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The order topology for a von Neumann algebra

by

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Abstract. The order topology $\tau_o(P)$ (resp. the sequential order topology $\tau_{os}(P)$) on a poset P is the topology that has as its closed sets those that contain the order limits of all their order convergent nets (resp. sequences). For a von Neumann algebra M we consider the following three posets: the self-adjoint part M_{sa}, the self-adjoint part of the unit ball M_{sa}^1, and the projection lattice $P(M)$. We study the order topology (and the corresponding sequential variant) on these posets, compare the order topology to the other standard locally convex topologies on M, and relate the properties of the order topology to the underlying operator-algebraic structure of M.

1. Introduction. Order convergence has been studied in the context of posets and lattices by various authors [6, 7, 17] (see also [15, 20, 19]). The order topology on a poset is defined to be the finest topology preserving order convergence.

In [22, 9] the order topology for the lattice of projections acting on a Hilbert space was studied. It is the aim of the present paper to give a first systematic treatment of various order topologies associated with a von Neumann algebra. We show that the properties of these topologies are nicely connected with the inner structure of the underlying algebra and with the locally convex topologies living on it.

We first consider the self-adjoint part M_{sa} of a von Neumann algebra M and study the order topology $\tau_o(M_{sa})$ induced by the standard operator order. We prove that when M is σ-finite, sequential convergence with respect to $\tau_o(M_{sa})$ coincides with sequential convergence with respect to the σ-strong topology. The proof is based on the Noncommutative Egoroff Theorem. As a consequence, one finds that on bounded parts of M_{sa} the order topology coincides with any of the locally convex topologies on M that is

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compatible with the duality $\langle M, M_* \rangle$ where M_* is the unique predual of M. Our result is sharp in the sense that the σ-strong topology coincides with the order topology $\tau_o(M_{sa})$ if and only if M is finite-dimensional.

The fact that the order topology on ordered vector spaces is in general far from being a linear topology makes this coincidence rather surprising. Another interesting feature of this result is the possibility to recover (on bounded parts) the locally convex topologies arising from the duality $\langle M, M_* \rangle$ (a component of the von Neumann structure) only from the order (a component of the C^*-structure). More precisely, we are saying that if M and N are σ-finite von Neumann algebras such that M_{sa} and N_{sa} are order-isomorphic (i.e. there exists a bijection preserving the order in both directions), then the unit balls M^1 and N^1 are homeomorphic with respect to the σ-strong topologies.

We then compare the Mackey topology $\tau(M, M_*)$ with the order topology $\tau_o(M_{sa})$ and the sequential variant $\tau_{os}(M_{sa})$. The Mackey topology is coarser than $\tau_{os}(M_{sa})$ and we characterize von Neumann algebras for which $\tau(M, M_*) = \tau_{os}(M_{sa})$. Indeed, we prove that this happens if and only if M is $*$-isomorphic to a countable direct sum of finite-dimensional full matrix algebras. From a topological point of view this happens exactly when any of the following conditions is satisfied:

(i) M is σ-finite and M^1 is compact with respect to the σ-strong* topology,
(ii) the Mackey topology is sequential,
(iii) M is σ-finite and $\tau_o(M_{sa})$ is a linear topology.

The proofs of these results rest heavily on the technique of mixed topologies. That is why we study mixed topologies and develop results that we believe can be of independent interest. Using [2] we show that the Mackey topology is equal to the mixed topology of the norm topology and the σ-strong* topology. This is in fact a noncommutative extension of the interesting result of M. Nowak [21] saying that the Mackey topology on L^∞ coincides with the mixed topology of the norm topology and the topology of convergence in measure. Although not investigated here, we believe that this equality can contribute to the problem studied by J. F. Aarnes [1] of whether the Mackey topology of a von Neumann subalgebra coincides with the restriction of the Mackey topology of the ambient algebra.

In the last section we consider as posets the projection lattice $P(M)$ and the self-adjoint part of the unit ball M^1_{sa}. Unless the algebra is abelian, the order topology on neither of these posets coincides with the restriction of the global order topology $\tau_o(M_{sa})$. In fact, we show that if M is σ-finite then the following conditions are equivalent:

(i) M is of finite type,
(ii) the order topology on M_{sa}^1 and the σ-strong operator topology restricted to M_{sa}^1 have the same null sequences ([1]),
(iii) the order topology on the projection lattice $P(M)$ and the σ-strong operator topology restricted to $P(M)$ have the same null sequences.

This gives a new characterization of finite von Neumann algebras.

The paper is organized as follows. Section 2 collects basic facts on the order topology on posets and ordered vector spaces needed later. In Section 3 results on mixed topologies are isolated. Section 4 deals with the relationship between the standard locally convex topologies and the order topology on M_{sa}. Section 5 deals with the order topologies of the projection lattice and the unit ball of a von Neumann algebra.

2. Preliminary results

2.1. The order topology and sequential order topology. Let $(P, \leq)$ be a partially ordered set. A net $(x_\gamma)_{\gamma\in\Gamma}$ is said to *order converge* to x in $(P, \leq)$ (in symbols $x_\gamma \xrightarrow{o} x$) if there exist nets $(y_\gamma)_{\gamma\in\Gamma}$ and $(z_\gamma)_{\gamma\in\Gamma}$ in P such that $y_\gamma \leq x_\gamma \leq z_\gamma$ for all $\gamma \in \Gamma$, $y_\gamma \uparrow x$ and $z_\gamma \downarrow x$; i.e. (y_γ) is increasing, (z_γ) is decreasing and ([2]) $\bigvee_{\gamma\in\Gamma} y_\gamma = x = \bigwedge_{\gamma\in\Gamma} z_\gamma$.

It is easy to see that the order limit of an order convergent net is uniquely determined. A subset X of P is called *order closed* (resp. *sequentially order closed*) if no net (resp. sequence) in X order converges to a point outside of X. The collection of all order closed sets (resp. sequentially order closed sets) comprises the closed sets for some topology, the *order topology* $\tau_o(P)$ (resp. the *sequential order topology* $\tau_{os}(P)$) of P. The order topology of P is the finest topology on P that preserves order convergence of nets; i.e. if τ is a topology on P such that $x_\gamma \xrightarrow{o} x$ in P implies $x_\gamma \xrightarrow{\tau} x$, then $\tau \subseteq \tau_o(P)$. The sequential order topology of P is the finest topology on P that preserves order convergence of sequences. Clearly, $\tau_o(P) \subseteq \tau_{os}(P)$ and we recall that both topologies satisfy T_1 but in general are not Hausdorff [12, 13].

Although convergence with respect to $\tau_o(P)$ does not necessarily imply order convergence, for a sequence converging with respect to $\tau_{os}(P)$ we have the following useful observation (well-known in a less general setting).

PROPOSITION 2.1 ([9, Proposition 2]). *Let $(P, \leq)$ be a partially ordered set, $x \in P$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in P. Then $(x_n)_{n\in\mathbb{N}}$ converges to x with*

([1]) A sequence in a topological vector space is said to be a *null sequence* if it is convergent to 0.

([2]) For a subset $X = \{x_\gamma : \gamma \in \Gamma\}$ of P we shall denote by $\bigvee_{\gamma\in\Gamma} x_\gamma$ and $\bigwedge_{\gamma\in\Gamma} x_\gamma$ the least upper bound and the greatest lower bound of X, respectively.

respect to $\tau_{os}(P)$ *if and only if any subsequence of* $(x_n)_{n\in\mathbb{N}}$ *has a subsequence order converging to* x.

The sequential order topology is in general strictly finer than the order topology; however, the two topologies coincide when P is monotone order separable. We call $(P,\leq)$ *monotone order separable* if for every increasing (or decreasing) net $(x_\gamma)_{\gamma\in\Gamma}$ in P that has a supremum (resp. infimum) in P there exists an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $\bigvee_{n\in\mathbb{N}} x_{\gamma_n} = \bigvee_{\gamma\in\Gamma} x_\gamma$ (resp. $\bigwedge_{n\in\mathbb{N}} x_{\gamma_n} = \bigwedge_{\gamma\in\Gamma} x_\gamma$).

Proposition 2.2 ([9, Proposition 3]). *Let* $(P,\leq)$ *be a partially ordered set. Then* $\tau_{os}(P) = \tau_o(P)$ *if and only if* $(P,\leq)$ *is monotone order separable.*

Every order convergent sequence is order bounded, and every order convergent net is eventually order bounded. Therefore in the definition of order closed sets it is enough to consider order bounded nets. We recall that $(P,\leq)$ is *Dedekind complete* if every subset having an upper bound (or a lower bound) has a supremum (resp. an infimum). $(P,\leq)$ is *conditional monotone complete* if every monotone increasing net (or monotone decreasing net) having an upper bound (resp. a lower bound) has a supremum (resp. an infimum). Dedekind σ-completeness (resp. conditional monotone σ-completeness) is defined analogously, requiring the condition to hold for countable subsets (resp. sequences). It is easily seen that when $(P,\leq)$ is Dedekind complete, an order bounded net $(x_\gamma)_{\gamma\in\Gamma}$ order converges to x in $(P,\leq)$ if and only if $\limsup_\gamma x_\gamma = \liminf_\gamma x_\gamma = x$. When $(P,\leq)$ is only assumed to be Dedekind σ-complete, a similar assertion holds for sequences.

If P_0 is a subset of P it can very well happen that $\tau_o(P_0)$ and $\tau(P)|P_0$ are incomparable. However, we have the following easily seen observations which we state as a proposition for better reference.

Proposition 2.3. *Let* $(P,\leq)$ *be a partially ordered set and let* P_0 *be a subset of* P.

- (i) *If* $(P,\leq)$ *is conditional monotone complete and* P_0 *is* $\tau_o(P)$*-closed then* $\tau_o(P)|P_0 \subseteq \tau_o(P_0)$.
- (ii) *If* $(P,\leq)$ *is Dedekind complete and* P_0 *is a* $\tau_o(P)$*-closed sublattice of* P *then* $\tau_o(P)|P_0 = \tau_o(P_0)$.

An analogous proposition holds for the sequential order topology: Proposition 2.3 remains true if one replaces the order topology by the sequential order topology, conditional monotone completeness by monotone σ-completeness and Dedekind completeness by Dedekind σ-completeness.

We shall now consider the case when the underlying poset carries also a linear structure. Let X be an ordered vector space with positive cone $X^+ = \{x \in X : x \geq 0\}$. For basic results and terminology on ordered vector spaces the reader may wish to consult [4, 18, 24]. It is clear that the order topology $\tau_o(X)$ and the sequential order topology $\tau_{os}(X)$ are translation invariant and homogeneous, i.e. if A is a subset of X closed with respect to $\tau_o(X)$ (or $\tau_{os}(X)$) then $A+x$ and λA are closed with respect to $\tau_o(X)$ (resp. $\tau_{os}(X)$) for every $x \in X$ and $\lambda \in \mathbb{R}$. In general, however, these topologies fail to be linear topologies, as the following example shows: Let $\mathfrak{A}$ be the complete Boolean algebra of all regular open subsets of $[0,1]$, and $B(\mathfrak{A})$ be the closed linear span of the set of characteristic functions χ_A, $A \in \mathfrak{A}$, in the space $(B[0,1], \|\cdot\|_\infty)$ of all bounded real functions on $[0,1]$ with respect to the supremum norm $\|\cdot\|_\infty$. Then $B(\mathfrak{A})$ is a monotone order separable, Dedekind complete Riesz space. Thus $\tau_{os}(B(\mathfrak{A})) = \tau_o(B(\mathfrak{A}))$. Since $\tau_o(B(\mathfrak{A}))$ satisfies T_1 and $\tau_o(B(\mathfrak{A}))$ is not Hausdorff, it follows that $\tau_o(B(\mathfrak{A}))$ is not a group topology (3).

PROPOSITION 2.4. *Let X be an ordered vector space, let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in X such that $a_n \xrightarrow{\tau_{os}(X)} a$ and $b_n \xrightarrow{\tau_{os}(X)} b$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\lambda_n \to \lambda$. Then:*

(i) $a_n + b_n \xrightarrow{\tau_{os}(X)} a + b$.
(ii) *If $(n^{-1}a)_{n\in\mathbb{N}}$ order converges to 0 (in particular if X is an Archimedian Riesz space), then $\lambda_n a_n \xrightarrow{\tau_{os}(X)} \lambda a$.*

Proof. We will apply Proposition 2.1. Passing to suitable subsequences we may assume that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ order converge to a and b, respectively, and moreover that $|\lambda - \lambda_n| \le 1/n$ and either $\lambda_n - \lambda \ge 0$ for each n or $\lambda_n - \lambda < 0$ for each n. Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in X such that

$$x_n \le a_n \le y_n, \qquad u_n \le b_n \le v_n, \qquad \text{for all } n \in \mathbb{N},$$

and $x_n \uparrow a$, $y_n \downarrow a$, $u_n \uparrow b$ and $v_n \downarrow b$. Then $x_n + u_n \le a_n + b_n \le y_n + v_n$, $x_n + u_n \uparrow a + b$ and $y_n + v_n \downarrow a + b$; thus $(a_n + b_n)$ order converges to $a + b$.

To prove (ii) first suppose that $\mu_n := \lambda_n - \lambda \ge 0$ for every n. Observing that

$$x_n - a \le \mu_n(x_n - a) \le \mu_n(a_n - a) \le \mu_n(y_n - a) \le y_n - a,$$

we deduce that $(\mu_n(a_n - a))_{n\in\mathbb{N}}$ order converges to 0. The additional assumption that $(n^{-1}a)_{n\in\mathbb{N}}$ order converges to 0 implies that there exist sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ satisfying $s_n \le n^{-1}a \le t_n$ for every $n \in \mathbb{N}$, $s_n \uparrow 0$ and $t_n \downarrow 0$. Observing that

$$s_n - t_n \le n\mu_n(s_n - t_n) \le \mu_n a \le n\mu_n(t_n - s_n) \le t_n - s_n,$$

we deduce that $(\mu_n a)_{n\in\mathbb{N}}$ order converges to 0. Thus $\lambda_n a_n = \mu_n(a_n - a) + \mu_n a + \lambda a_n$ order converges to λa.

(3) In [12] it is shown that $\tau_o(\mathfrak{A})$ is not Hausdorff.

If $\lambda_n - \lambda < 0$ for every n, then the above implies that $(-\lambda_n a_n)_{n\in\mathbb{N}}$ order converges to $-\lambda a$ and thus $(\lambda_n a_n)_{n\in\mathbb{N}}$ order converges to λa. ∎

Let us recall that a linear functional f on X is said to be *positive* if $x \geq 0$ implies $f(x) \geq 0$. If $f(x) > 0$ for every nonzero positive element x of X then f is said to be a *faithful* positive linear functional. A linear functional f is said to be *normal* (or order continuous) if $f(x_\gamma) \to f(x)$ whenever $x_\gamma \xrightarrow{o} x$ in X. Clearly, a positive linear functional f on X is normal if and only if $x_\gamma \downarrow 0$ implies $f(x_\gamma) \downarrow 0$.

In the proof of the following proposition we use the fact that an ordered vector space X is monotone order separable if and only if for every net $(x_\gamma)_{\gamma\in\Gamma}$ in X satisfying $x_\gamma \downarrow 0$ there exists an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $\bigwedge_{n\in\mathbb{N}} x_{\gamma_n} = 0$.

PROPOSITION 2.5. *Let X be a conditional monotone σ-complete ordered vector space admitting a faithful normal positive linear functional f. Then X is monotone order separable and therefore $\tau_{os}(X) = \tau_o(X)$.*

Proof. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in X satisfying $x_\gamma \downarrow 0$. The normality of f implies that $f(x_\gamma) \to 0$. Thus we can select an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $f(x_{\gamma_n}) \to 0$. Then $s := \inf_{n\in\mathbb{N}} x_{\gamma_n} \geq 0$ and by the normality of f we deduce that $f(s) = \lim_n f(x_{\gamma_n}) = 0$. Faithfulness of f implies $s = 0$. ∎

For Riesz spaces the mere existence of a faithful positive linear functional (without assuming normality) is sufficient for the order topology and the sequential order topology to coincide.

PROPOSITION 2.6. *Let X be a Riesz space admitting a faithful positive linear functional f. Then X is monotone order separable and therefore $\tau_{os}(X) = \tau_o(X)$.*

Proof. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in X satisfying $x_\gamma \downarrow 0$. Set $\alpha := \inf_{\gamma\in\Gamma} f(x_\gamma)$. Note that for all $\gamma, \gamma' \in \Gamma$ there is a $\gamma'' \in \Gamma$ with $\gamma'' \geq \gamma$ and $\gamma'' \geq \gamma'$, hence $f(x_\gamma \wedge x_{\gamma'}) \geq f(x_{\gamma''}) \geq \alpha$. Choose an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $f(x_{\gamma_n}) \to \alpha$. We show that $0 = \bigwedge_{n\in\mathbb{N}} x_{\gamma_n}$. To this end let x be a lower bound of $\{x_{\gamma_n} : n \in \mathbb{N}\}$ and let $\gamma \in \Gamma$. Then

$$0 \leq x \vee x_\gamma - x_\gamma \leq x_{\gamma_n} \vee x_\gamma - x_\gamma = x_{\gamma_n} - x_{\gamma_n} \wedge x_\gamma,$$

and therefore

$$0 \leq f(x \vee x_\gamma - x_\gamma) \leq f(x_{\gamma_n}) - f(x_{\gamma_n} \wedge x_\gamma) \leq f(x_{\gamma_n}) - \alpha \to 0$$

as $n \to \infty$, i.e. $f(x \vee x_\gamma - x_\gamma) = 0$. Faithfulness of f implies $x_\gamma = x \vee x_\gamma$ and so $x \leq x_\gamma$. We conclude that x is a lower bound for $\{x_\gamma : \gamma \in \Gamma\}$. Consequently, $x \leq \inf_{\gamma\in\Gamma} x_\gamma = 0$. It follows that $\bigwedge_{n\in\mathbb{N}} x_{\gamma_n} = 0$. This proves that X is monotone order separable and hence by Proposition 2.2 we deduce that $\tau_o(X) = \tau_{os}(X)$. ∎

Our main interest in this paper will be the ordered vector space X given by the self-adjoint part of a von Neumann algebra. In general this is far from being a Riesz space. However, it is interesting to note that in this case the assertion of Propositions 2.5 and 2.6 holds under the hypothesis that X admits a faithful positive linear functional. Indeed, if X admits a faithful positive linear functional then any family of pairwise orthogonal projections is necessarily countable, i.e. the corresponding von Neumann algebra must be σ-finite. As such it must admit a faithful normal positive linear functional and therefore Proposition 2.5 applies.

2.2. Preliminaries on von Neumann algebras. We first recall a few notions and fix the notation. We refer to [8, 16, 23, 26, 27] for more details. Let us recall that a *C^*-algebra* A is a complex Banach $*$-algebra satisfying $\|x^*x\| = \|x\|^2$ for every $x \in A$. We denote by A_{sa} the self-adjoint part of A, that is, $A_{sa} = \{x \in A : x = x^*\}$. A_{sa} is a real vector space, and when endowed with the partial order $\leq$ induced by the cone $A^+ := \{x^*x : x \in A\}$, it gets the structure of an ordered vector space. In general A_{sa} is far from being a Riesz space. In [25] it is shown that if A_{sa} is a lattice then A is abelian. Let A^1 denote the closed unit ball of A and let $A^1_{sa} := A_{sa} \cap A^1$. An element p of a C^*-algebra is called a *projection* if $p = p^* = p^2$. A C^*-algebra may have no nontrivial projections. A linear functional φ on A is *positive* (resp. *faithful*) if $\varphi|A_{sa}$ is positive (resp. faithful) in the sense of Subsection 2.1.

A *von Neumann algebra* M is a C^*-algebra that is simultaneously a dual as a Banach space. In this case M is the dual of a unique Banach space, called the *predual* of M and denoted by M_*. A linear functional φ on M is *normal* if $\varphi|M_{sa}$ is normal in the sense described for ordered vector spaces ([4]). It is known that we can identify the elements of M_* with the normal linear functionals in the continuous dual M^*. The set of normal positive linear functionals on M is denoted by M_*^+. M always has an identity element $\mathbb{1}$ and this element is an order-unit for M_{sa}. We recall that $(M_{sa}, \leq)$ is conditional monotone order complete. A von Neumann algebra is always rich in projections. In fact, a von Neumann algebra is the closure of the span of its projections. The set $P(M)$ of all projections in M is a complete orthomodular lattice under the partial order $\leq$ inherited from M_{sa}. M is called *σ-finite* if every set of nonzero pairwise orthogonal projections in M is at most countable. M is σ-finite if and only if it admits a faithful normal positive linear functional. For a Hilbert space H we denote by $B(H)$ the von Neumann algebra of all bounded operators acting on H.

([4]) Note that this is equivalent to requiring that $\varphi(x_\gamma) \to \varphi(x)$ for every net (x_γ) in M_{sa} satisfying $x_\gamma \uparrow x$. This follows because every normal linear functional can be expressed as a linear combination of four normal positive linear functionals.

For the rest of the paper M is always a von Neumann algebra. We shall primarily consider the order topology (and the corresponding sequential variant) on the following three posets: M_{sa}, M^1_{sa} and $P(M)$. We shall study the properties of the order topology of these posets, compare the order topology to other standard locally convex topologies on M and relate the properties of the order topology to the underlying algebraic structure of M.

We recall that the weak* topology $\sigma(M, M_*)$ on M is the coarsest locally convex topology compatible with the duality $\langle M_*, M\rangle$. The finest locally convex topology on M compatible with this duality is the Mackey topology $\tau(M, M_*)$. Lying between these topologies we have the σ-strong topology $s(M, M_*)$ determined by the family of seminorms $\{\varrho_\psi : \psi \in M_*^+\}$ where $\varrho_\psi(x) = \sqrt{\psi(x^*x)}$, and the σ-strong* topology $s^*(M, M_*)$ determined by the family $\{\eta_\psi : \psi \in M_*^+\}$ of seminorms where $\eta_\psi(x) = \sqrt{\psi(x^*x) + \psi(xx^*)}$. M can be faithfully represented on a Hilbert space H, i.e. M can be identified with a subalgebra of $B(H)$ closed with respect to the weak operator topology, and therefore one can endow M with the strong operator topology τ_s and the weak operator topology τ_w. These are the topologies of pointwise convergence with respect to the norm topology or the weak topology on H, respectively. Note however that τ_s and τ_w in general depend on the particular representation. It is well known that

$$\begin{gathered}\sigma(M, M_*) \subseteq s(M, M_*) \subseteq s^*(M, M_*) \subseteq \tau(M, M_*), \\ \tau_w \subseteq \tau_s, \qquad \tau_w \subseteq \sigma(M, M_*), \qquad \tau_s \subseteq s(M, M_*).\end{gathered} \tag{2.1}$$

By the uniform boundedness principle it follows that if A is a set of bounded linear operators on a Hilbert space that is bounded with respect to the weak operator topology then A is uniformly bounded. Hence, in view of (2.1) a subset K of M that is bounded with respect to any of the above locally convex topologies is uniformly bounded. Furthermore, we recall that if $x, y \in M_{sa}$ then:

(i) $-y \le x \le y$ implies $\|x\| \le \|y\|$;
(ii) $-\|x\|\mathbb{1} \le x \le \|x\|\mathbb{1}$;

i.e. if $K \subseteq M_{sa}$ then K is bounded (with respect to any of the above locally convex topologies) if and only if it is order bounded.

On bounded parts of M the σ-strong topology coincides with the strong operator topology, and the weak* topology coincides with the weak operator topology. Moreover, a deep classical result by C. Akemann [2] says that

$$s^*(M, M_*)|K = \tau(M, M_*)|K \tag{2.2}$$

for every bounded subset K of M. Since $s(M, M_*)$ and $s^*(M, M_*)$ coincide on M_{sa}, it follows that

$$\tau_s|K = s(M, M_*)|K = \tau(M, M_*)|K \tag{2.3}$$

for every bounded subset K of M_{sa}.

Let τ_u denote the uniform topology (i.e. $\|\cdot\|$-topology) on M. We show that $\tau_u|M_{sa}$ is finer than the sequential order topology (and hence than the order topology) of M_{sa}. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in M_{sa} such that $\|x_n\| \to 0$. If we set $\lambda_n := \sup_{k\ge n} \|x_k\|$ then

$$-\lambda_n \mathbb{1} \le -\|x_n\|\mathbb{1} \le x_n \le \|x_n\|\mathbb{1} \le \lambda_n \mathbb{1}$$

and $\lambda_n \mathbb{1} \downarrow 0$ in M_{sa}, i.e. $x_n \xrightarrow{o} 0$ in M_{sa}.

We shall now compare the order topology $\tau_o(M_{sa})$ with the σ-strong topology $s(M, M_*)$. If $(x_\gamma)_{\gamma\in\Gamma}$ is a net in M_{sa}^+ and $x_\gamma \xrightarrow{o} 0$ in M_{sa} then $\psi(x_\gamma) \to 0$ for every $\psi \in M_*^+$. The Cauchy–Schwarz inequality yields $\psi(x_\gamma^2) \le \sqrt{\psi(x_\gamma)\psi(x_\gamma^3)}$ and therefore one obtains $x_\gamma \xrightarrow{s(M,M_*)} 0$ by observing that the net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually bounded. Now suppose that $y_\gamma \xrightarrow{o} y$ in M_{sa}. Let $(a_\gamma)_{\gamma\in\Gamma}$ and $(b_\gamma)_{\gamma\in\Gamma}$ be nets in M_{sa} such that $a_\gamma \le y_\gamma \le b_\gamma$, $a_\gamma \uparrow y$ and $b_\gamma \downarrow y$. Then $y_\gamma - a_\gamma \ge 0$ for every $\gamma \in \Gamma$ and $y_\gamma - a_\gamma \xrightarrow{o} 0$. The above observation implies that $y_\gamma - a_\gamma \xrightarrow{s(M,M_*)} 0$ and $y - a_\gamma \xrightarrow{s(M,M_*)} 0$. The linearity of $s(M, M_*)$ implies $y_\gamma \xrightarrow{s(M,M_*)} y$. Thus, we conclude that

$$s(M, M_*)|M_{sa} \subseteq \tau_o(M_{sa}). \tag{2.4}$$

In particular $\tau_o(M_{sa})$ is Hausdorff and M_{sa}^1 is $\tau_o(M_{sa})$-closed. The inclusion in (2.4) together with the equality of (2.3) imply that $\tau(M, M_*)|K \subseteq \tau_o(M_{sa})|K$ for every bounded subset K of M_{sa}. Using the fact that an order convergent net of M_{sa} is eventually bounded, it is easy to see that a subset X of M_{sa} is closed with respect to $\tau_o(M_{sa})$ if and only if $X \cap rM_{sa}^1$ is closed with respect to $\tau_o(M_{sa})$ for every $r > 0$. Hence, if $X \subseteq M_{sa}$ is $\tau(M, M_*)$-closed then $X \cap rM_{sa}^1$ is $\tau(M, M_*)$-closed and therefore $X \cap rM_{sa}^1$ is $s(M, M_*)$-closed, by applying (2.3) to the $s(M, M_*)$-closed set $K := rM_{sa}^1$. Then (2.4) implies that $X \cap rM_{sa}^1$ is $\tau_o(M_{sa})$-closed. This holds for every $r > 0$ and therefore

$$\tau(M, M_*)|M_{sa} \subseteq \tau_o(M_{sa}). \tag{2.5}$$

We summarize the above observations in (2.6) below. Since M_{sa}^1 and $P(M)$ are $s(M, M_*)$-closed, (2.7)–(2.9) follow from (2.4) and Proposition 2.3(i).

PROPOSITION 2.7. *The following inclusions hold:*

$$s(M, M_*)|M_{sa} \subseteq \tau(M, M_*)|M_{sa} \subseteq \tau_o(M_{sa}) \subseteq \tau_{os}(M_{sa}) \subseteq \tau_u|M_{sa}, \tag{2.6}$$

$$\tau_o(M_{sa})|M_{sa}^1 \subseteq \tau_o(M_{sa}^1), \tag{2.7}$$

(2.8) $\tau_o(M_{sa}^1)|P(M) \subseteq \tau_o(P(M))$,

(2.9) $\tau_o(M_{sa})|P(M) \subseteq \tau_o(P(M))$.

LEMMA 2.8. *Let $0 \le x \le \mathbb{1}$ in M.*

(i) *If p is a projection, then $x \ge p$ if and only if $px = xp = p$.*
(ii) *If $\{p_\lambda : \lambda \in \Lambda\}$ is a set in $P(M)$ and $x \in M_{sa}^1$ satisfies $x \ge p_\lambda$ for every $\lambda \in \Lambda$, then $x \ge p$ where $p = \bigvee_{\lambda\in\Lambda} p_\lambda$ in $P(M)$.*

Proof. Suppose that M acts on a Hilbert space H.

(i) If $x \ge p$ then for any unit vector ξ in H that lies in the range of the projection p we have $1 \ge (x\xi, \xi) \ge (\xi, \xi) = 1$. So by the Cauchy–Schwarz inequality we deduce that $x\xi = \xi$. Consequently, $xp = p$. Conversely, if $xp = p$ then p and x commute and therefore $(\mathbb{1} - p)x = (\mathbb{1} - p)x(\mathbb{1} - p) \ge 0$. Hence $x = p + (\mathbb{1} - p)x \ge p$.

(ii) If $x \ge p_\lambda$ for every $\lambda \in \Lambda$, then $x\xi = \xi$ for every ξ in the range of p. Hence $xp = p = px$ and thus $x \ge p$ by (i). ∎

PROPOSITION 2.9. *The following statements are equivalent:*

(i) *M is abelian.*
(ii) *$\tau_o(M_{sa})|P(M) = \tau_o(M_{sa}^1)|P(M)$.*
(iii) *$\tau_o(M_{sa})|P(M) = \tau_o(P(M))$.*
(iv) *$\tau_o(M_{sa})|M_{sa}^1 = \tau_o(M_{sa}^1)$.*

Proof. When M is abelian, $(M_{sa}, \le)$ is a Dedekind complete lattice, and since M_{sa}^1 and $P(M)$ are $s(M, M_*)$-closed sublattices of M_{sa}, it follows by (2.4) and Proposition 2.3(ii) that $\tau_o(M_{sa})|M_{sa}^1 = \tau_o(M_{sa}^1)$ and $\tau_o(M_{sa})|P(M) = \tau_o(M_{sa}^1)|P(M) = \tau_o(P(M))$.

When M is not abelian, it contains a von Neumann subalgebra N (not necessarily unital) that is $*$-isomorphic to $B(H_2)$ where H_2 is a two-dimensional Hilbert space. We will identify N with $B(H_2)$. We show that $\tau_o(M_{sa}^1)|P(N)$ is discrete. To this end we suppose that $(p_\gamma)_{\gamma\in\Gamma}$ is a net of projections in N that order converges in $(M_{sa}^1, \le)$, say to p. (Note that p is also a projection in N because $P(N)$ is $s(M, M_*)$-closed and order convergence in M_{sa}^1 implies convergence with respect to $s(M, M_*)$.) Suppose, for contradiction, that $(p_\gamma)_{\gamma\in\Gamma}$ is not eventually constant. The inclusions

$$\tau_o(M_{sa})|N_{sa} \supseteq s(M, M_*)|N_{sa} = \tau_u|N_{sa}$$

and (2.7) imply that $p_\gamma \xrightarrow{\tau_u} p$ and therefore $p \notin \{0, \mathbb{1}_N\}$. We can thus assume that the range of p_γ is one-dimensional for every $\gamma \in \Gamma$. Lemma 2.8 implies that if $x \in M_{sa}^1$ satisfies $x \ge p_\gamma$ for every $\gamma \ge \gamma'$ then $x \ge \bigvee_{\gamma\ge\gamma'} p_\gamma = \mathbb{1}_N$. This implies that $p = \mathbb{1}_N$, a contradiction. Thus, every subset of $P(N)$ is $\tau_o(M_{sa}^1)$-closed, i.e. $\tau_o(M_{sa}^1)|P(N)$ is discrete. On the other hand, observe that $\tau_o(M_{sa})|P(N) \subseteq \tau_u|P(N)$, i.e. $\tau_o(M_{sa})|P(N)$ is not discrete. Thus, we have proved that if (ii) is true then M is abelian.

If (iii) is true, then we combine (2.7) and (2.8) to obtain

$$\tau_o(P(M)) = \tau_o(M_{sa})|P(M) \subseteq \tau_o(M_{sa}^1)|P(M) \subseteq \tau_o(P(M)),$$

i.e. (iii) implies (ii). The implication (iv)⇒(ii) is trivial. ∎

Proposition 2.9 implies that the inclusions in (2.7) and (2.9) are proper for nonabelian von Neumann algebras. In contrast, in the proof of Proposition 2.9 it is shown that when $M = B(H_2)$ then the inclusion in (2.8) is an equality. The question of when we get an equality in (2.8) will be dealt with in Section 5; in fact, we shall prove that for σ-finite von Neumann algebras this characterizes finiteness.

Remark 2.10. When M has an infinite linear dimension, it contains a sequence of pairwise orthogonal projections $(p_n)_{n\in\mathbb{N}}$ and then:

(i) Using the fact that every order convergent net is eventually bounded it is easy to see that the set $\{\sqrt{n}\,p_n : n \in \mathbb{N}\}$ is closed with respect to $\tau_o(M_{sa})$. On the other hand, 0 lies in the $s(M, M_*)$-closure of $\{\sqrt{n}\,p_n : n \in \mathbb{N}\}$. So $s(M, M_*)|M_{sa} \subsetneq \tau_o(M_{sa})$.

(ii) The sequence $(p_n)_{n\in\mathbb{N}}$ satisfies $\limsup_n p_n = \liminf_n p_n = 0$, i.e. it order converges to 0 in $(P(M), \le)$. Thus, (2.9) implies that $(kp_n)_{n\in\mathbb{N}}$ converges to 0 with respect to $\tau_o(M_{sa})$ for every $k \in \mathbb{N}$. For every $\tau_o(M_{sa})$-neighbourhood U of 0 there exists $n(k,U) \in \mathbb{N}$ such that $kp_n \in U$ for every $n \ge n(k,U)$. Define

$$\mathscr{N} := \{(k,U) : k \in \mathbb{N},\ U \text{ is a } \tau_o(M_{sa})\text{-neighbourhood of } 0\}$$

and equip it with the partial order defined by $(k_1, U_1) \le (k_2, U_2)$ if and only if $k_1 \le k_2$ and $U_2 \subseteq U_1$. Then $\mathscr{N}$ is an upward directed set. We can define a net $(x_{(k,U)})_{(k,U)\in\mathscr{N}}$ by setting $x_{(k,U)} := kp_{n(k,U)}$. It is clear that this net is not eventually bounded despite being convergent to 0 with respect to $\tau_o(M_{sa})$. Observe further that no subnet of this net is eventually bounded and therefore no subnet is order convergent in M_{sa}.

In contrast to the example exhibited in (ii) of the previous remark let us observe that any sequence converging in the order topology is bounded. Item (ii) of the previous remark suggests (in particular in view of Proposition 2.1) that a favoured case occurs when the sequential order topology coincides with the order topology because in this case—at least for sequences—convergence with respect to the order topology can be described by order convergent subsequences. The following proposition says that this occurs precisely when M is σ-finite.

Proposition 2.11. *The following three statements are equivalent:*

(i) *M is σ-finite.*

(ii) $\tau_{os}(M_{sa}) = \tau_o(M_{sa})$.

(iii) $\tau_{os}(M_{sa}^1) = \tau_o(M_{sa}^1)$.

(iv) $\tau_{os}(P(M)) = \tau_o(P(M))$.

Proof. We recall that bounded monotone nets in M_{sa} converge with respect to $s(M, M_*)$ to their supremum/infimum. Since M_{sa}^1 and $P(M)$ are $s(M, M_*)$-closed, it follows that if M_{sa} is monotone order separable then M_{sa}^1 is monotone order separable; and if M_{sa}^1 is monotone order separable then $P(M)$ is monotone order separable. If M is σ-finite then it admits a faithful normal positive linear functional and so, by Proposition 2.5, we have (i)⇒(ii)⇒(iii)⇒(iv). When M is not σ-finite, $P(M)$ contains an uncountable family of nonzero orthogonal projections and thus it is not monotone order separable, i.e. (iv)⇒(i). ∎

3. Vector spaces with mixed topology. Now we consider the mixed topology on a vector space introduced and studied in detail in [28]. We first list some of its basic known properties and then we add some new facts needed in what follows.

In this section let X be a real vector space endowed with two linear Hausdorff topologies τ and τ'. For each sequence $(U'_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in (X, τ') and for each 0-neighbourhood U in (X, τ) define

$$\gamma((U'_n)_{n\in\mathbb{N}}, U) := \bigcup_{n\in\mathbb{N}} \sum_{i=1}^{n} (U'_i \cap iU).$$

Then the family of these sets is a basis of 0-neighbourhoods for some linear Hausdorff topology $\gamma[\tau, \tau']$ called the *mixed topology* determined by τ and τ'. It is clear that if X is a complex vector space and the Hausdorff topologies τ and τ' are linear over $\mathbb{C}$ then $\gamma[\tau, \tau']$ is also linear over $\mathbb{C}$.

Proposition 3.1 ([28, 2.1.1]).

(i) $\tau' \subseteq \gamma[\tau, \tau']$.
(ii) *If* $\tau' \subseteq \tau$ *then* $\gamma[\tau, \tau'] \subseteq \tau$.
(iii) *If* τ *and* τ' *are locally convex, then* $\gamma[\tau, \tau']$ *is locally convex.*

Proposition 3.2 ([28, 2.2.1, 2.2.2]).

(i) $\gamma[\tau, \tau']|Z = \tau'|Z$ *for every* τ*-bounded subset* Z *of* X.
(ii) *If* (X, τ) *is locally bounded, then* $\gamma[\tau, \tau']$ *is the finest of all linear topologies agreeing with* τ' *on every* τ*-bounded subset of* X.

Proposition 3.2(ii) implies that $\gamma[\tau, \tau_1] = \gamma[\tau, \tau_2]$ when (X, τ) is locally bounded and τ_1 and τ_2 are Hausdorff linear topologies on X such that $\tau_1|Z = \tau_2|Z$ for every τ-bounded subset Z; in particular

$$\gamma[\tau, \tau'] = \gamma\big[\tau, \gamma[\tau, \tau']\big].$$

Proposition 3.3 ([28, 2.4.1]). *If* $\|\cdot\|$ *is a norm on* X *inducing* τ *and the unit ball of* $(X, \|\cdot\|)$ *is* τ'*-closed then a set* $A \subseteq X$ *is* $\gamma[\tau, \tau']$*-bounded if and only if it is simultaneously* $\|\cdot\|$*-bounded and* τ'*-bounded.*

The following two theorems will be of great use in Section 4.

THEOREM 3.4. *Assume that*

(i) *τ is induced by a norm $\|\cdot\|$ on X,*
(ii) *the unit ball X^1 of $(X,\|\cdot\|)$ is τ'-closed, but not τ'-compact, and*
(iii) *$\tau'|X^1$ is metrizable and strictly coarser than $\tau|X^1$.*

Then $(X,\gamma[\tau,\tau'])$ is not a sequential space.

Proof. By (ii) and (iii) it follows that X^1 contains a sequence $(a_n)_{n\in\mathbb{N}}$ without a τ'-cluster point. By (iii) there is an integer $m_0>1$ and a sequence $(b_n)_{n\in\mathbb{N}}$ in X^1 converging to 0 with respect to τ' such that $\|b_n\|>1/m_0$ for $n\in\mathbb{N}$.

We will show that

$$F:=\left\{\frac{1}{m}a_n+mb_n : n,m\in\mathbb{N},\, m\geq m_0\right\}$$

is sequentially closed, but not closed in $(X,\gamma[\tau,\tau'])$.

We have $0\notin F$ since $\|m^{-1}a_n\|<1<\|mb_n\|$ for $n,m\in\mathbb{N}$ with $m\geq m_0$. We show that on the other hand 0 is a $\gamma[\tau,\tau']$-limit point of F. Let $W:=\gamma\big((U_k')_{k\in\mathbb{N}},U\big)$ be a 0-neighbourhood in $\gamma[\tau,\tau']$ where U_k' and U are 0-neighbourhoods in τ' and τ, respectively. By (i) and (iii) it follows that $\tau'\subseteq\tau$. Since $B:=\bigcup_{n\in\mathbb{N}}\{a_n,b_n\}\subseteq X^1$, we have $m^{-1}B\subseteq U_1'\cap U$ for some $m\geq m_0$. Then $m^{-1}a_n\in U_1'\cap U$ for every $n\in\mathbb{N}$. Now, let $l\in\mathbb{N}$ be such that $(m/l)B\subseteq U$. Then $mb_n\in lU$ for every $n\in\mathbb{N}$. Since $(mb_n)_{n\in\mathbb{N}}$ converges to 0 with respect to τ', there exists $n_0\in\mathbb{N}$ such that $mb_{n_0}\in U_l'$. Then $m^{-1}a_{n_0}+mb_{n_0}\in U_1'\cap U+U_l'\cap lU\subseteq W$.

We now show that F is sequentially closed with respect to $\gamma[\tau,\tau']$. Let $(g_j)_{j\in\mathbb{N}}$ be a sequence in F converging to g with respect to $\gamma[\tau,\tau']$. We can write $g_j=m_j^{-1}a_{n_j}+m_jb_{n_j}$ where $m_j,n_j\in\mathbb{N}$ and $m_j\geq m_0$. The set $\{g_j : j\in\mathbb{N}\}$ is $\gamma[\tau,\tau']$-bounded and therefore τ-bounded in virtue of Proposition 3.3. Hence $\{m_jb_{n_j} : j\in\mathbb{N}\}$ is τ-bounded. But since $\|b_n\|\geq 1/m_0$ for all $n\in\mathbb{N}$, this can only happen if $\{m_j : j\in\mathbb{N}\}$ is finite. Thus, passing to a subsequence, we may assume that m_j is constant $(=m)$, i.e. $g_{n_j}=m^{-1}a_{n_j}+mb_{n_j}$. Suppose that $\{n_j : j\in\mathbb{N}\}$ is not finite. Passing to a subsequence we may assume that the sequence n_j is strictly increasing. Since $b_{n_j}\xrightarrow{\tau'}0$ and $\tau'\subseteq\gamma[\tau,\tau']$ we deduce that

$$a_{n_j}=mg_{n_j}-m^2b_{n_j}\xrightarrow{\tau'}mg\quad\text{ as } j\to\infty,$$

in contradiction to the fact that $(a_n)_{n\in\mathbb{N}}$ has no τ'-cluster point. Therefore $\{n_j : j\in\mathbb{N}\}$ is finite. But this implies that g belongs to F. ∎

Theorem 3.5. *Let τ' be induced by a pointwise bounded family $\{\rho_\lambda : \lambda \in \Lambda\}$ of seminorms on X and let τ be the topology induced by the norm*

$$\|x\| := \sup_{\lambda \in \Lambda} \rho_\lambda(x).$$

Assume further that the unit ball X^1 of $(X, \|\cdot\|)$ is τ'-compact. Then a subset C of X is $\gamma[\tau,\tau']$-closed if and only if $C \cap rX^1$ is $\gamma[\tau,\tau']$-closed for every $r > 0$.

Proof. Let C be $\gamma[\tau,\tau']$-closed and $r > 0$. Since rX^1 is τ'-compact, it is τ'-closed, hence $\gamma[\tau,\tau']$-closed since $\tau' \subseteq \gamma[\tau,\tau']$. Therefore $C \cap rX^1$ is $\gamma[\tau,\tau']$-closed.

The proof of the reverse implication is based on the following two lemmas. We use therein the notation

$$B(f) := \{x \in X : \rho_\lambda(x) \le f(\lambda) \text{ for all } \lambda \in \Lambda\} \quad \text{if } f : \Lambda \to (0,\infty].$$

Moreover, since the seminorms ρ_λ are not assumed to be different, we may assume that Λ is infinite.

Lemma 3.6. *Let $f : \Lambda \to (0,\infty)$ and $\sup_{\lambda\in\Lambda} f(\lambda) \le s < \infty$. Assume that $A \subseteq X$ is such that $A \cap B(f) = \emptyset$ and $A \cap sX^1$ is τ'-closed. Then there exists a finite subset F of Λ such that $A \cap B(g) = \emptyset$ where $g(\lambda) = f(\lambda)$ for $\lambda \in F$ and $g(\lambda) = s$ for $\lambda \in \Lambda \setminus F$.*

Proof. Otherwise for any finite subset F of Λ there exists $x_F \in A$ with $\rho_\lambda(x_F) \le f(\lambda)$ for $\lambda \in F$, and $\rho_\lambda(x_F) \le s$ for all $\lambda \in \Lambda$. Since $A \cap sX^1$ is τ'-compact, $(x_F)_{F\subseteq\Lambda, |F|<\infty}$ has a subnet τ'-converging to an element x in $A \cap sX^1$. For every fixed $\lambda \in \Lambda$ we have $\rho_\lambda(x_F) \le f(\lambda)$ eventually. Therefore $\rho_\lambda(x) \le f(\lambda)$ for all $\lambda \in \Lambda$. It follows $x \in A \cap B(f)$, a contradiction. ■

Lemma 3.7. *Let $A \subseteq X \setminus \{0\}$ be such that $A \cap rX^1$ is τ'-closed for every $r > 0$. Then there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ and a real sequence $(a_n)_{n\in\mathbb{N}}$ with $0 < a_n \uparrow \infty$ such that $A \cap B(g) = \emptyset$ where $g(\lambda_n) = a_n$ for $n \in \mathbb{N}$ and $g(\lambda) = \infty$ otherwise.*

Proof. By assumption $A \cap X^1$ is τ'-closed, therefore τ-closed since $\tau' \subseteq \tau$. Hence there is an $\varepsilon > 0$ such that $A \cap \varepsilon X^1 = \emptyset$.

Let $F_0 := \emptyset$. We define inductively a strictly increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Λ such that $A \cap B(g_n) = \emptyset$ where g_n is defined by

$$g_n(\lambda) = \begin{cases} i\varepsilon & \text{if } \lambda \in F_i \setminus F_{i-1} \text{ and } 1 \le i \le n, \\ (n+1)\varepsilon & \text{if } \lambda \in \Lambda \setminus F_n. \end{cases}$$

By the choice of ε, $A \cap B(g_n) = \emptyset$ is satisfied for $n = 0$ defining $g_0(\lambda) = \varepsilon$ for all $\lambda \in \Lambda$.

For the inductive step $[n-1 \to n]$ we apply Lemma 3.6 with $f := g_{n-1}$ and $s := (n+1)\varepsilon$. Choose F according to Lemma 3.6 and let F_n be a finite subset of Λ with $F \cup F_{n-1} \subseteq F_n$ and $F_{n-1} \neq F_n$. If we set $g_n(\lambda) = g_{n-1}(\lambda)$ for $\lambda \in F_n$ and $g_n(\lambda) = (n+1)\varepsilon$ for $\lambda \in \Lambda \setminus F_n$ then $A \cap B(g_n) = \emptyset$.

Let $g := \sup_{n\in\mathbb{N}} g_n$. Then $g(\lambda) = n\varepsilon$ whenever there exists $n \in \mathbb{N}$ such that $\lambda \in F_n \setminus F_{n-1}$. Otherwise $g(\lambda) = \infty$. Since the sequence (g_n) is increasing and $\{\rho_\lambda : \lambda \in \Lambda\}$ is pointwise bounded, $B(g) = \bigcup_{n\in\mathbb{N}} B(g_n)$ and therefore $A \cap B(g) = \emptyset$.

To complete the proof let $k_n := |F_n|$, choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with $F_n = \{\lambda_i : i \leq k_n\}$ and set $a_i := g(\lambda_i)$. ∎

To conclude the proof of Theorem 3.5 first we recall that the sets $\{x \in X : \rho_{\lambda_n}(x) \leq a_n \text{ for all } n \in \mathbb{N}\}$ where $\lambda_n \in \Lambda$ and $0 < a_n \uparrow \infty$ form a 0-neighbourhood base of $(X, \gamma[\tau, \tau'])$ (see [28, Theorem 3.1.1]). Suppose that $C \subseteq X$ and $C \cap rX^1$ is $\gamma[\tau, \tau']$-closed for every $r > 0$. Since $\gamma[\tau, \tau']$ and τ' induce on rX^1 the same topology (see Proposition 3.2(i)) and since rX^1 is τ'-closed, it follows that $C \cap rX^1$ is also τ'-closed. Let $x \notin C$ and $A := C - x$. Then $0 \notin A$. It follows from Lemma 3.7 that 0 does not belong to the $\gamma[\tau, \tau']$-closure $\overline{A}$ of A, i.e. $x \notin \overline{C}$. ∎

Corollary 3.8. *Under the assumptions of Theorem* 3.5, *if τ'' is a* (*not necessarily linear*) *topology on X such that $\tau''|rX^1 = \tau'|rX^1$ for every $r > 0$ then $\tau'' \subseteq \gamma[\tau, \tau']$.*

Proof. Let C be a τ''-closed subset of X. By $\tau''|rX^1 = \tau'|rX^1$ it follows that $C \cap rX^1$ is $\tau'|rX^1$-closed. This implies that $C \cap rX^1$ is τ'-closed, since rX^1 is τ'-closed. Thus, $C \cap rX^1$ is $\gamma[\tau, \tau']$-closed, since $\tau' \subseteq \gamma[\tau, \tau']$. The assertion now follows by Theorem 3.5. ∎

Corollary 3.9. *Under the assumptions of Theorem* 3.5, *if Λ is countable then $(X, \gamma[\tau, \tau'])$ is a sequential space.*

Proof. Let C be a sequentially closed subset of $(X, \gamma[\tau, \tau'])$ and let $r > 0$. Then since rX^1 is τ'-closed (and therefore $\gamma[\tau, \tau']$-closed), $C \cap rX^1$ is sequentially closed with respect to $\gamma[\tau, \tau']$. But on the τ'-closed subset rX^1 the topology τ' agrees with $\gamma[\tau, \tau']$. Therefore $C \cap rX^1$ is sequentially closed with respect to τ'. By our assumption on Λ we see that $C \cap rX^1$ is closed with respect to τ' and therefore $C \cap rX^1$ is $\gamma[\tau, \tau']$-closed. Hence C is $\gamma[\tau, \tau']$-closed by Theorem 3.5. ∎

We now give a first application of Theorem 3.4 and Corollary 3.9. Let (X, Σ, μ) be a σ-finite measure space, τ_∞ the topology of the Banach space $(L^\infty, \|\|_\infty)$ and τ_μ the topology of convergence in measure (on sets of finite measure), i.e. the Hausdorff linear topology induced by the family of F-seminorms $\rho_F : L^\infty \ni [f] \mapsto \int_X \min(|f|, \chi_F)\, d\mu$ ($F \in \Sigma$ with $\mu(F) < \infty$). (See [14, Proposition 245A, p. 172].) If μ is not purely atomic, then Theorem 3.4 implies that $(L^\infty, \gamma[\tau_\infty, \tau_\mu])$ is not a sequential space. If μ is purely atomic, then L^∞ can be identified with the sequence space ℓ^∞, and τ_μ with the topology of pointwise convergence on ℓ^∞, which is generated by the seminorms $p_n : \ell^\infty \ni (x_i) \mapsto x_n$ ($n \in \mathbb{N}$). It follows therefore from Corollary 3.9 that $(L^\infty, \gamma[\tau_\infty, \tau_\mu])$ is a sequential space. Combining these two results with Nowak's result [21, Theorem 5], saying that the mixed topology $\gamma[\tau_\infty, \tau_\mu]$ coincides with the Mackey topology $\tau(L^\infty, L^1)$ on L^∞ induced by the dual pairing (L^∞, L^1), we obtain:

THEOREM 3.10. *Let (X, Σ, μ) be a σ-finite measure space. Then the space $(L^\infty, \tau(L^\infty, L^1))$ is sequential if and only if μ is purely atomic.*

This theorem will be generalized in Theorem 4.8 for von Neumann algebras.

4. The order topology and the sequential order topology on M_{sa}. On M we consider the mixed topology $\gamma[\tau_u, s^*(M, M_*)]$ determined by τ_u and $s^*(M, M_*)$.

THEOREM 4.1. *The Mackey topology $\tau(M, M_*)$ coincides with the mixed topology $\gamma[\tau_u, s^*(M, M_*)]$.*

Proof. Proposition 3.2(ii) with $\tau := \tau_u$ and $\tau' := s^*(M, M_*)$ and Akemann's Theorem (2.2) already imply $\tau(M, M_*) \subseteq \gamma[\tau_u, s^*(M, M_*)]$. For the converse observe that if φ is a linear functional on M continuous with respect to $\gamma[\tau_u, s^*(M, M_*)]$ then [23, Corollary 1.8.10, p. 21] implies that φ is $\sigma(M, M_*)$-continuous. Hence the inclusion $\gamma[\tau_u, s^*(M, M_*)] \subseteq \tau(M, M_*)$ follows because $\tau(M, M_*)$ is the finest locally convex topology on M compatible with the duality (M, M_*). ∎

REMARK 4.2. Theorem 4.1 is a generalization of [21, Theorem 5]. Let (X, Σ, μ) be a localisable measure space. Then L^∞ is an abelian von Neumann algebra (see [14, Theorem 243G, p. 154]). It is easy to verify that on bounded parts of L^∞ the topology τ_μ of convergence in measure agrees with $s^*(L^\infty, L^1)$ $(= s(L^\infty, L^1))$ and therefore $\tau(L^\infty, L^1) = \gamma[\tau_u, \tau_\mu]$ by Theorem 4.1. In [21, Theorem 5] it is shown that when (X, Σ, μ) is σ-finite then $\tau(L^\infty, L^1) = \gamma[\tau_u, \tau_\mu]$.

THEOREM 4.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M_{sa} and let $x \in M_{sa}$.*

(i) $x_n \xrightarrow{\tau(M,M_*)} x \Leftrightarrow x_n \xrightarrow{s(M,M_*)} x \Leftarrow x_n \xrightarrow{\tau_o(M_{sa})} x$.

(ii) *If M is σ-finite then* $x_n \xrightarrow{s(M,M_*)} x \Leftrightarrow x_n \xrightarrow{\tau_{os}(M_{sa})} x$.

Proof. (i) First we show that $x_n \xrightarrow{\tau(M,M_*)} x \Leftrightarrow x_n \xrightarrow{s(M,M_*)} x$. One direction follows from (2.1). For the other direction note that if $x_n \xrightarrow{s(M,M_*)} x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded. Hence $x_n \xrightarrow{\tau(M,M_*)} x$ in view of (2.3). The implication $x_n \xrightarrow{\tau_o(M_{sa})} x \;\Rightarrow\; x_n \xrightarrow{s(M,M_*)} x$ follows from the fact that $\tau_o(M_{sa})$ is the finest topology that preserves order convergence.

(ii) In virtue of Propositions 2.11 and 2.1, it suffices to prove that for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to x with respect to $s(M,M_*)$ it is possible to extract a subsequence that order converges to x in $(M_{sa},\leq)$. By the translation invariance of $\tau_o(M_{sa})$ we can suppose that $x=0$, and since $(x_n)_{n\in\mathbb{N}}$ is necessarily bounded we can further suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in M_{sa}^1. The proof is based on a recursive application of the Noncommutative Egoroff Theorem [26, Theorem 4.13, p. 85]: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a von Neumann algebra M converging to 0 with respect to $s(M,M_*)$. Then, for every projection e in M, and all $\varphi\in M_*^+$ and $\varepsilon>0$, there exists a projection $e_0\leq e$ and a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\varphi(e-e_0)<\varepsilon$ and $\|a_{n_k}e_0\|<2^{-k-1}$.

First we suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ is positive. Since M is σ-finite, it admits a faithful normal state ψ. Applying Egoroff's Theorem with $e:=\mathbb{1}$, $\varphi:=\psi$ and $\varepsilon=2^{-1}$, we obtain a projection e_1 and a subsequence $(x_k^{(1)})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\|x_k^{(1)}e_1\|<2^{-k-1}$ for each $k\in\mathbb{N}$ and $\psi(\mathbb{1}-e_1)<2^{-1}$. The sequence $(x_k^{(1)})_{k\in\mathbb{N}}$ converges to 0 with respect to $s(M,M_*)$, and so we can apply Egoroff's Theorem again for this sequence with $e:=\mathbb{1}-e_1$, $\varphi:=\psi$ and $\varepsilon=2^{-2}$ to obtain a projection $e_2\leq\mathbb{1}-e_1$, and a subsequence $(x_k^{(2)})_{k\in\mathbb{N}}$ of $(x_k^{(1)})_{k\in\mathbb{N}}$ such that $\|x_k^{(2)}e_2\|<2^{-k-2}$ and $\psi(\mathbb{1}-e_1-e_2)<2^{-2}$.

An inductive application of Egoroff's Theorem yields a sequence of orthogonal projections $(e_n)_{n\in\mathbb{N}}$ satisfying $\psi(\mathbb{1}-\sum_{i=1}^n e_i)<2^{-n}$; and a nested sequence of subsequences $(x_k^{(j)})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ where $(x_k^{(j+1)})_{k\in\mathbb{N}}$ is a subsequence of $(x_k^{(j)})_{k\in\mathbb{N}}$ such that $\|x_k^{(j)}e_j\|<2^{-k-j}$.

Let $p_n:=\sum_{i=1}^n e_i$. Then $(\mathbb{1}-p_n)_{n\in\mathbb{N}}$ is a decreasing sequence of projections and $\psi(\bigwedge(\mathbb{1}-p_n))=0$ and thus, since ψ is faithful, $\mathbb{1}-p_n\downarrow 0$.

From the way the nested array $(x_k^{(j)})_{k\in\mathbb{N}}$ is constructed, one can check that if $j\geq i$ then $x_j^{(j)}=x_p^{(i)}$ for some $p\geq j$. Thus $\|x_j^{(j)}e_i\|=\|x_p^{(i)}e_i\|<2^{-p-i}\leq 2^{-j-i}$.

The sequence $(x_j^{(j)})_{j\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. We claim that $(x_j^{(j)})_{j\in\mathbb{N}}$ order converges to 0. To this end we observe that

$$\begin{aligned}
x_j^{(j)} &= x_j^{(j)}p_j + p_jx_j^{(j)}(\mathbb{1}-p_j) + (\mathbb{1}-p_j)x_j^{(j)}(\mathbb{1}-p_j)\\
&\leq \|x_j^{(j)}p_j + p_jx_j^{(j)}(\mathbb{1}-p_j)\|\mathbb{1} + \mathbb{1}-p_j \leq (1+2\|x_j^{(j)}p_j\|)\mathbb{1}-p_j\\
&\leq \Big(1+2\sum_{i=1}^{j}\|x_j^{(j)}e_i\|\Big)\mathbb{1}-p_j \leq (1+2^{-j+1})\mathbb{1}-p_j.
\end{aligned}$$

Since $(1+2^{-j+1})\mathbb{1}-p_j\downarrow 0$, it follows that $(x_j^{(j)})_{j\in\mathbb{N}}$ order converges to 0.

To complete the proof we consider the case when $(x_n)_{n\in\mathbb{N}}$ is not assumed to be in M_+. If $x_n \xrightarrow{s(M,M_*)} 0$ then $|x_n| \xrightarrow{s(M,M_*)} 0$ (where $|x_n| = \sqrt{x_n^2}$) and therefore $(|x_n|)_{n\in\mathbb{N}}$ converges to 0 with respect to $\tau_{os}(M_{sa})$ by the above. Thus $(|x_n|)_{n\in\mathbb{N}}$ has a subsequence $(|x_{n_k}|)_{k\in\mathbb{N}}$ that order converges to 0, i.e. one can find a sequence $(y_k)_{k\in\mathbb{N}}$ in M_+ such that $0 \le |x_{n_k}| \le y_k$ and $y_k \downarrow 0$. The result then follows from $-y_k \le -|x_{n_k}| \le x_{n_k} \le |x_{n_k}| \le y_k$. ∎

COROLLARY 4.4. *Assume that M is σ-finite and K is a bounded subset of M_{sa}. Then the $s(M,M_*)$-closure of K coincides with the $\tau_{os}(M_{sa})$-closure of K.*

Proof. Let $r > 0$ be such that $K \subseteq rM^1$. When M is σ-finite, the topology $s(M,M_*)|rM^1$ is metrisable. Since rM^1 is $s(M,M_*)$-closed, the assertion follows by Theorem 4.3. ∎

COROLLARY 4.5. *Assume that M is σ-finite. Then*

$$\tau_s|K = s(M,M_*)|K = \tau(M,M_*)|K = \tau_o(M_{sa})|K = \tau_{os}(M_{sa})|K$$

for every bounded subset K of M_{sa}.

Note that $s(M,M_*)|M_{sa}$ and $\tau_o(M_{sa})$ are different unless M is finite-dimensional (see Remark 2.10(i)). The aim of the rest of this section is to study when the order topology $\tau_o(M_{sa})$ coincides with $\tau(M,M_*)|M_{sa}$.

LEMMA 4.6. $\gamma[\tau_u|M_{sa}, s(M,M_*)|M_{sa}] = \gamma[\tau_u, s^*(M,M_*)]|M_{sa}$.

Proof. To simplify the notation let $\gamma := \gamma[\tau_u|M_{sa}, s(M,M_*)|M_{sa}]$. Since $s(M,M_*)|K = \gamma[\tau_u, s^*(M,M_*)]|K$ for every bounded subset K of M_{sa} we get $\gamma[\tau_u, s^*(M,M_*)]|M_{sa} \subseteq \gamma$ by Proposition 3.2(ii). For the reverse inclusion we consider $\tilde{M} := M_{sa} \times M_{sa}$ as a real vector space. The mapping $\tilde{M} \ni (x,y) \mapsto x + iy \in M$ is an isomorphism of $\tilde{M}$ onto M (as real vector spaces). $(\tilde{M}, \gamma \times \gamma)$ is a Hausdorff topological vector space over $\mathbb{R}$. Denote by τ the Hausdorff real-linear topology induced on M by $\gamma \times \gamma$. Then $\tau|K = s^*(M,M_*)|K$ for every bounded subset K of M. Hence $\tau \subseteq \gamma[\tau_u, s^*(M,M_*)]$ by Proposition 3.2(ii) and therefore $\tau|M_{sa} \subseteq \gamma[\tau_u, s^*(M,M_*)]|M_{sa}$. Finally, observe that $\tau|M_{sa} = \gamma$, and hence the required inclusion holds. ∎

Let p be a nonzero projection in M. We recall that p is said to be a *minimal projection* if whenever e is a nonzero projection such that $0 \ne e \le p$ then $e = p$. Equivalently, p is minimal if $pMp = \mathbb{C}p$. If pMp is abelian then p is said to be an *abelian projection.* Every minimal projection is obviously abelian. M is said to be of type I if every nonzero central projection of M majorizes a nonzero abelian projection. We recall that every *type I* factor is $*$-isomorphic to a $B(H)$ for some Hilbert space H. A von Neumann algebra M is said to be *atomic* if every nonzero projection majorizes a minimal projection. Obviously if M is atomic then M is of type I. Moreover, we say that M is *purely atomic* if every von Neumann subalgebra of M is atomic. (When talking about subalgebras we do not require that a subalgebra contains the unit of its superalgebra.) Observe that M can be atomic without being purely atomic. For example, when H is infinite-dimensional and separable, $B(H)$ is atomic but not purely atomic because $L^\infty[0,1]$ can be identified with a von Neumann subalgebra of $B(H)$.

THEOREM 4.7. *The following statements are equivalent:*

(i) *The unit ball M^1 is $s^*(M,M_*)$-compact* (*and therefore on bounded parts of M the σ-strong* topology coincides with the weak* topology*).
(ii) *M is purely atomic.*
(iii) *M is $*$-isomorphic to a direct sum of finite-dimensional matrix algebras.*

Proof. (i)⇒(ii). Suppose that M is not purely atomic and let N be a von Neumann subalgebra of M that is not atomic. Without any loss of generality we can assume that N has no minimal projections and that it is σ-finite. Let φ be a faithful normal state on N. Using the noncommutative version of the Lyapunov Theorem [3] (or [11, 5]) it is possible to define projections like the Rademacher functions: $\{p_{n,i} : n \in \mathbb{N},\ i = 1,\dots,2^n\}$ such that $\mathbb{1}_N = p_{1,1} + p_{1,2}$ and $p_{n-1,i} = p_{n,2i-1} + p_{n,2i}$; and moreover $\varphi(p_{n,i}) = 2^{-n}$. Set $e_n = \sum_{i=1}^{2^n}(-1)^i p_{n,i}$. Then $(e_n)_{n\in\mathbb{N}}$ is a sequence of self-adjoint elements in the unit ball of N and $\eta_\varphi(e_n - e_m) = 2$ for every $n \neq m$. This implies that the unit ball of N is not $s^*(N,N_*)$-compact and thus the result follows.

(ii)⇒(iii). Let $\mathscr{Z}(M)$ denote the centre of M. If z is a minimal projection of $\mathscr{Z}(M)$ then zM is a type I factor and therefore zM is $*$-isomorphic to $B(H)$ for some Hilbert space H. Note that H cannot be infinite-dimensional because M is purely atomic. The result then follows by taking a family of pairwise orthogonal minimal projections in $\mathscr{Z}(M)$, say $\{z_\lambda : \lambda \in \Lambda\}$, such that $\sum_{\lambda\in\Lambda} z_\lambda = \mathbb{1}$.

(iii)⇒(i). Let $M = \sum_{\lambda\in\Lambda} \oplus B(H_{n_\lambda})$ where Λ is an indexing set and $n_\lambda \in \mathbb{N}$ for every $\lambda \in \mathbb{N}$. Denote by $\|\cdot\|_\lambda$ the norm on $B(H_{n_\lambda})$ and by B_λ its unit ball. Then $(B_\lambda, \|\cdot\|_\lambda)$ is compact for every $\lambda \in \Lambda$ and therefore the product space $\prod_{\lambda\in\Lambda}(B_\lambda, \|\cdot\|_\lambda)$ is compact by the Tychonoff Theorem. Observe that $(M^1, s^*(M,M_*)|M^1)$ is homeomorphic to $\prod_{\lambda\in\Lambda}(B_\lambda, \|\cdot\|_\lambda)$ and therefore the result follows. ∎

THEOREM 4.8. *The following statements are equivalent:*

(i) $\tau(M,M_*)|M_{sa} = \tau_{os}(M_{sa})$.
(ii) *$\tau(M,M_*)|M_{sa}$ is sequential.*
(iii) *$\tau(M,M_*)$ is sequential.*
(iv) *M is σ-finite and $\tau_o(M_{sa})$ is a linear topology.*

(v) *M is σ-finite and satisfies one* (*and therefore all*) *of the equivalent conditions of Theorem* 4.7.

Proof. (i)⇒(ii). If $\tau_{os}(M_{sa}) = \tau(M, M_*)|M_{sa}$ then $\tau(M, M_*)|M_{sa}$ is sequential since $\tau_{os}(M_{sa})$ is obviously sequential.

(ii)⇒(v). Suppose that $\tau(M, M_*)|M_{sa}$ is sequential. Observe that M must be σ-finite because otherwise it contains an uncountable family $\{p_\gamma : \gamma \in \Gamma\}$ of nonzero orthogonal projections, and then the set

$$N := \Big\{x \in M_{sa} : \exists \Gamma_0 \subseteq \Gamma,\ |\Gamma_0| \le \aleph_0,\ 0 \le x \le \bigvee_{\gamma\in\Gamma_0} p_\gamma\Big\}$$

is sequentially $\tau(M, M_*)$-closed but not $\tau(M, M_*)$-closed. We recall that $s(M, M_*)|M^1_{sa}$ is metrisable when M is σ-finite. Let us show that M^1 is $s^*(M, M_*)$-compact. Since $M^1 \subseteq M^1_{sa} + iM^1_{sa}$ is $s^*(M, M_*)$-closed, it suffices to show that M^1_{sa} is $s(M, M_*)$-compact. If M^1_{sa} is not $s(M, M_*)$-compact then we can apply Theorem 3.4 with $X := M_{sa}$, $\tau' := s(M, M_*)|M_{sa}$ and $\tau := \tau_u|M_{sa}$ to deduce that the mixed topology $\gamma\big[\tau_u|M_{sa}, s(M, M_*)|M_{sa}]$ is not sequential and therefore (v) follows by Lemma 4.6 and Theorem 4.1.

(v)⇒(iii). Assume that M is σ-finite and $*$-isomorphic to a direct sum of finite-dimensional matrix algebras, say $M = \sum_{\lambda\in\Lambda} \oplus B(H_{n_\lambda})$ where Λ is countable. We can apply Corollary 3.9 with $X := M$ and $(\rho_\lambda)_{\lambda\in\Lambda}$ defined by $\rho_\lambda(x) := \|x_\lambda\|_\lambda$ where $x = (x_\lambda)_{\lambda\in\Lambda}$ and $\|\cdot\|_\lambda$ denotes the norm on $B(H_{n_\lambda})$ to deduce that $\gamma[\tau, \tau']$ is sequential. Obviously τ coincides with τ_u and it is easily seen that on bounded parts of M the topology τ' (= product topology) coincides with $s^*(M, M_*)$. Hence (see the comment following Proposition 3.2) $\gamma[\tau, \tau'] = \gamma\big[\tau_u, s^*(M, M_*)\big]$. Thus (iii) follows by Theorem 4.1.

(iii)⇒(i). If $\tau(M, M_*)$ is sequential then $\tau(M, M_*)|M_{sa}$ is sequential and therefore M is σ-finite. Therefore we get $\tau(M, M_*)|M_{sa} = \tau_{os}(M_{sa})$ in virtue of Theorem 4.3.

(i)⇔(iv). The implication (i)⇒(iv) is trivial. In virtue of Lemma 4.6 and Theorem 4.1 we have

$$\tau(M, M_*)|M_{sa} = \gamma[\tau_u, s^*(M, M_*)]|M_{sa} = \gamma[\tau_u|M_{sa}, s(M, M_*)|M_{sa}],$$

i.e. $\tau(M, M_*)|M_{sa}$ is the finest linear topology on M_{sa} that agrees with $s(M, M_*)|M_{sa}$ on bounded subsets of M_{sa}. Thus in view of Corollary 4.5 we get $\tau_{os}(M_{sa}) \subseteq \tau(M, M_*)|M_{sa}$ when $\tau_o(M_{sa})$ is linear. ∎

5. The order topology and the sequential order topology on M^1_{sa} and $P(M)$. The order topology on the projection lattice of a Hilbert space was studied in [22] and [9]. Let $\mathscr{L}$ denote the lattice of projections on a separable Hilbert space H. Using (2.4) and (2.9) we immediately get $\tau_s|\mathscr{L} \subseteq \tau_o(\mathscr{L})$. [22, Example 2.4] shows that if $\dim H \ge 2$ then $\tau_o(\mathscr{L}) \nsubseteq \tau_u|\mathscr{L}$ and therefore $\tau_s|\mathscr{L} \ne \tau_o(\mathscr{L})$. (This is in contrast with Corollary 4.5.)

In this connection we mention that in [9, Theorem 20] the authors show that when B is a maximal Boolean sublattice of $\mathscr{L}$ then $\tau_o(\mathscr{L})|B = \tau_s|B$. Let us point out that in fact this follows from Proposition 2.9 and Corollary 4.5. Indeed, if B is a maximal Boolean sublattice of $\mathscr{L}$ then B is the projection lattice of a maximal abelian $*$-subalgebra M of $B(H)$ and therefore

$$\tau_s|B = s(M, M_*)|B = \tau_o(M_{sa})|B = \tau_o(B).$$

When $\dim H < \infty$ the order topology on $\mathscr{L}$ coincides with the discrete topology and therefore it is finer than the restriction of the uniform topology, but [22, Example 2.3] shows that if $\dim H = \infty$ then $\tau_u|\mathscr{L} \nsubseteq \tau_o(\mathscr{L})$. In [22] V. Palko conjectured that $\tau_s|\mathscr{L} = \tau_o(\mathscr{L}) \cap \tau_u|\mathscr{L}$. This is obviously true when $\dim H < \infty$, and in full agreement with the conjecture he proved that a sequence of atoms in $\mathscr{L}$ converges to 0 with respect to τ_s if and only if it converges to 0 with respect to $\tau_u|\mathscr{L} \cap \tau_o(\mathscr{L})$. [9, Example 16] however shows that $\tau_s|\mathscr{L} = \tau_o(\mathscr{L}) \cap \tau_u|\mathscr{L}$ only when $\dim H < \infty$.

In this section we study the order topology and the sequential order topology on M_{sa}^1 and $P(M)$. Proposition 2.7 already implies $\tau_o(M_{sa})|M_{sa}^1 \subseteq \tau_o(M_{sa}^1)$ and $\tau_o(M_{sa})|P(M) \subseteq \tau_o(P(M))$. (Similar inclusions hold for the sequential order topology in view of the comment following Proposition 2.3.)

We shall now exhibit an example that will be used later. It is in fact a construction given in [9, Example 26].

Example 5.1. Let $(\xi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a separable Hilbert space H. For each n let p_n denote the projection of H onto $\overline{\text{span}}\{n^{-1}\xi_1 + \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots\}$. Let $M = B(H)$. Then:

(i) $(p_n)_{n\in\mathbb{N}}$ converges to 0 with respect to τ_s,
(ii) $(p_n)_{n\in\mathbb{N}}$ converges to 0 with respect to $\tau_{os}(M_{sa})$,
(iii) $(p_n)_{n\in\mathbb{N}}$ does not converge to 0 with respect to $\tau_{os}(M_{sa}^1)$,
(iv) $(p_n)_{n\in\mathbb{N}}$ does not converge to 0 with respect to $\tau_{os}(P(M))$.

Proof. (i) and (iv) were proved in [9, Example 26]. (ii) follows from Theorem 4.3 and (2.3). To prove (iii) suppose that $(a_{n_k})_{k\in\mathbb{N}}$ is a decreasing sequence in M_{sa}^1 such that $p_{n_k} \le a_{n_k}$ for every $k \in \mathbb{N}$. Then Lemma 2.8(ii) yields $a_{n_k} \ge \bigvee_{i\ge k} p_{n_i} \ge p$ for every $k \in \mathbb{N}$ where p denotes the projection of H onto the one-dimensional subspace spanned by ξ_1. ∎

We recall that two projections e and f in M are said to be *equivalent* (in symbols $e \sim f$) if there exists $u \in M$ such that $uu^* = e$ and $u^*u = f$. A projection e is said to be *finite* if whenever f is a projection such that $e \sim f$ and $f \le e$ then $e = f$. If e is not finite then it is *infinite*. Moreover, e is said to be *properly infinite* if ze is infinite or 0 for every $z \in \mathscr{Z}(M)$. M is said to be finite, infinite or properly infinite according to the property of the identity projection $\mathbb{1}$. Moreover, there are two orthogonal projections z_f and z_i in $\mathscr{Z}(M)$ such that z_f is finite, z_i is properly infinite and $z_f + z_i = \mathbb{1}$.

We further recall that if M is properly infinite then there is a sequence $(e_n)_{n\in\mathbb{N}}$ of mutually equivalent and pairwise orthogonal projections such that $\bigvee_{n\in\mathbb{N}} e_n = \mathbb{1}$. These projections, together with the partial isometries implementing their equivalence, generate a type I subfactor (i.e. a unital von Neumann subalgebra that is a factor) N of M that is $*$-isomorphic to $B(H)$ for some separable infinite-dimensional Hilbert space H. Observing that $\tau_o(P(M))|P(N) = \tau_o(P(N)) = \tau_{os}(P(N))$, it follows by Example 5.1 that a properly infinite von Neumann algebra M contains a sequence of projections which is σ-strongly null but not $\tau_o(P(M))$-null. This observation is in part a motivation for Theorem 5.3 in which we give a new characterization of finite von Neumann algebras.

We further recall that in the proof of Proposition 2.9 we have seen that when $N = B(H_2)$ then $\tau_o(N^1_{sa})|P(N) = \tau_o(P(N))$, i.e. unlike (2.7) and (2.9), in (2.8) we can have an equality without the algebra being abelian.

LEMMA 5.2. *Let (p_i) be a decreasing sequence of projections in M. Then the sequence $(2^{-i}(\mathbb{1} - p_i) + p_i)$ is decreasing.*

Proof. Indeed,

$$\begin{aligned}2^{-i-1}(\mathbb{1} - p_{i+1}) + p_{i+1} &= 2^{-1}(2^{-i}(\mathbb{1} - p_i)) + p_i - (1 - 2^{-i-1})(p_i - p_{i+1})\\ &\le 2^{-i}(\mathbb{1} - p_i) + p_i. \ \blacksquare\end{aligned}$$

Let us recall that if p and q are projections in M then $p\vee q - p \sim q - p\wedge q$ [26, p. 292, Proposition V.1.6]. Hence, if a state ψ on M is *tracial* (i.e. $\psi(x^*x) = \psi(xx^*)$ for all $x \in M$), then its restriction to $P(M)$ is a *valuation*, i.e. $\psi(p \vee q) + \psi(p \wedge q) = \psi(p) + \psi(q)$ for any $p, q \in P(M)$. Consequently, ψ is *subadditive*, i.e. $\psi(p\vee q) \le \psi(p) + \psi(q)$ for any $p, q \in P(M)$; if moreover ψ is normal, then it is even *σ-subadditive*, i.e. $\psi\big(\bigvee_n p_n\big) \le \sum_n \psi(p_n)$ for every sequence $(p_n)_{n\in\mathbb{N}}$ in $P(M)$. (In [10] it is shown that, conversely, every subadditive probability measure on $P(M)$ arises in this way.)

THEOREM 5.3. *Let M be a σ-finite von Neumann algebra. Then the following statements are equivalent:*

- (i) *M is finite.*
- (ii) *Every sequence $(x_n)_{n\in\mathbb{N}}$ in M^1_{sa} converging σ-strongly to 0 converges to 0 with respect to $\tau_{os}(M^1_{sa})$.*
- (iii) *If $(p_n)_{n\in\mathbb{N}}$ is a sequence in $P(M)$ converging σ-strongly to 0, then there exists a subsequence (p_{n_i}) such that*
$$\limsup_i p_{n_i} = 0.$$
- (iv) $\tau_o(M^1_{sa})|P(M) = \tau_o(P(M))$.

Proof. (i)⇒(ii). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M^1_{sa} converging σ-strongly to 0. We shall first suppose that $x_n \ge 0$ for each n. We need to exhibit a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that order converges to 0. Since M is finite and σ-finite, M admits a faithful, normal, tracial state ψ. Since $\psi(x_n^2) = \rho_\psi(x_n)^2 \to 0$, we can extract a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\psi(x_{n_i}) \leq \sqrt{\psi(x_{n_i}^2)} < 4^{-i}$. For each $i \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, let $e^i(\lambda)$ be the projection in M corresponding to the characteristic function associated with $\mathrm{sp}(x_{n_i}) \cap (-\infty, \lambda]$, i.e. $\{e^i(\lambda)\}_{\lambda\in\mathbb{R}}$ is the spectral resolution of x_{n_i}. Then

$$\begin{aligned} 0 \leq x_{n_i} &\leq 2^{-i}e^i(2^{-i}) + (\mathbb{1} - e^i(2^{-i})) \\ &= 2^{-i}\Big(\bigwedge_{j\geq i} e^j(2^{-j}) + e^i(2^{-i}) - \bigwedge_{j\geq i} e^j(2^{-j})\Big) + (\mathbb{1} - e^i(2^{-i})) \\ &\leq 2^{-i}\bigwedge_{j\geq i} e^j(2^{-j}) + e^i(2^{-i}) - \bigwedge_{j\geq i} e^j(2^{-j}) + \mathbb{1} - e^i(2^{-i}) \\ &= 2^{-i}\bigwedge_{j\geq i} e^j(2^{-j}) + \bigvee_{j\geq i}(\mathbb{1} - e^j(2^{-j})). \end{aligned}$$

Let

$$y_i = 2^{-i}\bigwedge_{j\geq i} e^j(2^{-j}) + \bigvee_{j\geq i}(\mathbb{1} - e^j(2^{-j})).$$

Then $0 \leq x_{n_i} \leq y_i \leq \mathbb{1}$ and by Lemma 5.2 the sequence $(y_i)_{i\in\mathbb{N}}$ is decreasing. Thus, $\bigwedge_{i\in\mathbb{N}} y_i$ exists in M_{sa} and $\bigwedge_{i\in\mathbb{N}} y_i \geq 0$. The normality of ψ entails that $\psi\big(\bigwedge_{i\in\mathbb{N}} y_i\big) = \lim_{i\to\infty}\psi(y_i)$. Since $2^{-j}(\mathbb{1} - e^j(2^{-j})) \leq x_{n_j}$, it follows that $\psi(\mathbb{1} - e^j(2^{-j})) \leq 2^j\psi(x_{n_j}) < 2^{-j}$.

Since ψ is σ-subadditive we can estimate

$$\psi(y_i) \leq 2^{-i} + \sum_{j\geq i}\psi(\mathbb{1} - e^j(2^{-j})) < 2^{-i} + \sum_{j\geq i} 2^{-j} = 3\cdot 2^{-i}.$$

Thus, $\psi(\bigwedge_{i\in\mathbb{N}} y_i) = 0$ and therefore, since ψ is faithful, it follows that $\bigwedge_{i\in\mathbb{N}} y_i = 0$, i.e. $(x_{n_i})_{i\in\mathbb{N}}$ is order convergent to 0.

If not every element x_n is positive, then we can consider the sequence $(|x_n|)_{n\in\mathbb{N}}$ which is again σ-strongly convergent to 0. Then $(|x_n|)_{n\in\mathbb{N}}$ has a subsequence $(|x_{n_i}|)_{i\in\mathbb{N}}$ that order converges to 0, i.e. there is a sequence $(y_i)_{i\in\mathbb{N}}$ in M_+^1 such that $0 \leq |x_{n_i}| \leq y_i$ and $y_i \downarrow 0$. The result then follows from

$$-\mathbb{1} \leq -y_i \leq -|x_{n_i}| \leq x_{n_i} \leq |x_{n_i}| \leq y_i \leq \mathbb{1} \quad \text{(for all } i \in \mathbb{N}).$$

(ii)⇒(iii). If $(p_n)_{n\in\mathbb{N}}$ is a sequence of projections that converges σ-strongly to 0 then $p_n \xrightarrow{\tau_{os}(M_{sa}^1)} 0$ by (ii). Therefore, by Proposition 2.1, $(p_n)_{n\in\mathbb{N}}$ has a subsequence $(p_{n_i})_{i\in\mathbb{N}}$ order converging to 0 in $(M_{sa}^1, \leq)$. Thus there is a sequence $(y_i)_{i\in\mathbb{N}}$ in M_{sa}^1 such that $0 \leq p_{n_i} \leq y_i$ and $y_i \downarrow 0$. From Lemma 2.8 we deduce that $\bigvee_{j\geq i} p_{n_j} \leq y_i$ for every $i \in \mathbb{N}$ and therefore $0 \leq \bigwedge_{i\in\mathbb{N}}\bigvee_{j\geq i} p_{n_j} \leq \bigwedge_{i\in\mathbb{N}} y_i = 0$, i.e. $\limsup_i p_{n_i} = 0$.

(iii)⇒(i). This follows from Example 5.1 and the discussion that follows it.

(iv)⇒(i). If N is a W^*-subalgebra of M then N^1_{sa} and $P(N)$ are $s(M,M_*)$-closed. The hypothesis together with Propositions 2.3 and 2.7 implies that

$$\begin{aligned}\tau_o(N^1_{sa})|P(N) \supseteq \tau_o(M^1_{sa})|P(N) &= \tau_o(P(M))|P(N)\\ &= \tau_o(P(N)) \supseteq \tau_o(N^1_{sa})|P(N).\end{aligned}$$

Hence, in view of the discussion in the paragraph before Lemma 5.2 it is enough to show that $\tau_o(M^1_{sa})|P(M) \neq \tau_o(P(M))$ when $M = B(H)$ for a separable infinite-dimensional Hilbert space H.

Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence in $(\pi/4, \pi/2)$ such that $\theta_n \uparrow \pi/2$, and let $\sigma_n := \sin\theta_n$ and $\gamma_n := \cos\theta_n$. Fix an orthonormal basis $(\xi_n)_{n\in\mathbb{N}}$ of H and define $\eta_n := \sigma_n\xi_1 + \gamma_n\xi_n$. Denote by e_n the projection of H onto $\operatorname{span}\{\xi_n\}$, q_n the projection of H onto $\operatorname{span}\{\eta_n\}$, and f_n the projection of H onto $\overline{\operatorname{span}}\{\xi_n, \xi_{n+1}, \dots\}$.

We show that $\sigma_n^4 e_1 - e_n \le q_n$ for every $n \in \mathbb{N}$. It is enough to consider vectors in the two-dimensional subspace spanned by ξ_1 and ξ_n. Thus we can express e_1, e_n, q_n in matrix form (relative to the vectors ξ_1 and ξ_n). Writing $q_n - \sigma_n^4 e_1 + e_n$ in matrix form:

$$\begin{pmatrix}\sigma_n^2 - \sigma_n^4 & \gamma_n\sigma_n\\ \gamma_n\sigma_n & \gamma_n^2 + 1\end{pmatrix} = \begin{pmatrix}\gamma_n^2\sigma_n^2 & \gamma_n\sigma_n\\ \gamma_n\sigma_n & \gamma_n^2 + 1\end{pmatrix}$$

one sees that $\sigma_n^4 e_1 - e_n \le q_n$. Thus

$$x_n := \sigma_n^4 e_1 - f_n \le q_n + f_{n+1} \le e_1 + f_n =: y_n,$$

the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in M^1_{sa}, $x_n \uparrow e_1$ and $y_n \downarrow e_1$. Hence $p_n := q_n + f_{n+1} \to e_1$ with respect to $\tau_o(M^1_{sa})$. On the other hand, observe that if $(p_{n_i})_{i\in\mathbb{N}}$ is a subsequence of $(p_n)_{n\in\mathbb{N}}$ and $p \in P(M)$ satisfies $p \le p_{n_i}$ for every $i \in \mathbb{N}$, then $p = 0$. In view of Proposition 2.1, this shows that $p_n \nrightarrow e_1$ with respect to $\tau_{os}(P(M))$ $(= \tau_o(P(M))$ by Proposition 2.11).

(i)⇒(iv). As observed before, M admits a faithful, normal, tracial state ψ. Then $\psi|(P(M)$ is a valuation. Thus $d(p,q) := \psi(p\vee q) - \psi(p\wedge q)$ defines by [7, p. 230, Theorem X.1.1] a metric on $P(M)$. We first prove the following estimate: If $x, y \in M_{sa}$ and $p, q \in P(M)$ with $x \le p \le y \le 1$ and $x \le q \le y$, then

$$d(p,q) \le 2\psi(y - x).$$

In fact,

$$\begin{aligned}d(p,q) &= (\psi(p\vee q) - \psi(p)) + (\psi(p) - \psi(p\wedge q))\\ &= (\psi(p\vee q) - \psi(p)) + (\psi(p\vee q) - \psi(q))\\ &\le 2(\psi(y) - \psi(x)) = 2\psi(y - x).\end{aligned}$$

In the last inequality we have used the fact that $p \vee q \le y$ by Lemma 2.8.

In view of (2.8) and Propositions 2.11 and 2.1, for the proof of (iv) it suffices to show that any sequence $(p_n)_{n\in\mathbb{N}}$ in $P(M)$ order converging in

$(M^1_{sa}, \leq)$ to $p \in P(M)$ has a subsequence order converging in $(P(M), \leq)$. Let now $p_n, p \in P(M)$ and $x_n, y_n \in M^1_{sa}$ be such that $x_n \uparrow p$, $y_n \downarrow p$ and $x_n \leq p_n \leq y_n$. Then $y_n - x_n \downarrow 0$ and $d(p_n, p) \leq 2\psi(y_n - x_n) \to 0$. Therefore $(p_n)_{n\in\mathbb{N}}$ converges to p in the metric lattice $(P(M), d)$. It follows from the proof of [7, p. 246, Theorem X.10.16] that $(p_n)_{n\in\mathbb{N}}$ has a subsequence order converging to p in $(P(M), \leq)$. ∎

We remark that Theorem 5.3 does not imply that for finite, σ-finite algebras the restriction of $s(M, M_*)$ to M^1_{sa} (resp. $P(M)$) coincides with the order topology $\tau_o(M^1_{sa})$ (resp. $\tau_o(P(M))$)—see Proposition 2.9. We also note that in the proof of the implications (ii)⇒(iii), (iii)⇒(i) and (iv)⇒(i) the assumption that M is σ-finite is not used.

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