviously the columns of A form a basis for the lattice

$$\mathcal{L} = \{A\boldsymbol{\xi} : \boldsymbol{\xi} \in \mathbb{Z}^N\} \subseteq \mathbb{R}^N. \tag{4.10}$$

By Schinzel's inequality,

$$|\det A| \le \prod_{n=1}^{N} \delta(\boldsymbol{a}_n).$$

Using the geometry of numbers, we will establish the existence of linearly independent points $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_N$ in $\mathcal{L}$ for which $\prod_{n=1}^{N} \delta(\boldsymbol{\ell}_n)$ is not much larger than $|\det A|$. An explicit bound on such a product follows immediately from Minkowski's theorem on successive minima and our formula (4.5) for the volume of K_N.

THEOREM 4.2. *Let $\mathcal{L} \subseteq \mathbb{R}^N$ be defined by* (4.10). *Then there exist linearly independent points $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_N$ in $\mathcal{L}$ such that*

$$\prod_{n=1}^{N} \delta(\boldsymbol{\ell}_n) \le \frac{2^N(N!)^3}{(2N)!}|\det A|. \tag{4.11}$$

Proof. Let $0 < \lambda_1 \le \cdots \le \lambda_N < \infty$ be the successive minima of $\mathcal{L}$ with respect to the convex symmetric set K_N. Then there exist linearly independent points $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_N$ in $\mathcal{L}$ such that

$$\delta(\boldsymbol{\ell}_n) = \lambda_n \quad \text{for each } n = 1, \ldots, N.$$

By Minkowski's theorem on successive minima (see [9, Section VIII.4.3]),

$$\mathrm{Vol}_N(K_N)\lambda_1 \cdots \lambda_N \le 2^N |\det A|.$$

From Lemma 4.1 we get (4.11). ∎

5. Proof of Theorems 1.1 and 1.2. We require the following lemma, which connects the Schinzel norm (4.1) with the Weil height.

Lemma 5.1. *Let $\widehat{v}$ be a place of the algebraic number field k, and let $\alpha \in k^\times \setminus \{0\}$. Then*

$$\max\Big\{\sum_{v \ne \widehat{v}} \log^+ |\alpha|_v, \sum_{v \ne \widehat{v}} \log^- |\alpha|_v\Big\} = h(\alpha). \tag{5.1}$$

Proof. The product formula implies that

$$h(\alpha) = \sum_v \log^+ |\alpha|_v = \sum_v \log^- |\alpha|_v.$$

If $\log |\alpha|_{\widehat{v}} \le 0$ then

$$\max\Big\{\sum_{v \ne \widehat{v}} \log^+ |\alpha|_v, \sum_{v \ne \widehat{v}} \log^- |\alpha|_v\Big\} = \sum_v \log^+ |\alpha|_v = h(\alpha).$$

On the other hand, if $\log |\alpha|_{\widehat{v}} \ge 0$ then

$$\max\Big\{\sum_{v \ne \widehat{v}} \log^+ |\alpha|_v, \sum_{v \ne \widehat{v}} \log^- |\alpha|_v\Big\} = \sum_v \log^- |\alpha|_v = h(\alpha). \ ∎$$

Proof of Theorem 1.1. First we combine (2.3), (2.4), (2.7) and (2.9) to obtain

$$\mathrm{Reg}_S(k)[\mathfrak{U}_S(k) : \mathfrak{A}] = [k : \mathbb{Q}]^r |\det(\log |\alpha_j|_v)|, \tag{5.2}$$

where v in $S \setminus \{\widehat{v}\}$ indexes rows and $j = 1, \ldots, r$ indexes columns in the matrix on the right of (5.2). We estimate the determinant in (5.2) by applying Schinzel's inequality (4.2). Using (4.1) and (5.1) we get

$$|\det(\log |\alpha_j|_v)| \le \prod_{j=1}^r \max\Big\{\sum_{v \ne \widehat{v}} \log^+ |\alpha_j|_v, \sum_{v \ne \widehat{v}} \log^- |\alpha_j|_v\Big\} = \prod_{j=1}^r h(\alpha_j). \tag{5.3}$$

The inequality (1.4) follows from (5.2) and (5.3). ∎

Proof of Theorem 1.2. Let $\eta_1, \ldots, \eta_r$ be multiplicatively independent elements that form a basis for $\mathfrak{U}_S(k)$ as a free abelian group of rank r. Let $\widehat{v}$

be a place of k contained in S, and

$$M^{(\widehat{v})} = (d_v \log \|\eta_j\|_v)$$

the $r\times r$ real matrix defined in (2.3). By hypothesis, $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$ is a subgroup of rank r. Let $\alpha_1, \ldots, \alpha_r$ be multiplicatively independent elements in $\mathfrak{A}$ that form a basis for $\mathfrak{A}$. As in (2.6), there exists an $r \times r$ nonsingular matrix $B = \big(b_{ij}\big)$ with entries in $\mathbb{Z}$ such that

$$\log \|\alpha_j\|_v = \sum_{i=1}^{r} b_{ij} \log \|\eta_i\|_v \tag{5.4}$$

for each place v in S and each $j = 1, \ldots, r$. Alternatively, if we define the $r \times r$ real matrix

$$A^{(\widehat{v})} = (d_v \log \|\alpha_j\|_v),$$

where $v \in S \setminus \{\widehat{v}\}$ and $j = 1, \ldots, r$, then (5.4) is equivalent to the matrix identity

$$A^{(\widehat{v})} = M^{(\widehat{v})} B. \tag{5.5}$$

We use the nonsingular $r\times r$ real matrix $A^{(\widehat{v})}$ to define a lattice $\mathcal{L}^{(\widehat{v})} \subseteq \mathbb{R}^r$ by

$$\mathcal{L}^{(\widehat{v})} = \{A^{(\widehat{v})}\boldsymbol{\xi} : \boldsymbol{\xi} \in \mathbb{Z}^r\}.$$

Then (2.4), (2.9) and (5.5) imply that

$$\mathrm{Reg}_S(k)[\mathfrak{U}_S(k) : \mathfrak{A}] = |\det M^{(\widehat{v})}|\, |\det B| = |\det A^{(\widehat{v})}|, \tag{5.6}$$

which is independent of the choice of $\widehat{v}$ in S, and is also the determinant of $\mathcal{L}^{(\widehat{v})}$. By Theorem 4.2 and (5.6), there exist linearly independent vectors $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_r$ in $\mathcal{L}^{(\widehat{v})}$ such that

$$\prod_{j=1}^{r} \delta(\boldsymbol{\ell}_j) \leq \frac{2^r (r!)^3}{(2r)!} \mathrm{Reg}_S(k)[\mathfrak{U}_S(k) : \mathfrak{A}]. \tag{5.7}$$

As each (column) vector $\boldsymbol{\ell}_j$ belongs to $\mathcal{L}^{(\widehat{v})}$, it has rows indexed by $v \in S\setminus\{\widehat{v}\}$. Thus $\boldsymbol{\ell}_j$ can be written as

$$\boldsymbol{\ell}_j = \Big(d_v \sum_{i=1}^{r} f_{ij} \log \|\alpha_i\|_v\Big) = (d_v \log \|\beta_j\|_v),$$

where $F = (f_{ij})$ is an $r \times r$ nonsingular matrix with entries in $\mathbb{Z}$, and $\beta_1, \ldots, \beta_r$ are multiplicatively independent elements in $\mathfrak{A}$. By Lemma 5.1,

$$\begin{aligned}\delta(\boldsymbol{\ell}_j) &= \max\Big\{\sum_{v\neq\widehat{v}} d_v \log^+ \|\beta_j\|_v, \sum_{v\neq\widehat{v}} d_v \log^- \|\beta_j\|_v\Big\} \\ &= [k:\mathbb{Q}] \max\Big\{\sum_{v\neq\widehat{v}} \log^+ |\beta_j|_v, \sum_{v\neq\widehat{v}} \log^- |\beta_j|_v\Big\} = [k:\mathbb{Q}] h(\beta_j).\end{aligned} \tag{5.8}$$

The inequality (1.9) follows from (5.7) and (5.8). ∎

6. Proof of Theorem 3.1. Let $\eta_1, \dots, \eta_{r(l/k)}$ be a basis for the free abelian group $E_{l/k}$. Then there exists a nonsingular $r(l/k) \times r(l/k)$ matrix $C = (c_{ij})$ with entries in $\mathbb{Z}$ such that

$$\log \|\varepsilon_j\|_w = \sum_{i=1}^{r(l/k)} c_{ij} \log \|\eta_i\|_w \tag{6.1}$$

at each archimedean place w of l. As in our derivation of (2.7) and (2.9), the equations (6.1) can be written as the matrix equation

$$([l_w : \mathbb{Q}_w] \log \|\varepsilon_j\|_w) = ([l_w : \mathbb{Q}_w] \log \|\eta_j\|_w)C, \tag{6.2}$$

where w is an archimedean place of l, and w indexes the rows of the matrices on both sides of (6.2). Let $\mathfrak{E}$ be the subgroup of $E_{l/k}$ generated by $\varepsilon_1, \dots, \varepsilon_{r(l/k)}$. It follows from (6.2) that

$$[E_{l/k} : \mathfrak{E}] = |\det C|. \tag{6.3}$$

At each archimedean place v of k let $\widehat{w}_v$ be a place of l such that $\widehat{w}_v \mid v$. As in (3.4), we write

$$M_{l/k} = ([l_w : \mathbb{Q}_w] \log \|\eta_j\|_w),$$

for the $r(l/k) \times r(l/k)$ matrix where w is an archimedean place of l, but $w \neq \widehat{w}_v$ for each $v \mid \infty$, and $j = 1, \dots, r(l/k)$. Let

$$L(\mathfrak{E}) = ([l_w : \mathbb{Q}_w] \log \|\varepsilon_j\|_w)$$

be the analogous $r(l/k) \times r(l/k)$ matrix where again w is an archimedean place of l, but $w \neq \widehat{w}_v$ for each $v \mid \infty$, and $j = 1, \dots, r(l/k)$. From (6.2),

$$L(\mathfrak{E}) = M_{l/k} C. \tag{6.4}$$

Then we combine (3.5) and (6.2)–(6.4) to conclude that

$$\mathrm{Reg}(E_{l/k})[E_{l/k} : \mathfrak{E}] = |\det L(\mathfrak{E})|. \tag{6.5}$$

To complete the proof we apply Schinzel's inequality (4.2) to the determinant on the right of (6.5). We find that

$$\begin{aligned}
&[l : \mathbb{Q}]^{-r(l/k)} |\det L(\mathfrak{E})| \\
&\quad \le \prod_{j=1}^{r(l/k)} \biggl\{ \frac{1}{2} \Bigl| \sum_{w \neq \widehat{w}_v} \log |\varepsilon_j|_w \Bigr| + \frac{1}{2} \sum_{w \neq \widehat{w}_v} \bigl| \log |\varepsilon_j|_w \bigr| \biggr\} \\
&\quad = \prod_{j=1}^{r(l/k)} \biggl\{ \frac{1}{2} \Bigl| \sum_{v \mid \infty} \log |\varepsilon_j|_{\widehat{w}_v} \Bigr| + \frac{1}{2} \sum_{w \neq \widehat{w}_v} \bigl| \log |\varepsilon_j|_w \bigr| \biggr\} \\
&\quad \le \prod_{j=1}^{r(l/k)} \biggl\{ \frac{1}{2} \sum_{v \mid \infty} \bigl| \log |\varepsilon_j|_{\widehat{w}_v} \bigr| + \frac{1}{2} \sum_{w \neq \widehat{w}_v} \bigl| \log |\varepsilon_j|_w \bigr| \biggr\} = \prod_{j=1}^{r(l/k)} h(\varepsilon_j).
\end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6) leads to (3.6). ∎

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