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EVOLUTION DIFFERENTIAL EQUATIONS IN FRÉCHET SEQUENCE SPACES

BY

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Abstract. We consider evolution differential equations in Fréchet spaces with unconditional Schauder basis, and construct a version of the majorant functions method to obtain existence theorems for Cauchy problems. Applications to PDE are also considered.

1. Introduction. Countable systems of ordinary differential equations appear in different areas of differential equations and applications: see, for example, [10], [2].

The most famous problem which leads to such a system is the Cauchy–Kovalevskaya problem in the case nonanalytic in time. To reduce this problem to a countable system of ODEs one must expand the solution in the Taylor series in spatial variables and substitute this expansion to the corresponding initial value problem; then the Taylor coefficients satisfy infinite system of ODEs.

The Cauchy–Weierstrass–Kovalevskaya method of majorant functions can be modified for the case nonanalytic in time to obtain the corresponding existence theorem [17]. Generally, when applied to the Cauchy–Kovalevskaya problem, this modification does not give anything different from the results of Nirenberg and Nishida [8]. Nevertheless, in some cases this method allows one to obtain global in time existence theorems or at least effective estimates for the solution's existence time [12].

Another application of the majorant functions method is to initial value problems with non-Lipschitz right hand side. It is well known that in infinite-dimensional spaces such problems in general do not have solutions. But the majorant functions method allows one to prove existence theorems in some special cases.

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This article is devoted to a generalisation of this method to countable systems of ODEs in Fréchet spaces with Schauder bases.

For example, $\mathcal{D}(\mathbb{T}^m)$, $\mathbb{T}^m = \mathbb{R}^m/(2\pi\mathbb{Z})^m$, is a Fréchet space with the unconditional Schauder basis $\{e^{i(k,x)}\}$, $k \in \mathbb{Z}^m$. Other examples are given below.

Note that our theorems are closely related to the results of Müller [7] and Uhl [13]. Theorem 2.4 when applied to $\mathbb{R}^m$ turns out to be a special case of Müller's result.

A generalisation of Müller's theorem is presented in [13]. In that article ODEs in partially ordered Banach spaces are considered, and the main conditions on the right side of the equation $\dot{x} = f(t,x)$ are formulated in terms of a measure of noncompactness (see also [14]). The results of [13], [14] do not allow one to consider partial differential equations, in particular they do not imply the Cauchy–Kovalevskaya theorem.

2. Main theorems. Let E stand for a Fréchet space. Its topology is defined by a collection $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of seminorms.

Recall that such a space is completely metrizable by the metric

$$\rho(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x-y\|_k\}.$$

DEFINITION 2.1. A sequence $\{e_k\}_{k\in\mathbb{N}} \subset E$ is called a *Schauder basis* in E if for every $x \in E$ there is a unique sequence $\{x_k\}_{k\in\mathbb{N}}$ of scalars such that

$$x = \sum_{k=1}^{\infty} x_k e_k, \tag{2.1}$$

where the series is convergent in the topology of E.

We shall say that $\{e_k\}_{k\in\mathbb{N}}$ is an *unconditional basis* if for any $x \in E$ and any permutation $\pi : \mathbb{N} \to \mathbb{N}$ the sum

$$\sum_{k=1}^{\infty} x_{\pi(k)} e_{\pi(k)}$$

is convergent.

In what follows we assume that E has an unconditional Schauder basis.

We write $I_T = [0,T]$, $T > 0$, and $I_\infty = [0,\infty)$. If not explicitly stated otherwise, we assume that $T < \infty$.

DEFINITION 2.2. We shall say that an element $y = \sum_{k=1}^{\infty} y_k e_k$ is a *majorant* for an element $x = \sum_{k=1}^{\infty} x_k e_k$, and write $x \ll y$, if

$$|x_k| \le y_k, \quad k \in \mathbb{N}.$$

DEFINITION 2.3. We shall say that $x(\cdot) \in C^1(I_T, E)$ if for each $t \in I_T$ there exists an element $\dot{x}(t) \in E$ such that for all i one has

$$\lim_{h\to 0} \left\| \frac{x(t+h) - x(t)}{h} - \dot{x}(t) \right\|_i = 0, \tag{2.2}$$

and $\dot{x} \in C(I_T, E)$.

In (2.2) it is assumed that if $t = 0$ then $h > 0$, and $h < 0$ provided $t = T$.

Fix $y \in E$ and let $\mathcal{X}_j[y] : E \to E$ stand for the following affine mappings:

$$\mathcal{X}_1[y]x = y_1 e_1 + \sum_{k=2}^{\infty} x_k e_k,$$

$$\mathcal{X}_j[y]x = \sum_{k=1}^{j-1} x_k e_k + y_j e_j + \sum_{k=j+1}^{\infty} x_k e_k, \quad j > 1.$$

Let $X(\cdot) = \sum_{k=1}^{\infty} X_k(\cdot) e_k \in C(I_T, E)$ be such that

$$X_k(t) \geq 0, \qquad k \in \mathbb{N},\ t \in I_T,$$

and $X_k(\cdot) \in C^1(I_T)$. Set

$$W_X = \{(t, x) \in I_T \times E \mid x \ll X(t)\}.$$

Consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(0) = \hat{x}, \tag{2.3}$$

$$f(t, x) = \sum_{k=1}^{\infty} f_k(t, x) e_k, \qquad f \in C(W_X, E).$$

THEOREM 2.4. *Suppose that $X_k(t) > 0$ for all $k \in \mathbb{N}$ and $t \in I_T$, and for each $(t, x) \in W_X$ one has*

$$\pm f_k(t, \mathcal{X}_k[\pm X(t)]x) \leq \dot{X}_k(t), \quad \hat{x} \ll X(0).$$

(*Here and below, this means that for each k two inequalities hold.*) *Then problem* (2.3) *has a solution $x \in C^1(I_T, E)$ such that*

$$x(t) \ll X(t), \quad t \in I_T.$$

REMARK 2.5. The function X satisfying the conditions of Theorem 2.4 is called a *majorant function* for problem (2.3).

This theorem develops corresponding results of [17] and, like the theorems from that article, implies the classical Cauchy–Kovalevskaya theorem and a number of its generalisations.

THEOREM 2.6. *Suppose that $T = \infty$ and the function f is ω-periodic ($\omega > 0$) in t. Suppose also that $X_k(t) > 0$ for all $k \in \mathbb{N}$ and $t \in I_T$, and for each $(t, x) \in W_X$ one has*

$$\pm f_k(t, \mathcal{X}_k[\pm X(t)]x) \leq \dot{X}_k(t),$$

and $X(\omega) \ll X(0)$. Then problem (2.3) *has a solution $\tilde{x} \in C^1(I_\infty, E)$ such that*

$$\tilde{x}(t) \ll X(t), \quad \tilde{x}(t+\omega) = \tilde{x}(t), \qquad t \in I_\infty.$$

Theorems 2.4 and 2.6 are proved in Section 4.

The following technical proposition is useful for proving continuity of some mappings.

Proposition 2.7. *Let $A = \sum_{k=1}^\infty A_k e_k \in E$ with $A_k \geq 0$ be a fixed element. Assume that a sequence $x_n = \sum_{k=1}^\infty x_{kn} e_k$ is contained in*

$$K_A = \Big\{ y = \sum_{k=1}^\infty y_k e_k \in E \;\Big|\; |y_k| \leq A_k \Big\}$$

and is weakly convergent: for all k, we have $x_{kn} \to x_k$ as $n \to \infty$. Then $x = \sum_{k=1}^\infty x_k e_k \in K_A$ and the sequence $\{x_n\}$ is convergent in E, i.e. $\rho(x_n, x) \to 0$.

This is proved by the methods developed in Section 4.

2.1. Nonnegative solutions. In this section we formulate another pair of theorems which belong to the same range of ideas. We do not give their proofs since they repeat the argument of Section 4 up to evident modifications.

Endow the space E with a partial order $\prec$ by the following rule.

Definition 2.8. We shall write $x = \sum_{k=1}^\infty x_k e_k \prec y = \sum_{k=1}^\infty y_k e_k$ iff

$$x_k \leq y_k, \qquad k \in \mathbb{N}.$$

Set

$$W_X^+ = \{(t,x) \in I_T \times E \mid 0 \prec x \prec X(t)\}.$$

Assume that $f \in C(W_X^+, E)$.

Theorem 2.9. *Suppose that $X_k(t) > 0$ for all $k \in \mathbb{N}$ and $t \in I_T$, and for each $(t,x) \in W_X^+$ one has*

$$f_k(t, \mathcal{X}_k[X(t)]x) \leq \dot{X}_k(t), \quad 0 \prec \hat{x} \prec X(0),$$

and $f_k(t, \mathcal{X}_k[0]x) \geq 0$. Then problem (2.3) *has a solution $x \in C^1(I_T, E)$ such that*

$$0 \prec x(t) \prec X(t), \qquad t \in I_T.$$

Theorem 2.10. *Suppose that $T = \infty$ and the function f is ω-periodic ($\omega > 0$) in t. Suppose also that $X_k(t) > 0$ for all $k \in \mathbb{N}$ and $t \in I_T$, and for each $(t,x) \in W_X^+$ one has*

$$f_k(t, \mathcal{X}_k[X(t)]x) \leq \dot{X}_k(t),$$

and $f_k(t,\mathcal{X}_k[0]x) \geq 0$. *Moreover suppose that* $X(\omega) \ll X(0)$. *Then problem* (2.3) *has a solution* $\tilde{x} \in C^1(I_\infty, E)$ *such that*

$$0 \prec \tilde{x}(t) \prec X(t), \quad \tilde{x}(t+\omega) = \tilde{x}(t), \quad t \in I_\infty.$$

3. Applications

3.1. Linear PDEs. To avoid technical details we restrict ourselves to the case of PDEs with one-dimensional spatial variable. However the propositions below can easily be proved for systems with multidimensional spatial variable.

3.1.1. *The existence theorem.* Let $\mathcal{O}(\mathbb{C})$ stand for the space of entire functions $u : \mathbb{C} \to \mathbb{C}$. This is a Fréchet space with seminorms

$$\|v\|_n = \max_{|z|\leq n} |v(z)|, \quad n \in \mathbb{N},$$

and with Schauder basis $e_j = z^j$, $j \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

Let $E \subset \mathcal{O}(\mathbb{C})$ be the space of entire functions $v : \mathbb{C} \to \mathbb{C}$ such that $\overline{v(z)} = v(\overline{z})$.

Fix $T > 0$ and let $a, b \in C(I_T, \mathbb{R})$. Consider the initial value problem

$$v_t(t,z) = b(t)v(t,z) + a(t)z^m \frac{\partial^N v(t,z)}{\partial z^N}, \quad v(0,z) = \hat{v}(z). \tag{3.1}$$

Here $N > m \geq 0$ are some integers. Set

$$q_{jmN} = \frac{(j-m+N)!}{(j-m)!}, \quad j \geq m, \quad a^* = \|a\|_{C(I_T)}.$$

We assume that $a^* \neq 0$. Take arbitrary positive constants $U_0, \dots, U_{N-1}$ and define other constants recurrently by

$$U_{j-m+N} = \frac{U_j}{a^* q_{jmN}}, \quad j \geq m.$$

It is not hard to show that $U(z) := \sum_{j=0}^\infty U_j z^j \in E$.

Proposition 3.1. *Suppose that* $\hat{v} \ll U$. *Then problem* (3.1) *has a solution* $v(\cdot, z) \in C^1(I_T, E)$ *and* $v(t,z) \ll e^{\int_0^t b(s)\,ds + t} U(z)$ *for all* $t \in I_T$ *and all* $z \in \mathbb{C}$.

Note that this proposition does not follow from the results of [1].

Indeed, after the change of variable $v = e^{\int_0^t b(s)\,ds+t} u$ problem (3.1) takes the form

$$u_t(t,z) = -u(t,z) + a(t)z^m \frac{\partial^N u(t,z)}{\partial z^N}, \quad u(0,z) = \hat{v}(z). \tag{3.2}$$

In coordinate notation problem (3.2) has the form $u(t,z)=\sum_{j=0}^{\infty}u_j(t)z^j$, where

$$\begin{aligned}\dot u_j &= -u_j, & u_j(0)&=\hat v_j, \quad j<m,\\ \dot u_j &= -u_j+q_{jmN}a(t)u_{j-m+N}, & u_j(0)&=\hat v_j, \quad j\ge m.\end{aligned} \tag{3.3}$$

To apply Theorem 2.4 to problem (3.3) observe that whenever $|u_l|\le U_l$ for all $l\ge m$, we have

$$\pm\big(-(\pm U_j)+a(t)q_{jmN}u_{j-m+N}\big) \\ \le -U_j+a^*q_{jmN}U_{j-m+N}=0=\dot U_j, \qquad j\ge m,$$

completing the proof.

3.1.2. *Periodic solutions.* Let us redefine the sequence $\{U_k\}$. Take a sequence $F_k\ge 0$, $k\in\mathbb{N}$, and let $U_0,\dots,U_{N-1}$ be arbitrary positive constants. Then set

$$U_{j-m+N}=\frac{U_j}{a^*q_{jmN}+F_j}, \qquad j\ge m.$$

It is not hard to show that $U(z):=\sum_{j=0}^{\infty}U_jz^j\in E$.

Consider the system

$$u_t(t,z)=-u(t,z)+a(t)z^m\frac{\partial^N u(t,z)}{\partial z^N}+f(t,z). \tag{3.4}$$

Assume that

$$f(t,z):=\sum_{k=0}^{\infty}f_k(t)z^k\in C(I_\infty,E)$$

and the f_k are ω-periodic functions.

PROPOSITION 3.2. *Suppose that for $j\ge m$,*

$$\max_{t\in I_\omega}|f_j(t)|\le F_jU_{j-m+N}.$$

Then system (3.4) *has an ω-periodic solution $u(\cdot,z)\in C^1(I_\infty,E)$.*

In coordinate notation problem (3.4) has the form

$$\begin{aligned}\dot u_j &= -u_j+f_j, & j<m,\\ \dot u_j &= -u_j+q_{jmN}a(t)u_{j-m+N}+f_j, & j\ge m.\end{aligned} \tag{3.5}$$

To apply Theorem 2.6 to problem (3.5) observe that whenever $|u_l|\le U_l$ for all $l\ge m$, we have

$$\pm\big(-(\pm U_j)+a(t)q_{jmN}u_{j-m+N}+f_j(t)\big) \\ \le -U_j+a^*q_{jmN}U_{j-m+N}+F_jU_{j-m+N}=0=\dot U_j, \qquad j\ge m,$$

completing the proof.

3.2. Periodic solutions to the Smoluchowski equation. In this section we consider the IVP

$$\dot{x}_k = c_k + \frac{1}{2}\sum_{i+j=k} b_{ij}x_ix_j - x_k\sum_j b_{kj}x_j, \qquad x_k(0) = \hat{x}_k,\ i,j,k \in \mathbb{N}. \tag{3.6}$$

The functions $c_i, b_{ij} \in C(I_T)$ are nonnegative, and

$$b_{ij} = b_{ji}, \qquad \hat{x}_k \geq 0.$$

For this IVP, nonnegative solutions x_k are of interest.

In [5], [16] existence theorems have been proved under the following assumptions: $b_{ij}(t) \leq (i+j)^\alpha$ with some $\alpha \in [0,1]$ and $\hat{x}_k, c_k = O(c/k^p)$ as $k \to \infty$, with some $p > \alpha$. The solutions obtained are bounded in certain norms on bounded intervals.

We do not assume anything about the growth of the coefficients b_{ij} for $i \neq j$. On the other hand, we assume that the coefficients b_{kk} grow fast enough. Under these assumptions we prove the existence of global in time bounded solutions and the existence of a periodic solution when the coefficients b_{ij}, c_k are periodic.

Assume that

$$c_k(t) \leq C_k, \quad b_{kk}(t) \geq \beta_k, \quad b_{ij}(t) \leq B_{ij}$$

for all $i,j,k \in \mathbb{N}$ and $t \in I_T$, with some nonnegative constants C_k, B_{ij}, β_k. Set

$$X_1 = \sqrt{C_1/\beta_1}, \quad X_k = \sqrt{\frac{C_k + \frac{1}{2}\sum_{i+j=k} B_{ij}X_iX_j}{\beta_k}}, \qquad k = 2,3,\ldots.$$

We assume that the constants β_k are so large that

$$b_k = \sum_{j=1}^{\infty} B_{kj}X_j < \infty.$$

Set

$$A_k = C_k + \frac{1}{2}\sum_{i+j=k} B_{ij}X_iX_j + X_kb_k, \qquad F_k = \max\{1, A_k, X_k\}.$$

We introduce a Banach space E of all sequences $x = \{x_k\}$ with

$$\|x\| = \sum_{k=1}^{\infty} \frac{1}{k^2F_k}|x_k| < \infty.$$

Evidently, this space possesses an unconditional Schauder basis $e_j = \{\delta_{ij}\}_{i\in\mathbb{N}}$. Note that

$$A = \sum_{k\in\mathbb{N}} A_ke_k, \qquad X = \sum_{k\in\mathbb{N}} X_ke_k \in E.$$

Proposition 3.3. *Assume that* $\hat{x} \prec X$. *Then problem* (3.6) *has a solution* $x \in C^1(I_T, E)$ *such that*

$$0 \le x_k(t) \le X_k, \quad t \in I_T.$$

Moreover, if the functions b_{ij} *and* c_j *are* ω*-periodic then there is a solution* $\tilde{x} \in C^1(I_\infty, E)$ *that is also* ω*-periodic and* $0 \le \tilde{x}_k(t) \le X_k$ *for all* k.

Proof. We wish to apply Theorems 2.9 and 2.10.

For $0 \le x_s \le X_s$, $s \in \mathbb{N}$, and $x_k = 0$ one assumption of the theorems is satisfied identically:

$$c_k + \frac{1}{2}\sum_{i+j=k} b_{ij}x_ix_j \ge 0.$$

Another assumption to check is for $0 \le x_s \le X_s$, $s \in \mathbb{N}$, and $x_k = X_k$:

$$c_k + \frac{1}{2}\sum_{i+j=k} b_{ij}x_ix_j - x_k\sum_j b_{kj}x_j \le C_k + \frac{1}{2}\sum_{i+j=k} B_{ij}X_iX_j - X_k^2\beta_k \le \dot{X}_k = 0.$$

It remains to show that the mapping

$$f(t,x) = \sum_{k=1}^\infty f_k(t,x)e_k, \quad f_k = c_k + \frac{1}{2}\sum_{i+j=k} b_{ij}x_ix_j - x_k\sum_j b_{kj}x_j,$$

is continuous from W_X^+ to E. Observe that $f(t,x) \ll A$ for each $(t,x) \in W_X^+$. Now the continuity follows from Proposition 2.7.

4. Proof of main theorems

4.0.1. *A short digression in functional analysis.* Let $\mathcal{P}_n : E \to E$ be the projection

$$\mathcal{P}_n\Big(\sum_{k=1}^\infty x_ke_k\Big) = \sum_{k=1}^n x_ke_k.$$

Set $\mathcal{Q}_n = \mathrm{id} - \mathcal{P}_n$.

Theorem 4.1. *Let* $\lambda = \{\lambda_j\}_{j\in\mathbb{N}} \in \ell_\infty$ *and define*

$$\mathcal{M}_\lambda x = \sum_{k=1}^\infty \lambda_k x_k e_k.$$

Then for any $i' \in \mathbb{N}$ *there exist* $i \in \mathbb{N}$ *and a positive constant* c*, both independent of* λ*, such that*

$$\|\mathcal{M}_\lambda x\|_{i'} \le c\|\lambda\|_\infty \cdot \|x\|_i.$$

In particular, Theorem 4.1 implies that the operators $\mathcal{P}_n, \mathcal{Q}_n$ are continuous. Theorem 4.1 is proved in Section 6.

LEMMA 4.2. *The set W_X is compact in $I_T \times E$.*

Proof. Consider the continuous mappings

$$v_n : I_T \to E, \qquad v_n(t) = \mathcal{Q}_n X(t).$$

This sequence is pointwise convergent to zero: $v_n(t) \to 0$ as $n \to \infty$ for any fixed $t \in I_T$. On the other hand, this sequence is uniformly continuous on I_T.

Indeed, by Theorem 4.1 the mappings $\mathcal{Q}_n$ are uniformly continuous, so for any i' there exist a constant $c > 0$ and i such that

$$\|v_n(t') - v_n(t'')\|_{i'} \le c\|X(t') - X(t'')\|_i.$$

But the mapping X is uniformly continuous on the compact set I_T.

Consequently, $v_n(t) \to 0$ uniformly in I_T [11].

Evidently, the set W_X is closed. We will prove the lemma if we show that the sets

$$A_n = \{(t, \mathcal{P}_n x) \in I_T \times E \mid x \ll X(t)\}$$

form a sequence of compact ϵ-nets in W_X.

Indeed, each set A_n is closed and bounded in $\mathbb{R}^{n+1}$.

Let $(t, x) \in W_X$ and employ Theorem 4.1 with

$$\lambda_k(t) = x_k / X_k(t), \qquad |\lambda_k(t)| \le 1.$$

Then it follows that for any $i' \in \mathbb{N}$ there exist $i \in \mathbb{N}$ and a constant c such that

$$\|x - \mathcal{P}_n x\|_{i'} = \|\mathcal{M}_{\lambda(t)} \mathcal{Q}_n X(t)\|_{i'} \le c\|v_n(t)\|_i.$$

Therefore $\sup_{(t,x)\in W_X} \|x - \mathcal{P}_n x\|_{i'} \to 0$ as $n \to \infty$, proving the lemma.

By an analogous argument one obtains the following lemma.

LEMMA 4.3. *Let $U = \sum_{k=1} U_k e_k$ with $U_k \ge 0$. Then the set*

$$K_U = \{u \in E \mid u \ll U\}$$

is compact.

THEOREM 4.4 (Arzelà–Ascoli, [11]). *Let $K \subset C(I_T, E)$. Suppose that*

- *for any $t \in I_T$ the set $K_t = \{x(t) \mid x(\cdot) \in K\} \subset E$ is compact;*
- *for any $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists a constant $\delta > 0$ such that if $t', t'' \in I_T$ with $|t' - t''| < \delta$ then*

$$\|x(t') - x(t'')\|_n \le \epsilon.$$

Then K is compact.

4.0.2. *Back to the proof of Theorem 2.4.* We approximate problem (2.3) by the following finite-dimensional ones:

$$\dot{y}^n = \mathcal{P}_n f(t, y^n), \quad y^n(0) = \hat{y}^n = \mathcal{P}_n \hat{x}, \qquad y^n = \sum_{j=1}^{n} y_j e_j. \tag{4.1}$$

By Theorem 5.1 all the problems (4.1) have solutions $y^n \in C^1(I_T, \mathbb{R}^n)$ and

$$(t, y^n(t)) \in W_X, \quad t \in I_T. \tag{4.2}$$

By Theorem 4.1 and Lemma 4.2 for any $i' \in \mathbb{N}$ there is $i \in \mathbb{N}$ and a constant c such that

$$\begin{aligned}\sup\{\|\dot y^n(t)\|_{i'} \mid n \in \mathbb{N},\, t \in I_T\} &\le \sup\{\|\mathcal{P}_n f(t,x)\|_{i'} \mid (t,x) \in W_X,\, n \in \mathbb{N}\}\\ &\le c \sup_{(t,x)\in W_X} \|f(t,x)\|_i \le C_{i'} < \infty.\end{aligned}$$

For any $t', t'' \in I_T$ this implies

$$\|y^n(t') - y^n(t'')\|_{i'} = \Big\| \int_{t'}^{t''} \dot y^n(s)\, ds \Big\|_{i'} \le C_{i'} |t' - t''|.$$

By Theorem 4.4 and Lemma 4.3 the sequence $\{y^n\}$ contains a subsequence (not relabelled) that is convergent in $C(I_T, E)$:

$$y^n(\cdot) \to x(\cdot) \quad \text{in } C(I_T, E).$$

Since the operators $\mathcal{P}_n$ are continuous, (4.2) implies $x(t) \ll X(t)$ for $t \in I_T$.

Our next goal is to show that $x(\cdot)$ is the desired solution to problem (2.3). Rewrite problem (4.1) as

$$y^n(t) - \hat y^n = \int_0^t \mathcal{P}_n f(s, y^n(s))\, ds. \tag{4.3}$$

Letting $n \to \infty$ in the left hand side we obtain $x(t) - \hat x$.

Consider the right hand side of (4.3).

Lemma 4.5. *For all $i \in \mathbb{N}$ and $t \in I_T$,*

$$\Big\| \int_0^t \mathcal{P}_n f(s, y^n(s))\, ds - \int_0^t f(s, x(s))\, ds \Big\|_i \to 0$$

as $n \to \infty$. The integrals are understood in the sense of Millionshchikov [6].

Proof. Estimate this expression by parts

$$\begin{aligned}\Big\| \int_0^t \mathcal{P}_n f(s, y^n(s))\, ds - \int_0^t f(s, x(s))\, ds \Big\|_i &\le \int_0^t \big\| \mathcal{P}_n \big(f(s, y^n(s)) - f(s, x(s)) \big) \big\|_i\, ds\\ &\quad + \int_0^t \|\mathcal{Q}_n f(s, x(s))\|_i\, ds.\end{aligned}$$

Then due to Theorem 4.1 we have

$$\big\| \mathcal{P}_n \big(f(s, y^n(s)) - f(s, x(s)) \big) \big\|_i \le c_i \|f(s, y^n(s)) - f(s, x(s))\|_{i'} \to 0.$$

Since f is uniformly continuous in the compact set W_X, this limit is uniform in $s \in I_T$.

The set $f(W_X)$ is a compact set as the continuous image of a compact set. The operators $\mathcal{Q}_n$ are uniformly continuous (Theorem 4.1). Consequently, the convergence $\|Q_n f(s, x(s))\|_i \to 0$ is uniform in $s \in I_T$ [11], proving the lemma.

From Lemma 4.5 and formula (4.3) it follows that

$$x(t) - \hat{x} = \int_0^t f(s, x(s))\, ds.$$

Consequently, $x(\cdot) \in C^1(I_T, E)$ and $\dot{x}(t) = f(t, x(t))$ [6].

In coordinate notation this implies

$$x_k(t) - \hat{x}_k = \int_0^t f_k(s, x(s))\, ds,$$

or

$$\dot{x}_k(t) = f_k(t, x(t)), \qquad x_k(0) = \hat{x}_k.$$

This in particular implies that the series $\sum_{k=1}^{\infty} \dot{x}_k(t) e_k$ is convergent for each t.

Theorem 2.4 is proved.

4.0.3. *Proof of Theorem 2.6.* By Theorem 5.2 all the problems (4.1) have ω-periodic solutions $\tilde{y}^n(t)$ such that

$$\tilde{y}^n(t) \ll X(t).$$

By the same argument as above, the set $\{\tilde{y}^n(\cdot)\}$ is relatively compact in $C(I_\omega, E)$. Let $y_*(\cdot)$ be an accumulation point of this set. Then the function

$$\tilde{x}(t) = y_*(\tau), \qquad \tau \in I_\omega, \qquad t = \tau \pmod{\omega}$$

is a periodic solution, completing the proof.

5. Finite-dimensional case. In this section we consider ordinary differential equations in $\mathbb{R}^m$ and formulate several known results needed in the proof of infinite-dimensional theorems.

Set

$$\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_k \geq 0 \text{ for } k = 1, \ldots, m\}, \qquad I_T = [0, T].$$

We shall say that a vector $X = (X_1, \ldots, X_m) \in \mathbb{R}^m_+$ *majorizes* a vector $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, written $x \ll X$, if

$$|x_k| \leq X_k, \qquad k = 1, \ldots, m.$$

Let $X(\cdot) \in C^1(I_T, \mathbb{R}^m)$ be such that $X(t) \in \mathbb{R}^m_+$ for all $t \in I_T$. Define

$$W_X = \{(t, x) \in I_T \times \mathbb{R}^m \mid x \ll X(t)\}.$$

In this section we assume that $f \in C(W_X, \mathbb{R}^m)$ and consider the problem

$$\dot{x} = f(t, x), \qquad x(0) = \hat{x}, \qquad x = (x_1, \ldots, x_m). \tag{5.1}$$

Theorem 5.1 ([7]). *Suppose that $X_k(t) > 0$ for all $k = 1, \dots, m$ and $t \in I_T$, and for each $(t, x) \in W_X$,*

$$\pm f_k(t, x_1, \dots, x_{k-1}, \pm X_k(t), x_{k+1}, \dots, x_m) \le \dot{X}_k(t), \quad \hat{x} \ll X(0).$$

Then problem (5.1) *has a solution $x(\cdot) \in C^1(I_T, \mathbb{R}^m)$ such that $x(t) \ll X(t)$ for all $t \in I_T$.*

The following theorem is a consequence of Theorem 5.1, Brouwer's fixed point theorem and uniform approximation of f by locally Lipschitz continuous functions.

Theorem 5.2. *Suppose that $T = \infty$ and in addition to conditions of Theorem* 5.1 *assume the function f is ω-periodic $(\omega > 0)$ in t:*

$$f(t + \omega, x) = f(t, x)$$

and $X(\omega) \ll X(0)$. Then problem (5.1) *has a solution $\tilde{x}(\cdot) \in C^1(I_T, \mathbb{R}^m)$ such that $\tilde{x}(t) \ll X(t)$ for all $t \in I_T$ and $\tilde{x}(t + \omega) = \tilde{x}(t)$.*

6. Appendix: Proof of Theorem 4.1. Let us note that Theorem 4.1 remains valid for the space E over the field $\mathbb{C}$, with $\lambda = \{\lambda_j\}$, $\lambda_j \in \mathbb{C}$. This case can be reduced to the real one by considering the realification of the space E with the Schauder basis $\{e_k, ie_k\}$, $i^2 = -1$.

For the Banach space version of Theorem 4.1 see [3]. The Fréchet space version follows from a lemma of L. Weill.

Let S stand for the set $\{\pm 1\}^{\mathbb{N}}$.

Lemma 6.1 (L. Weill, [15]). *For any $i \in \mathbb{N}$ there are a constant $c > 0$ and $i' \in \mathbb{N}$ such that for all $x \in E$,*

$$\sup_{\theta \in S} \|\mathcal{M}_\theta x\|_i \le c\|x\|_{i'}.$$

Now let us prove the theorem. Let us show that the operator $\mathcal{M}_\lambda$ is defined for all $x \in E$. Set

$$b_{nm} = \sum_{n \le k \le m} \lambda_k x_k e_k, \qquad a_{nm} = \sum_{n \le k \le m} x_k e_k.$$

We wish to show that for each $j \in \mathbb{N}$,

$$\|b_{nm}\|_j \to 0 \quad \text{as } n, m \to \infty.$$

There exists $f \in E^*$ (E^* stands for the algebraic dual space of E) such that $f(b_{nm}) = \|b_{nm}\|_j$ and $|f(x)| \le \|x\|_j$ for all $x \in E$ [9]. The element f depends on b_{nm}, j.

Then $f(b_{nm}) = \sum_{n \le k \le m} \lambda_k x_k f(e_k)$. Define a sequence $\theta \in S$ as follows. Set $\theta_k = 1$ for $x_k f(e_k) \ge 0$, and $\theta_k = -1$ otherwise.

Thus

$$\|b_{nm}\|_j \le \sup_k |\lambda_k| \sum_{n\le k\le m} \theta_k x_k f(e_k).$$

From this formula it follows that

$$\|b_{nm}\|_j \le \|\lambda\|_\infty f(\mathcal{M}_\theta a_{nm}) \le \|\lambda\|_\infty \|\mathcal{M}_\theta a_{nm}\|_j.$$

By Lemma 6.1 there is $i \in \mathbb{N}$ and a constant $c > 0$ such that

$$\|\mathcal{M}_\theta a_{nm}\|_j \le c\|a_{nm}\|_i.$$

The parameters i, c are independent of a_{nm} and $\theta \in S$. Since the series (2.1) is convergent, $a_{nm} \to 0$ as $n, m \to \infty$, and so $\mathcal{M}_\theta a_{nm} \to 0$. Thus $\mathcal{M}_\lambda x$ is defined for all $x \in E$ and $\lambda \in \ell_\infty$.

Now replacing b_{nm} with the partial sums $b_n = \sum_{k=1}^n \lambda_k x_k e_k$ and repeating the previous argument we obtain the assertion of the theorem.

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