BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 63, No. 3, 2015

NUMBER THEORY

On the Behavior of Power Series with Completely Additive Coefficients

by

Oleg PETRUSHOV

Presented by Jerzy KACZOROWSKI

Summary. Consider the power series $\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n) z^n$, where $\alpha(n)$ is a completely additive function satisfying the condition $\alpha(p) = o(\ln p)$ for prime numbers p. Denote by $e(l/q)$ the root of unity $e^{2\pi i l/q}$. We give effective omega-estimates for $\mathfrak{A}(e(l/p^k)r)$ when $r \to 1-$. From them we deduce that if such a series has non-singular points on the unit circle, then it is a zero function.

1. Introduction. In this paper we study power series with completely additive coefficients. Power series with coefficients that have some arithmetical structure possess interesting properties. All known power series with nontrivial arithmetical coefficients have no continuation beyond the unit circle. Moreover they have interesting properties when z tends to the unit circle along a radius.

In 1981 L. G. Lucht [L] proved that for an extensive set of multiplicative $\alpha(n)$ the unit circle is the natural boundary of the series $\sum_{n=1}^{\infty} \alpha(n) z^n$.

In [P1] we studied the power series $\mathfrak{M}(z) = \sum_{n=1}^{\infty} \mu(n) z^n$ where $\mu(n)$ is the Möbius function and proved that for each $\beta \in \mathbb{Q}$,

$$\mathfrak{M}(re(\beta)) = \Omega((1-r)^{-a})$$

when $r \to 1-$ for some $a > 0$ depending on β.

In [P2] we obtained nontrivial estimates for $\mathfrak{M}_0(z) = \sum_{n=1}^{\infty} \mu^2(n) z^n$. The behavior of this series when $z = e(\beta)r$ with $r \to 1-$ depends on Diophantine

---

2010 *Mathematics Subject Classification*: Primary 11N37; Secondary 30B30.

Key words and phrases: power series, additive functions, growth conditions, omega-estimates, boundary behavior, arithmetic functions.
Received 2 April 2015.
Published online 18 January 2016.

DOI: 10.4064/ba8018-1-2016

approximation properties of β. We proved that if the irrationality exponent of β equals 2 then

$$\mathfrak{M}_0(z) = O((1-r)^{-1/2-\varepsilon}).$$

An arithmetical function $\alpha(n)$ is *completely additive* if

$$\alpha(mn) = \alpha(m) + \alpha(n)$$

for all m and n.

As usual for $s \in \mathbb{C}$ we denote $\sigma = \Re s$, $t = \Im s$.

We will prove that some conditions on the growth of the coefficients of the power series with completely additive coefficients give us a class of power series that have the unit circle as the natural boundary. Moreover we prove that if $\alpha(n)$ is completely additive and $\alpha(p) = o(\ln p)$ then $\alpha(n) = 0$ for each n. Moreover we give some omega-estimates of such series for z tending to a root of unity.

Denote $\exp(2\pi i\beta)$ by $e(\beta)$. For any sequence $\alpha(n)$, real β and Dirichlet character χ we define

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}, \\ F[\beta](s) &= \sum_{n=1}^{\infty} \alpha(n)e(\beta n)n^{-s}, \\ F(s,\chi) &= \sum_{n=1}^{\infty} \alpha(n)\chi(n)n^{-s}, \\ A(x,\beta) &= \sum_{n<x} \alpha(n)e(\beta n). \end{aligned} \tag{1.1}$$

Denote by $\mathfrak{A}(z)$ where $z \in \mathbb{C}$ the power series

$$\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n)z^n.$$

For a Dirichlet character χ modulo q, $\bar{\chi}$ is the character conjugate to χ, $\tau(\chi,l) = \sum_{k=1}^{q} \chi(k)e(lk/q)$, and $L(s,\chi)$ is a Dirichlet L-series.

We study the Dirichlet series $F[\beta](s)$ and from its properties we deduce the properties of the series $\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n)z^n$.

We will always denote by p prime numbers.

THEOREM 1.1. *Let $\alpha(n)$ be a completely additive function satisfying*

$$\alpha(p) = o(\ln p). \tag{1.2}$$

Then for $\beta = l/p^k$ with $(l,p) = 1$ we have

$$\overline{\lim_{r\to 1-}} |\mathfrak{A}(e(\beta)r)(1-r)| \geq \frac{|\alpha(p)|p^{1-k}}{p-1}, \tag{1.3}$$

$$\overline{\lim_{x\to\infty}} \left|\frac{A(x,\beta)}{x}\right| \geq \frac{|\alpha(p)|p^{1-k}}{p-1}. \tag{1.4}$$

Moreover if $\mathfrak{A}(z)$ has nonsingular points on the unit circumference then $\alpha(n) \equiv 0$.

Hence all series $\mathfrak{A}(z)$ with completely additive coefficients that satisfy (1.2) and do not equal 0 identically have the unit circle as the natural boundary.

Theorem 1.1 is proved in Sections 2–4. In Section 5 we also note that the growth condition (1.2) in Theorem 1.1 cannot be significantly weakened.

2. Preliminary results. From the definition (1.1) of $F(s,\chi)$ we easily obtain the following properties.

Lemma 2.1. *Let F and G be Dirichlet series absolutely convergent at a point s. Then*

$$(F+G)(s,\chi) = F(s,\chi) + G(s,\chi),$$
$$(FG)(s,\chi) = F(s,\chi)G(s,\chi).$$

Lemma 2.2. *Let $G_p(s) = \sum_{k=1}^{\infty} c_{p^k} p^{-ks}$. Let the double series $\sum_p G_p(s)$ be absolutely convergent. Let $F(s)$ be a Dirichlet series that is derived from $\sum_p G_p(s)$ by regulating its summands. Then*

$$F(s,\chi) = \sum_p G_p(s,\chi).$$

The following lemma detects the structure of the Dirichlet series with completely additive coefficients.

Lemma 2.3. *Let $\alpha(n)$ be a completely additive function with $\alpha(n) = O(\ln n)$. Then*

$$F(s) = \zeta(s) \sum_p \alpha(p) \frac{p^{-s}}{1-p^{-s}} \tag{2.1}$$

for $\Re s > 1$.

Proof. If the series $\sum_p \alpha(p)\frac{p^{-s}}{1-p^{-s}}$ is considered as a double series, for $\sigma > 1$ it converges absolutely. We have

$$\sum_p \alpha(p) \frac{p^{-s}}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \tag{2.2}$$

where

$$b_n = \begin{cases} \alpha(p) & \text{if } n = p^k, \\ 0 & \text{if } n \neq p^k. \end{cases}$$

For s with $\Re s > 1$,

$$\zeta(s) \sum_p G_p(s) = \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s},$$

where for $n = p_1^{l_1} \dots p_r^{l_r}$ we have

$$\gamma(n) = \sum_{d|n} b_d = \sum_{p^k|n} \alpha(p) = \sum_{i=1}^{r} \alpha(p^{l_i}) = \alpha(p_1^{l_1} \dots p_r^{l_r}) = \alpha(n). \ \blacksquare$$

Let $I(s)$ be a Dirichlet integral,

$$I(s) = \int_0^{\infty} u^{s-1} f(u)\, du, \qquad s \in \mathbb{C},$$

where $f \in L^1[r, R]$ for any $0 < r < R < \infty$. Let

$$I_1(s) = \int_0^1 u^{s-1} f(u)\, du,$$
$$I_2(s) = \int_1^{\infty} u^{s-1} f(u)\, du.$$

The following lemma relates $F[l/q](s)$ to $\mathfrak{A}(e(l/q)r)$.

LEMMA 2.4. *Let $\alpha(n)$ be any sequence of complex numbers, and $l \in \mathbb{Z}$. Suppose the Dirichlet series $F(s) = \sum_{n=1}^{\infty} \alpha(n) e(ln/q) n^{-s}$ is convergent for $\sigma = \Re s > \sigma_0 > 0$. Then for each s with $\Re s > \sigma_0$,*

$$\Gamma(s) \sum_{n=1}^{\infty} \alpha(n) e(ln/q) n^{-s} = \int_0^{\infty} t^{s-1} \mathfrak{A}(e(l/q)e^{-t})\, dt.$$

Proof. Follows from the results of [H]. $\blacksquare$

The following lemma relates the behavior of $\mathfrak{A}(e(l/q)r)$ as $r \to 1-$ to the behavior of $F[l/q](s)$ near 1.

THEOREM 2.5. *Let $\alpha(n)$ be any sequence of complex numbers, and let $q \in \mathbb{N}$, $q > 1$, $(l, q) = 1$. Let $F(s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s}$. Assume the series $F[l/q](s)$ is convergent in the domain $\{\Re s > 1\}$. Suppose that*

$$\overline{\lim_{x \to 0+}} \frac{|F[l/q](1+x)|}{x^{-1}} \geq c_1. \tag{2.3}$$

Then

$$\overline{\lim_{u\to 0+}} \frac{|\mathfrak{A}(e(l/q)e^{-u})|}{u^{-1}} \geq c_1, \tag{2.4}$$

$$\overline{\lim_{x\to \infty}} \frac{|A(x, e(l/q))|}{x} \geq c_1. \tag{2.5}$$

Proof. By Lemma 2.4,

$$\Gamma(s)F[l/q](s) = \int_0^\infty u^{s-1}\mathfrak{A}(e(l/q)e^{-u})\, du$$

where the integral is also convergent in the domain $\{\Re s > 1\}$. Assume that there exists $c_0 < c_1$ such that for $u \to 0+$,

$$\frac{|\mathfrak{A}(e(l/q)e^{-u})|}{u^{-1}} < c_0.$$

Then for $0 < u < u_0$ we have $|\mathfrak{A}(e(l/q)e^{-u})| < c_0 u^{-1}$. From this inequality we obtain

$$|I(1+x)| \leq O(1) + c_0\Big|\int_0^1 u^{x-1}\, du\Big| \leq O(1) + c_0 x^{-1}, \qquad x \to 0+.$$

This inequality contradicts (2.3). Thus we obtain (2.4). Using Abel transform we deduce the inequality (2.5) of the theorem. ∎

Let $\beta = l/p^k$. Each $n \in \mathbb{N}$ has a unique representation $n = mk$ where $k = p^j$ and $(m,p) = 1$. For integer $j \geq 0$ denote by A_j the set $\{n : n = mp^j, (m,p) = 1\}$. From the above it follows that these sets are pairwise disjoint and $\mathbb{N} = \bigcup_j A_j$. Let $u(n) = e(ln/p^k)$ if $(n,p) = 1$, and $u(n) = 0$ if $(n,p) \neq 1$. From the orthogonal relations for Dirichlet characters we derive

$$u(n) = \frac{1}{\phi(p^k)} \sum_{\chi \,(\mathrm{mod}\, p^k)} \tau(\bar{\chi}, l)\chi(n). \tag{2.6}$$

Let $\alpha(n)$ be a completely additive function of integer argument with $\alpha(p) = o(\ln p)$. Let us represent $F[\beta](s)$ in terms of $F(s,\chi)$. For $j \geq 0$ consider

$$\begin{aligned} S_j &= \sum_{n\in A_j} \alpha(n) n^{-s} e(\beta n) = \sum_{(m,p)=1} \alpha(mp^j)(mp^j)^{-s} e(mlp^j/p^k) \\ &= \sum_{(m,p)=1} \frac{\alpha(m) + j\alpha(p)}{(mp^j)^s} e(mlp^j/p^k) \\ &= \frac{1}{p^{js}} \sum_{(m,p)=1} \alpha(m) e(mlp^j/p^k) m^{-s} + j\frac{\alpha(p)}{p^{js}} \sum_{(m,p)=1} e(mlp^j/p^k) m^{-s}. \end{aligned}$$

From (2.6) we obtain

$$S_j = \frac{1}{\phi(p^k)}\Bigg(\frac{1}{p^{js}}\sum_{\chi\,(\mathrm{mod}\,p^k)}\tau(\bar\chi, lp^j)\sum_{m=1}^{\infty}\frac{\alpha(m)\chi(m)}{m^s} + j\frac{\alpha(p)}{p^{js}}\sum_{\chi\,(\mathrm{mod}\,p^k)}\tau(\bar\chi, lp^j)\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}\Bigg)$$

$$= \frac{1}{\phi(p^k)}\sum_{\chi\,(\mathrm{mod}\,p^k)}\bigg(\frac{\tau(\bar\chi, lp^j)}{p^{js}}F(s,\chi) + j\tau(\bar\chi, lp^j)\frac{\alpha(p)}{p^{js}}L(s,\chi)\bigg).$$

Summing S_j with respect to $j \in \mathbb{N}\cup 0$ we obtain

Lemma 2.6. *Let $\alpha(n)$ be a completely additive function. Then*

$$F[\beta](s) = \frac{1}{\phi(p^k)}\sum_{\chi\,(\mathrm{mod}\,p^k)} C_\chi(s)F(s,\chi) + D_\chi(s)L(s,\chi), \tag{2.7}$$

where

$$C_\chi(s) = \sum_{j=0}^{\infty}\frac{\tau(\bar\chi, lp^j)}{p^{js}}, \tag{2.8}$$

$$D_\chi(s) = \alpha(p)\sum_{j=0}^{\infty} j\frac{\tau(\bar\chi, lp^j)}{p^{js}}. \tag{2.9}$$

This lemma relates the behavior of $F[\beta](s)$ to the behavior of $F(s,\chi)$ and $L(s,\chi)$.

3. The behavior of some useful Dirichlet series. In this section, for any function A and any positive function B, $A \ll B$ means $A = O(B)$.

Let us recall the properties of the Ramanujan sum. For prime p, integers k, l, a with $(l,p) = 1$ and a principal character χ_0 modulo p^k,

$$\tau(\chi_0, lp^{k-1}) = -p^{k-1},$$
$$\tau(\chi_0, lp^a) = p^{k-1}(p-1), \qquad a \ge k,$$
$$\tau(\chi_0, lp^a) = 0, \qquad a < k-1.$$

Let p_0 be a fixed prime number. Let χ_0 be a principal character modulo $q = p_0^k$. Then from (2.8), (2.9),

$$C_{\chi_0}(s) = -\frac{p_0^{k-1}}{p_0^{(k-1)s}} + p_0^{k-1}(p_0-1)\sum_{n=k}^{\infty}\frac{1}{p_0^{ns}} = -\frac{p_0^{k-1}}{p_0^{(k-1)s}} + p_0^{k-1}(p_0-1)\frac{p_0^{-ks}}{1-p_0^{-s}}, \tag{3.1}$$

$$D_{\chi_0}(s) = \left(-(k-1)\frac{p_0^{k-1}}{p_0^{(k-1)s}} + p_0^{k-1}(p_0-1)\sum_{n=k}^{\infty} \frac{n}{p_0^{ns}} \right) \alpha(p_0). \tag{3.2}$$

Note that $\tau(\chi, lp_0^a) = 0$ if $a > k$ for a nonprincipal character χ modulo q. Hence if $\chi \neq \chi_0 \pmod q$ the sums in (2.8) and (2.9) are finite. Thus $C_\chi(s) \ll 1$ and $D_\chi(s) \ll 1$ when $s \to 1+$ if $\chi \neq \chi_0 \pmod q$. Further we will use the simple asymptotic equality

$$L(1+\sigma, \chi_0) \sim \frac{p_0 - 1}{p_0}\sigma^{-1}, \qquad \sigma \to 0+. \tag{3.3}$$

Consider the function $F(s, \chi)$. By Lemma 2.3 for s with $\Re s > 1$,

$$F(s) = \zeta(s) \sum_p \alpha(p) \frac{p^{-s}}{1-p^{-s}}.$$

Hence by Lemmas 2.1 and 2.2,

$$F(s, \chi_0) = \sum_{p \neq p_0} \alpha(p) \frac{p^{-s}}{1-p^{-s}}. \tag{3.4}$$

Since for every $c > 0$ there exists a number $P(c)$ such that $|\alpha(p)| \leq c \ln p$ if $p > P(c)$, we obtain

$$\begin{aligned} |F(1+\sigma, \chi_0)| &\leq L(1+\sigma, \chi_0)\Big(O(1) + 2c \sum_{p > P(c)} p^{-1-\sigma} \ln p \Big) \\ &= L(1+\sigma, \chi_0)\Big(O(1) + 2c \sum_p p^{-1-\sigma} \ln p \Big) \\ &\leq L(1+\sigma, \chi_0)\left(O(1) + 2c \frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)} \right) \sim 2c \frac{p_0}{p_0 - 1}\sigma^{-2}. \end{aligned}$$

From (3.1), $C_{\chi_0}(s)$ has a simple zero at $s = 1$. Hence

$$|C_{\chi_0}(1+\sigma) F(1+\sigma, \chi_0)| \leq c_1 \sigma^{-1}, \qquad \sigma \to 0+,$$

where $c_1 > 0$ is an arbitrarily small number. Hence

$$C_{\chi_0}(1+\sigma) F(1+\sigma, \chi_0) = o(\sigma^{-1}), \qquad \sigma \to 0+. \tag{3.5}$$

From Lemmas 2.1 and 2.2,

$$F(s, \chi) = L(s, \chi) \sum_p \alpha(p) \frac{p^{-s}\chi(p)}{1 - p^{-s}\chi(p)}.$$

Hence for each $c > 0$, as $\sigma \to 0+$,

$$\begin{aligned} |F(s, \chi)| &\leq \sum_p c \ln p \left| \frac{p^{-s}\chi(p)}{1 - p^{-s}\chi(p)} \right| + O(1) \\ &\leq H + c \sum_p \ln p \frac{p^{-\sigma}}{1 - p^{-\sigma}} = H + c \frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)}. \end{aligned}$$

Hence if $\chi \neq \chi_0 \pmod q$ then

$$C_\chi(1+\sigma)F(1+\sigma,\chi) = o(\sigma^{-1}). \tag{3.6}$$

Thus from (3.5) and (3.6) we obtain

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} C_\chi(1+\sigma)F(1+\sigma,\chi) = o(\sigma^{-1}). \tag{3.7}$$

Let us estimate $D_{\chi_0}(s)$.

From the series on the right-hand side of (3.2) we see that $D_{\chi_0}(s)$ is an analytic function in the half-plane $\{\Re s > 0\}$.

Since for nonprincipal χ the functions $L(s,\chi)$ are holomorphic at $s=1$ and $D_\chi(s)$ are bounded when $s \to 1$, we have

$$\sum_{\chi \neq \chi_0 \pmod q} D_\chi(1+\sigma)L(1+\sigma,\chi) \ll 1, \qquad \sigma \to 0+. \tag{3.8}$$

From (3.2) we deduce

$$\begin{aligned} D_{\chi_0}(1) &= \alpha(p_0)p_0^{k-1}\Big(-(k-1)p_0^{-(k-1)} + \sum_{j=k-1}^{\infty} \frac{j+1}{p_0^j} - \sum_{j=k}^{\infty} \frac{j}{p_0^j}\Big) \\ &= \alpha(p_0)p_0^{k-1}\Big(-(k-1)p_0^{-(k-1)} + kp_0^{-(k-1)} + \sum_{j=k}^{\infty} p_0^{-j}\Big) \\ &= \alpha(p_0)\frac{p_0}{p_0-1} \end{aligned} \tag{3.9}$$

when $\sigma \to 0+$.

Thus if $\alpha(p_0) \neq 0$, from (3.3) and (3.9) we obtain

$$\frac{1}{\phi(q)} D_{\chi_0}(1+\sigma)L(1+\sigma,\chi_0) \sim \frac{\alpha(p_0)p_0^{1-k}}{p_0-1}\sigma^{-1}. \tag{3.10}$$

From (3.7), (3.8), (3.10) and Lemma 2.6 we deduce

$$F[l/p_0^k](1+\sigma) \sim \frac{\alpha(p_0)p_0^{1-k}}{p_0-1}\sigma^{-1}, \qquad \sigma \to 0+. \tag{3.11}$$

4. Proof of Theorem 1.1. Applying Theorem 2.5 to $\alpha(n)$, $F(s)$, l, $q = p^k$ from (3.11) we obtain the inequalities (1.3) and (1.4) of the theorem. Let $\alpha(p) \neq 0$ for some p. Then singular points are dense on the unit circle. Hence $\mathfrak{A}(z)$ has the unit circle as its natural boundary. If $\mathfrak{A}(z)$ is continuable beyond the unit circle we have $\alpha(p) = 0$ for each p. Hence $\mathfrak{A}(z) = 0$. ∎

5. Some examples. Let $n = p_1^{l_1} \dots p_r^{l_r}$ be the canonical representation of n, $\Omega(n) = l_1 + \dots + l_r$ and

$$W(x) = \sum_{n<x} \Omega(n), \quad \mathfrak{W}(z) = \sum_{n=1}^{\infty} \Omega(n) z^n, \quad W(x,\beta) = \sum_{n<x} \Omega(n) e(\beta n).$$

Applying Theorem 1.1 to the classical function $\Omega(n)$ we obtain the following estimates. For prime p and integer $k \ge 0$, l,

$$\varlimsup_{r\to 1-} |\mathfrak{W}(e(l/p^k)r)(1-r)| \ge \frac{p^{1-k}}{p-1},$$

$$\varlimsup_{x\to\infty} \frac{|W(x, e(l/p^k))|}{x} \ge \frac{p^{1-k}}{p-1}.$$

We note that the growth condition (1.2) cannot be strongly weakened.

EXAMPLE 5.1. If $\alpha(n) = c \ln n$ then $\mathfrak{A}(z)$ has an analytic continuation to the domain $\mathbb{C} \setminus [1,\infty)$.

Proof. Note that $\sum_{n=1}^{\infty} (\ln n) z^n$ is the s-derivative of the polylogarithm function $\sum_{n=1}^{\infty} n^s z^n$ at $s = 0$ (see [LS]). ■

References

[H] G. H. Hardy, *On a case of term-by-term integration of an infinite series*, Messenger Math. 39 (1910), 136–139.

[L] L. G. Lucht, *Power series with multiplicative coefficients*, Math. Z. 177 (1981), 359–374.

[LS] L. G. Lucht and A. Schmalmack, *Polylogarithms and arithmetic function spaces*, Acta Arith. 95 (2000), 361–382.

[P1] O. A. Petrushov, *On the behavior close to the unit circle of the power series with Möbius function coefficients*, Acta Arith. 164 (2014), 119–136.

[P2] O. A. Petrushov, *On the behavior close to the unit circle of the power series whose coefficients are squared Möbius function values*, Acta Arith. 168 (2015), 17–30.

Oleg Petrushov
Moscow State University
Moscow, Russia
E-mail: olegAP86@yandex.ru