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The Tree Property at ω_2 and Bounded Forcing Axioms

by

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Summary. We prove that the Tree Property at ω_2 together with BPFA is equiconsistent with the existence of a weakly compact reflecting cardinal, and if BPFA is replaced by BPFA(ω_1) then it is equiconsistent with the existence of just a weakly compact cardinal. Similarly, we show that the Special Tree Property for ω_2 together with BPFA is equiconsistent with the existence of a reflecting Mahlo cardinal, and if BPFA is replaced by BPFA(ω_1) then it is equiconsistent with the existence of just a Mahlo cardinal.

1. Introduction. In this article we discuss some consistency results concerning the conjunction of forcing axioms with the Tree Property for ω_2. We say that a regular cardinal κ has the *Tree Property* (TP(κ)) if every tree T of height κ with levels of size $< \kappa$ has a branch of length κ. Erdős and Tarski [5] showed that if κ is weakly compact, then κ has the tree property. They also proved that if κ is inaccessible and has the tree property, then κ is weakly compact.

We recall a result of Silver stating that if TP(ω_2) holds then ω_2 is weakly compact in L [12, Theorem 5.9]. Mitchell proved that if κ is weakly compact then there is a generic extension where $\kappa = \omega_2 = 2^\omega$ and TP(ω_2) holds (see [12]). So in particular, TP(ω_2) is equiconsistent with the existence of a weakly compact cardinal.

Our motivation for the results of this paper was to see how consistency proofs for the Tree Property for ω_2 and for forcing axioms can be combined. It is not clear how the standard consistency proofs of TP(ω_2) due to Mitchell or to Baumgartner and Laver via iterated Sacks forcing (see [3]) can

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be merged with consistency proofs of forcing axioms such as MA_{ω_1} or the Bounded Proper Forcing Axiom. Our approach solves this problem through use of Baumgartner's method for specializing ω_1-trees together with a weakly compact diamond-sequence as a bookkeeping method.

In this paper we prove that the existence of a weakly compact cardinal is equiconsistent with the conjunction of $\mathrm{TP}(\omega_2)$ and MA_{ω_1}, or even with $\mathrm{TP}(\omega_2)$ and $\mathrm{BPFA}(\omega_1)$. Also we prove that $\mathrm{TP}(\omega_2)$ together with BPFA is equiconsistent with the existence of a weakly compact cardinal which is also reflecting.

We also work with similar results involving the Special Tree Property. Trees of height κ with levels of size $< \kappa$ and no branches of length κ are called *κ-Aronszajn*, in reference to Aronszajn's construction of a tree of height ω_1 each of whose levels is countable but with no uncountable branch (see [11]). An ω_2-Aronszajn tree T is *special* if there is a function $f : T \to \omega_1$ such that for any $s, t \in T$, if $s <_T t$ then $f(s) \neq f(t)$. We say that ω_2 has the *Special Tree Property*, $\mathrm{SpTP}(\omega_2)$, if there are no special ω_2-Aronszajn trees. Recall that an inaccessible cardinal κ is *Mahlo* if the set of all regular cardinals below κ is stationary, and so the set of all inaccessible cardinals below κ is also stationary. Also in [12], Mitchell proved that the existence of a Mahlo cardinal is equiconsistent with $\mathrm{SpTP}(\omega_2)$.

Using similar methods, we establish the same results for $\mathrm{SpTP}(\omega_2)$ with "weakly compact" replaced by "Mahlo", i.e. we prove that the existence of a Mahlo cardinal is equiconsistent with the conjunction of $\mathrm{SpTP}(\omega_2)$ and $\mathrm{BPFA}(\omega_1)$. Also we prove that $\mathrm{SpTP}(\omega_2)$ together with BPFA is equiconsistent with the existence of a Mahlo cardinal which is also reflecting ([1]).

2. Preliminaries and basic definitions. Recall that a cardinal κ is *weakly compact* if it is uncountable and for every function $F : [\kappa]^2 \to 2$, there is $H \subseteq \kappa$ of cardinality κ such that $F \restriction [H]^2$ is constant. We use a characterization of weak compactness due to Hanf–Scott [7]. A formula is Π^1_1 if it is of the form $\forall X\ \psi$, where X is a second-order variable and ψ has only first-order quantifiers. A cardinal κ is Π^1_1-*indescribable* if whenever $U \subseteq V_\kappa$ and φ is a Π^1_1-sentence such that $(V_\kappa, \in, U) \models \varphi$ then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \models \varphi$. As shown in [7], a cardinal κ is Π^1_1-indescribable if and only if it is weakly compact.

([1]) Sakai and Veličković [14] showed that the Weak Reflection Principle (WRP) together with $\mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ implies that ω_2 has the Super Tree Property. It is implicit in their proof that $\mathrm{WRP}(\omega_2) + \mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ implies $\mathrm{TP}(\omega_2)$. This leads to an alternative proof of the consistency of $\mathrm{TP}(\omega_2) + \mathrm{BPFA}(\omega_1)$ from a weakly compact cardinal. Our construction is flexible enough to yield further results, such as the results mentioned regarding the Special Tree Property.

We also recall the following definitions:

DEFINITION 2.1 (Shelah [15]). A notion of forcing $\mathbb{P}$ is *proper* if for every uncountable cardinal κ, all stationary subsets of $[\kappa]^\omega$ remain stationary in $\mathbb{P}$-generic extensions.

DEFINITION 2.2. (PFA) := For every proper notion of forcing $\mathbb{P}$ and for every collection $\langle D_\xi : \xi < \omega_1 \rangle$ of maximal antichains of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for all $\xi < \omega_1$.

DEFINITION 2.3. (BPFA) := For every proper notion of forcing $\mathbb{P}$ and for every collection $\langle D_\xi : \xi < \omega_1 \rangle$ of maximal antichains of $\mathbb{P}$, each of size at most ω_1, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for all $\xi < \omega_1$.

Bagaria and Stavi [1, Theorem 5] showed that BPFA is equivalent to the following statement: For every proper forcing $\mathbb{P}$, every Σ_1 formula with parameters from H_{ω_2} that holds in a $\mathbb{P}$-generic extensions also holds in V.

DEFINITION 2.4. An uncountable regular cardinal κ is *reflecting* if for every $a \in H_\kappa$ and any formula $\varphi(x)$, if there is a regular cardinal θ such that $H_\theta \models \varphi(a)$, then there is a regular $\theta' < \kappa$ such that $a \in H_{\theta'} \models \varphi(a)$.

M. Goldstern and S. Shelah [6] proved that BPFA is equiconsistent with the existence of a reflecting cardinal.

BPFA(ω_1) is the statement of BPFA restricted to forcings of size at most ω_1. BPFA(ω_1) is only slightly stronger than MA(ω_1); it is easy to force it by starting with GCH, and in ω_2 steps hitting every proper forcing of size ω_1 via a countable support iteration.

We recall some basic properties of forcing notions used in our constructions. Given two sets I, J, and a cardinal λ, let $\mathbb{P}_\lambda(I, J)$ be the set of all partial functions p from I to J such that $|\mathrm{dom}(p)| < \lambda$. The order in $\mathbb{P}_\lambda(I, J)$ is given by $\supseteq$.

$\mathbb{P}_\kappa(\kappa \times \lambda, 2)$ is usually denoted by $\mathrm{Add}(\kappa, \lambda)$, and $\mathbb{P}_\kappa(\kappa, \lambda)$ is usually denoted by $\mathrm{Col}(\kappa, \lambda)$.

We say that a notion of forcing is *ω-closed* if every countable descending sequence of conditions $p_0 \geq p_1 \geq \cdots$ has a lower bound.

We recall that ω-closed and c.c.c. forcings are also proper (see for example [10, Lemma V.7.2]). A two-step iteration of proper forcing is proper [10, Lemma V.7.4]. Even more, Shelah showed that a countable support iteration of proper forcing notions is proper (see for example [8, Theorem 31.15]).

In our forcing constructions we will use the following forcing notion due to Baumgartner [2] which specializes any tree of height ω_1 with no uncountable branches (the tree may have uncountable levels).

DEFINITION 2.5. Given a tree T of height ω_1 with no uncountable branches we define a partial order $\mathbb{P}_{\mathrm{sp}}(T)$ by $a \in \mathbb{P}_{\mathrm{sp}}(T)$ if and only if a

is a function from a finite subset of T into ω such that $a(t_0) \neq a(t_1)$ whenever t_0, t_1 are comparable in T.

Baumgartner [2] showed that the forcing $\mathbb{P}_{\mathrm{sp}}(T)$ defined above has the countable chain condition. Furthermore, Silver showed that if T is an ω_2-Aronszajn tree then T still has no cofinal branch after forcing with

$$\mathrm{Add}(\omega, \omega_2) * \mathrm{Col}(\omega_1, \omega_2).$$

Therefore Baumgartner's specializing forcing can be applied to the restriction of T to a cofinal set of levels in this model; we still refer to this forcing as $\mathbb{P}_{\mathrm{sp}}(T)$.

Given an uncountable cardinal λ, recall that a $\square_\lambda$-*sequence* is a sequence $\langle c_\alpha : \alpha \in \mathrm{Lim}(\lambda^+)\rangle$ such that for all $\alpha \in \mathrm{Lim}(\lambda^+)$:

(1) c_α is club in α,
(2) $\mathrm{ot}(c_\alpha) \leq \lambda$,
(3) $c_\alpha \cap \beta = c_\beta$ whenever $\beta \in \mathrm{Lim}(c_\alpha)$.

Let λ be an uncountable cardinal. We define $\mathbb{P}(\square_\lambda)$ as follows: $p \in \mathbb{P}$ iff

- $\mathrm{dom}(p) = (\beta + 1) \cap \mathrm{Lim}(\lambda^+)$ for some $\beta \in \mathrm{Lim}(\lambda^+)$;
- $p(\alpha)$ is a club set in α and $\mathrm{ot}(p(\alpha)) \leq \lambda$ for all $\alpha \in \mathrm{dom}(p)$;
- if $\alpha \in \mathrm{dom}(p)$, then $p(\alpha) \cap \beta = p(\beta)$ for every $\beta \in \mathrm{Lim}(p(\alpha))$.

We order $\mathbb{P}(\square_\lambda)$ by letting $p \leq q$ if and only if $q = p{\restriction}_{\mathrm{dom}(q)}$ for $p, q \in \mathbb{P}(\square_\lambda)$.

$\mathbb{P}(\square_\lambda)$ adds a $\square_\lambda$-sequence in the generic extension. It is due to Jensen and does not add λ-sequences (see [4]).

3. The Tree Property and forcing axioms. In this section we prove that $\mathrm{TP}(\omega_2) + \mathrm{BPFA}(\omega_1)$ is equiconsistent with the existence of a weakly compact cardinal. In our proof we use a weakly compact $\diamondsuit$-sequence (Definition 3.2) to code objects during the iteration. We first discuss some of the properties of these weakly compact diamond sequences.

Given a cardinal κ and $S \subseteq \kappa$, recall Jensen's Diamond Principle $\diamondsuit_\kappa(S)$: There is a sequence $\langle D_\alpha : \alpha \in S\rangle$ such that for every $X \subseteq \kappa$, the set $\{\alpha \in S : X \cap \alpha = D_\alpha\}$ is stationary. We recall the following (see Lemma 6.5 in [9]):

LEMMA 3.1. *Suppose $V = L$. Given a regular cardinal κ, $\diamondsuit_\kappa(S)$ holds for every stationary set $S \subseteq \kappa$.*

Actually, if κ is a weakly compact cardinal, we can have in L a stronger form of a diamond sequence.

DEFINITION 3.2. A *weakly compact $\diamondsuit$-sequence* for a cardinal κ is a sequence $\langle D_\alpha : \alpha < \kappa\rangle$ such that:

(1) $D_\alpha \subseteq \alpha$,
(2) for every $A \subseteq V_\kappa$ and every Π^1_1-formula φ such that $(V_\kappa, A) \models \varphi(A)$, and every $D \subseteq \kappa$, the set

$$S(A,\varphi,D) = \{\alpha < \kappa : (V_\alpha, A \cap V_\alpha) \models \varphi\,(A \cap V_\alpha) \text{ and } D \cap \alpha = D_\alpha\}$$

is stationary in κ.

Observe that the existence of a weakly compact diamond sequence can hold only if κ is weakly compact, due to the characterization of Hanf–Scott given in the introduction.

Lemma 3.3. *In L, there is a weakly compact $\diamondsuit$-sequence for κ whenever κ is a weakly compact cardinal.*

Proof. See [16, Theorem 2.13]. ∎

In this paper, in order to code some objects of the universe, we would like to deal with subsets of V_α rather than just subsets of α. We have the following:

Lemma 3.4. *For a given cardinal κ, suppose there is a weakly compact $\diamondsuit$-sequence $\langle D_\alpha : \alpha < \kappa\rangle$ for κ. Then there is a sequence $\langle D^*_\alpha : \alpha < \kappa\rangle$ such that:*

(1) $D^*_\alpha \subseteq V_\alpha$,
(2) *for every $D^* \subseteq V_\kappa$ and every Π^1_1-formula φ with $(V_\kappa, D^*) \models \varphi(D^*)$, the set*

$$S^*(D^*,\varphi) = \{\alpha < \kappa : (V_\alpha, D^* \cap V_\alpha) \models \varphi(D^* \cap V_\alpha) \text{ and } D^* \cap V_\alpha = D^*_\alpha\}$$

is stationary.

Proof. Fix a weakly compact $\diamondsuit$-sequence $\langle D_\alpha : \alpha < \kappa\rangle$ for κ. As already mentioned, the existence of a weakly compact $\diamondsuit$-sequence for κ implies that κ is weakly compact due to the characterization of Hanf–Scott mentioned in the introduction. In particular, κ is inaccessible, so there is a bijection $f : \kappa \to V_\kappa$ (see for example [10, Lemmas I.13.26 and I.13.31]). Observe that the set

$$C = \{\alpha < \kappa : f{\restriction}_\alpha : \alpha \to V_\alpha \text{ is a bijection}\}$$

is a club set in κ. Define $D^*_\alpha = f\,[D_\alpha]$ if $\alpha \in C$ and empty otherwise. Let $D^* \subseteq V_\kappa$ and φ be a Π^1_1-formula such that $(V_\kappa, D^*) \models \varphi(D^*)$. We need to show that the set $S^*(D^*,\varphi)$ defined above is stationary. Since $\langle D_\alpha : \alpha < \kappa\rangle$ is a weakly compact $\diamondsuit$-sequence for κ, the set

$$S = S(D^*, \varphi, f^{-1}\,[D^*]) \cap C$$

is stationary (see Definition 3.2).

Now it is not hard to see that $S \subseteq S^*(D^*,\varphi)$, and therefore $S^*(D^*,\varphi)$ is stationary as desired. ∎

THEOREM 3.5. *Suppose $V = L$ and let κ be a weakly compact cardinal in L. Then there is a forcing iteration $\mathbb{P}$ of countable support and length κ such that in $L^{\mathbb{P}}$, both* $\mathrm{TP}(\omega_2)$ *and* $\mathrm{BPFA}(\omega_1)$ *hold.*

Proof. We remark that we can find a Π^1_1-sentence ψ (with no parameter) such that L_α satisfies ψ iff α is inaccessible. For example, let ψ be the Π^1_1-sentence expressing: "There is no cofinal function from an ordinal into the class of ordinals, ω exists and the Power Set Axiom holds". Then ψ holds in L_α iff α is inaccessible.

Therefore, we can fix a weakly compact diamond sequence concentrated on inaccessible cardinals and with the properties of Lemma 3.4. Let

$$\langle D_\alpha : \alpha \text{ inaccessible}, \alpha < \kappa\rangle$$

be such a sequence.

Also observe that our weakly compact sequences can be concentrated on inaccessible cardinals, and in L we have $L_\alpha = V_\alpha$ whenever α is inaccessible.

We will perform a countable support iteration $\langle\langle\mathbb{P}_\alpha : \alpha \le \kappa\rangle, \langle\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle\rangle$ in which at L-inaccessible stages α we will use our weakly compact diamond sequence to ensure that there is no ω_2-Aronszajn tree, and at L-accessible stages we will ensure $\mathrm{BPFA}(\omega_1)$.

Choose an enumeration $\langle\dot{\mathbb{R}}_\alpha : \alpha < \kappa, \alpha \text{ not inaccessible}\rangle$ of all nice $\mathbb{S}$-names for forcings with universe ω_1 as $\mathbb{S}$ ranges over forcings in L_κ. Moreover assume that this bookkeeping is redundant in the sense that each such $\mathbb{S}$-name appears cofinally often in this list.

We define our countable support iteration as follows. $\dot{\mathbb{Q}}_0$ is the trivial forcing. If α is not inaccessible in L and $\dot{\mathbb{R}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing in $L[G_\alpha]$ (where G_α denotes the $\mathbb{P}_\alpha$-generic) then declare $\dot{\mathbb{Q}}_\alpha$ to be $\dot{\mathbb{R}}_\alpha * \mathrm{Col}(\omega_1, \alpha)$; otherwise take $\dot{\mathbb{Q}}_\alpha$ to be the forcing $\mathrm{Col}(\omega_1, \alpha)$.

Now suppose α is inaccessible in L. Then α is the ω_2 of $L[G_\alpha]$. See if D_α is a $\mathbb{P}_\alpha$-name for an Aronszajn tree T_α in $L[G_\alpha]$. If not, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. Otherwise let $\dot{\mathbb{Q}}_\alpha$ be

$$\mathrm{Add}(\omega, \alpha) * \mathrm{Col}(\omega_1, \alpha) * \mathbb{P}_{\mathrm{sp}}(T),$$

i.e. add α many Cohen reals followed by a Lévy collapse of α to ω_1 followed by a specialization of T (more precisely, of the restriction of T to cofinally many levels).

Now after κ steps, κ becomes ω_2 as the forcing is proper, κ-cc and collapses each $\alpha < \kappa$ to ω_1.

Suppose that σ were a $\mathbb{P}$-name for an ω_2-Aronszajn tree in $L[G]$ (where $\mathbb{P}$ is the final iteration and G denotes the $\mathbb{P}$-generic).

Observe that σ can be regarded as a subset of V_κ. The statement "σ is a κ-Aronszajn tree" is a Π^1_1-statement about V_κ with σ as a predicate (in addition to basic first-order properties about (V_κ, σ) the key second-order property is the nonexistence of a cofinal branch). Now if ϕ is a Π^1_1 sentence then the statement "p forces $\phi(\sigma)$" is a Π^1_1-statement about (V_κ, σ). (The forcing relation for a first-order statement is first-order; from this it follows that the forcing relation for Π^1_1-statements is Π^1_1.) (Note: $\mathbb{P}_\kappa$ is another predicate in the sentence to be reflected; however $\mathbb{P}_\kappa$ is actually first-order definable over V_κ, using the weakly compact diamond sequence, which can be chosen to be first-order definable over V_κ).

Apply Diamond to get an inaccesible α such that $D_\alpha = \sigma \cap L_\alpha$ and D_α is forced to be a name for an Aronszajn tree in $\mathbb{P}_\alpha$. But then at stage α, T_α, the interpretation of D_α, is specialized and therefore has no branch of length α (as ω_1 is preserved). This contradicts the fact that T_α is an initial segment of T, the interpretation of σ, and therefore must have branches of length α.

Finally, observe that in $L[G]$ we also have BPFA(ω_1) since any proper forcing $\mathbb{Q}$ with universe ω_1 in $L[G]$ is proper in $L[G_\alpha]$ at cofinally many stages α where we forced with $\mathbb{Q}$, so surely we have a generic filter hitting ω_1 many dense sets for $\mathbb{Q}$. ∎

Observe that the above yields another proof of the consistency of TP(ω_2) from a weakly compact cardinal:

COROLLARY 3.6. *The following are equiconsistent:*

(1) *There exists a weakly compact cardinal.*
(2) TP(ω_2) *holds.*
(3) TP(ω_2) + MA$_{\omega_1}$ *holds.*
(4) TP(ω_2) + BPFA(ω_1) *holds.*

DEFINITION 3.7. We say that a cardinal κ is *weakly compact relative to subsets of* ω_1 whenever κ is weakly compact in $L[A]$ for every $A \subseteq \omega_1$.

We also have the following:

PROPOSITION 3.8. *If there is a weakly compact cardinal κ, there is a model where* BPFA *holds, ω_2 is weakly compact relative to subsets of ω_1, but ω_2 does not have the Tree Property.*

Proof. Start with a weakly compact cardinal κ, force BPFA with a forcing $\mathbb{P}$, and then let $\mathbb{P}(\square_{\omega_1})$ be the forcing which adds a $\square_{\omega_1}$-sequence. Then TP(ω_2) fails in the final model as $\square_{\omega_1}$ is sufficient to yield the existence of an ω_2-Aronszajn tree (see [4]).

CLAIM 3.9. *$\mathbb{P}(\square_{\omega_1})$ preserves* BPFA *over $V^{\mathbb{P}}$.*

Proof. Observe that all subsets of ω_1 in $V^{\mathbb{P}*\mathbb{P}(\square_{\omega_1})}$ are in $V^{\mathbb{P}}$, and any proper extension of $V^{\mathbb{P}*\mathbb{P}(\square_{\omega_1})}$ is also a proper extension of $V^{\mathbb{P}}$ as $\mathbb{P}(\square_{\omega_1})$ is proper. ∎

Claim 3.10. *ω_2 is weakly compact relative to subsets of ω_1 in $V^{\mathbb{P}*\mathbb{P}(\square_{\omega_1})}$.*

Proof. Any subset of ω_1 is added by a forcing of size less than κ, and any such forcing preserves the weak compactness of κ. ∎

This ends the proof of Proposition 3.8. ∎

So BPFA plus ω_2 weakly compact relative to subsets of ω_1 is not enough to get $\mathrm{TP}(\omega_2)$. Obviously BPFA alone is not enough because its consistency strength, a reflecting cardinal, is less than that of $\mathrm{TP}(\omega_2)$, a weakly compact cardinal.

However, we have the following:

Theorem 3.11. $\mathrm{TP}(\omega_2) + \mathrm{BPFA}$ *is equiconsistent with the existence of a weakly compact cardinal which is also reflecting.*

Proof. Suppose that κ is a weakly compact reflecting cardinal. Repeat the proof above, forcing κ to be ω_2, $\mathrm{TP}(\omega_2)$ and $\mathrm{BPFA}(\omega_1)$, but instead of hitting proper forcings of size ω_1, use the consistency proof of BPFA to force with proper forcings of size less than κ which witness Σ_1-sentences with subsets of ω_1 as parameters. The only small change is that α will not necessarily be the ω_2 of $L[G_\alpha]$ whenever α is L-inaccessible, but this will be the case for all L-inaccessible α in a closed unbounded subset of κ. The fact that κ is reflecting implies that the latter forcings may be chosen to have size less than κ. After κ steps, we again have $\mathrm{TP}(\omega_2)$, and the extra forcing we have done ensures that we also have BPFA.

Conversely, suppose that we have $\mathrm{TP}(\omega_2) + \mathrm{BPFA}$. Then by [6], ω_2 is reflecting in L, and by a result of Silver (see [12]), ω_2 is also weakly compact in L. ∎

We have some further open questions:

(1) $\mathrm{Con}(\mathrm{TP}(\omega_2) + \mathrm{MA} + \mathfrak{c} = \omega_3)$?
(2) $\mathrm{Con}(\mathrm{TP}(\omega_3) + \mathrm{MA})$?

Of course $\mathrm{Con}(\mathrm{TP}(\omega_4) + \mathrm{BPFA})$ is no problem because when forcing $\mathrm{TP}(\omega_4)$ one does not need to add subsets of ω_1. Further, $\mathrm{TP}(\omega_3) + \mathrm{BPFA}$ is inconsistent as BPFA implies that GCH holds at ω_1 (see [13]) whereas $\mathrm{TP}(\omega_3)$ implies the opposite.

4. The Special Tree Property and forcing axioms. The proof is similar to that of our previous theorem. Therefore, we only give a sketch of the proof, just pointing out the differences. This time we use a simple $\diamondsuit$-sequence to code the names of special Aronszajn trees during the iteration.

Theorem 4.1. *Assume $V = L$ and κ is a Mahlo cardinal. Then there is a forcing iteration $\mathbb{P}$ of countable support and length κ such that in $L^{\mathbb{P}}$, both* $\mathrm{SpTP}(\omega_2)$ *and* $\mathrm{BPFA}(\omega_1)$ *hold.*

Proof. This time we consider a name for an ω_2-tree together with a specializing function (into ω_1) for it. Using a diamond sequence $\langle D_\alpha : \alpha$ inaccessible$\rangle$, find an inaccessible $\alpha < \kappa$ where the name restricted to α is a name for an α-tree together with a specializing function for it, where α is the ω_2 of $V[G_\alpha]$ and where we guessed that name using the diamond sequence. This α-tree has no cofinal branch because it is specialized (into ω_1). Then in the construction we added α-many Cohen reals followed by an ω-closed Levy collapse of alpha to ω_1 (the tree still has no cofinal branch) and specialized the tree (into ω). But this is a contradiction because any node on level α of the original ω_2-tree yields a cofinal branch through the α-tree and then an injection of α into ω, contradicting the fact that ω_1 is preserved. ∎

As in the previous section (now using the result in [12] that $\mathrm{SpTP}(\omega_2)$ implies that ω_2 is Mahlo in L), we have:

THEOREM 4.2. $\mathrm{SpTP}(\omega_2) + \mathrm{BPFA}$ *is equiconsistent with the existence of a Mahlo cardinal which is also reflecting.*

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